Some solvable stochastic control problems with delay

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Abstract

We consider optimal harvesting of systems described by stochastic differential equations with delay. We focus on those situations where the value function of the harvesting problem depends on the initial path of the process in a simple way, namely through its value at 0 and through some weighted averages.

A verification theorem of variational inequality type is proved. This is applied to solve explicitly some classes of optimal harvesting delay problems.

1 Introduction

Consider a 1-dimensional stochastic differential delay equation (SDDE) of the form

$$\begin{aligned} dX(t) &= b(X(t), Y(t), Z(t))dt \\ &\quad + \sigma(X(t), Y(t), Z(t))dB(t)\,, \qquad t \geq 0 \end{aligned} \tag{1.1}$$

where

$$Y(t) = \int_{-\delta}^{0} e^{\lambda s} X(t+s)ds \qquad \text{and} \qquad Z(t) = X(t-\delta)$$

and $b : \mathbf{R}^3 \to \mathbf{R}$ and $\sigma : \mathbf{R}^3 \to \mathbf{R}$ are given functions, $\delta > 0$ is the (constant) *delay*, $\lambda \in \mathbf{R}$ is a constant and $(\Omega, \mathcal{F}, \mathcal{F}_t, B(t) = B(t,\omega); t \geq 0, \omega \in \Omega)$ is a 1-dimensional Brownian motion.

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For example, $X(t)$ can model the size of a population or the value of an investment at time t, in situations where not only the present value of $X(t)$ but also $X(t-\delta)$ and some (sliding) average of previous values has effect on the growth at time t. By allowing for such delays δ in the equation we can obtain more realistic mathematical models for such situations.

For such systems it is necessary to specify the whole *initial path* $\xi(s)$; $-\delta \le s \le 0$. I.e., we set

$$X(s) = \xi(s) \ge 0 \qquad \text{for} \quad -\delta \le s \le 0 \,. \tag{1.2}$$

The solution of (1.1) with initial path (1.2) is denoted by $X^{(\xi)}(t)$, if it exists. See e.g. [M1], [M2] for conditions for existence and uniqueness of solutions of such equations.

Suppose we introduce *harvesting* of such a system. For example, the harvesting could be fishing from a fish population or paying of dividends from an investment. Let $\gamma(t) = \gamma(t,\omega)$ be an $\mathcal{F}_t$-adapted, right-continuous, nondecreasing stochastic process modelling the total amount taken out of the system up to time t. The corresponding population process $X(t) = X^{(\xi,\gamma)}(t)$ will then satisfy the equation

$$dX(t) = b(X(t),Y(t),Z(t))dt + \sigma(X(t),Y(t),Z(t))dB(t) - d\gamma(t) \,; \qquad t \ge 0 \,. \tag{1.3}$$

Let Γ denote the set of all such harvesting processes γ. Let $S \subseteq \mathbf{R}^3$ be a given Borel set (our *survival* set or *solvency* set) with the property that

$$\bar{S} = \overline{S^0}$$

where S^0 denotes the interior of S, $\bar{S}$ the closure of S, and define

$$T = \inf\{t > 0; (s+t, X(t), Y(t)) \notin S\} \tag{1.4}$$

i.e., T is a time of extinction of the harvested population (or a time of bankruptcy for the wealth).

Suppose the harvestor or investor obtains a price/utility rate $u(t,x,y)$ when the size of the population/wealth and its average at time t is x and y, respectively, where $u : \mathbf{R}^3 \to \mathbf{R}$ is a given continuous, increasing concave function. Let $\pi(t) \ge 0$ be a given price/utility per unit harvested at time t. Then the total utility obtained by using the harvesting strategy $\gamma \in \Gamma$ is given by

$$J^\gamma(s,\xi) = E^{s,\xi}\Big[\int_0^T u(s+t, X(t), Y(t))dt + \int_0^T \pi(s+t)d\gamma(t)\Big], \tag{1.5}$$

where $E^{s,\xi}$ denotes the expectation with respect to the law $P^{s,\xi,\gamma}$ of the time-space harvested process

$$W(t) = W^{(\xi,\gamma)}(t) = (s+t, X^{(\xi,\gamma)}(t)) \,. \tag{1.6}$$

We assume that

$$E^{s,\xi}\left[\int_0^T |u(s+t,X(t),Y(t))|dt\right] < \infty \qquad \text{for all} \quad s,\xi,\gamma\ . \tag{1.7}$$

We consider the problem of finding $\Phi(s,\xi)$ and $\gamma^* \in \Gamma$ such that

$$\Phi(s,\xi) = \sup_{\gamma\in\Gamma} J^{\gamma}(s,\xi) = J^{\gamma^*}(s,\xi)\ . \tag{1.8}$$

For more information about SDDE's in general we refer to [M1] and [M2].

For stochastic systems *without* delay optimal harvesting problems of this type have been studied in [A], [AS], [JS], [LØ1] and [LØ2]. To the best of our knowledge this is the first time such singular stochastic control problems have been considered for delay systems.

In general one would expect that the value function Φ of problem (1.8) depends on the initial path ξ in a complicated way. Indeed, even if we restrict ourselves to consider initial paths $\xi \in C[-\delta,0]$, the set of continuous real functions on $[-\delta,0]$, the problem is infinite-dimensional and therefore the usual variational inequality approach does not work. However, the purpose of this paper is to show that for a certain class of systems (1.1) the function Φ depends only on the initial path ξ through the three linear functionals

$$x = x(\xi) := \xi(0)\ , \quad y = y(\xi) := \int_{-\delta}^{0} e^{\lambda s}\xi(s)ds \quad \text{and} \quad z = z(\xi) := \xi(-\delta)\ . \tag{1.9}$$

If this is the case we can write

$$\Phi(s,\xi) = \varphi(s,x,y,z) \qquad \text{where} \quad \varphi : \mathbf{R}^4 \to \mathbf{R}\ . \tag{1.10}$$

In fact, we will show that in the cases we consider with $\pi(t) = e^{-\rho t}$ we have

$$\varphi(s,x,y,z) = e^{-\rho s}\psi(x,y) \tag{1.11}$$

for some function $\psi : \mathbf{R}^2 \to \mathbf{R}$.

Our approach is inspired by [KM], where a (nonsingular) stochastic control problem for a certain linear delay system with quadratic cost functional is solved. See also [KS].

2 A variational inequality formulation

In this section we establish a set of sufficient variational inequalities for the problem (1.7), in the case when (1.8) and (1.9) hold. We first introduce some notation and establish some useful auxiliary results.

For $t \geq 0$ let $X_t(\cdot)$ be the function defined by

$$X_t(s) = X(t+s) \; ; \qquad -\delta \leq s \leq 0 \; , \tag{2.1}$$

i.e. X_t is the segment of the path of X from $t-\delta$ to t. Define

$$G(t) = f(s+t, X(t), Y(t)) \tag{2.2}$$

where f is a given function in $C^{1,2,1}(\mathbf{R}^3)$ and

$$Y(t) = \int_{-\delta}^{0} e^{\lambda s} X(t+s) ds \tag{2.3}$$

as in (1.1). Then we have

Lemma 2.1 (The Ito formula)

$$dG(t) = Lf \; dt + \frac{\partial f}{\partial x} \cdot \sigma(x,y,z) dB(t) + \frac{\partial f}{\partial y} \cdot [x - e^{-\lambda\delta} z - \lambda y] dt \tag{2.4}$$

where

$$Lf = Lf(u,x,y,z) = \frac{\partial f}{\partial u} + b(x,y,z)\frac{\partial f}{\partial x} + \tfrac{1}{2}\sigma^2(x,y,z)\frac{\partial^2 f}{\partial x^2} \tag{2.5}$$

and $Lf(u,x,y,z)$ and the other functions are evaluated at

$$u = s+t \; , \quad x = x(X_t^{(\xi)}(\cdot)) = X^{(\xi)}(t) \; ,$$

$$y = y(X_t^{(\xi)}(\cdot)) = \int_{-\delta}^{0} e^{\lambda s} X^{(\xi)}(t+s) ds =: Y(t) \quad \textit{and} \tag{2.6}$$

$$z = z(X_t^{(\xi)}(\cdot)) = X^{(\xi)}(t-\delta) =: Z(t) \; . \tag{2.7}$$

Proof. First note that by (2.3) we have, for $\eta \in C[-\delta, 0]$,

$$\begin{aligned}
\frac{d}{dt}[y(\eta_t(\cdot))] &= \frac{d}{dt}\Big[\int_{-\delta}^{0} e^{\lambda s}\eta(t+s)ds\Big] \\
&= \frac{d}{dt}\Big[\Big|_{-\delta}^{0} e^{\lambda s} H(t+s) - \int_{-\delta}^{0} \lambda e^{\lambda s} H(t+s) ds\Big] \\
&= \eta(t) - e^{-\lambda\delta}\eta(t-\delta) - \int_{0}^{\delta} \lambda e^{\lambda s}\eta(t+s) ds \\
&= x(\eta_t) - e^{-\lambda\delta} z(\eta_t) - \lambda y(\eta_t) \; ,
\end{aligned} \tag{2.8}$$

where H denotes an antiderivative of η. Therefore, since $G(t)=f(s+t,X^{(\xi)}(t),y(X_t^{(\xi)}(\cdot)))$, the result follows from the classical Ito formula. □

From Lemma 2.1 we immediately get

Lemma 2.2 (The Dynkin formula) *Let $f\in C_0^{1,2,1}(\mathbf{R}^3)$. Then for $t\geq 0$ we have*

$$E^{s,\xi}[f(t+s,X^{(\xi)}(t),y(X_t^{(\xi)}(\cdot)))]=f(s,\xi(0),y(\xi))$$

$$+E^{s,\xi}\Big[\int_0^t\Big\{Lf+\frac{\partial f}{\partial y}\cdot[x-e^{-\lambda\delta}z-\lambda y]\Big\}dr\Big], \tag{2.9}$$

where $Lf(u,x,y,z)$ and the other functions in the curly bracket are evaluated at

$$u=s+r\ ,\quad x=X^{(\xi)}(r)\ ,\quad y=y(X_r^{(\xi)}(\cdot))\ ,\quad z=X^{(\xi)}(r-\delta)\ .$$

We can now proceed as in the proof of Theorem 3.3 in [LØ1] and obtain the following variational inequality verification theorem for optimal control of stochastic systems with delay. Note that if $X=X^{(\xi,\gamma)}$ satisfies (1.3) then $X(t)$ could possibly jump at $t=0$, which would imply that $X(0)$ is different from the starting point x, which we will denote by $X(0^-)$.

Theorem 2.3 a) *Suppose $\varphi(s,x,y)$ is a nonnegative function in $C^{1,2,1}(S^0)\cap C(\overline{S}\,)$ with the following properties, (2.10)–(2.11):*

$$\frac{\partial\varphi}{\partial x}(s,x,y)\geq\pi(s)\qquad \textit{everywhere on } S^0 \tag{2.10}$$

$$\mathcal{L}\varphi(s,x,y,z):=\frac{\partial\varphi}{\partial s}+b(x,y,z)\frac{\partial\varphi}{\partial x}+\tfrac{1}{2}\sigma^2(x,y,z)\frac{\partial^2\varphi}{\partial x^2}$$
$$+[x-e^{-\lambda\delta}z-\lambda y]\frac{\partial\varphi}{\partial y}+u(s,x,y)\leq 0\qquad \textit{for all } z\in\mathbf{R},\ (s,x,y)\in S^0\ . \tag{2.11}$$

Then

$$\varphi(s,\xi(0),y(\xi))\geq\Phi(s,\xi) \tag{2.12}$$

for all $(s,\xi)\in\mathbf{R}\times C[-\delta,0]$.

b) *Define the* non-intervention region D *by*

$$D=\{(s,x,y)\in S^0;\ \frac{\partial\varphi}{\partial x}(s,x,y)>\pi(s)\} \tag{2.13}$$

Suppose, in addition to (2.10)–(2.11), that

$$\mathcal{L}\varphi=0\qquad \textit{for all } z\in\mathbf{R} \textit{ if } (s,x,y)\in D \tag{2.14}$$

and that there exists a harvesting strategy $\widehat{\gamma} \in \Gamma$ such that the following, (2.15)–(2.17), hold:

$$(s+t, X^{(\xi,\widehat{\gamma})}(t), y(X_t^{(\xi,\widehat{\gamma})}(\cdot))) \in \overline{D} \qquad \text{for all } t > 0 \tag{2.15}$$

$$\left(\frac{\partial \varphi}{\partial x}(s,x,y) - \pi(s)\right) d\widehat{\gamma}(s) = 0 \tag{2.16}$$

(i.e. $\widehat{\gamma}$ increases only when $\frac{\partial \varphi}{\partial x}(s,x,y) = \pi(s)$)

$$E^{s,\xi}[\varphi(s+T_R, X^{(\xi,\widehat{\gamma})}(T_R), y(X_{T_R}^{\xi,\widehat{\gamma}}(\cdot))] \to 0 \qquad \text{as } R \to \infty \tag{2.17}$$

for all (s,ξ), where

$$T_R = T \wedge R \wedge \inf\{t > 0;\ |X^{(\xi,\widehat{\gamma})}(t)| \geq R\}\ . \tag{2.18}$$

Then

$$\varphi(s, \xi(0), y(\xi)) = \Phi(s,\xi) \qquad \text{for all } (s,\xi)$$

and

$$\widehat{\gamma} \quad \text{is an optimal harvesting strategy}\ .$$

Proof. The proof follows the proof of Theorem 3.3 in [LØ1]. For completeness we give the details:

a) Choose $\gamma \in \Gamma$ and assume $\varphi \in C^{1,2,1}(S^0) \cap C(\bar{S})$ satisfies (2.10)–(2.11). Then by Dynkin's formula Lemma 2.2, extended to the semimartingale case (see e.g. [P, Th. II.7.33]) we get

$$\begin{aligned} E^{s,\xi}[\varphi(s+T_R, X^{(\xi,\gamma)}(T_R), y(X_{T_R}^{(\xi,\gamma)}(\cdot)))] &= E^{s,\xi}\Bigg[\varphi(s, X^{(\xi,\gamma)}(s), y(\xi)) \\ &- \int_0^{T_R} \frac{\partial \varphi}{\partial x} \cdot d\gamma(t) + \int_0^{T_R} \mathcal{L}\varphi dt - \int_0^{T_R} u\, dt \\ &+ \sum_{0<t_k\leq T_R} \left\{\Delta\varphi(t_k) - \frac{\partial \varphi}{\partial x}(s+t_k, X^{(\xi,\gamma)}(t_k^-), y(x_{t_k}^{(\xi,\gamma)}(\cdot))) \cdot \Delta X^{(\xi,\gamma)}(t_k)\right\}\Bigg]\ , \end{aligned} \tag{2.19}$$

where the sum is taken over all jumping times $t_k \in (0, T_R]$ and

$$\Delta\varphi(t_k) = \varphi(s+t_k, X^{(\xi,\gamma)}(t_k), y(X_{t_k}^{(\xi,\gamma)}(\cdot))) - \varphi(s+t_k, X^{(\xi,\gamma)}(t_k^-), y(X_{t_k}^{(\xi,\gamma)}(\cdot))) \tag{2.20}$$

and

$$\Delta X^{(\xi,\gamma)}(t_k) = X^{(\xi,\gamma)}(t_k) - X^{(\xi,\gamma)}(t_k^-) \tag{2.21}$$

are the jumps of φ and X at time $t = t_k$ (caused by γ). As in Lemma 2.2 we evaluate $\mathcal{L}\varphi(u, x, y, z)$ at

$$u = s + t\ , \quad x = X^{(\xi,\gamma)}(t)\ , \quad y = y(X_t^{(\xi,\gamma)}(\cdot))\ , \quad z = X^{(\xi,\gamma)}(t - \delta) \tag{2.22}$$

Using (2.11) this gives

$$\begin{aligned}
&E^{s,\xi}[\varphi(s + T_R, X^{(\xi,\gamma)}(T_R), y(X_{T_R}^{(\xi,\gamma)}(\cdot)))] \\
&\le E^{s,\xi}\Big[\varphi(s, X^{(\xi,\gamma)}(s), y(\xi)) - \int_{0^+}^{T_R} \frac{\partial\varphi}{\partial x} \cdot d\gamma(t) - \int_0^{T_R} u\, dt \\
&+ \sum_{0<t_k\le T_R} \Big\{\Delta\varphi(t_k) + \frac{\partial\varphi}{\partial x}(s + t_k, X^{(\xi,\gamma)}(t_k^-), y(X_{t_k}^{(\xi,\gamma)}(\cdot))) \cdot \Delta\gamma(t_k)\Big\}\Big]\ .
\end{aligned} \tag{2.23}$$

Let $\gamma^c(t)$ denote the continuous part of $\gamma(t)$, i.e.

$$\gamma^c(t) = \gamma(t) - \sum_{0<t_k\le t} \Delta\gamma(t_k)\ .$$

Then (2.23) implies that

$$\begin{aligned}
&E^{s,\xi}[\varphi(s + T_R, X^{(\xi,\gamma)}(T_T), y(X_{T_R}^{(\xi,\gamma)}(\cdot)))] \\
&\le \varphi(s, \xi(0), y(\xi)) - E^{s,\xi}\Big[\int_0^{T_R} u\, dt + \int_0^{T_R} \frac{\partial\varphi}{\partial x} \cdot d\gamma^c(t) - \sum_{0\le t_k\le T_R} \Delta\varphi(t_k)\Big]\ .
\end{aligned} \tag{2.24}$$

By the mean value property we have

$$\Delta\varphi(t_k) = -\frac{\partial\varphi}{\partial x}(s + t_k, \widehat{x}_k, y(X_{t_k}^{(\xi,\gamma)}(\cdot))) \cdot \Delta\gamma(t_k)$$

for some $\widehat{x}_k$ on the interval between $X^{(\xi,\gamma)}(t_k^-)$ and $X^{(\xi,\gamma)}(t_k)$. Hence by combining (2.24) with (2.10) we get

$$\begin{aligned}
\varphi(s, \xi(0), y(\xi)) &\ge E^{s,\xi}\Big[\int_0^{T_R} u\, dt + \int_0^{T_R} \pi(s + t) d\gamma(t) + \varphi(s + T_R, X^{(\xi,\gamma)}(T_R), y(X_{T_R}^{(\xi,\gamma)}(\cdot)))\Big] \\
&\ge E^{s,\xi}\Big[\int_0^{T_R} \pi(s + t) d\gamma(t)\Big]
\end{aligned} \tag{2.25}$$

Therefore

$$\varphi(s, \xi(0), y(\xi)) \ge \lim_{R\to\infty} E^{s,\xi}\Big[\int_0^{T_R} u\, dt + \int_0^{T_R} \pi(s + t) d\gamma(t)\Big] = J^\gamma(s, \xi) \tag{2.26}$$

Since $\gamma \in \Gamma$ was arbitrary this proves (2.12).

b) Next, assume D is defined by (2.13) and that (2.14)–(2.17) hold. Then the above calculations with γ replaced by $\widehat{\gamma}$ give *equality* everywhere and we end up with equality in (2.26), viz.

$$\varphi(s,\xi(0),y(\xi)) = J^{\widehat{\gamma}}(s,\xi) \; . \tag{2.27}$$

Combining this with (2.12) we obtain that

$$\varphi(s,\xi(0),y(\xi)) = J^{\widehat{\gamma}}(s,\xi) = \Phi(s,\xi)$$

and hence $\widehat{\gamma}$ is optimal. □

3 A deterministic example

To illustrate Theorem 2.3 let us first consider the following example:

Suppose the equation for the harvested population $X(t) = X^{(\xi,\gamma)}(t)$ is of the form (with θ, α, β constants)

$$dX(t) = [K + \theta X(t) + \alpha Y(t) + \beta Z(t)]dt - d\gamma(t) \tag{3.1}$$

$$X(s) = \xi(s) \geq 0 \; ; \qquad -\delta \leq s \leq 0 \; , \tag{3.2}$$

where, as before,

$$Y(t) = \int_{-\delta}^{0} e^{\lambda s} X(t+s)ds \qquad \text{and} \qquad Z(t) = X(t-\delta) \; .$$

Put

$$S = \{(s,x,y); \;\; \min(x,y) \geq 0\}$$

and

$$T = \inf\{t > 0; \; \min(X(t),Y(t)) < 0\} \; .$$

Assume $\pi(t) = e^{-\rho t}$ for some constant $\rho > 0$. We want to find $\varphi(s,x,y) \in C^{1,2,1}(S^0) \cap C(S)$ and $\gamma^* \in \Gamma$ such that (see (1.8))

$$\varphi(s,x(\xi),y(\xi)) = \Phi(s,\xi) = \sup_{\gamma \in \Gamma} J^{\gamma}(s,\xi) = J^{\gamma^*}(s,\xi) \; , \tag{3.3}$$

where

$$J^\gamma(s,\xi) = E^{s,\xi}\Big[\int_0^T e^{-\rho(s+t)} d\gamma(t)\Big], \qquad \text{as in (1.5) (with } u=0) \,. \tag{3.4}$$

Let us try

$$\varphi(s,x,y) = e^{-\rho s}\Big(\frac{K}{\rho} + x + \beta e^{\lambda\delta} y\Big) \,. \tag{3.5}$$

Note that

$$\frac{\partial\varphi}{\partial x} = e^{-\rho s} = \pi(s)$$

and, with $\mathcal{L}$ as in (2.11) and $u=0$,

$$\begin{aligned}
\mathcal{L}\varphi(s,x,y,z) &= e^{-\rho s}\Big\{ -\rho\Big(\frac{K}{\rho} + x + \beta e^{\lambda\delta} y\Big) + \\
&\quad + [K + \theta x + \alpha y + \beta z]\cdot 1 + [x - e^{-\lambda\delta} z - \lambda y]\beta e^{\lambda\delta}\Big\} \\
&= e^{-\rho s}\{(\theta + \beta e^{\lambda\delta} - \rho)x + (\alpha - (\lambda+\rho)\beta e^{\lambda\delta})y\} \,.
\end{aligned}$$

Hence

$$\mathcal{L}\varphi(s,x,y) \le 0 \qquad \text{for all } \; s,x,y \ge 0 \,, \; z \in \mathbf{R}$$

if and only if

$$\rho \ge \theta + \beta e^{\lambda\delta} \qquad \text{and} \qquad (\lambda+\rho)\beta e^{\lambda\delta} \ge \alpha \,. \tag{3.6}$$

Therefore, if (3.6) holds then, with $x = x(\xi)$, $y = y(\xi)$,

$$\varphi(s,x,y) = e^{-\rho s}\Big(\frac{K}{\rho} + x + \beta e^{\lambda\delta} y\Big) \ge \Phi(s,\xi) \,. \tag{3.7}$$

Do we have equality here?

To answer this, let us compute the expected discounted total income obtained by choosing $\gamma = \widehat{\gamma}$ to be delay analogue of the "*take the money and run*" strategy, i.e. $\widehat{\gamma}$ is the strategy which empties the system as quickly as possible (still by harvesting from X only). If the current state of the system is (s,x,y,z), then $\widehat{\gamma}$ immediately brings x to 0 by harvesting all of x. After that $\widehat{\gamma}$ harvests exactly at the rate money is coming in from the reserves, i.e.

$$d\widehat{\gamma}(t) = \Big[K + \alpha\int_{-\delta}^0 e^{\lambda r} X(t+r)dr + \beta X(t-\delta)\Big]dt \tag{3.8}$$

(see (3.1)).

This gives the total harvested income

$$I = I_0 + x + \alpha I_2 + \beta I_3 \ , \tag{3.9}$$

where

$$I_0 \colon = \int_0^\infty e^{-\rho t} K \, dt = \frac{K}{\rho} \tag{3.10}$$

$$I_3 \colon = \int_0^\delta e^{-\rho u} X(u-\delta) du \overset{(u-\delta=v)}{=} \int_{-\delta}^0 e^{-\rho(v+\delta)} X(v) dv = e^{-\rho\delta} \int_{-\delta}^0 e^{-\rho v} X(v) dv \ . \tag{3.11}$$

Finally, using integration by parts we get,

$$\begin{aligned}
I_2 \colon &= \int_0^\delta e^{-\rho u} \Big(\int_{-\delta}^{-u} e^{\lambda s} X(u+s) ds \Big) du \\
&\overset{(v=u+s)}{=} \int_0^\delta e^{-\rho u} \Big(\int_{u-\delta}^0 e^{\lambda(v-u)} X(v) dv \Big) du \\
&= \int_0^\delta e^{-(\rho+\lambda) u} \Big(\int_{u-\delta}^0 e^{\lambda v} X(v) dv \Big) du \\
&= \Big|_0^\delta - \frac{1}{\rho+\lambda} e^{-(\rho+\lambda)u} \Big(\int_{u-\delta}^0 e^{\lambda v} X(v) dv \Big) \\
&\quad - \int_0^\delta \Big(- \frac{1}{\rho+\lambda} e^{-(\rho+\lambda)u} \Big) (-e^{\lambda(u-\delta)} X(u-\delta)) du \\
&= \frac{1}{\rho+\lambda} \Big[\int_{-\delta}^0 e^{\lambda v} X(v) dv - \int_0^\delta e^{-\rho u} \cdot e^{-\lambda\delta} X(u-\delta) du \Big] \\
&\overset{(w=u-\delta)}{=} \frac{1}{\rho+\lambda} \Big[\int_{-\delta}^0 e^{\lambda v} X(v) dv - e^{-(\rho+\lambda)\delta} \int_{-\delta}^0 e^{-\rho w} X(w) dw \Big], \qquad (3.12)
\end{aligned}$$

assuming

$$\rho + \lambda \neq 0 \ .$$

If

$$\rho + \lambda = 0$$

then a similar, but simpler, computation gives

$$I_2 = \int_{-\delta}^{0} (v+\delta)e^{\lambda v} X(v)dv \tag{3.13}$$

Combining (3.9)–(3.12) we get, if $\rho + \lambda \neq 0$,

$$\begin{aligned} I &= \frac{K}{\rho} + x + \frac{\alpha}{\rho+\lambda} \int_{-\delta}^{0} e^{\lambda v} X(v)dv \\ &\quad - \frac{\alpha}{\rho+\lambda} e^{-(\rho+\lambda)\delta} \int_{-\delta}^{0} e^{-\rho v} X(v)dv \\ &\quad + \beta e^{-\rho\delta} \int_{-\delta}^{0} e^{-\rho v} X(v)dv \\ &= \frac{K}{\rho} + x + \frac{\alpha}{\rho+\lambda} y \\ &\quad + \frac{e^{-(\rho+\lambda)\delta}}{\rho+\lambda} [\beta(\rho+\lambda)e^{\lambda\delta} - \alpha] \int_{-\delta}^{0} e^{-\rho v} X(v)dv \; . \end{aligned} \tag{3.14}$$

Similarly, if $\rho + \lambda = 0$ we get, using (3.13),

$$\begin{aligned} I &= \frac{K}{\rho} + x + \alpha \int_{-\delta}^{0} (v+\delta)e^{\lambda v} X(v)dv \\ &\quad + \beta e^{\lambda\delta} \int_{-\delta}^{0} e^{\lambda v} X(v)dv \; . \end{aligned} \tag{3.15}$$

In any case we see that I can be expressed in terms of $x = x(\xi)$ and $y = y(\xi)$ if and only if

$$\alpha = (\lambda + \rho)\beta e^{\lambda\delta} \tag{3.16}$$

and if this is the case then

$$I = I(x,y) = \frac{K}{\rho} + x + \beta e^{\lambda\delta} y = e^{\rho s} \varphi(s,x,y) \; .$$

We summarize what we have found in the following:

Theorem 3.1 *Suppose the equation for the harvested population $X(t) = X^{(\xi,\gamma)}(t)$ is of the form (3.1)–(3.2) with*

$$\alpha = (\lambda+\rho)\beta e^{\lambda\delta} \qquad \text{and} \qquad \rho \geq \theta + \beta e^{\lambda\delta} \; . \tag{3.17}$$

Then the solution of the optimal harvesting problem (3.3)–(3.4) is

$$\Phi(s,\xi) = e^{-\rho s}\left(\frac{K}{\rho} + x(\xi) + \beta e^{\lambda\delta} y(\xi)\right) \tag{3.18}$$

with $x(\xi)$, $y(\xi)$ as before (see (1.9)), and this optimal value is achieved by applying the "take the money and run"*-strategy $\widehat{\gamma}$ (see (3.8) and above). Thus $\gamma^* : = \widehat{\gamma}$ is an optimal harvesting strategy.*

4 Optimal harvesting from a geometric Brownian motion with delay

The following example may be regarded as a delay version of an example studied in [A] in the no delay case. Suppose the harvested system is given by

$$\begin{aligned} dX(t) &= [\theta X(t) + \alpha Y(t) + \beta Z(t))]dt \\ &+\sigma\left[X(t) + \beta e^{\lambda\delta}\int_{-\delta}^{0} e^{\lambda s} X(t+s)ds\right]dB(t) - d\gamma(t) \end{aligned} \tag{4.1}$$

$$X(s) = \xi(s) \; , \qquad -\delta \leq s \leq 0 \; , \tag{4.2}$$

where θ, α, β and σ are constants and, as before,

$$Y(t) = \int_{-\delta}^{0} e^{\lambda s} X(t+s)ds \qquad \text{and} \qquad Z(t) = X(t-\delta) \; .$$

Suppose the price per unit harvested at time t is

$$\pi(t) = e^{-\rho t} \tag{4.3}$$

where $\rho > 0$ is a constant and that the utility rate obtained when the size of the population at time t is x is given by

$$u(t,x,y) = e^{-\rho t}(x + \beta e^{\lambda\delta} y)^k \tag{4.4}$$

where $k \in (0,1)$ is a constant.

Let $S=\{(s,x,y);x+\beta e^{\lambda\delta}y>0\}$, so that

$$T=\inf\{t>0;\ X(t)+\beta e^{\lambda\delta}Y(t)\le 0\}\ .$$

We want to find $\varphi(s,x,y)\in C^{1,2,1}(S^0)\cap C(\bar{S})$ and $\gamma^*\in\Gamma$ such that (see (1.8)

$$\varphi(s,x(\xi),y(\xi))=\Phi(s,\xi)=\sup_{\gamma\in\Gamma}J^\gamma(s,\xi)=J^{\gamma^*}(s,\xi)\ , \tag{4.5}$$

where

$$J^\gamma(s,\xi)=E^{s,\xi}\Big[\int_0^T e^{-\rho(s+t)}(X(t)+\beta e^{\lambda\delta}Y(t))^k dt+\int_0^T e^{-\rho(s+t)}d\gamma(t)\Big] \tag{4.6}$$

With $\mathcal{L}$ as in (2.11) and with

$$\varphi(s,x,y)=e^{-\rho s}\psi(x,y)$$

we get

$$\begin{aligned}
e^{\rho s}\mathcal{L}\varphi(s,x,y,z)&=-\rho\psi(x,y)+(\theta x+\alpha y+\beta z)\frac{\partial\psi}{\partial x}\\
&\quad+\tfrac{1}{2}\sigma^2(x+\beta e^{\lambda\delta}y)^2\frac{\partial^2\psi}{\partial x^2}+(x-e^{-\lambda\delta}z-\lambda y)\frac{\partial\psi}{\partial y}+(x+\beta e^{\lambda\delta}y)^k\\
&=z\Big[\beta\frac{\partial\psi}{\partial x}-e^{-\lambda\delta}\frac{\partial\psi}{\partial y}\Big]-\rho\psi+(\theta x+\alpha y)\frac{\partial\psi}{\partial x}\\
&\quad+\tfrac{1}{2}\sigma^2(x+\beta e^{\lambda\delta}y)^2\frac{\partial^2\psi}{\partial y^2}+(x-\lambda y)\frac{\partial\psi}{\partial y}+(x+\beta e^{\lambda\delta})^k\ .
\end{aligned}$$

Therefore $\mathcal{L}\varphi(s,x,y)=0$ for all z iff

$$\beta\frac{\partial\psi}{\partial x}-e^{-\lambda\delta}\frac{\partial\psi}{\partial y}=0 \tag{4.7}$$

and

$$\begin{aligned}
\mathcal{L}_0\psi:&=-\rho\psi+(\theta x+\alpha y)\frac{\partial\psi}{\partial x}+\tfrac{1}{2}\sigma^2(x+\beta e^{\lambda\delta}y)^2\frac{\partial^2\psi}{\partial x^2}\\
&\quad+(x-\lambda y)\frac{\partial\psi}{\partial y}+(x+\beta e^{\lambda\delta}y)^k=0\ .
\end{aligned} \tag{4.8}$$

Equation (4.6) holds iff

$$\psi(x,y)=g(v)\qquad\text{for some }\ g:\mathbf{R}\to\mathbf{R} \tag{4.9}$$

where

$$v = v(x,y) = x + \beta e^{\lambda\delta} y \tag{4.10}$$

Substituting (4.9)–(4.10) into (4.8) we get

$$\begin{aligned}\mathcal{L}_0\psi(x,y) = -\rho g(v) + [(\theta + \beta e^{\lambda\delta})x + (\alpha - \lambda\beta e^{\lambda\delta})y]g'(v) \\ + \tfrac{1}{2}\sigma^2 v^2 g''(v) + v^k = 0\ .\end{aligned} \tag{4.11}$$

Suppose

$$\alpha = \beta e^{\lambda\delta}(\lambda + \theta + \beta e^{\lambda\delta})\ . \tag{4.12}$$

Then (4.11) gets the form

$$\mathcal{L}_0\psi(x,y) = -\rho g(v) + (\theta + \beta e^{\lambda\delta})vg'(v) + \tfrac{1}{2}\sigma^2 v^2 g''(v) + v^k = 0\ . \tag{4.13}$$

The general solution of (4.13) is

$$g(v) = C_1 v^{r_1} + C_2 v^{r_2} + K v^k \tag{4.14}$$

where C_1, C_2 are arbitrary constants,

$$r_i = \sigma^{-2}\Big[\tfrac{1}{2}\sigma^2 - \theta - \beta e^{\lambda\delta} \pm \sqrt{(\tfrac{1}{2}\sigma^2 - \theta - \beta e^{\lambda\delta})^2 + 2\rho\sigma^2}\,\Big]; \qquad i = 1, 2 \tag{4.15}$$

are the solutions of the equation

$$\tfrac{1}{2}\sigma^2 r^2 + (\theta + \beta e^{\lambda\delta} - \tfrac{1}{2}\sigma^2)r - \rho = 0\ ; \qquad r_1 < 0 < r_2 \tag{4.16}$$

and

$$K = -[\tfrac{1}{2}\sigma^2 k^2 + (\theta + \beta e^{\lambda\delta} - \tfrac{1}{2}\sigma^2)k - \rho]^{-1}\ . \tag{4.17}$$

Assume that

$$\rho > \theta + \beta e^{\lambda\delta}\ . \tag{4.18}$$

Then

$$r_2 > 1\ , \tag{4.19}$$

which implies that $K > 0$ (since $0 < k < 1$).

We now guess that the value function $\Phi(s,\xi)$ has the form

$$\varphi(s,x,y) = e^{-\rho s}\psi(x,y) = e^{-\rho s} g(v) \tag{4.20}$$

where

$$v = x + \beta e^{\lambda\delta} y \tag{4.21}$$

and

$$g(v) = \begin{cases} C_1 v^{r_1} + C_2 v^{r_2} + K v^k & \text{for } 0 < v < v^* \\ v - v^* + g(v^*) & \text{for } v \geq v^* \end{cases} \tag{4.22}$$

for some $v^* > 0$. Since $|g|$ must be bounded as $v \to 0^+$ we put $C_1 = 0$. To determine C_2 and v^* we require that g be twice continuously differentiable at $v = v^*$. This gives the two equations

$$r_2 C_2 (v^*)^{r_2 - 1} + kK(v^*)^{k-1} = 1 \tag{4.23}$$

$$r_2(r_2-1) C_2 (v^*)^{r_2 - 2} + k(k-1)K(v^*)^{k-2} = 0 \tag{4.24}$$

By (4.19) we have $r_2 \neq k$ so we can solve (4.24) for v^* and get

$$v^* = \left[\frac{k(1-k)K}{r_2(r_2-1)C_2} \right]^{\frac{1}{r_2-k}} > 0 \; . \tag{4.25}$$

Substituting this into (4.23) we obtain

$$C_2 = \left[r_2 \Big(\left\{ \frac{k(1-k)K}{r_2(r_2-1)} \right\}^{\frac{r_2-1}{r_2-k}} + kK \left\{ \frac{k(1-k)K}{r_2(r_2-1)} \right\}^{\frac{k-1}{r_2-k}} \Big) \right]^{-\frac{r_2-k}{1-k}} > 0 \; . \tag{4.26}$$

We proceed to verify that with this choice of C_1, C_2 and v^* the function φ given by (4.20)–(4.22) satisfies all conditions of Theorem 2.3:

Verification of (2.10): We have

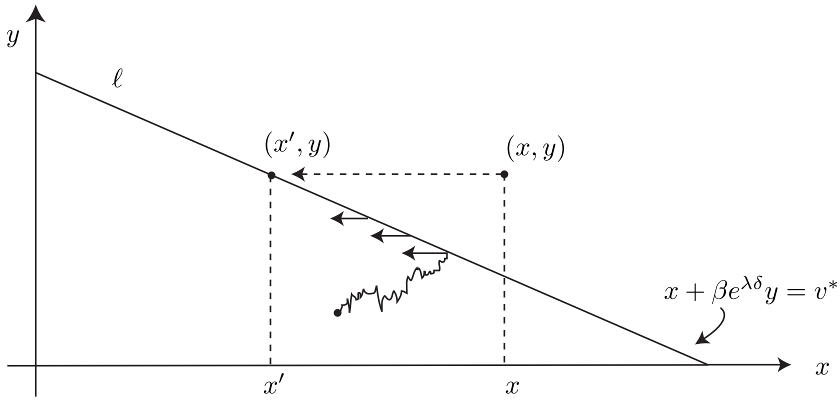
$$\frac{\partial \psi}{\partial x} = r_2 C_2 v^{r_2-1} + kK\, v^{k-1} \qquad \text{and}$$

$$\frac{\partial^2 \psi}{\partial x^2} = r_2(r_2-1) C_2 v^{r_2-2} + k(k-1)K\, v^{k-2}$$

Since $\frac{\partial^2 \psi}{\partial x^2} = 0$ for $v = v^*$ and $r_2 > k$ we conclude that $\frac{\partial^2 \psi}{\partial x^2} < 0$ for $v < v^*$ and hence $\frac{\partial \psi}{\partial x} > 1$ for $v < v^*$.

Verification of (2.11): If $\psi(x,y) = v - v^* + g(v^*)$ we get from (4.13)

$$\begin{aligned} \mathcal{L}_0 \psi(x,y) &= -\rho(v - v^* + g(v^*)) + (\theta + \beta e^{\lambda\delta})v + v^k \\ &= (\theta + \beta e^{\lambda\delta} - \rho)v + \rho(v^* - g(v^*)) + v^k \end{aligned}$$



Since $\mathcal{L}_0\psi(x,y) = 0$ for $v = v^*$ and $0 < k < 1$ we see that $\mathcal{L}_0\psi(x,y) \le 0$ for $v > v^*$ iff

$$\rho > \theta + \beta e^{\lambda\delta} , \quad \text{which is (4.18)} .$$

Now let $\widehat{\gamma}$ be the harvesting strategy which corresponds to *local time* at the line

$$\ell \colon = \{(x,y) \in \mathbf{R}^2;\ x + \beta e^{\lambda\delta} y = v^*\} \tag{4.27}$$

of the process $(\widehat{X}(t), \widehat{Y}(t))$ obtained by reflecting $(X(t), Y(t))$ horizontally to the left at ℓ. Define

$$D = \{(x,y) \in \mathbf{R}^2;\ 0 < x + \beta e^{\lambda\delta} y < v^*\} \tag{4.28}$$

Then if $(x,y) \in D$ we have $d\widehat{\gamma} = 0$. If $(x,y) \notin \bar{D}$ we harvest exactly enough to bring the x-level down to the value x' given by $v(x', y) = v^*$, i.e.

$$x' = v^* - \beta e^{\lambda\delta} y . \tag{4.29}$$

Note that if $v > v^*$ then

$$v - v^* = x + \beta e^{\lambda\delta} y - v^* = x - x' , \tag{4.30}$$

so this stragegy $\widehat{\gamma}$ gives exactly the value of φ stated in (4.22) for $v > v^*$.

In short: $\widehat{\gamma}$ harvests (horizontally) exactly what is necessary to keep the process $(X^{\hat{\gamma}}(t), Y^{\hat{\gamma}}(t))$ below or on the line ℓ.

We conclude that (2.14) holds, as well as (2.15), (2.16) and (2.17). Hence $\varphi = \Phi$ and $\widehat{\gamma}$ is optimal.

The precise construction of $\widehat{\gamma}$ goes as follows:

Consider the system $(X(t), Y(t)) \in \mathbf{R}^2$, where, as before,

$$Y(t) = \int_{-\delta}^{0} e^{\lambda s} X(t+s) ds = \int_{t-\delta}^{t} e^{\lambda(r-t)} X(r) dr . \tag{4.31}$$

In other words,

$$dX(t) = [\theta X(t) + \alpha Y(t) + \beta Z(t)]dt + \sigma[X(t) + \beta e^{\lambda\delta} Y(t)]dB(t) \tag{4.32}$$

$$dY(t) = [X(t) - \lambda Y(t) - e^{-\lambda\delta} Z(t)]dt \tag{4.33}$$

For $f, g \in C([0,\infty))$ define

$$\Lambda(f,g)(t) = f(t) - \max_{0\le s\le t}(f(s) + \beta e^{\lambda\delta} g(s) - v^*)^+ \; ; \qquad t \ge 0 \; . \tag{4.34}$$

Let $U(t), V(t)$ be the solution of the stochastic delay equations

$$\begin{aligned} dU(t) = {} & [\theta\Lambda(U,V) + \alpha y(\Lambda(U,V)) + \beta z(\Lambda(U,V))](t)dt \\ & + [\sigma\Lambda(U,V) + \beta e^{\lambda\delta} y(\Lambda(U,V))](t)dB(t) \; , \end{aligned} \tag{4.35}$$

$$dV(t) = [\Lambda(U,V) - \lambda y(\Lambda(U,V)) - e^{-\lambda\delta} z(\Lambda(U,V))](t)dt \; , \tag{4.36}$$

where (see (2.7))

$$y(\Lambda(U,V))(t) = \int_{-\delta}^{0} e^{\lambda s}\Lambda(U,V)(t+s)ds \; , \qquad z(\Lambda(U,V))(t) = \Lambda(U,V)(t-\delta) \; .$$

Now define

$$\widehat{X}(t) := \Lambda(U,V)(t), \qquad \widehat{Y}(t) = V(t) \; , \qquad \widehat{Z}(t) = \Lambda(U,V)(t-\delta) \tag{4.37}$$

and

$$\widehat{\gamma}(t) = \max_{0\le s\le t}(U(s) + \beta e^{\lambda\delta} V(s) - v^*)^+ \; . \tag{4.38}$$

Then by (4.34)

$$\widehat{X}(t) = \Lambda(U,V)(t) = U(t) - \widehat{\gamma}(t)$$

and therefore, by (4.35), (4.36),

$$d\widehat{X}(t) = [\theta\widehat{X}(t) + \alpha\widehat{Y}(t) + \beta\widehat{Z}(t)]dt + [\sigma\widehat{X}(t) + \beta e^{\lambda\delta}\widehat{Y}(t)]dB(t) - d\widehat{\gamma}(t) \tag{4.39}$$

and

$$d\widehat{Y}(t) = [\widehat{X}(t) - \lambda\widehat{Y}(t) - e^{-\lambda\delta}\widehat{Z}(t)]dt \; . \tag{4.40}$$

Moreover,

(4.41) $\qquad \widehat{\gamma}(t)$ is a nondecreasing $\mathcal{F}_t$-adapted process

(4.42) $\qquad \widehat{X}(t) + \beta e^{\lambda\delta}\widehat{Y}(t) \le v^*$ for all t

(4.43) $\qquad \widehat{\gamma}(t)$ increases only when $\widehat{X}(t) + \beta e^{\lambda\delta}\widehat{Y}(t) = v^*$.

The proof is similar to the proof of Theorem 6.1 in [F, p. 89]. We omit the proof.

We summarize what we have found in the following:

Theorem 4.1 *Let the harvested system $X(t) = X^\gamma(t)$ be on the form (4.1)–(4.2). Define*

$$J^\gamma(s,\xi) = E^{s,\xi}\Big[\int_0^T e^{-\rho(s+t)}(X(t)+\beta e^{\lambda\delta}Y(t))^k dt + \int_0^T e^{-\rho(s+t)}d\gamma(t)\Big]$$

and

$$\Phi(s,\xi) = \sup_{\gamma\in\Gamma} J^\gamma(s,\xi)\ .$$

Assume that

$$\alpha = \beta e^{\lambda\delta}(\lambda+\theta+\beta e^{\lambda\delta}) \tag{4.44}$$

and

$$\rho > \theta + \beta e^{\lambda\delta}\ . \tag{4.45}$$

Then, with $v = v(x,y) = x + \beta e^{\lambda\delta}y$,

$$\Phi(s,\xi) = \varphi(s,x,y) = \varphi(s,v) = \begin{cases} e^{-\rho s}(C_2 v^{r_2} + Kv^k) & \text{for} \quad 0 < v < v^* \\ e^{-\rho s}(v - v^*) + \varphi(s,v^*) & \text{for} \quad v \geq v^* \end{cases} \tag{4.46}$$

where C_2, K and v^* are given by (4.22), (4.18) and (4.25) respectively.

Moreover, the local time $\widehat{\gamma}$ at the line ℓ given by (4.27), as described in (4.37)–(4.43), is a corresponding optimal harvesting strategy.

Remark 4.2 If we let the delay δ approach 0 then the system $X(t)$ approaches the limit $X_0(t)$ given by

$$dX_0(t) = (\theta+\beta)X_0(t)dt + \sigma X_0(t)dB(t) - d\gamma(t)\ . \tag{4.47}$$

The corresponding no delay problem

$$\Phi_0(s,x):\ = \sup_\gamma E^{s,x}\Big[\int_0^T e^{-\rho(s+t)}X_0^\gamma(t)dt + \int_0^T e^{-\rho(s+t)}d\gamma(t)\Big] \tag{4.48}$$

will then be the limit of $\Phi(s,\xi) = \Phi_\delta(s,\xi)$ as $\delta \to 0^+$. The problem (4.48) is solved in [A].

Remark 4.3 It is possible to see more directly why the example studied in this section is finite-dimensional:

Define

$$W(t) = X(t) + \beta e^{\lambda\delta} Y(t) \; ; \qquad t \geq 0 \tag{4.49}$$

Then by (4.1) and (4.32) we have

$$\begin{aligned} dW(t) &= [\theta X(t) + \alpha Y(t) + \beta Z(t) + \beta e^{\lambda\delta} X(t) - \lambda\beta e^{\lambda\delta} Y(t) \\ &\quad - \beta Z(t)]dt + \sigma[X(t) + \beta e^{\lambda\delta} Y(t)]dB(t) \\ &= [(\theta + \beta e^{\lambda\delta})X(t) + (\alpha - \lambda\beta e^{\lambda\delta})Y(t)]dt + \sigma W(t) dB(t) \; . \end{aligned}$$

If we assume that (4.44) holds, then this can be written

$$dW(t) = (\theta + \beta e^{\lambda\delta}) W(t) dt + \sigma W(t) dB(t) \; ; \qquad t > 0 \; . \tag{4.50}$$

Moreover,

$$W(0) = X(0) + \beta e^{\lambda\delta} Y(0) = \xi(0) + \beta e^{\lambda\delta} \int_{-\delta}^{0} e^{\lambda\delta} \xi(s) ds =: w \; . \tag{4.51}$$

So $W(t)$ is an ordinary (no delay) geometric Brownian motion.

However, this in itself does not imply that the original delay harvesting problem for $X(t)$ can be reduced to a corresponding no-delay harvesting problem for $W(t)$, because we have a priori assumed harvesting from $X(t)$, not from $W(t) = X(t) + \beta e^{\lambda\delta} Y(t)$. On the other hand, the associated variational inequalities, culminating in Theorem 4.1, *proves* that the two problems have the same value function Φ. Moreover, if we harvest from $X(t)$ as described in Theorem 4.1 then we get the same result as when we harvest from $W(t)$ according to local time for $W(t)$ reflected downwards at $W(t) = v^*$. However, the latter harvesting strategy for $W(t)$ is not admissible for $X(t)$, because it implies harvesting from $X(t)$ and $Y(t)$ simultaneously (corresponding to a *normal* and not *horizontal* reflection of $(X(t), Y(t))$ at the line $\ell = \{(x, y); x + \beta e^{\lambda\delta} = v^*\}$).

References

[A] L. Alvarez: Optimal upward reflection and singular stochastic control of linear diffusions. Manuscript, University of Turku, October 11, 1999.

[F] M. Freidlin: Functional Integration and Partial Differential Equations. Princeton University Press 1985.

[AS] L. Alvarez and L. Shepp: Optimal harvesting of stochastically fluctuating populations. *J. Math. Biology* **37** (1998), 155–177.

[JS] M. Jeanblanc and A. Shiryaev: Optimization of the flow of dividends. *Russian Math. Surveys* **50** (1995), 257–277.

[KM] V.B. Kolmanovskii and T.L. Maizenberg: Optimal control of stochastic systems with aftereffect. In Stochastic Systems. Translated from Avtomatika i Telemekhanika, No. 1 (1973), 47–61.

[LØ1] E. Lungu and B. Øksendal: Optimal harvesting from a population in a stochastic crowded environment. *Math. Biosciences* **145** (1997), 47–75.

[LØ2] E. Lungu and B. Øksendal: Optimal harvesting from interacting populations in a stochastic environment. Preprint, University of Oslo 1999.

[KS] V.B. Kolmanovskii and L.E. Shaikhet: Control of Systems with Aftereffect. Translations of Mathematical Monographs, Vol. 157, American Mathematical Society 1996.

[M1] S. Mohammed: Stochastic Functional Differential Equations. Pitman 1984.

[M2] S. Mohammed: Stochastic differential systems with memory. Theory, examples and applications. In L. Decreusefond et al: Stochastic Analysis and Related Topics VI. The Geilo Workshop 1996. Birkhäuser 1998, pp. 1–77.

[P] P. Protter: Stochastic Integration and Differential Equations. Springer-Verlag 1990.