

SOME SMALL ASPHERICAL SPACES

Dedicated to the memory of Hanna Neumann

ELDON DYER and A. T. VASQUEZ

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Let S^n denote the sphere of all points in Euclidean space \mathbf{R}^{n+1} at a distance of 1 from the origin and D^{n+1} the ball of all points in \mathbf{R}^{n+1} at a distance not exceeding 1 from the origin. The space X is said to be *aspherical* if for every $n \geq 2$ and every continuous mapping $f: S^n \rightarrow X$, there exists a continuous mapping $g: D^{n+1} \rightarrow X$ with restriction to the subspace S^n equal to f . Thus, the only homotopy group of X which might be non-zero is the fundamental group $\pi_1(X, *) \cong G$. If X is also a cell-complex, it is called a $K(G, 1)$. If X and Y are $K(G, 1)$'s, then they have the same homotopy type, and consequently

$$H_i(X; M) \cong H_i(Y; M) \text{ and } H^i(X; M) \cong H^i(Y; M)$$

for all coefficients M and integers i . If M is a left G -module, it determines a local coefficient system over X . The groups

$$H_i(X; M) \text{ and } H^i(X; M)$$

are thus invariants of the group G and module M . They are called the homology and cohomology groups of G with coefficients M , written

$$H_i(G; M) \text{ and } H^i(G; M)$$

(These observations, essentially due to H. Hopf, led to the development of homological algebra, which initiated the study of these groups by algebraic means.)

For a given discrete group G , it may be that there is an integer n such that for all left G -modules M and integers $i > n$, $H^i(G; M) = 0$. If so, then G is said to be of finite cohomological dimension and the minimum of all such integers n is defined to be the *cohomological dimension* of G , written cdG .

It might also be that there exists a $K(G, 1)$ which has finite topological dimension. If so, the minimum such dimension is called the *geometric dimension* of G , written $\dim G$.

It is easy to show that $cdG \leq \dim G$. In [3], Eilenberg and Ganea showed that when $3 \leq cdG$, $cdG = \dim G$. In [11] and [13], Stallings and Swan showed if $1 = cdG$, then $cdG = \dim G$. The case $2 = cdG$ is unresolved at this time.

This paper presents explicit cellular constructions of 2-dimensional $K(G, 1)$'s for certain families of groups G . The techniques involve elementary rewriting arguments and covering space constructions. This recovers and casts new light on the Lyndon Identity Theorem. Based on a group described by Higman [4], a 2-dimensional $K(G, 1)$ is given which has curious topological properties. Also, conjecture I' of Papakyriakopoulos [8] is answered affirmatively.

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1. Preliminaries

We present several constructions and lemmas which will be used subsequently.

Let X denote a pathwise connected space and A denote a subspace of X which is simply and pathwise connected. Each path β in X induces an isomorphism

$$\beta_*: \pi_1(X, \beta(1)) \rightarrow \pi_1(X, \beta(0))$$

given by $\beta_*\omega = \beta\omega\beta^{-1}$ for ω a loop at $\beta(1)$. The isomorphism β_* depends only on the homotopy class of β relative to its end points. Since A is both pathwise connected and simply connected, for any two points a_0 and a_1 of A , there is a path α from a_0 to a_1 and any two such paths are homotopic relative to their end points. Thus, there is a well defined isomorphism

$$\pi_1(X, a_0) \cong \pi_1(X, a_1)$$

for any two points a_0 and a_1 of A . We define $\pi_1(X, A)$ to be the subgroup of the group $\prod_{a \in A} \pi_1(X, a)$ defined by $n \in \pi_1(X, A)$ if and only if for every two points a_0 and a_1 of A , $n(a_0) \in \pi_1(X, a_0)$ and $n(a_1) \in \pi_1(X, a_1)$ correspond under the above isomorphism. Clearly, for any $a \in A$ projection of $\pi_1(X, A)$ on the a th-coordinate is an isomorphism $\pi_1(X, A) \cong \pi_1(X, a)$. The advantage of $\pi_1(X, A)$ is that it is determined independently of base points.

LEMMA 1.1: *Let X be a pathwise connected, locally pathwise connected space, A be a pathwise connected, simply connected subspace of X , and H be a group of homeomorphisms of X which acts properly discontinuously on X and each of which transforms A onto itself.*

Then the fundamental group G of the orbit space X/H is isomorphic to the semi-direct product determined by the split extension

$$1 \rightarrow N \rightarrow G \rightleftarrows H \rightarrow 1,$$

where $N = \pi_1(X, A)$ and the action of H on N is the topologically induced action.

PROOF: The action of H on $\pi_1(X, A)$ is defined by $h(n)(a) = \pi_1(h)(n(a'))$, where $h(a') = a$ and $\pi_1(h): \pi_1(X, a') \rightarrow \pi_1(X, a)$, for $h \in H$ and $n \in \pi_1(X, A)$.

Let $B = X/H$ be the space of orbits and $p: X \rightarrow B$ be the quotient map. It is well-known that (X, p, B) is a regular covering space with H being the group of covering transformations. Thus, for any $x_0 \in X$, $b_0 = p(x_0)$, there is an exact sequence

$$1 \rightarrow \pi_1(X, x_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{p} H \rightarrow 1.$$

Let $x_0 = a_0 \in A$. We shall show this sequence splits. For $h \in H$ and α a path in A from a_0 to $h(a_0)$, set $s(h) = [p\alpha] \in \pi_1(B, b_0)$. This is well-defined, it is a homomorphism, and $\rho s = 1_H$.

Thus, $\pi_1(B, b_0)$ is the semi-direct product of $\pi_1(X, a_0)$ and H with H acting on $\pi_1(X, a_0)$ defined by

$$p_*(h(\gamma)) = s(h) p_*(\gamma) s(h)^{-1} \in \pi_1(B, b_0).$$

Identifying $\pi_1(X, a_0)$ with $\pi_1(X, A)$ as above, we have left to show only that this action coincides with the induced action of H on $\pi_1(X, A)$. Let $n \in \pi_1(X, A)$ and $h \in H$; let α be a path in A from a_0 to $h(a_0)$ and β be a loop at a_0 in the class of $n(a_0)$. We must show that

$$s(h) p_*(n(a_0)) s(h)^{-1} = p_* h(n)(a_0).$$

The loop $p\alpha \cdot p\beta \cdot p\alpha^{-1}$ is in the homotopy class of $s(h) p_*(n(a_0)) s(h)^{-1}$ and $\alpha \cdot h\beta \cdot \alpha^{-1}$ is the lift of this loop starting at a_0 . Thus, $p_*(\alpha \cdot h\beta \cdot \alpha^{-1}) = s(h) p_*(n(a_0)) s(h)^{-1}$. We need show only that $h(n)(a_0) = [\alpha \cdot h\beta \cdot \alpha^{-1}] \in \pi_1(X, a_0)$. Let n' be the element of $\pi_1(X, A)$ with $n'(a_0) = [\alpha \cdot h\beta \cdot \alpha^{-1}]$. Since α lies in A , $n'(a_1) = [h\beta]$, where $a_1 = h(a_0)$. But $[h\beta] = \pi_1(h)[\beta] = \pi_1(h)n(a_0) = h(n)(a_1)$. Hence, $n'(a_1) = h(n)(a_1)$; consequently, $n' = h(n)$ and, in particular, they are equal at a_0 .

Let $P: \{x_1, \dots, x_n; R_1, \dots, R_m\}$ be a presentation of the discrete group G ; i.e., R_1, \dots, R_m are words in the x_1, \dots, x_n and G is the quotient of the free group $F(x_1, \dots, x_n)$ on x_1, \dots, x_n by the normal closure in F of the elements R_1, \dots, R_m . The wedge $\bigvee_{i=1}^n S_i^1$ of n 1-spheres has fundamental group

$$\pi_1\left(\bigvee_{i=1}^n S_i^1, *\right) \cong F(x_1, \dots, x_n),$$

with the generator x_i corresponding to a map $S^1 \rightarrow S_i^1$ of degree 1. For each j , $1 \leq j \leq m$, let D_j^2 be a 2-ball, B_j^1 be its boundary (which is a 1-sphere) and

$$f_j: B_j^1 \rightarrow \bigvee_{i=1}^n S_i^1$$

be a map such that if t_j is a generator of $\pi_1(B_j^1) \cong \mathbb{Z}$, then

$$f_{j*}: \pi_1(B_j^1) \rightarrow \pi_1\left(\bigvee_{i=1}^n S_i^1\right)$$

maps t_j into the word R_j . Form the quotient space $C(P)$ from the disjoint union

$$\bigvee_{i=1}^n S_i^1 \cup \bigcup_{j=1}^m D_j^2$$

by identifying $y \in B_j^1$ with $f_j(y)$ for every point in $\bigcup_{j=1}^m B_j^1$. (This is sometimes described as attaching the cells D_j^2 by the maps f_j .) This quotient space is called the *canonical cell-complex of the presentation P*, written $C(P)$; it is a 2-dimensional cell-complex and an application of the Siefert-van Kampen Theorem shows

$$\pi_1(C(P), *) \cong G.$$

LEMMA 1.2: *If*

$$P: \{x_1, \dots, x_n; R_1, \dots, R_m\}$$

and

$$P': \{x_1, \dots, x_n; \alpha_1 R_1^{\varepsilon_1} \alpha_1^{-1}, \dots, \alpha_m R_m^{\varepsilon_m} \alpha_m^{-1}\}, \varepsilon_i = \pm 1,$$

are presentations, then $C(P)$ and $C(P')$ have the same unpointed homotopy type; one is aspherical if and only if the other is.

This is true basically because the attaching maps are freely homotopic.

We recall a theorem of J. H. C. Whitehead [14]:

THEOREM W: *Suppose X, Y and $X \cap Y$ are aspherical cell-complexes and both $\pi_1(X \cap Y, *) \rightarrow \pi_1(X, *)$ and $\pi_1(X \cap Y, *) \rightarrow \pi_1(Y, *)$ are injections. Then $X \cup Y$ is aspherical.*

This theorem has a number of useful immediate consequences. Let G, H and A denote the fundamental groups of X, Y and $X \cap Y$, respectively. Then by the Seifert-van Kampen Theorem, the fundamental group of $X \cup Y$ is the free product with amalgamation $G *_A H$. Letting M be a left $G *_A H$ -module viewed as local coefficients over $X \cup Y$ and applying the Mayer-Vietoris Sequences for $(X, Y; A)$, one obtains the Mayer-Vietoris Sequences for homology and cohomology of groups of free products with amalgamation: (This argument is essentially that of [10]; the result is indicated in [6] and a homological algebra proof appears in [13].)

It is immediate from these sequences that

$$\max \{cdG, cdH\} \leq cd G *_A H \leq 1 + \max \{cdG, cdH\},$$

with equality on the left applying if either $cdA < cdG$ or $cdA < cdH$. Using mapping cylinders for cellular maps $K(A, 1) \rightarrow K(G, 1)$ and $K(A, 1) \rightarrow K(H, 1)$ which induce the appropriate injections of fundamental groups, we can conclude from Whitehead's Theorem that

$$\max \{ \dim G, \dim H \} \leq \dim G *_A H \leq 1 + \max \{ \dim G, \dim H \},$$

with equality on the left applying if either $\dim A < \dim G$ or $\dim A < \dim H$.

In particular, we have the generalized free product (free product amalgamated over a free group) of 2-dimensional groups is a 2-dimensional group. Of course, the same is true for cohomology dimension.

Using the homological algebra definition of cohomology of groups, since a projective resolution over a group remains projective over a subgroup, we have

$$\text{if } H \text{ is a subgroup of } G, \text{cd } H \leq \text{cd } G.$$

Also, if X is an n -dimensional $K(G, 1)$, the covering space of X with fundamental group H is an n -dimensional aspherical cell-complex. Thus,

$$\text{if } H \text{ is a subgroup of } G, \dim H \leq \dim G.$$

In light of the Eilenberg-Ganea Theorem, this parallelism of conclusions is of interest only at 2.

2. One relator presentations

Let $P: \{x_1, \dots, x_n; R\}$ be a one relator presentation for the group G . We assume R is fully reduced, $R \neq 1$; if for some word W in the x_i 's and integer t , $R = w^t$, let s be the maximal such integer and Q the corresponding word, $R = Q^s$. If $s > 1$, R is said to be a proper power. Clearly, Q is not a proper power.

For s a positive integer, let Z_s denote the cyclic group of order s , Z/sZ . There is a cell-complex $K(Z_s, 1)$ with 1-sphere S^1 as 1-skeleton; for $s > 1$ $K(Z_s, 1)$ can be formed with exactly one cell in each dimension (using infinite dimensional lens spaces {p. 67 of [12]}), and its homology and cohomology have been well understood for several decades. For $s = 1$, $K(0, 1)$ can be formed by adding the 2-cell D^2 to S^1 along its boundary. In any case the inclusion $S^1 \rightarrow K(Z_s, 1)$ induces an epimorphism of fundamental groups $Z \rightarrow Z_s$. As in Section I, let $f: S^1 \rightarrow \bigvee_{i=1}^n S_i^1$ induce the homomorphism on fundamental groups sending a generator of $\pi_1(S^1)$ to the word Q in $\pi_1(\bigvee_{i=1}^n S_i^1)$. From the disjoint union $K(Z_s, 1) \cup \bigvee_{i=1}^n S_i^1$ form the quotient space $D(P)$ by identifying each $x \in S^1 \subset K(Z_s, 1)$ with $f(x) \in \bigvee_{i=1}^n S_i^1$. For $s = 1$ (that is, R not a proper power, $R = Q$), $D(P)$ is exactly the cell-complex $C(P)$ described in the first section with n 1-cells and 1 2-cell. For $s > 1$, $D(P)$ is an infinite dimensional cell-complex, with n 1-cells and 1 i -cell for each $i \geq 2$. By the Siefert-van Kampen Theorem, $\pi_1(D(P))$ is the pushout of the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & Z_s \\ \downarrow \beta & & \\ F(x_1, \dots, x_n) & & \end{array}$$

where α is an epimorphism and $\beta(t) = Q$. Thus, the pushout is the quotient of $F(x_1, \dots, x_n)$ by the normal closure of the relation $Q^g = 1$, that is, G .

THEOREM 2.1: *Let P be a one relator presentation of the group G . Then the cell-complex $D(P)$ is a $K(G, 1)$. In particular, if the relator is not a proper power, then the geometric dimension of G is less than or equal to two.*

Before giving our proof of this theorem, we observe several consequences of it and relations between it and previously known results.

If $P: a_1, b_1, \dots, a_g, b_g; \prod_{i=1}^g [a_i, b_i]$, then the relator is not a proper power and $C(P)$ is a compact, orientable surface of genus g . Theorem 2.1 asserts the asphericity of the surface. This familiar fact is thus proved in an unfamiliar manner. Similarly, this argument applies to the non-orientable surfaces except for the projective plane — for it the relator is a proper power.

COROLLARY 2.2: *If in the presentation P of G , R is not a proper power, and N denotes the normal closure in $F(x_1, \dots, x_n)$ of R , then there is an isomorphism $\psi: \mathbf{Z}(G) \rightarrow N/[N, N]$ of $\mathbf{Z}(G)$ -modules with $\psi(1)$ the coset containing R .*

(This is a special case of the Lyndon Identity Theorem [6].)

PROOF: Let $X = C(P)$ and X^1 be its 1-skeleton ($= \bigvee_{i=1}^n S_i^1$); let $\tilde{X} \rightarrow X$ be the universal covering of X and $p^1: Y \rightarrow X^1$ be the restriction of p to the 1-skeleta. We have the diagram

$$\begin{array}{ccc} Y & \longrightarrow & \tilde{X} \\ \downarrow p^1 & & \downarrow p \\ X^1 & \xrightarrow{i} & X \end{array}$$

with p^1 and p both covering maps. Any two points of Y are end points of a path w in \tilde{X} ; pw is homotopic relative to its end points to a path \hat{w} in X^1 . Covering this homotopy in \tilde{X} yields a path in Y with the same end points as w ; thus, Y is patwise connected. A loop in X^1 which is null-homotopic in X can be lifted to a loop in Y . Thus,

$$p^1_* \pi_1(Y) = \text{Kernel}(i_*: \pi_1(X^1) = F(x_1, \dots, x_n) \rightarrow \pi_1(X) = F/N) = N.$$

Since p^1_* is an injection, $\pi_1(Y) \cong N$. By the Hurewicz Theorem $H_1(Y; \mathbf{Z}) \cong N/[N, N]$. Since X is aspherical, \tilde{X} is both simply connected and aspherical. Thus, \tilde{X} is contractible. This implies it is acyclic. Thus,

$$H_2(\tilde{X}, Y) \xrightarrow[\cong]{\partial} H_1(Y).$$

But Y is the 1-skeleton of \tilde{X} ; and so, $H_2(\tilde{X}, Y)$ is the free abelian group on the 2-cells of \tilde{X} . These are permuted simply transitively by G , the group of deck

transformations. Hence, $H_2(\tilde{X}, Y) \cong \mathbf{Z}(G)$ as G -modules; and $\mathbf{Z}(G) \cong N/[N, N]$ as G -modules.

It remains to describe the image of a generator. Let e^2 be the 2-cell containing the base point of Y . $\partial(e^2)$ is the homology class represented by the loop on the boundary. But p_*^1 of this loop is the class of the attaching map R .

NOTE: In this case the argument can be reversed; Lyndon's Identity Theorem implies that ∂ is an isomorphism and hence that $H_2(\tilde{X}) = 0$. Since \tilde{X} is simply connected and 2-dimensional, this implies it is contractible and $C(P)$ is aspherical. A similar argument was made by Cockcroft in [2].

COROLLARY 2.3: *The group G with a one relator presentation $P: \{x_1, \dots, x_n; R\}$ is torsion free if R is not a proper power. If $R = Q^s$, $s > 1$ and maximal, the order of Q in G is s .*

This is a special case of results of Karrass, Magnus and Solitar [5].

PROOF: If R is not a proper power, by Theorem 2.1 $C(P)$ is a $K(G, 1)$; and so, $cd(G) \leq 2$. If H is a non-trivial 1-generator subgroup of G , since $cd(H) \leq cd(G)$, H is not finite; i.e., G is torsion free.

If $R = Q^s$, $s > 1$ and maximal, since $D(P)$ has no cells out of $K(\mathbf{Z}_s, 1)$ in dimensions greater than two and the boundary of the 3-cell of $K(\mathbf{Z}_s, 1)$ lies in the 2-cell of $K(\mathbf{Z}_s, 1)$ it follows from standard computations of $H_*(\mathbf{Z}_s; \mathbf{Z})$, that $H_3(D(P); \mathbf{Z}) \cong \mathbf{Z}_s$. For a presentation $P': \{x_1, \dots, x_n; Q^t\}$, we have that $H_3(D(P'); \mathbf{Z}) \cong \mathbf{Z}_t$. If Q^t is trivial in G , and t is minimal, it lies in the normal closure of R ; by minimality, R lies in the normal closure of Q^t . Thus, P and P' present isomorphic groups, which have isomorphic homology groups. But this implies $t = s$.

There are some simple relations among the homology (and cohomology) of one relator groups immediately derivable from the geometry in Theorem 2.1.

Suppose $P: \{x_1, \dots, x_n; R\}$ is a one relator presentation of G and $R = Q^s$, $s > 1$ and maximal. Then $P': \{x_1, \dots, x_n; Q\}$ is a one relator presentation of a group G' . Let $\theta: G \rightarrow G'$ be the surjection induced by the identity on generators.

COROLLARY 2.4: *For M a left G' -module, there is an exact sequence*

$$\begin{aligned} 0 \rightarrow H_2(\mathbf{Z}_s; M) \rightarrow H_2(G; M) \xrightarrow{\theta_*} H_2(G'; M) \\ \rightarrow H_1(\mathbf{Z}_s; M) \rightarrow H_1(G; M) \xrightarrow{\theta_*} H_1(G'; M) \\ \rightarrow H_0(\mathbf{Z}_s; M) \rightarrow H_0(G; M) \xrightarrow{\theta_*} H_0(G'; M) \rightarrow 0. \end{aligned}$$

If S is the kernel of θ , then for $q \geq 3$ and any left G -module L , there is an isomorphism

$$H_q(\mathbf{Z}_s; L) \otimes \mathbf{Z}(G') \xrightarrow{\cong} H_q(S; L)$$

of left G' -modules.

PROOF: We modify the construction of $D(P)$ slightly. Instead of identifying the S^1 of $K(\mathbf{Z}_s, 1)$ with its image under the map into $\bigvee_{i=1}^n S_i^1$ spelling Q , first attach to S^1 the cylinder $S^1 \times I$ along $S^1 \times (0)$ and then identify $S^1 \times 1$ into $\bigvee_{i=1}^n S_i^1$ to spell Q . Call this space $D'(P)$; it obviously has the same homotopy type as $D(P)$. The advantage of $D'(P)$ is that $K(\mathbf{Z}_s, 1)$ is a subspace of $(D'P)$ (in fact it is imbedded as a cofibration). Let C denote the cone of $K(\mathbf{Z}_s, 1)$. Then C is contained in $C \cup D'(P)$ as a cofibration and is contractible in itself to a point. Thus, $C \cup D'(P)$ and $C \cup D'(P)/C$ have the same homotopy type. But $C \cup D'(P)/C$ is homeomorphic to $D(P') = C(P')$. Hence, we have the inclusions

$$K(\mathbf{Z}_s, 1) < D'(P) < C \cup D'(P),$$

where $D'(P)$ is a $K(G, 1)$ and $C \cup D'(P)$ is a $K(G', 1)$. Since

$$\begin{aligned} H_i(D'(P), K(\mathbf{Z}_s, 1); M) & \\ \cong H_i(C \cup D'(P), C; M) & \quad (\text{by excision}) \\ \cong H_i(C(P'); M) & \\ \cong H_i(G'; M), & \end{aligned}$$

the exact sequence of the corollary is just the exact sequence of the pair $(D'(P), K(\mathbf{Z}_s, 1))$. It is trivial on the left as indicated since G' has geometric dimension ≤ 2 .

Let us abbreviate $K(\mathbf{Z}_s, 1)$ by K and $D'(P)$ by X . Let $X_1 \rightarrow X$ be the regular covering of X (a $K(G, 1)$) corresponding to the subgroup S of G . Since X is aspherical, so is X_1 ; hence, X_1 is a $K(S, 1)$. The group of deck transformations of the covering $X_1 \rightarrow X$ is the quotient group G' . This gives $H_q(X_1; L)$ the structure of a G' -module.

Since $\pi_1(K) \rightarrow \pi_1(X)$ factors through $\pi_1(X_1) \rightarrow \pi_1(X)$, the inclusion $K \rightarrow X$ factors through $X_1 \rightarrow X$. The inverse \hat{K} of K in X_1 is then the disjoint union of copies of K , which are simply transitively permuted by the group G' of deck transformations. Hence,

$$H_q(\hat{K}; L) \cong H_q(K; L) \otimes \mathbf{Z}(G')$$

for all q . However, $\dim X - K = 2$; and so, $\dim X_1 - \hat{K} = 2$. Thus,

$$H_q(\hat{K}; L) \cong H_q(X_1; L) \text{ for } q \geq 3.$$

Combining these two isomorphisms and the information that X_1 is a $K(S, 1)$ gives the conclusion

$$H_q(\mathbf{Z}_s; L) \otimes \mathbf{Z}(G') \cong H_q(S; L) \text{ for } q \geq 3.$$

(Using results of Cohen and Lyndon [1], it can be shown that $S \cong *_{n' \in G'} (\mathbf{Z}_{n'})_{g'}$ with the obvious G' action.)

We conclude this list of implications of Theorem 2.1 with an alternate proof of the Lyndon Identity Theorem

$P: \{x_1, \dots, x_n; R\}$, $R = Q^s$, $N =$ normal closure of R in $F = F(x_1, \dots, x_n)$ is a presentation of G as before. The image of Q in G generates a cyclic subgroup H of G of order s .

COROLLARY 2.5. (The Lyndon Identity Theorem): *There is an isomorphism*

$$N/[N, N] \cong \mathbf{Z}[G/H]$$

of G -modules, where $\mathbf{Z}[G/H]$ is the free abelian group on the set of cosets G/H of H in G with obvious G -action. It can be arranged so that the coset H corresponds to the coset of R .

PROOF: Let $p: X \rightarrow D(P)$ be the universal covering space of $D(P)$. X has a cell-complex structure induced from that of $D(P)$: the cells of X lying over a single cell of $D(P)$ are permuted simply transitively among themselves by the group G of deck transformations. Since $H_k(X^{(k)}, X^{(k-1)})$ is the free abelian group on generators corresponding to the k -cells of X (thus, the cells of $X^{(k)} - X^{(k-1)}$), it is thus a free $\mathbf{Z}(G)$ -module with generators corresponding to the cells of $D(P)$.

Since X is contractible, its homology is trivial; also, it is easy to verify that

$$H_i(X) = H_i(X^{(k)}) \text{ for } i < k.$$

Thus, there are exact sequences

$$0 \rightarrow H_2(X^{(2)}) \rightarrow H_2(X^{(2)}, X^{(1)}) \rightarrow H_1(X^{(1)}) \rightarrow 0$$

$$0 \rightarrow H_3(X^{(3)}) \rightarrow H_3(X^{(3)}, X^{(2)}) \rightarrow H_2(X^{(2)}) \rightarrow 0,$$

etc. Splicing these together gives an exact sequence

$$\dots \rightarrow H_4(X^{(4)}, X^{(3)}) \rightarrow H_3(X^{(3)}, X^{(2)}) \xrightarrow{\partial_3} H_2(X^{(2)}, X^{(1)}) \xrightarrow{\partial_2} H_1(X^{(1)}) \rightarrow 0$$

of $\mathbf{Z}(G)$ -modules.

We can assume $s > 1$. (Corollary 2.2 established the conclusion for $s = 1$.) We also assume the $K(\mathbf{Z}_s, 1)$ used to construct $D(P)$ has only one cell in each dimension and that the attaching map

$$f: S^1 \rightarrow \bigvee_{i=1}^n S_i^1$$

is cellular. As in Corollary 2.2, $H_1(X^{(1)}) \cong N/[N, N]$. Thus cokernel $\partial_3 \cong N/[N, N]$. As in Corollary 2.2, the 2-cell e^2 containing the base point of X can be taken as the generator for $H_2(X^{(2)}, X^{(1)})$ and $\partial_2 e_2$ is the coset of R in $N/[N, N]$.

It remains only to describe ∂_3 . Let $K = K(\mathbf{Z}_s, 1)$ and $q: \tilde{K} \rightarrow K$ be the universal covering map. There is a map $\tilde{f}: \tilde{K} \rightarrow X$ so that the diagram

$$\begin{array}{ccc}
 \tilde{K} & \xrightarrow{i} & X \\
 \downarrow q & & \downarrow p \\
 K & \xrightarrow{i} & D(P)
 \end{array}$$

commutes. Let e'_2 and e'_3 be the 2- and 3-cells of \tilde{K} containing the base point. It is a standard computation (p. 67 of [12]) that for

$$\partial_3 : H_3(\tilde{K}^3, \tilde{K}^2) \rightarrow H_2(\tilde{K}^2, \tilde{K}^1),$$

$\partial_3 e'_3 = (1 - \mu)e'_2$, where μ is the generator of $\pi_1(K) = H$. Thus,

$$\partial_3 i e'_3 = (1 - \mu) i e'_2.$$

As K contains all of the cells of $D(P)$ of dimensions 2 and greater, $i e'_3 = e_3$ and $i e'_2 = e_2$. Thus, the $\mathbf{Z}(G)$ -homomorphism ∂_3 involves only the $\mathbf{Z}(H)$ -module structure. Since $\mathbf{Z}(G)$ is a direct sum of copies of $\mathbf{Z}(H)$, one for each element of G/H , it is clear that $\text{cokernel } \partial_3 \cong \mathbf{Z}[G/H]$.

3. Proof of Theorem 2.1.

The proof of Theorem 2.1 is inductive on k , the length of the word Q . The method we use to analyze a one-relator presentation goes back to Magnus [7].

Let us first briefly restate the construction. A group G is given by a one-relator presentation $P: \{x_1, \dots, x_n; R\}$ with $R = Q^s$, s maximal, R being fully reduced. To the wedge of n circles with base point is attached the circle with base point of a $K(\mathbf{Z}_s, 1)$ by a base point preserving map which “spells” the word Q . The resultant space is denoted $D(P)$. By the Seifert-van Kampen Theorem we know the fundamental group of G to have the presentation P , and we need to verify only that $D(P)$ is aspherical to know that it is a $K(G, 1)$.

The case $s = 1$, that is the case in which R is not a proper power, can be read independently of the more general case. In this case, $D(P) = C(P)$ is two-dimensional, and certain complications, particularly in Sublemma 3.3.1, do not arise.

Being aspherical is an invariant of free homotopy type (homotopies not preserving the base point can be used) and altering the attaching map by a free homotopy does not change the free homotopy type of the space constructed. By using as attaching map a pointed map which “spells” a conjugate of Q , we are only altering by a free homotopy. Thus, to check $D(P)$ is aspherical, we can use $D(P')$ where

$$R' = \alpha R \alpha^{-1} = (\alpha Q \alpha^{-1})^s.$$

In particular, we can assume R and Q are cyclically reduced.

LEMMA 3.1: *If Q has length 1, $D(P)$ is aspherical.*

PROOF: Permuting the x_i change $D(P)$ only by a homeomorphism. We will assume $Q = x_1^{\pm 1}$ and $R = x_1^{\pm s}$. The attaching map of the S^1 in $K(\mathbf{Z}_s, 1)$ to

S_1^1 can be taken to be a homeomorphism (it has degree ± 1 as needed), and $D(P) = K(\mathbf{Z}, 1) \vee \bigvee_{i=2}^n S_i^1$, which by Theorem *W* is aspherical.

We shall use the following notations: for $Q \in F(x_1, \dots, x_n)$ and a one of the x_i , $\#_Q(a)$ is the total number of occurrences of a in Q ; i.e., the sum of the absolute values of the exponents of a in Q , assuming Q to be fully reduced; $\sigma_Q(a)$ is the exponent sum for a in Q ; i.e., the sum of the values of the exponents of a in Q ; the length of Q , $l(Q) = \sum_{i=1}^n \#_Q(x_i)$.

We assume that $k = l(Q) \geq 2$ and that $D(P')$ is aspherical for any presentation P' for which $l(Q') < k$. This is our inductive hypothesis.

LEMMA 3.2: *Let $R = Q^s$, s maximal, be a cyclically reduced word in $F(x_1, \dots, x_n)$. Suppose there is an $a = x_I$ for which*

$$0 \neq \#_Q(a), 0 = \sigma_Q(a), \text{ and}$$

$$\sum_{i \neq I} \#_Q(x_i) < k.$$

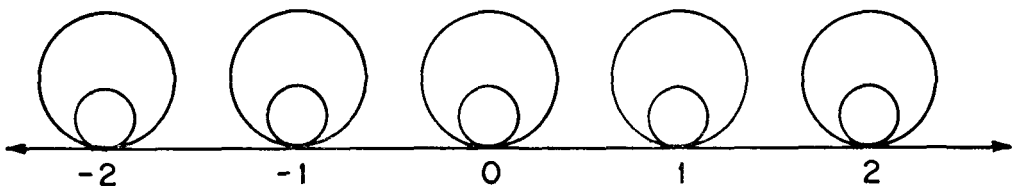
Then the inductive hypothesis implies that $D(P)$ is aspherical.

COROLLARY OF LEMMA 3.2: *Let $R = Q^s$, s maximal, be a cyclically reduced word in $F(x_1, \dots, x_n)$. If Q has length k and contains a generator with exponent sum 0, then the inductive hypothesis implies $D(P)$ is aspherical.*

LEMMA 3.3: *Let $R = Q^s$, s maximal, be a cyclically reduced word in $F(x_1, \dots, x_n)$. Suppose Q has length k and contains no generator with exponent sum 0. Then the inductive hypothesis implies $D(P)$ is aspherical.*

PROOF OF LEMMA 3.2: Since $\sigma_Q(a) = 0$, the function $F(x_1, \dots, x_n) \rightarrow \mathbf{Z}$ defined by $W \rightarrow \sigma_W(a)$ induces a homomorphism of G to the integers. Let N be the kernel of this homomorphism. This proof will construct a regular covering space X of $D(P)$ corresponding to the subgroup N and prove X is aspherical. The general structure of Lemma 1.1 will be reflected in these particular constructions. The construction of X is motivated by the Reidemeister-Schreier rewriting process for N .

Let $\mathcal{B} = \{x_i \mid 1 \leq i \leq n, i \neq I\}$ and $W(\mathcal{B})$ be the wedge of $n - 1$ circles labelled by the x_i in \mathcal{B} . For each integer i let $W(\mathcal{B})_i$ be a copy of the space $W(\mathcal{B})$. Let T be the space obtained by attaching to the real line at each integer point i the space $W(\mathcal{B})_i$ at its base point. For example, if \mathcal{B} had two elements, part of T would look like



Let $h: T \rightarrow T$ be the homeomorphism that carries a point of $W(\mathcal{B})_i$ into the corresponding point of $W(\mathcal{B})_{i+1}$ and takes a point x of the real line to the point $x + 1$. Clearly, h generates an infinite cyclic group H of self-homeomorphisms of T . The orbit space T/H contains $W(\mathcal{B})$ (the image of any $W(\mathcal{B})_i$) and one more S^1 (the image of any interval $[k, k + 1]$ of the real line for $k \in \mathbf{Z}$). Thus, T/H can be identified with

$$\bigvee_{i=1}^n S_i^1 \equiv W(\mathcal{A}),$$

where \mathcal{A} is the set $\{x_1, \dots, x_n\}$, and

$$p: T \rightarrow T/H = W(\mathcal{A})$$

is a regular covering space.

It is easy to see that $\pi_1(T, (0))$ is free. A convenient set of generators are the $x_{\beta,i}$ described as follows: for $\beta \in \mathcal{B}$, $i \in \mathbf{Z}$, $x_{\beta,i}$ is the sum of the paths $[0, i]$, $S_\beta^1 \in W(\mathcal{B})_i$, and $[i, 0]$. This is a loop at 0. Clearly,

$$p_*(x_{\beta,i}) = a^i \beta a^{-i} \in F(x_1, \dots, x_n) = \pi_1(W(\mathcal{A}), *).$$

Let S^1 be the 1-sphere of $K(\mathbf{Z}_s, 1)$ and $f: S^1 \rightarrow W(\mathcal{A})$ spell Q , i.e., $f_*(\mu) = Q$. Since $\sigma_Q(a) = 0$, $f_*(\mu) \in \text{Im}(p_*)$. Thus, there exists $f_0: S^1 \rightarrow T$ (taking base point to $0 \in T$) such that $pf_0 = f$.

Let $Q' = f_{0*}(\mu) \in \pi_1(T, (0))$. Recall that $\pi_1(T, (0))$ is free on the $x_{\beta,i}$. We claim

- (i) Q' is not a proper power,
- (ii) length $Q' < k$, and
- (iii) Q' is cyclically reduced.

The first is easy since $p_*(Q') = Q$ and Q is not a proper power. Let $N = \text{length } Q'$ and write

$$Q' = x_{\beta_1, i_1}^{\varepsilon_1} \cdot x_{\beta_2, i_2}^{\varepsilon_2} \cdot \dots \cdot x_{\beta_N, i_N}^{\varepsilon_N}$$

$\varepsilon_i = \pm 1$. Then

$$p_*Q' = Q = (a^{i_1} \beta_1^{\varepsilon_1} a^{-i_1}) (a^{i_2} \beta_2^{\varepsilon_2} a^{-i_2}) \cdot \dots \cdot (a^{i_N} \beta_N^{\varepsilon_N} a^{-i_N}).$$

Since $N = \text{length } Q'$, if $(\beta_j, i_j) = (\beta_{j+1}, i_{j+1})$, then $\varepsilon_j = \varepsilon_{j+1}$. Hence, the only cancellation taking place in this expression for Q is cancellation among the a 's. Thus, $N = \sum_{\beta \in \mathcal{B}} \#_Q \beta < k$.

If Q' were not cyclically reduced, then $(\beta_1, i_1) = (\beta_N, i_N)$ and $\varepsilon_1 + \varepsilon_N = 0$. Thus, Q starts with $a^{i_1} \beta_1^{\varepsilon_1}$ and ends with $(a^{i_1} \beta_1^{\varepsilon_1})^{-1}$; i.e., it is not cyclically reduced.

Let $u = \min \{i_j\}$ and $v = \max \{i_j\}$ for i_j appearing in Q' . Since we can alter f_0 by a pointed homotopy, we can assume

$$f_0(S^1) \subset \mathbf{R} \cup \bigcup_{i \in [u, v]} W(\mathcal{B})_i.$$

For each integer i let K_i be a copy of $K(\mathbb{Z}_3, 1)$ and C_i be the 1-sphere of K_i . Let X be the quotient space of the disjoint union of the K_i 's and T by identifying each point $P \in C_i$ with $h^i(f_0(P)) \in T$. It is clear that h extends to a self-homeomorphism of X , which we also call h (it takes a point of K_i to the corresponding point of K_{i+1}). Again, h generates an infinite cyclic group H of self-homeomorphisms of X . It is clear that the orbit space X/H is

$$T/H \cup_{pf_0} K = \bigvee_{i=1}^m S_i^1 \cup_f K = D(P).$$

Also, the action of H on X is properly discontinuous; and so,

$$X \rightarrow X/H = D(P)$$

is a covering map. Thus, it suffices to show that X is aspherical.

Let X_N be the subcomplex of X consisting of the real line, the $W(\mathcal{B})_i$ for $u + N \leq i \leq v + N$ and K_N . Clearly, $h(X_N) = X_{N+1}$. Our inductive hypothesis will show X_N is aspherical, and we will use this to conclude that X is aspherical.

SUBLEMMA 3.2.1: a) X_N is aspherical,

b) the elements $x_{\beta, i}, \beta \in \mathcal{B}, i \in [u + N + 1, v + N]$, are distinct and generate a free subgroup of $\pi_1(X_N, (0))$, and

c) the elements $x_{\beta, i}, \beta \in \mathcal{B}, i \in [u + N, v + N - 1]$, are distinct and generate a free subgroup of $\pi_1(X_N, (0))$.

PROOF: It suffices to consider the case $N = 0$. Since the real line is a contractible subcomplex of X_0 , collapsing it to a point does not change homotopy type. Call the resultant space Y_0 and $q: X_0 \rightarrow Y_0$ the collapsing map. Let W be the subcomplex of Y_0 which is the image of all $W(\mathcal{B})_i, u \leq i \leq v$. This is a wedge of circles, indexed by $\mathcal{B} \times (u, u + 1, \dots, v)$. Y_0 is of the form $W \cup_{qf_0} K$, for $qf_0: C \rightarrow K$. Let $Q' = (qf_0)_*(\mu) \in \pi_1(W)$. It is clear that Y_0 is $D(P')$ for the presentation $P': \{\mathcal{B} \times (u, u + 1, \dots, v); Q'\}$, where Q' is as above. The inductive hypothesis then implies that Y_0 , and hence X_0 , is aspherical.

The definition of u is such that b) is an immediate consequence of the Magnus Freeness Theorem. Similarly, c) follows from the definition of v .

SUBLEMMA 3.2.2: For $X_{0,N} = X_0 \cup X_1 \cup \dots \cup X_N, X_{0,N}$ is aspherical and $X_N \rightarrow X_{0,N}$ induces an injection of fundamental groups.

PROOF: The assertion is true for $N = 0$ by Sublemma 3.2.1. Assume it is true for N . Then $X_{0,N+1} = X_{0,N} \cup X_{N+1}$ and

$$X_{0,N} \cap X_{N+1} = (\text{real line}) \cup W(\mathcal{B})_{u+N+1} \cup \dots \cup W(\mathcal{B})_{v+N}.$$

By the Siefert-van Kampen Theorem, the following diagram is a pushout in the category of groups

$$\begin{array}{ccc}
 \pi_1(X_{0,N} \cap X_{N+1}) & \xrightarrow{i_1} & \pi_1(X_{0,N}) \\
 \downarrow i_2 & & \downarrow j_2 \\
 \pi_1(X_{N+1}) & \xrightarrow{j_1} & \pi_1(X_{0,N+1}).
 \end{array}$$

Sublemma 3.2.1.c) shows that i_2 is an injection. The map i_1 factors through $\pi_1(X_N)$ and is thus the composition of two injections, by sublemma 3.2.1.b) and induction. Hence, $\pi_1(X_{0,N+1})$ is a free product with amalgamation, and, in particular, j_1 is an injection. The conclusion that $X_{0,N+1}$ is aspherical then follows from Theorem W .

We now conclude the proof of Lemma 3.2 by showing X to be aspherical. Let $F: S^q \rightarrow X$ be a continuous map. Since S^q is compact, there is an integer N such that $F(S^q) \subset X_{-N} \cup X_{-N+1} \cup \dots \cup X_N$, a subspace homeomorphic to $X_{0,2N}$. Since $X_{0,2N}$ is aspherical, F is null-homotopic if $q \geq 2$. Thus, X is aspherical.

PROOF OF LEMMA 3.3: Since s is maximal and $k \geq 2$, Q must involve at least two generators. Let a and b be two such and \mathcal{C} be set of all other x_i , $1 \leq i \leq n$. Let A and B be two objects distinct from any in $F(X_1, \dots, x_n)$ and G . Let $\mathcal{A}' = \mathcal{C} \cup \{A\} \cup \{b\}$ and $\mathcal{A}'' = \mathcal{C} \cup [A] \cup [B]$ and let $F(\mathcal{A}')$ and $F(\mathcal{A}'')$ be the free groups generated by the elements of \mathcal{A}' and \mathcal{A}'' , respectively. Define homomorphisms

$$\theta_1: F(x_1, \dots, x_n) \rightarrow F(\mathcal{A}')$$

and

$$\theta_2: F(\mathcal{A}') \rightarrow F(\mathcal{A}'')$$

by $\theta_1(a) = A^q$, $\theta_1(b) = b$, $\theta_1(c) = c$ for all $c \in \mathcal{C}$, and $\theta_2(A) = A$, $\theta_2(b) = BA^{-p}$, $\theta_2(c) = c$ for all $c \in \mathcal{C}$, where $q = \sigma_Q(b)$ and $p = \sigma_Q(a)$. Let $Q' = \theta_1(Q) \in F(\mathcal{A}')$ and $Q'' = \theta_2(Q') \in F(\mathcal{A}'')$. It is immediate that Q' and Q'' are cyclically reduced and are not proper powers. Also,

$$\begin{aligned}
 \sigma_{Q''}(A) &= \sigma_{Q'}(A) - p\sigma_{Q'}(b) \\
 &= q\sigma_{Q'}(a) - p\sigma_{Q'}(b) \\
 &= 0,
 \end{aligned}$$

and

$$\#_{Q''}B = \#_Q b \text{ and } \#_{Q''}c = \#_Q c \text{ for all } c \in \mathcal{C}.$$

Lemma 3.2 applies to the presentation

$$P'' = \{\mathcal{A}''; Q''^s\};$$

i.e., $D(P'')$ is aspherical. Let P' be the presentation $\{\mathcal{A}', Q'^s\}$.

SUBLEMMA 3.3.1. $D(P')$ and $D(P'')$ have the same homotopy type.

PROOF OF SUBLEMMA 3.3.1: Let $S^1 \subset K(\mathbb{Z}_s, 1)$ and

$$f': S^1 \rightarrow S_A^1 \vee S_b^1 \vee \bigvee_{c \in \mathcal{G}} S_c^1$$

be a map ‘‘spelling’’ Q' . Let

$$h: S_A^1 \vee S_b^1 \vee \bigvee_{c \in \mathcal{G}} S_c^1 \rightarrow S_A^1 \vee S_B^1 \vee \bigvee_{c \in \mathcal{G}} S_c^1$$

induce the homomorphism θ_2 on fundamental groups. Then hf' ‘‘spells’’ Q'' . Since θ_2 is an isomorphism, h is a homotopy equivalence; and so $D(P')$ and $D(P'')$ are homotopy equivalent.

We next compare $D(P)$ and $D(P')$ in order to deduce the asphericity of $D(P)$. Let M be the mapping cylinder of a map from S^1 to S^1 of degree q . Recall that M has in it circles S_d^1 and S_r^1 (domain and range) such that S_r^1 is a strong deformation retract of M , and the inclusion of S_d^1 in M followed by the retraction of M onto S_r^1 is a map of degree q . Let $\bar{W} = M \vee S_b^1 \vee \bigvee_{c \in \mathcal{G}} S_c^1$; then $W(\mathcal{A}) = S_d^1 \vee S_b^1 \vee \bigvee_{c \in \mathcal{G}} S_c^1$ and $W(\mathcal{A}') = S_r^1 \vee S_b^1 \vee \bigvee_{c \in \mathcal{G}} S_c^1$ are contained in \bar{W} . Clearly, $W(\mathcal{A}')$ is a strong deformation retract of \bar{W} and the composition of $i: W(\mathcal{A}) \rightarrow \bar{W}$ followed by the retraction of \bar{W} onto $W(\mathcal{A}')$ is a map $k: W(\mathcal{A}) \rightarrow W(\mathcal{A}')$ which induces the homomorphism $\theta_1: F(\mathcal{A}) \rightarrow F(\mathcal{A}')$ on fundamental groups.

Let $f: S^1 \rightarrow W(\mathcal{A})$ spell Q . Then $D(P) = W(\mathcal{A}) \cup_f K$ is a subcomplex of $X = \bar{W} \cup_{i \circ f} K$, where $S^1 \subset K = K(\mathbb{Z}_s, 1)$. Let $r: \bar{W} \rightarrow W(\mathcal{A}')$ be the retraction. Then $W(\mathcal{A}') \cup_{r \circ (i \circ f)} K$ has the same homotopy type as X . (This uses the general proposition that if $W \rightarrow D$ is a cofibration and $k: W \rightarrow W'$, then $D \cup_k W'$ and $D \cup M(k)$ have the same homotopy type. In our case, $M(k) = \bar{W}$.) The map $r \circ i \circ f: S^1 \rightarrow W(\mathcal{A}')$ spells Q' . Thus, $W(\mathcal{A}') \cup_{r \circ (i \circ f)} K$ is $D(P')$. Since this is aspherical, so is X .

Let X_1 be that part of X which is in M and has cylinder coordinate $\geq \frac{1}{2}$. Let X_2 be the closure in X of the complement of X_1 . Then X_1 has S_r^1 as strong deformation retract, X_2 has $D(P)$ as strong deformation retract, and $X_0 = X_1 \cap X_2$ is a circle.

By the Siefert-van Kampen Theorem, the following is a pushout diagram in the category of groups:

$$\begin{array}{ccc} \pi_1(X_0) & \xrightarrow{i_2} & \pi_1(X_2) \\ \downarrow i_1 & & \downarrow j_1 \\ \pi_1(X_1) & \xrightarrow{j_2} & \pi_1(X). \end{array}$$

From remarks above, the groups on the left are both infinite cyclic and i_1 is raising to the q th power. Since X_2 has the homotopy type of $D(P)$, $\pi_1(X_2) = G$, the

group presented by P . By construction i_2 takes a generator into the element of G represented by a . By the Freeness Theorem (recall that R is cyclically reduced and involves b) i_2 is an injection. Thus, $\pi_1(X)$ is a free product with amalgamation and j_1 and j_2 are injections.

Consider the universal covering $p: \tilde{X} \rightarrow X$ of X . Since X is aspherical, \tilde{X} is contractible, and has trivial homology groups. Thus, by the Mayer-Vietoris Theorem

$$H_i(p^{-1}X_0) \rightarrow H_i(p^{-1}X_1) \oplus H_i(p^{-1}X_2)$$

is an isomorphism for $i \geq 1$. For $\alpha = 0, 1, 2$, $\pi_1(X_\alpha) \rightarrow \pi_1(X)$ is an injection; hence each component of $p^{-1}X_\alpha$ is homomorphic to the universal covering space of X_α . In particular, X_0 being a circle, each component of $p^{-1}X_0$ is the real line, and $H_i(p^{-1}X_0) = 0$ for $i \geq 1$. Thus, $H_i(p^{-1}X_2) = 0$ for $i \geq 1$; i.e., the universal covering space of X_2 is acyclic. Since it is simply connected, it is contractible. Hence, X_2 is aspherical, and so is $D(P)$.

4. An amusing space

In [4] Higman presented a group \mathbf{H} on four generators and four relators which has a number of interesting properties. We observe here that the canonical two-dimensional cell-complex of his presentation is an aspherical space and translate various of the group-theoretic properties of \mathbf{H} into topological properties of the associated cell-complex.

Let $H_{x,y}$ be the group presented by $P_{x,y}: \{x, y; xyx^{-1}y^{-2}\}$. Since $xyx^{-1}y^{-2}$ has exponent sum equal -1 , it is not a proper power and so by Theorem 2.1, $C(P_{x,y})$ is aspherical. Let $H_{x,y,z}$ be the free product with amalgamation defined by

$$\begin{array}{ccc} F(y) & \longrightarrow & H_{x,y} \\ \downarrow & & \\ & & H_{y,z} \end{array}$$

where each homomorphism carries y into its class in the appropriate group. By Theorem W , $C(P_{x,y}) \cup_y C(P_{y,z})$ is aspherical. As seen in [4], the homomorphism

$$F(x, z) \rightarrow H_{x,y,z},$$

defined by taking each of x and z into its class in $H_{x,y,z}$, is an injection. Thus letting \mathbf{H} be the free product with amalgamation defined by

$$\begin{array}{ccc} F(x, z) & \longrightarrow & H_{x,y,z} \\ \downarrow & & \\ & & H_{z,w,x} \end{array}$$

the space $(C(P_{x,y}) \cup_y C(P_{y,z})) \cup_{x,z} (C(P_{z,w}) \cup_w C(P_{w,x}))$ is a $K(\mathbf{H}, 1)$. This space is $C(P)$ for the presentation

$$P: \{w, x, y, z; wxw^{-1}x^{-2}, xyx^{-1}y^{-2}, yzy^{-1}z^{-2}, z wz^{-1}w^{-2}\}$$

of \mathbf{H} . It has one vertex, four 1-cells and four 2-cells; it is aspherical and $\pi_1(C(P), \star) \cong \mathbf{H}$. Since the abelianization of \mathbf{H} is trivial, $H_1(C(P); \mathbf{Z}) = 0$. Since the boundary of a non-zero 2-chain corresponding to the cell attached by the relation in $P_{x,y}$ is a non-zero multiple of the 1-chain corresponding to y , it is easy to see that the only 2-cycle on $C(P)$ is the 0 cycle. Thus, $H_2(C(P); \mathbf{Z}) = 0$. Clearly the higher dimensional homology groups of $C(P)$ are all zero. *This implies that the suspension of $C(P)$ is contractible.*

The group \mathbf{H} has no non-trivial finite dimensional representations over any field, see [4]. It follows that if \mathcal{B} is any local coefficient system over $C(P)$ of finitely generated abelian groups, then

$$\tilde{H}_*(C(P); \mathcal{B}) = 0 = \tilde{H}^*(C(P); \mathcal{B}).$$

Also, any finite dimensional real or complex vector bundle over $C(P)$ is trivial since obstructions to finding a section are elements of suitable cohomology groups.

Summarizing, $C(P)$ is a finite 2-dimensional cell-complex (nine cells) $C(P)$ is a $K(\mathbf{H}, 1)$, for \mathbf{H} the Higman group, the suspension of $C(P)$ is contractible, homology and cohomology of $C(P)$ with finitely generated local coefficients is trivial, all finite dimensional real or complex vector bundles over $C(P)$ are trivial. In short, it is difficult to detect that $C(P)$ does not have the homotopy type of a point by the usual abelian invariants of algebraic topology. (If one performs the same construction on the analogous presentation, cycling around in three steps, instead of four, as here, the complex obtained does have the homotopy type of a point.)

If N is a regular neighborhood of an imbedded $C(P)$ in \mathbf{R}^4 and S is the union of two copies of N identified along the boundary of N , the ‘‘double’’ of N , then S is an orientable 4-manifold and is a homology 4-sphere. It can be shown that $\pi_1(S) \cong \mathbf{H}$. Except for infinitely generated local coefficient systems, the manifold S is hard to distinguish from S^4 by usual invariants of algebraic topology.

5. Conjecture I'

In 1961 Papakyriakopoulos conjectured [8] that for the presentation

$$\tilde{P}: \{a, b, x_1, y_1, \dots, x_n, y_n; [a, b] \prod_{i=1}^n [x_i, y_i], [a, b\tau]\},$$

where τ is in the commutator subgroup of the free group on a, b, x_1, \dots, y_n , the complex $C(\tilde{P})$ is aspherical.

In 1963 Rapaport showed [9] that for the presentation

$$P: \{a, b, x_1, y_1, \dots, x_n, y_n; [a, b] \prod_{i=1}^n [x_i, y_i], [a, w]\},$$

where w is any word in a, b, \dots, y_n , the group defined by P is torsion free. (This established a generalization of Conjecture I of Papakyriakopoulos; the statement above for \tilde{P} is his Conjecture I’.)

We show here, using specific results in Rapaport’s argument, that if $[a, w]$ is not in the normal closure in $F(a, b, x_1, \dots, y_n)$ of the word $[a, b] \prod_{i=1}^n [x_i, y_i]$, then $C(P)$ is aspherical. Of course, if $[a, w]$ were in that normal closure, then $C(P)$ would be the homotopy type of the wedge of a surface of genus $n + 1$ and a 2-sphere; thus, $C(P)$ would not be aspherical. The word $[a, b\tau]$ in Papakyriakopoulos Conjecture is not in the normal closure of $[a, b] \prod_{i=1}^n [x_i, y_i]$, (see Theorem 6.2 of [8]), and the above assertion then implies that Conjecture I’ is true.

Before giving this argument, we would like to make two comments. There is a vast difference between asking whether the group defined by a presentation P is torsion free and asking if $C(P)$ is aspherical. The latter conclusion implies the group has geometric dimension two, a rare property even among groups without torsion. G. Baumslag has given (unpublished) an example of an infinite dimensional, finitely presented group without torsion; it is known that for each integer n , there exists a finitely presented group with geometric dimension n . Of course, all of them are torsion free. Secondly, although our argument establishes Conjecture I’, it is not in the geometric spirit sought by Papakyriakopoulos. Indeed, it relies heavily on the specific rewriting obtained by Rapaport. While this is a failing from the viewpoint of [8] and even though there may be some geometric argument applicable for \tilde{P} , we would like to point out that the combination of rewriting and covering space techniques employed here has a general character which may be applicable to a wider class of groups.

We first modify the presentation P to obtain a new presentation P' , differing from P only in that a word $[w^*, a]$ with $\sigma_{w^*}a = 0$ is substituted for the word $[a, w]$ in P . As we have seen $C(P)$ is aspherical if and only if the corresponding construction with the conjugate $w^{-1}(awa^{-1}w^{-1})w$ is aspherical (Lemma 1.2). Writing

$$w^{-1}awa^{-1} = w^{-1}a^h a^{-h}wa^{-1},$$

let $w^* = w^{-1}a^h$ for that h with $\sigma_{w^{-1}a^h} a = 0$; then $w^{-1}awa^{-1} = [w^*, a]$ and $\sigma_{w^*}a = 0$. Letting $n = g - 1$ and writing

$$\begin{aligned} \chi_{2i-1} &= x_i \\ \chi_{2i} &= y_i, \quad i = 1, \dots, g - 1, \end{aligned}$$

we have precisely the presentation in [9].

Continuing with the notation and results of [9], let

$$\begin{aligned}
 b_k &= a^{-k} b a^k \\
 x_{i,k} &= a^{-k} \chi_i a^k \\
 [jj] &= \prod_{i=1}^{g-1} [x_{2i-1} j, x_{2i,j}].
 \end{aligned}$$

The relator $G^* = [a, b] \cdot [00]$ becomes

$$G^* = G_0 = b_{-1} b_0^{-1} [00],$$

and

$$G_k = a^{-k} G^* a^k = b_{k-1} b_k^{-1} [k k].$$

Let

$$R = w^* a w^{*-1} a^{-1} = w_0 w_{-1}^{-1},$$

and

$$R_k = w_k w_{k-1}^{-1},$$

where $R_k = a^{-k} R a^k$ and $w_k = a^{-k} w_0 a^k$.

If R is not in the normal closure of G^* , there are integers (u, v) with either $u = v = 0$ or $u \leq 0$ and $1 \leq v$ and a word $A = A_0$ in $F_0 = F(x_{i,k}, b \text{ for } u \leq k \leq v)$ such that, letting

$$H_r : \{F_r; A_r\}$$

$$S_r : F(x_{i,k}, b_r \text{ for } r + u + 1 \leq k \leq r + v),$$

($= F(b_r)$ if $u = v = 0$),

$\alpha_r : S_r \rightarrow H_r$ be induced by the identity on generators,

$\beta_r : S_r \rightarrow H_{r+1}$ be induced by the identity on the $x_{i,k}$ and $\beta_r(b_r) = [r + 1, r + 1]^{-1} b_{r+1}$, then α_r and β_r are injections,

A_r is not a proper power in F_r ,

$H = \cdots * H_{-1}^* H_0^* S_0 H_1^* \cdots$ is the kernel of $G(P) \rightarrow \mathbf{Z}(a)$ defined by exponent sum of a ,

$\mathbf{Z}(a)$ acts as group of automorphisms on H with the generator a translating the above displayed generalized free product one step to the right (by conjugation), and

the modified presentation P' of G is precisely that defined by this extension.

As in Section 3 we will construct an aspherical, 2-dimensional cell-complex for H with a properly discontinuous action of $\mathbf{Z}(a)$ on it inducing the above automorphisms on H ; the quotient space defined by this action will be (CP') . As it has an aspherical covering space, it will also be aspherical.

Let X_0 be the cell-complex constructed as follows: to the real line adjoin at each integer point k , $u \leq k \leq v$, a wedge of $2g - 2$ circles corresponding to the generators $x_{i,k}$, $i = 1, \dots, 2g - 2$, and one more circle at 0 corresponding to b . Let 0 be the base point and describe the generator in $\pi_1(X_0, 0)$ corresponding to $x_{i,k}$ by the sum $[0, k] + S_{i,k}^1 + [k, 0]$. Attach a 2-cell to X_0 , forming Y_0 , to spell

the word A_0 with the above description of F_0 . As A_0 is not a proper power in F_0 , Y_0 with the line identified to a point is $C(H_0)$ and is aspherical by Theorem 2.1. Thus, Y_0 is aspherical.

For each integer r let Y_r be a copy of Y_0 translated r to the right. Take all of the Y_r to be disjoint. Attach the cylinder $(S^1 \times I)_{r+1}$ to Y_{r+1} along $S^1 \times 1$ by a mapping β_r which spells $[r + 1 r + 1]^{-1} b_{r+1}$. Since Y_{r+1} is a strong deformation retract of $Y_{r+1} \cup_{\beta_r} (S^1 \times 1)_{r+1} = W_{r+1}$, the space W_{r+1} is aspherical; it is a $K(H_{r+1}, 1)$. Let Z_r be the complex consisting of the real line with circles corresponding to $x_{i,k}$, $1 \leq i \leq 2g - 2$, $r + u + 1 \leq k \leq r + v$, adjoined at k , together with a circle corresponding to b_r adjoined at r .

Define $\bar{\alpha}_r: Z_r \rightarrow W_r$ to be the identity on the line and to carry the circle in Z_r corresponding to $x_{i,k}$ (to b_r) homeomorphically into the circle in W_r corresponding to $x_{i,k}$ (to b_r). Define $\bar{\beta}_r: Z_r \rightarrow W_{r+1}$ to be the identity on the line, to carry the circle in Z_r corresponding to $x_{i,k}$ homeomorphically into the circle in W_{r+1} corresponding to $x_{i,k}$, and to carry the circle corresponding to b_r homeomorphically into the circle $(S^1 \times 0)_{r+1}$ in W_{r+1} . Then $\bar{\alpha}_r$ and $\bar{\beta}_r$ induce the homomorphisms α_r and β_r on fundamental groups. As these are injections, by Theorem $W_r \cup_{Z_r} W_{r+1}$ has fundamental group $H_r *_S H_{r+1}$ and is aspherical. Since $\pi_1(W_r) \rightarrow \pi_1(W_r \cup_{Z_r} W_{r+1})$, $i = r, r + 1$, is injective, so are the compositions

$$\pi_1(Z_{r-1}) \xrightarrow{\beta_{r-1}} \pi_1(W_r) \longrightarrow \pi_1(W_r \cup_{Z_r} W_{r+1})$$

and

$$\pi_1(Z_{r+1}) \xrightarrow{\alpha_{r+1}} \pi_1(W_{r+1}) \longrightarrow \pi_1(W_r \cup_{Z_r} W_{r+1}).$$

Thus, continuing inductively, the process can be iterated to form a space

$$X = \dots \cup_{Z_{-1}} W_{-1} \cup_{Z_1} W_0 \cup_{Z_1} W_1 \cup \dots$$

which is a $K(H, 1)$. There is a properly discontinuous action of $\mathbf{Z}(a)$ defined on X by letting a move each point one unit to the right or into its correspondent with one higher subscript r or k , as the case may be. The quotient space $X/\mathbf{Z}(a)$ has

- a base point, (0)
- $2g - 2$ circles, $(x_{i,k})$, $i = 1, \dots, 2g - 2$
- 1 circle, (b_k)
- 1 circle, image $[0, 1]$
- 1 2-cell, attached by A_0 , and
- 1 2-cell, attached by $b_1 = [1 1]b_0$.

These correspond precisely to the generators and relators of the presentation P' .

Thus, $X/\mathbf{Z}(a)$ is homeomorphic to $C(P')$. Since X is aspherical and the projection $X \rightarrow X/\mathbf{Z}(a) = C(P')$ is a covering map, $C(P')$ is aspherical.

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The Graduate School
 City University of New York,
 33 West 42 Street
 New York, 10036
 U. S. A.