Some Special Solutions of the Equations of Axially Symmetric Gravitational Fields.

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Introduction.

The problem of axially symmetric fields was first treated by Weyl,* who succeeded in obtaining solutions for a static field in terms of the Newtonian potential of a distribution of matter in an associated canonical space. He also solved the more general problem involving the electric field. Levi Civita,† by slightly different methods, obtained solutions differing from those of Weyl in one respect, and discussed fully the case in which the field is produced by an infinite cylinder. R. Bach‡ has discussed the special case of two spheres and has calculated their mutual attraction. Bach also considered the field of a slowly rotating sphere, and obtained approximate solutions, taking the Schwarzchild solution as his zero-th approximation. The same field was discussed earlier by Leuse and Thirring,§ who considered the linear terms, only, in the gravitational equation. Kornel Lanczos‖ has also considered a special case of stationary fields and applied the results to cosmological problems. The more general case of gravitational fields produced by matter in stationary rotation has been treated by W. R. Andress¶ and E. Akeley.** Both these authors obtain approximate solutions of the general problem, and the latter treats at length the field of a rotating fluid.

The object of this paper is to present some special, but exact, solutions which the author obtained some years ago and, also, two methods of successive approximation for obtaining solutions of a more general type, which behave in an assigned manner at infinity and on a surface of revolution enclosing the rotating matter to which the field is due. Our solutions include as special cases the solutions of Weyl, Levi Civita and others which pertain to static fields. Also, the approximate solutions for stationary fields obtained by Leuse

* 'Ann. Physik,' vol. 54, p. 117 (1917).

† 'R. Acc. Lincei,' 5, vol. 28, p. 101 (1919).

‡ 'Mat. Z.,' vol. 13, p. 134 (1922).

§ 'Phys. Z.,' vol. 19, p. 156 (1918).

‖ 'Z. Physik,' vol. 21, p. 73 (1924).

¶ 'Proc. Roy. Soc.,' A, vol. 126, p. 592 (1930).

** 'Phil. Mag.,' vol. 11, p. 322 (1931).

and Thirring, Bach and Andress are contained in our solutions when appropriate choice of boundary conditions is made and higher order terms are neglected.

The special feature of this paper is the simplification of the gravitational equations which results on the introduction of canonical co-ordinates. This is always admissible in space free of matter. In order to illustrate the advantage gained by working with canonical co-ordinates we express Andress' approximate equations in these co-ordinates and show that, to the approximation considered by him, they are equivalent to our equations. No loss of generality is involved in the use of canonical co-ordinates, which are connected with any other co-ordinates preserving the normal form of the line element by a transformation of the type

$$r + iz = \phi(x_1 + ix_2).$$

In fact, the canonical co-ordinates serve to remind one of the degree of arbitrariness involved in our solutions.

We do not concern ourselves with the problem of finding the gravitational field inside matter. Certain stresses, t_{ik}, of non-gravitational origin, are necessary to maintain the steady rotation of the field producing matter, so we will assume that inside matter the gravitational potentials have any reasonable values which are continuous on the surface, and regard the equations

$$R_{ik} - \tfrac{1}{2}g_{ik}R = -(T_{ik} + t_{ik}),$$

as equations to determine the t_{ik} (T_{ik} being the components of the energy-momentum tensor).

The gravitational equations will be derived from a *Variational Principle* after the manner of Weyl. The latter's work was criticised by Levi Civita on the grounds that he did not make full use of the principle. Weyl based his calculation of the *action function* on a normal form of the line element and thereby obtained a set of equations which are not complete, though certainly compatible with the complete set. In order to avoid this difficulty we shall base our calculation of the *action function* on a non-normalised line element and show that it can be normalised without violating the gravitational equations. The possibility of introduction of canonical co-ordinates is immediately suggested by the form of our equations.

Exact solutions will be given in the case of the field due to an infinite rotating cylinder in the canonical space, and, to illustrate one method of approximation in the general case, the field of a rotating sphere will be worked out to a second order of approximation.

§ 1. *The Hamiltonian Function and Gravitational Equations.*

The field will depend upon two variables x_1 and x_2, $x_1 = 0$ being the axis of symmetry of the field. x_0 will be interpreted as time co-ordinate and x_3 as an angular variable varying from 0 to 2π. The fundamental quadratic form may be written

$$ds^2 = fdx_0^2 - \{e^{\mu}dx_1^2 + e^{\nu}dx_2^2 + ldx_3^2\} - 2mdx_0\, dx_3, \tag{1.1}$$

in which ν will be put equal to μ after the gravitational equations have been deduced. The effect of the rotation is represented mainly by the last term.

With the usual notation we find that

$$-g = r^2 e^{\mu+\nu}, \quad \text{where} \quad r^2 = fl + m^2, \tag{1.2}$$

and

$$g^{00} = r^{-2}l, \quad g^{11} = -e^{-\mu}, \quad g^{22} = -e^{-\nu}, \quad g^{33} = -r^{-2}f, \quad g^{03} = -r^{-2}m.$$

The only 3-index symbols which concern us are the following:

$$\left\{{00 \atop 1}\right\} = \tfrac{1}{2}e^{-\mu} f_1,\ \left\{{11 \atop 1}\right\} = \tfrac{1}{2}\mu_1,\ \left\{{22 \atop 1}\right\} = -\tfrac{1}{2}e^{-\mu+\nu}\, \nu_1,\ \left\{{33 \atop 1}\right\} = -\tfrac{1}{2}e^{-\mu}\, l_1,\ \left\{{03 \atop 1}\right\} = -\tfrac{1}{2}e^{-\mu}\, m_1,$$

$$\left\{{00 \atop 2}\right\} = \tfrac{1}{2}e^{-\nu} f_2,\ \left\{{11 \atop 2}\right\} = -\tfrac{1}{2}e^{\mu-\nu}\mu_2,\ \left\{{22 \atop 2}\right\} = \tfrac{1}{2}\nu_2,\ \left\{{33 \atop 2}\right\} = -\tfrac{1}{2}e^{-\nu}l_2,\ \left\{{03 \atop 2}\right\} = -\tfrac{1}{2}e^{-\nu}\, m_2,$$

where the suffix 1 denotes differentiation with respect to x_1 and 2 with respect to x_2.

The action function from which the field equations are derived differs from $R\sqrt{-g}$ by the divergence of a function of the gravitational potentials and their derivatives. It is G, defined by

$$2G = \left\{{ik \atop r}\right\} \frac{\partial}{\partial x_r}(g^{ik}\sqrt{-g}) - \left\{{ir \atop r}\right\} \frac{\partial}{\partial x_k}(g^{ik}\sqrt{-g}).$$

Inserting the above expressions for the 3-index symbols we obtain, finally,

$$2G = e^{-(\mu-\nu)/2}\left(\frac{f_1 l_1 + m_1^2}{r} + 2r_1\nu_1\right) + e^{(\mu-\nu)/2}\left(\frac{f_2 l_2 + m_2^2}{r} + 2r_2\mu_2\right). \tag{1.3}$$

The gravitational equations are the necessary and sufficient set of conditions for a stationary value of the integral

$$\int G dx$$

for arbitrary small variations of the g_{ik} which vanish on the boundary of the region of integration.

On carrying out the variation and putting $\nu = \mu$ the following equations are obtained :—

$$2r_{22} + (r_1\mu_1 - r_2\mu_2) + \frac{1}{2r}\{(f_1l_1 + m_1^2) - (f_2l_2 + m_2^2)\} = 0. \qquad (1.4)$$

$$-2r_{11} + (r_1\mu_1 - r_2\mu_2) + \frac{1}{2r}\{(f_1l_1 + m_1^2) - (f_2l_2 + m_2^2)\} = 0, \qquad (1.5)$$

$$\frac{\partial}{\partial x_1}\left(\frac{f_1}{r}\right) + \frac{\partial}{\partial x_2}\left(\frac{f_2}{r}\right) + \frac{f}{2r}\left\{\frac{[f,l] + [m,m]}{r^2} + 2\nabla^2\mu\right\} = 0, \qquad (1.6)$$

$$\frac{\partial}{\partial x_1}\left(\frac{l_1}{r}\right) + \frac{\partial}{\partial x_2}\left(\frac{l_2}{r}\right) + \frac{l}{2r}\left\{\frac{[f,l] + [m,m]}{r^2} + 2\nabla^2\mu\right\} = 0, \qquad (1.7)$$

$$\frac{\partial}{\partial x_1}\left(\frac{m_1}{r}\right) + \frac{\partial}{\partial x_2}\left(\frac{m_2}{r}\right) + \frac{m}{2r}\left\{\frac{[f,l] + [m,m]}{r^2} + 2\nabla^2\mu\right\} = 0 \qquad (1.8)$$

where

$$[\phi, \psi] \equiv \phi_1\psi_1 + \phi_2\psi_2,$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

§ 2. *The Compatibility of our Equations and Introduction of Canonical Co-ordinates.*

We notice that the last three equations are invariant with respect to a transformation of the type

$$x_1 + ix_2 = \phi\,(x_1' + ix_2').$$

Again, subtracting (1.5) from (1.4) yields a simple equation involving r only, namely,

$$r_{11} + r_{22} = 0. \qquad (2.1)$$

Hence, if z be the conjugate of r, we can make the following identification

$$r + iz = x_1 + ix_2. \qquad (2.2)$$

This equation defines our canonical co-ordinates. The equations (1.4) and (1.5) now become identical.

Multiplying (1.6) by l, (1.7) by f, (1.8) by $2m$, and adding, we get, in virtue of (1.2)

$$\frac{\partial}{\partial r}\left(\frac{(r^2)_1}{r}\right) + \frac{\partial}{\partial z}\left(\frac{(r^2)_2}{r}\right) - \frac{1}{r}\{[f, l] + [m, m]\} + 2r\,\nabla^2\mu = 0.$$

But

$$\frac{\partial}{\partial r}\left(\frac{(r^2)_1}{r}\right) + \frac{\partial}{\partial z}\left(\frac{(r^2)_2}{r}\right) \equiv 0.$$

Hence

$$\nabla^2\mu = \frac{1}{2r^2}\{[f, l] + [m, m]\}. \tag{2.3}$$

The remaining equations now become

$$\mu_1 = -\frac{1}{2r}\{f_1 l_1 + m_1^2 - (f_2 l_2 + m_2^2)\}, \tag{2.4}$$

$$f_{11} + f_{22} - \frac{f_1}{r} = -\frac{f}{r^2}\{[f, l] + [m, m]\}, \tag{2.5}$$

$$l_{11} + l_{22} - \frac{l_1}{r} = -\frac{l}{r^2}\{[f, l] + [m, m]\}, \tag{2.6}$$

$$m_{11} + m_{22} - \frac{m_1}{r} = -\frac{m}{r^2}\{[f, l] + [m, m]\}. \tag{2.7}$$

The last three equations are not independent. If f and l, for example, have been found (1.2) determines m which will satisfy (2.7) identically.

We also notice that when f, l, m have been determined, μ can be found from (2.3) and (2.4). But instead of the latter equation it is convenient to use another equation which has been calculated separately, namely, the one resulting from

$$R_{12} = 0.$$

(This equation could have been obtained by including a term $g_{12}\, dx_1\, dx_2$ in our quadratic form and putting $g_{12} = 0$ after variation of G.)

In canonical co-ordinates the equation is

$$\mu_2 = -\frac{1}{2r}\{f_1 l_2 + f_2 l_1 + 2m_1 m_2\}. \tag{2.8}$$

If (2.5)–(2.7) are taken into account, one easily verifies that $\mu_{12} - \mu_{21} = 0$, and that (2.3) is a consequence of (2.4) and (2.8). We can therefore write

$$\mu = -\int \frac{1}{2r}\{f_1 l_1 + m_1^2 - (f_2 l_2 + m_2^2)\}\, dr + \frac{1}{2r}\{f_1 l_2 + f_2 l_1 + 2m_1 m_2\}\, dz. \tag{2.9}$$

Thus the determination of f, l, m completely determines μ except for an additive constant, and it follows that μ cannot assume an arbitrarily assigned value on the boundary.

§ 3. *Transformation of Equations* (2.5)–(2.7).

One can always find a linear transformation of the differentials of the co-ordinates such that the fundamental quadratic form (1.1) transforms into

$$ds^2 = \mathrm{F}\, dt'^2 - \{e^{\mu}(dr^2 + dz^2) + \mathrm{L}\, d\theta^2\}.$$

In general, this transformation is purely local, *i.e.*, non-integrable. For example, let

$$dx_0 = dt = dt' \cosh u - d\theta' \sinh u, \quad d\theta = dx_3 = d\theta' \cosh u - dt' \sinh u, \quad (3.1)$$

$$f = \mathrm{F} \cosh^2 u - \mathrm{L} \sinh^2 u, \quad l = \mathrm{L} \cosh^2 u - \mathrm{F} \sinh^2 u,$$
$$m = \tfrac{1}{2} (\mathrm{L} - \mathrm{F}) \sinh 2u. \quad (3.2)$$

It follows that

$$fl + m^2 = \mathrm{FL} = r^2. \quad (3.3)$$

This relation suggests the substitutions

$$\mathrm{F} = r e^{-\lambda}, \quad \mathrm{L} = r e^{\lambda}. \quad (3.4)$$

The action function now becomes

$$\mathrm{G} = \frac{1}{2r} - \tfrac{1}{2} r [\lambda, \lambda] + 2r \sinh^2 \lambda \, [u, u] + [r, \mu],$$

and variation gives the following equations for λ and u :—

$$\frac{\partial}{\partial r}\left(r \frac{\partial \lambda}{\partial r}\right) + \frac{\partial}{\partial z}\left(r \frac{\partial \lambda}{\partial z}\right) + 2r \sinh 2\lambda \, [u, u] = 0, \quad (3.5)$$

$$\frac{\partial}{\partial r}\left(r \sinh^2 \lambda \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(r \sinh^2 \lambda \frac{\partial u}{\partial z}\right) = 0, \quad (3.6)$$

A special set of solutions suggests itself at once, namely,

$$u = \text{constant}, \quad \lambda = \log r - 2\psi, \quad (3.7)$$

where ψ is the Newtonian potential of an arbitrary axially symmetric distribution of matter in the canonical space (r, z, θ). It follows that F and L are identical with the f and l found by Weyl in the static case. Our f and l are linear combinations of Weyl's, with constant coefficients. They admit of a very simple interpretation—the observer in the canonical space (r, z, θ) is using a system of reference which rotates with constant angular speed to describe the static field of the canonical space (r', z', θ').

§ 4. *Special Solutions involving Functional Relation between λ and u.*

It is possible to obtain solutions of (3.5) and (3.6) on the assumption that u is a function of λ. It is easily verified that the condition for this is

$$\frac{d^2 u}{d\lambda^2} + 2 \frac{du}{d\lambda} \coth \lambda - 2 \left(\frac{du}{d\lambda}\right)^3 \sinh 2\lambda = 0. \quad (4.1)$$

The general solution of this equation is

$$u = u_0 + \tfrac{1}{2}\log\frac{\cosh\lambda \mp \sqrt{k^2\sinh^2\lambda + 1}}{\sinh\lambda} \tag{4.2}$$

where u_0 and k are arbitrary constants, and k^2 need not be positive. Let us now put

$$\psi = \int^u \sinh^2\lambda\, du = \pm\tfrac{1}{2}\int^\lambda \frac{\sinh\lambda\, d\lambda}{\sqrt{k^2\sinh^2\lambda+1}}.$$

Equation (3.6) shows that ψ satisfies the equation for a Newtonian potential in the canonical space, namely,

$$\psi_{11} + \psi_{22} + \psi_1/r = 0. \tag{4.3}$$

Integration of the expression for ψ gives

$$\psi = \pm\frac{1}{2k}\log\left(k\cosh\lambda + \sqrt{k^2\sinh^2\lambda + 1}\right), \tag{4.4}$$

Solving this equation for $\cosh\lambda$ we get

$$\left.\begin{aligned} \cosh\lambda &= \frac{1}{2k}\{e^{\pm 2k\psi} - (1-k^2)\,e^{\mp 2k\psi}\} \\ \text{and}\qquad \sqrt{k^2\sinh^2\lambda+1} &= \tfrac{1}{2}\{e^{\pm 2k\psi} + (1-k^2)\,e^{\mp 2k\psi}\} \end{aligned}\right\}. \tag{4.5}$$

We can substitute these expressions in (4.2) and thus find u, and then find f, l, m from the formulæ (3.2) and (3.4). The calculation, however, which is long and tedious, will not be given here. But one can verify directly that

$$f = r(\alpha_1^2 e^{\psi} - \gamma_1^2 e^{-\psi}), \quad l = r(-\alpha_2^2 e^{\psi} + \gamma_2^2 e^{-\psi}),$$
$$m = r(-\alpha_1\alpha_2 e^{\psi} + \gamma_1\gamma_2 e^{-\psi}) \tag{4.6}$$

satisfy equations (2.5), (2.6) and (2.7), where the constants satisfy the equation

$$\alpha_1\gamma_2 - \alpha_2\gamma_1 = 1, \tag{4.7}$$

and ψ is any function which is a formal solution of (4.3). But since f, l, m must be real, ψ cannot be complex—it must be real or purely imaginary.

We notice that if ψ is a function of r only, the solutions (4.6) are the most general solutions, for they involve four arbitrary constants, the fourth being contained in ψ, which is now of the form

$$\psi = -k\log r/r_0, \tag{4.8}$$

where both k and r_0 are arbitrary. (r_0, however, plays no essential rôle, for it can be absorbed in the constants (α_1, α_2, γ_1, γ_2) without violating (4.7)).

In this special case

$$\mu_1 = -\frac{1}{2r}(f_1 l_1 + m_1^2) = -\frac{1}{2r}(1 - r^2\psi_1^2) = \frac{k^2-1}{2r},$$

which, on integration, gives

$$\mu = \tfrac{1}{2}(k^2 - 1)\log r + \text{constant}, \tag{4.9}$$

k is not necessary real. The solutions corresponding to an imaginary k will be given separately.

It is convenient to introduce new constants defined by the formulæ

$$\alpha_1 = \kappa\beta_1,\quad \gamma_1 = \kappa\omega\beta_2,\quad \alpha_2 = \kappa\omega\beta_2^{-1},\quad \gamma_2 = \kappa\beta_1^{-1},\quad 1 - k = \varepsilon \tag{4.10}$$

where $\kappa = (1 - \omega^2)^{-\frac{1}{2}}$.

The expressions (4.6) and (4.9) can now be written in the form

$$f = \kappa^2(\beta_1^2 r^\epsilon - \omega^2\beta_2^2 r^{2-\epsilon}),\quad l = \kappa^2(\beta_1^{-2} r^{2-\epsilon} - \omega^2\beta_2^{-2} r^\epsilon),$$

$$m = \kappa^2\omega(\beta_2^2 r^{2-\epsilon} - \beta_1^2 r^\epsilon)\,\beta_1^{-1}\beta_2^{-1} \tag{4.11}$$

$$e^\mu = (r/r_0)^{-\epsilon(2-\epsilon)/2}. \tag{4.12}$$

When $\omega = 0$, these solutions reduce to those discussed by Levi Civita, *i.e.*, those characteristic of the gravitational field of an infinite cylinder in the canonical space. The modification of the field due to the stationary rotation of the cylinder is thus represented by the terms involving ω, which is of zero dimensions and may be regarded as a measure of the angular velocity of rotation. ε is of zero dimensions and proportional to the mass per unit length of cylinder. β_1 is of zero dimensions and very nearly equal to unity and β_2 is the reciprocal of a length—Newtonian theory gives no indication of its magnitude.

One of the most interesting effects of the rotation is to disturb the radial character of the field. It can be shown from the equations of motion that, in general, a particle started at rest anywhere in the field will not move radially, as in the corresponding static field. (The exceptional case is given by $\omega = 1$, which makes κ infinite.) This result is consistent with Einstein's fundamental hypothesis that a gravitational field is equivalent to an acceleration field. In our case, the forces derived from the ω^2 terms in f and l are analogous to centrifugal forces, and the forces derived from m correspond to Coriolis forces of the classical theory of rotating axes.

The more general solutions (4.6) allow for a variation of the density of the cylinder as z varies, but they are essentially solutions associated with an *infinite* cylinder.

If we write $i\psi$ for ψ in (4.6), the corresponding f, l, m are still formal solutions of the gravitational equations. We easily verify that these solutions are real provided

$$\alpha_1^2 - \gamma_1^2, \quad \alpha_2^2 - \gamma_2^2, \quad \alpha_1\alpha_2 - \gamma_1\gamma_2 \text{ are real}$$

and

$$\alpha_1^2 + \gamma_1^2, \quad \alpha_2^2 + \gamma_2^2, \quad \alpha_1\alpha_2 + \gamma_1\gamma_2 \text{ imaginary.}$$

These conditions are satisfied if we introduce new real constants defined by

$$\sqrt{2}\alpha_1 = a_1 + ib_1, \quad \sqrt{2}\gamma_1 = b_1 + ia_1, \quad \sqrt{2}\alpha_2 = a_2 + ib_2, \quad \sqrt{2}\gamma_2 = b_2 + ia_2.$$

The new constants are arbitrary except for the condition

$$a_1b_2 - a_2b_1 = 1. \tag{4.7$'$}$$

The solutions may now be written

$$r^{-1}f = (a_1^2 - b_1^2)\cos\psi - 2a_1b_1\sin\psi, \quad -r^{-1}l = (a_2^2 - b_2^2)\cos\psi$$
$$- 2a_2b_2\sin\psi, \quad -r^{-1}m = (a_1a_2 - b_1b_2)\cos\psi - (a_1b_2 + a_2b_1)\sin\psi. \tag{4.6$'$}$$

If ψ is a function of r only, it is of the form

$$\psi = -k'\log(r/\nu_0), \tag{4.8$'$}$$

and

$$\mu_1 = -\frac{1}{2r}(1 + r^2\psi_1^2) = -\frac{k'^2 + 1}{2r}.$$

It follows that

$$\mu = -\tfrac{1}{2}(k'^2 + 1)\log r + \text{constant}. \tag{4.9$'$}$$

These solutions are interesting because there are no corresponding, real solutions of the static problem, *i.e.*, the constants a and b cannot be chosen so as to make m vanish everywhere.

The space-time defined by these solutions is entirely without resemblance to space-time, empty of matter, ordinarily available to physical exploration. Its deviation from *flat* space-time could be demonstrated without exploring very large tracts of it. If these solutions have any applications at all, it must be to the fields of vast astronomical distributions of matter. The discussion of such fields is safer in the hands of astronomers.

Some further special solutions of (3.5) and (3.6) may be obtained by assuming λ to be a function of r only and that

$$u = v(r) + az,$$

where a is a constant.

λ and v satisfy the differential equations

$$\frac{d}{dr}\left(r\frac{d\lambda}{dr}\right) + 2r \sinh 2\lambda\,(k^2/r^2 \sinh^4 \lambda + a^2) = 0$$

$$\frac{dv}{dr} = k/r \sinh^2 \lambda.$$

When a is zero, the solutions of these equations reduce to those already discussed. The equation for λ may be solved by a method of successive approximation, but as the solutions obtained in this way do not appear to have any obvious application they will not be pursued. A more general solution obtained by successive approximations is given in the next paragraph.

§ 5. *Approximate Solutions satisfying given Boundary Conditions.*

If we multiply (2.5) by l, (2.6) by $-f$, and add, we obtain the equation

$$\frac{\partial}{\partial r}\left\{\frac{1}{r}(lf_1 - fl_1)\right\} + \frac{\partial}{\partial z}\left\{\frac{1}{r}(lf_2 - fl_2)\right\} = 0.$$

Introducing new functions defined by the equations

$$2\psi = \log l/f, \qquad \tau = m/r, \tag{5.1}$$

the above equation becomes

$$\psi_{11} + \psi_{22} + \psi_1/r = \frac{2\tau}{1-\tau^2}[\tau, \psi]. \tag{5.2}$$

The equation for τ is obtained by writing τr for m in (2.7). It is

$$\tau_{11} + \tau_{22} + \tau_1/r - \tau/r^2 = -\frac{\tau}{r^2}\{[f, l] + [m, m]\}, \tag{5.3}$$

where

$$f = r\sqrt{1-\tau^2}\,e^{-\psi}, \qquad l = r\sqrt{1-\tau^2}\,e^{\psi}. \tag{5.4}$$

Provided m and the differential coefficients of f are small quantities, equations (5.2) and (5.3) are forms suitable for obtaining solutions by successive approximations. The first approximations are got by neglecting the right-hand sides. They are

$$\psi_0 = \log r - 2\mathrm{V}, \qquad \tau = \tau_0, \tag{5.5}$$

where V is the Newtonian potential of an arbitrary, axially symmetrical distribution of matter in the canonical space (r, z, θ), and τ_0 is the coefficient of

$\sin\theta$ in the expansion of $\phi(r, z, \theta)$, which is a second arbitrary Newtonian potential. It is of the form

$$\phi = \phi_0(r, z) + \tau_0 \sin\theta + \sum_{n=2}^{\infty} \phi_n \sin n\theta + \text{cosine terms}.$$

Each term of the expansion is a solution of Laplace's equation.

It will be assumed that all the matter producing the field is enclosed by a surface of revolution S and that on this surface f, l, m assume assigned values, while at infinity $f \to 1$, $l \to r^2$, $m \to 0$, that is, the metric approaches that of the *Special Theory of Relativity.*

We can always determine ψ_0 and τ_0 such that these conditions are satisfied.*

If we write

$$\psi' = \psi - \psi_0, \qquad \tau' = \tau - \tau_0, \tag{5.6}$$

ψ' and τ' are small quantities of the second order which vanish on S and at infinity. They satisfy the differential equations

$$\psi'_{11} + \psi'_{22} + \psi'_1/r = \frac{2\tau}{1-\tau^2}[\tau, \psi], \tag{5.2)'$$

$$\tau'_{11} + \tau'_{22} + \tau'_1/r - \tau'/r^2 = -\frac{\tau}{r^2}\{[f, l] + [m, m]\}. \tag{5.3)'$$

Approximate solutions of these equations can be obtained if on the right-hand sides quantities of order higher than the second are neglected, τ_0 and the differential coefficients of f being treated as small.

To this order, the right-hand side of (5.2)′ is

$$\rho_0 = 2\tau_0\tau_{01}/r, \tag{5.7}$$

and the right-hand side of (5.3)′, by means of (5.4) and (5.5)

$$\sigma_0 = -4\tau_0 V_1/r. \tag{5.8}$$

It is convenient to introduce the functions

$$\phi' = \tau' \sin\theta, \quad \sigma'_0 = \sigma_0 \sin\theta. \tag{5.9}$$

We have now to find functions ψ' and ϕ' which vanish on S and at infinity and satisfy the Poisson equations

$$\nabla^2\psi' = \rho_0, \quad \nabla^2\phi' = \sigma_0'.$$

The problem may be regarded as a purely geometric one and can be solved with the aid of Green's function, *i.e.*, a function G $(x, y, z;\ x', y', z')$, which

* Dirichlet's problem for space.

vanishes on S and at infinity, and satisfies Laplace's equation at all points except (x', y', z'), where it behaves like $1/4\pi\sqrt{(x-x')^2+\ldots}$. In terms of this function

$$\left.\begin{aligned} \psi' &= -\int \mathrm{G}(x, \ldots, x', \ldots)\, \rho_0(x', \ldots)\, dx' \ldots \\ \phi' &= -\int \mathrm{G}(x, \ldots, x', \ldots)\, \sigma_0'(x', \ldots)\, dx' \ldots \end{aligned}\right\}, \tag{5.10}$$

where the region of integration is the space bounded by S and the sphere at infinity.

Approximations of higher order can be obtained by similar processes.

Using the expressions (5.4) for f and l, one easily verifies the relations

$$f_1 l_1 + m_1^2 = 1 - r^2\{(1-\tau^2)\,\psi_1^2 - \tau_1^2/(1-\tau^2)\},$$
$$f_2 l_2 + m_2^2 = -r^2\{(1-\tau^2)\,\psi_2^2 - \tau_2^2/(1-\tau^2)\},$$
$$f_1 l_2 + f_2 l_1 + 2m_1 m_2 = -2r^2\{(1-\tau^2)\,\psi_1\psi_2 - \tau_1\tau_2/(1-\tau^2)\}.$$

If we substitute these expressions in (2.9) and neglect terms of order higher than the second, and bear in mind the equations satisfied by V and ψ', we get, eventually,

$$\mu + 2\mathrm{V} - \psi' = \int \tfrac{1}{2}r\{4(\mathrm{V}_1^2 - \mathrm{V}_2^2) - \tau_0^2/r^2 - (\tau_{01}^2 - \tau_{02}^2)\}\, dr + \tfrac{1}{2}r\{8\mathrm{V}_1\mathrm{V}_2 - 2\tau_{01}\tau_{02}\}\, dz. \tag{5.11}$$

Another method of approximation is available when S has certain forms. This method will be illustrated by an example in § 7.

§ 6. *Andress' Equations in Canonical Co-ordinates.*

In the second part of his paper Andress deduces the approximate equations of the stationary field with axial symmetry on the basis of the quadratic form

$$ds^2 = -e^{\lambda}(dx^2 + dr^2) - r^2 e^{-\rho+\epsilon} + e^{\rho+\epsilon}\, dt^2 + 2r\tau\, d\theta\, dt, \tag{6.1}$$

and the final forms of his differential equations are*

$$\tau_{11} + \tau_{22} + \tau_2/r - \tau/r^2 = 0, \tag{6.2}$$

$$\nabla^2(\rho + \epsilon) = -\tau_1^2 - (\tau_2 + \tau/r)^2, \tag{6.3}$$

$$\epsilon_{11} + \epsilon_{22} + 2\epsilon_2/r = -(\tau_1^2 + \tau_2^2 + \tau\tau_2/r + \tau^2/r^2), \tag{6.4}$$

* *Loc. cit.*, pp. 601, 602. In (6.5), Andress' (5.44), he has ϵ instead of $-\epsilon$ on the left-hand sides. But it is clear from his equations (2.11)–(2.14) and (5.11), (5.12) and (5.16) that $-\epsilon$ is correct. There are other minor misprints in the indices of the last equation.

$$\left.\begin{aligned}\frac{\partial}{\partial r}(\lambda+\rho-\varepsilon) &= \tfrac{1}{2}r(\rho_2{}^2-\rho_1{}^2)-r(\varepsilon_{11}-\varepsilon_{22}) \\ &\qquad -r\tau(\tau_{11}-\tau_{22}-\tau_2/r)-\tfrac{1}{2}r(\tau_1{}^2-\tau_2{}^2+\tau^2/r^2) \\ \frac{\partial}{\partial x}(\lambda+\rho-\varepsilon) &= r\rho_1\rho_2+2r\varepsilon_{12}+r\tau_1(\tau_2+\tau/r)+2r\tau\tau_{12}\end{aligned}\right\}, \quad (6.5)$$

where the suffix 1 means differentiation with respect to x and 2 with respect to r.

Comparing (6.1) with (1.1) of this paper we get

$$f = e^{\rho+\varepsilon}, \quad l = r^2 e^{-\rho+\varepsilon}, \quad m = -r\tau, \quad \mu = \lambda.$$

It follows from (5.1) that

$$2\psi = \log l/f = 2\log r - 2\rho \qquad (6.6)$$

and

$$fl + m^2 = r^2(e^{2\varepsilon} + \tau^2). \qquad (6.7)$$

In virtue of the existence of canonical co-ordinates we can write

$$e^{2\varepsilon} + \tau^2 = 1, \qquad (6.8)$$

i.e., we can identify r and x with the canonical co-ordinates without changing the form of (6.1). The relation (6.8) does not follow accurately from Andress' equations on account of his neglecting all but linear terms in (6.2).

However, to the order of approximation considered by Andress, we can write, in virtue of (6.8),

$$\varepsilon = -\tfrac{1}{2}\tau^2. \qquad (6.9)$$

On substituting this expression for ε in (6.4) we find that the latter is identically satisfied in virtue of (6.2). Making the same substitution in (6.3) it reduces to

$$\nabla^2\rho = -\frac{2\tau\tau_2}{r}, \qquad (6.10)$$

which is identical with the equation satisfied by ψ' of this paper when higher order terms are neglected.

The equations (6.5) reduce to

$$\left.\begin{aligned}\frac{\partial}{\partial r}(\lambda+\rho) &= \tfrac{1}{2}r(\rho_2{}^2-\rho_1{}^2)+\tfrac{1}{2}r(\tau_1{}^2-\tau_2{}^2)-\tau^2/2r, \\ \frac{\partial}{\partial x}(\lambda+\rho) &= r\rho_1\rho_2 - r\tau_1\tau_2\end{aligned}\right\}. \quad (6.11)$$

These last equations are equivalent to (5.11) of this paper if $(2V-\psi')$ is written for ρ and μ for λ on the left-hand sides and $2V$ for ρ and τ_0 for τ on the right-hand sides. The solution of (6.10) actually given by Andress is a particular one—the Poisson Integral. He omits the complementary function 2V which is necessary to make the solution reduce to his solution for the static case for a vanishing τ.

We thus see that the use of canonical co-ordinates greatly simplifies the differential equations to be solved. If we consider any transformation of co-ordinates which preserves the normal form of $e^{\mu}(dr^2 + dz^2)$, the expressions for f, l, m are obtained by direct substitution in the expressions for these potentials in the canonical system, and e^{μ} must be multiplied by the modulus of the transformation. When discussing the static case Andress actually introduces canonical co-ordinates by putting $\nu = -\rho$. In the stationary case, however, it is not so easy to spot canonical co-ordinates unless one proceeds from the *variational principle*.

§ 7. *The Field of a Rotating Sphere.*

Green's function is known for a spherical surface, but the integrations involved in the calculation of second order terms are very cumbersome, and labour is saved by using another method. We will show how to calculate second order terms in ψ, but the calculation for the second order terms in τ will not be given in detail. The method is essentially the same in the two cases, though the differential equations involved are different.

It is convenient to use spherical polar co-ordinates associated with the canonical space. They are defined by the transformation

$$x = \mathrm{R}\sin\theta'\cos\theta, \quad y = \mathrm{R}\sin\theta'\sin\theta, \quad z = \mathrm{R}\cos\theta', \quad r = \mathrm{R}\sin\theta'. \tag{7.1}$$

The first approximations in this case may be written

$$\left.\begin{aligned} \mathrm{V} &= -\frac{\kappa\mathrm{M}}{\mathrm{R}} + \sum_{n=1}^{\infty} \mathrm{A}_n\mathrm{P}_n(\cos\theta')/\mathrm{R}^{n+1} \\ \tau_0 &= \frac{\mathrm{B}_1\sin\theta'}{\mathrm{R}^2} + \sum_{n=2}^{\infty} \mathrm{B}_n\mathrm{P}_n{}^1(\cos\theta')/\mathrm{R}^{n+1} \end{aligned}\right\}. \tag{7.2}$$

$-\kappa\mathrm{M}/\mathrm{R}$ is the ordinary Newtonian potential for a spherically symmetrical distribution of matter. We will assume that the distribution in the canonical space deviates but slightly from spherical symmetry and that the angular speed is not too great. This amounts to assuming that the A_n are small quantities of the second order. We also assume, for simplicity, that the B_n ($n = 2, 3, \ldots$) are of the second order. The A_n and B_n have been chosen so as to satisfy boundary conditions on the surface of the sphere. As these boundary conditions do not affect the determination of terms of higher order, we need not limit ourselves to any particular set of boundary values.

Hence, disregarding all terms other than the dominant ones, we find that, by (5.7)

$$\rho_0 = 2\tau_0\tau_{01}/r = 2B_1^2(1 - 3\sin^2\theta')/R^6. \quad (7.3)$$

It is required to find a solution of the equation (5.2)′, which becomes

$$\frac{\partial}{\partial R}\left(R^2\frac{\partial\psi'}{\partial R}\right) + \frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\left(\sin\theta'\frac{\partial\psi'}{\partial\theta'}\right) = 2B_1^2(1 - 3\sin^2\theta')/R^4. \quad (7.4)$$

The right-hand side of this equation can be written in the form

$$\rho_0 = \Sigma k_n P_n(\cos\theta')/R^4, \quad (7.3)'$$

where

$$k_0 = -2B_1^2, \quad k_2 = 4B_1^2,$$

and all the other k_n are zero.

We can find a particular solution of (7.4) of the form

$$\Sigma f_n P_n(\cos\theta'),$$

where f_n satisfies the differential equation

$$\frac{d}{dR}\left(R^2\frac{df_n}{dR}\right) - n(n+1)f_n = k_n/R^4.$$

This equation has a particular integral

$$k_n/R^4\{12 - n^2 - n\}.$$

It follows that

$$B_1^2(-1 + 4P_2(\cos\theta'))/6R^4 \quad (7.5)$$

is a particular solution of (7.4).

But on the sphere $R = a$ (a is not to be confused with the gravitational radius) ψ' is zero. Hence to (7.5) one must add a term of the type

$$\frac{C_1}{R} + \frac{C_2P_2(\cos\theta')}{R^3}.$$

Substituting boundary conditions we find, ultimately, that

$$\psi' = B_1^2\left\{\frac{1}{a^3} - \frac{1}{R^3} + 4\left(\frac{1}{R^3} - \frac{1}{aR^2}\right)P_2(\cos\theta')\right\}\Big/6R. \quad (7.6)$$

Again, putting in their values for the dominant terms in (5.11), we get

$$\mu + 2V - \psi' = \int \frac{r}{2}\left\{\frac{4\kappa^2 M^2 (r^2 - z^2)}{R^6} - \frac{B_1^2 (5r^4 - 11r^2z^2 + 2z^4)}{R^{10}}\right\} dr + \frac{r}{2}\left\{\frac{8\kappa^2 M^2 rz}{R^6} + \frac{6B_1 (z^2 - 2r^2)\, rz}{R^{10}}\right\} dz$$

$$= -\kappa^2 M^2 r^2/R^4 + B_1^2\{-r^2/2R^6 + 9r^4/8R^8\}. \tag{7.7}$$

To the same order, we get from (5.8) and (7.2)

$$\sigma_0 = -4B_1\kappa M \sin\theta'/R^5 ; \tag{7.8}$$

and equation (5.3)′ for τ' may be written

$$\frac{\partial}{\partial R}\left(R^2 \frac{\partial \tau'}{\partial R}\right) + \frac{1}{\sin\theta'}\frac{\partial}{\partial \theta'}\left(\sin\theta' \frac{\partial \tau'}{\partial \theta'}\right) - \frac{\tau'}{\sin^2\theta'} = -4B_1\kappa M \sin\theta'/R^3. \tag{7.9}$$

This equation has a particular solution of the form

$$k \sin\theta'/R_3.$$

Evaluating the constant and adding a solution of the homogeneous equation, we finally get a function satisfying (7.9) and vanishing on the surface of the sphere and at infinity. It is

$$\tau' = -B_1 \kappa M \sin\theta' \left(\frac{1}{R^3} - \frac{1}{aR^2}\right). \tag{7.10}$$

It is assumed that both B_1 and κM are small. Approximations of higher order can be obtained in an analogous manner. The solutions will proceed in powers of κM and B_1. At no stage do we introduce a singularity other than the original one at $R = 0$, so there should be no difficulty about convergence, provided the above-named constants are small.

Disregarding A_1, B_2, etc., the values of f, l, m, as far as second order terms, are obtained by expanding (5.4) and putting in the expressions for V, ψ', τ_0 and τ'. These values are

$$\left.\begin{aligned} f &= 1 - \frac{2\kappa M + B_1^2/6a^3}{R} + \frac{2\kappa^2 M^2}{R^2} + \frac{2B_1^2 P_2(\cos\theta')}{3aR^3} - \frac{B_1^2(1 + 2P_2(\cos\theta'))}{6R^4} \\ r^{-2} l &= 1 + \frac{2\kappa M + B_1^2/6a^3}{R} + \frac{2\kappa^2 M^2}{R^2} - \frac{2B_1^2 P_2(\cos\theta')}{3aR^3} - \frac{B_1^2(1 - 2P_2(\cos\theta'))}{2R^4} \\ r^{-2} m &= \frac{B_1 \sin\theta'}{R^2}\{(1 + \kappa M/a) - \kappa M/R\} \end{aligned}\right\}. \tag{7.11}$$

If squares and products of κM and B_1 are neglected, the form (1.1) becomes

$$ds^2 = \left(1 - \frac{2\kappa M}{R}\right) dt^2 - \left\{\left(1 + \frac{2\kappa M}{R}\right)(dr^2 + dz^2 + r^2 d\theta^2)\right\} - \frac{2B_1 r^2}{R^3} d\theta\, dt.$$

This approximation is identical with the form obtained by Leuse and Thirring, and with Bach's first approximation. These authors write $-\kappa \frac{4}{5} Ml^2 w$ for B_1, *i.e.*, they assume B_1 to be proportional to the angular momentum.

If we examine the coefficient of $1/R$ in the first two of equations (7.11), we notice that at a sufficiently great distance from the sphere the effect of the rotation is to increase the effective mass of the sphere by $B_1^2/12\kappa a^3$.

We also notice that τ_0 is proportional to the magnetic potential of a uniformly magnetised sphere. The effect of this term on the motion of a material particle is analogous to the effect of a Coriolis force.

One cannot expect our second order terms to fit with the second order terms obtained by Bach. For, in fact, they do not represent the field of the same distribution of matter. A sphere in Bach's space is not a sphere in the canonical space. Weyl has shown, in the static case, that the canonical co-ordinates are connected with the co-ordinates of the Schwarzchild space by the transformation

$$r + iz = r' + iz' - \frac{(a/2)^2}{r' + iz'},$$

where a is the *gravitational radius* of the particle, in this case. The particle is transformed into a uniform rod of length $2a$ in the canonical space. Since the transformation involves the square of a (or κM), the first approximations of Bach are necessarily contained in our first approximations when boundary conditions are suitably chosen.

[*Note.*—When the field in question is due to a large body there is no comparison between the gravitational radius and the ordinary radius. For example, the gravitational radius of the sun is about 1·47 kilometres and the gravitational radius of the earth only 5 millimetres. Hence there is no danger of our method of approximation leading to a singularity of the gravitational potentials outside matter.]

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