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Some Spinor-Curvature Identities

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Abstract: We describe a class of spinor-curvature identities which exist for Riemannian or Riemann-Cartan geometries. Each identity relates an expression quadratic in the covariant derivative of a spinor field with an expression linear in the curvature plus an exact differential. Certain special cases in 3 and 4 dimensions which have been or could be used in applications to General Relativity are noted.
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Spinor techniques continue to provide new results in physics and mathematics. For any n-dimensional Riemann or Riemann-Cartan geometry with a spinor structure we have found a certain class of identities each of which relates an expression quadratic in the covariant differential of a spinor field with an expression linear in the curvature plus an exact differential. We describe these identities in general and consider in particular certain special cases in 3 and 4 dimensions which have application to Einstein's gravity theory.

An orthonormal coframe field ϑ^a (note: the metric tensor components g_{ab} are constant) and the metric compatible connection 1-form $\omega^{ab} = -\omega^{ba}$ are "potentials" for the *torsion* and *curvature* 2-forms:

$$\Theta^a := d\vartheta^a + \omega^a{}_b \wedge \vartheta^b, \qquad \Omega^a{}_b := d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{1}$$

The dual n-k forms $\eta^{ab\cdots} := *(\vartheta^a \wedge \vartheta^b \cdots)$ are sometimes convenient.

A Clifford algebra $\{1, \gamma_a, \cdots, \gamma_{ab\cdots c}, \cdots, \gamma\}$, where $\gamma_{ab\cdots c} := \gamma_{[a}\gamma_b \cdots \gamma_{c]}$ and $\gamma := \gamma^1\gamma^2 \cdots \gamma^n$, is generated by "Dirac matrices" satisfying $\gamma_a\gamma_b + \gamma_b\gamma_a = 2g_{ab}$. Some of our results are most succinctly presented in terms of the Clifford algebra valued forms:

$$\vartheta := \gamma_a\vartheta^a, \qquad \omega := \tfrac{1}{4}\gamma_{ab}\omega^{ab} \tag{2}$$

$$\Theta := \gamma_a\Theta^a, \qquad \Omega := \tfrac{1}{4}\gamma_{ab}\Omega^{ab}, \tag{3}$$

In this notation eqs (1) become

$$\Theta := D\vartheta := d\vartheta + \omega \wedge \vartheta + \vartheta \wedge \omega, \qquad \Omega := d\omega + \omega \wedge \omega. \tag{4}$$

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(A nice development of such "Clifforms" is given by Dimakis and Müller-Hoissen 1991.)

The identities discussed here are all linear combinations of expressions of the general type

$$2D(\overline{\psi}A)\wedge D(B\psi)\equiv d\{\overline{\psi}A\wedge D(B\psi)-\varepsilon D(\overline{\psi}A)\wedge B\psi\}-\varepsilon 2\overline{\psi}A\wedge\Omega\wedge B\psi, \tag{5}$$

where ψ is a Dirac spinor, A and B are matrix valued forms of rank a, b and $\varepsilon=(-1)^a$. The general identity is easily established from

$$d\{\overline{\psi}A\wedge D(B\psi)-\varepsilon D(\overline{\psi}A)\wedge B\psi\}\equiv 2D(\overline{\psi}A)\wedge D(B\psi)+\varepsilon\overline{\psi}A\wedge D^2(B\psi)-\varepsilon D^2(\overline{\psi}A)\wedge B\psi, \tag{6}$$

using $D^2(B\psi)=\Omega B\psi$ and $D^2(\overline{\psi}A)=-\overline{\psi}A\Omega$. The simplest case, $A=B=1$, yields an identity with the full curvature 2-form:

$$2D\overline{\psi}\wedge D\psi\equiv d(\overline{\psi}D\psi-D\overline{\psi}\psi)-\tfrac{1}{2}\Omega^{ab}\overline{\psi}\gamma_{ab}\psi. \tag{7}$$

It should be noted that identities of this general form considered here are not confined to spinor fields, for example for 2 vector fields we have

$$\begin{aligned} DW_\alpha\wedge DV^\alpha &\equiv d(W_\alpha DV^\alpha - V^\alpha DW_\alpha)-(W_\alpha D^2V^\alpha - V^\alpha D^2 W_\alpha)\\ &\equiv d(W_\alpha DV^\alpha - V^\alpha DW_\alpha)-2W_\alpha\Omega^\alpha{}_\beta V^\beta. \end{aligned} \tag{8}$$

Here, however, we consider spinor fields and focus on the cases where A and B in eq (4) are simple combinations naturally constructed from ϑ, their matrix-wedge products $\vartheta^K := \vartheta\wedge\vartheta\wedge\ldots$ (K factors) and the Hodge dual $*$ because, for suitable choices of such A and B, we can project out curvature combinations such as the Einstein tensor or the scalar curvature; we do not know how to get these using vectors or tensors.

The 3 and 4 dimensional versions of such identities have application to gravity theory. In particular they can be used to replace the curvature terms in Lagrangians and Hamiltonians. This can be advantageous because the quadratic spinor expressions have faster asymptotic fall off.

The first construction of this type contained the 4-dimensional Einstein tensor and was used in the Witten positive energy proof (Witten 1981, Nester 1981). Subsequently it was shown that the *Hamiltonian density* for Einstein gravity could be expressed as a 4-covariant quadratic spinor 3-form (Nester 1984):

$$\mathcal{H}(\psi):=2\{D(\overline{\psi}\gamma_5\vartheta)\wedge D\psi - D\overline{\psi}\wedge D(\gamma_5\vartheta\psi)\}\equiv 2N^\mu G^\nu_\mu\eta_\nu + dB, \tag{9}$$

where

$$B:=\overline{\psi}\gamma_5\vartheta\wedge D\psi + D(\overline{\psi}\gamma_5\vartheta)\psi - \overline{\psi}D(\gamma_5\vartheta\psi)+D\overline{\psi}\wedge(\gamma_5\vartheta\psi), \tag{10}$$

and $N^\mu=\overline{\psi}\gamma^\mu\psi$. The key is that

$$\Omega^{\mu\nu}\wedge\overline{\psi}\gamma_5(\vartheta\gamma_{\mu\nu}+\gamma_{\mu\nu}\vartheta)\psi\equiv\Omega^{\mu\nu}\wedge\eta_{\mu\nu\alpha}\overline{\psi}\gamma^\alpha\psi\equiv G^\nu{}_\mu\eta_\nu\overline{\psi}\gamma^\mu\psi. \tag{11}$$

The Hamiltonian density (9) can be decomposed, with respect to the normal to any space-like hypersurface, into positive and negative definite parts and is locally non-negative if ψ

satisfies the Witten equation $\gamma^a D_a \psi = 0$, thereby permitting a positive energy proof and more: a non-negative "localization" of gravitational energy.

More generally for an n-dimensional Riemanian space the Einstein tensor (for $n > 3$) appears in the "Hamiltonian" $n-1$ form (Mason and Frauendiener 1990):

$$\begin{aligned}\mathcal{H}(\psi) &:= 4D\overline{\psi} \wedge \gamma\vartheta^{(n-3)} \wedge D\psi \\ &\equiv 2\overline{\psi}\gamma^b\psi G^a{}_b\eta_a + d\{\overline{\psi}\gamma\vartheta^{(n-3)} \wedge D\psi + D\overline{\psi} \wedge \gamma\vartheta^{(n-3)}\psi\}.\end{aligned} \tag{12}$$

The corresponding Riemann-Cartan expressions are considerably more complicated as it is necessary to take into account non-vanishing contributions from $D\vartheta = \Theta$ on the left hand side and grade 5 Clifford algebra terms proportional to $D\Theta^a \equiv \Omega^a{}_b \wedge \vartheta^b$ on the right hand side; the general pattern is binomial, in particular $\overline{\psi}\gamma(\Omega\vartheta\vartheta + 2\vartheta\Omega\vartheta + \vartheta\vartheta\Omega)\psi$ projects out the Einstein 4-form in 5 dimensions; for 6 dimensions $\overline{\psi}\gamma(\Omega\vartheta\vartheta\vartheta + 3\vartheta\Omega\vartheta\vartheta + 3\vartheta\vartheta\Omega\vartheta + \Omega\vartheta\vartheta\vartheta)\psi$ gives the Einstein 5-form, etc. By the way, the rank of such identities is not as restricted as our discussion has indicated for we could take ψ to be a spinor valued differential form of some suitable rank.

The next application came up in a new gravitational energy positivity proof and localization using 3-dimensional spinors (Nester and Tung 1993). The *scalar curvature* term and the boundary term in the ADM Hamiltonian were replaced using the new *spinor identity*

$$2[\nabla(\varphi^\dagger i\vartheta) \wedge \nabla\varphi - \nabla\varphi^\dagger \wedge \nabla(i\vartheta\varphi)] \equiv dB - (\varphi^\dagger\varphi)\Omega^{ij} \wedge \eta_{ij}, \tag{13}$$

where

$$B := \varphi^\dagger i\vartheta \wedge \nabla\varphi - \varphi^\dagger\nabla(i\vartheta\varphi) + \nabla(\varphi^\dagger i\vartheta)\varphi + (\nabla\varphi^\dagger) \wedge i\vartheta\varphi. \tag{14}$$

The key detail is

$$\varphi^\dagger\varphi R\sqrt{g}d^3x \equiv \varphi^\dagger\varphi\Omega^{ij} \wedge \eta_{ij} \equiv \Omega^{ij} \wedge \varphi^\dagger i(\vartheta\gamma_{ij} + \gamma_{ij}\vartheta)\varphi. \tag{15}$$

Generalizing, we find an identity which contains the Riemannian scalar curvature n-form: $R\eta = \Omega^{ab} \wedge \eta_{ab}$,

$$2D\overline{\psi} \wedge \gamma\vartheta^{(n-2)} \wedge D\psi \equiv d\{\overline{\psi}\gamma\vartheta^{(n-2)} \wedge D\psi - D\overline{\psi} \wedge \gamma\vartheta^{(n-2)}\psi\} - \overline{\psi}\psi\Omega^{ab} \wedge \eta_{ab}. \tag{16}$$

The corresponding Reimann-Cartan identity is more complicated but again follows a binomial pattern, the general structure can be inferred from the special cases in low dimensions. For 3 dimensions we have eq (13), for 5 dimensions the Clifform combination which projects out purely the scalar curvature 5-form is again $\overline{\psi}\gamma(\Omega\vartheta\vartheta\vartheta + 3\vartheta\Omega\theta\vartheta + 3\vartheta\vartheta\Omega\vartheta + \vartheta\vartheta\vartheta\Omega)\psi$, for 4 dimensions it is $\overline{\psi}\gamma(\Omega\vartheta\vartheta + 2\vartheta\Omega\vartheta + \vartheta\vartheta\Omega)\psi$ so that we have in detail

$$\begin{aligned}D(\overline{\psi}\gamma\vartheta^2) \wedge D\psi - 2D(\overline{\psi}\gamma\vartheta) \wedge D(\vartheta\psi) + D\overline{\psi}\gamma \wedge D(\vartheta^2\psi) &\equiv -R\overline{\psi}\psi\eta \\ + d\{-D\overline{\psi} \wedge \gamma\vartheta^2\psi - D(\overline{\psi}\gamma\vartheta) \wedge \vartheta\psi - \overline{\psi}\gamma\vartheta \wedge D(\vartheta\psi) &+ \overline{\psi}\gamma\vartheta^2 \wedge D\psi\},\end{aligned} \tag{17}$$

for the 4-dimensional identity. This expression illustrates how much more complicated the identities are with torsion. The Riemannian version is simply

$$4D\overline{\psi} \wedge \gamma\vartheta^2 \wedge D\psi \equiv -R\overline{\psi}\psi\eta + 2d\{-D\overline{\psi} \wedge \gamma\vartheta^2\psi + \overline{\psi}\gamma\vartheta^2 \wedge D\psi\}. \tag{18}$$

These 4-dimensional scalar curvature identities invite applications where the Einstein-Hilbert Lagrangian is replaced by a quadratic spinor Lagrangian 4-form.

The higher dimensional identities could also be useful for gravity applications, e.g., unified field theories of the generalized Kaluza-Klein type. Moreover, a 5-dimensional geometry is useful for embedding 4 geometries with a cosmological constant. However, most gravity applications are in 3 and 4 dimensions.

In 3-dimensions, up to multiples of i there are only 2 interesting cases $D\varphi^\dagger \wedge D\varphi$, which gives the full curvature 2-form (7), and the combination (13) which gives the scalar curvature.

The basic 4-dimensional cases are as follows. First the 2-form $D\overline{\psi} \wedge D\psi$ is related to the full curvature tensor according to (7). Next we have the Einstein tensor (2). And the scalar curvature (17). Moreover we can insert an additional γ_5 into all of these.

We have found the Dirac spinor notation most suitable for the general case. However, for the important special case of 4 dimensions (and 3 dimensions) another spinor notation is also popular (Penrose and Rindler 1986). The basic identities in this notation have the form (Riemannian geometry):

$$2D\varphi_A \wedge D\chi^A \equiv d(\varphi_A D\chi^A - D\varphi_A \chi^A) + 2\varphi^A \chi^B \Omega_{AB}, \tag{19}$$

$$2D\varphi_A \wedge \theta^{A\dot{B}} \wedge D\chi_{\dot{B}} \equiv d[\varphi_A \theta^{A\dot{B}} \wedge D\chi_{\dot{B}} - D\varphi_A \wedge \theta^{A\dot{B}} \chi_{\dot{B}}] - 2\varphi_A \chi_{\dot{B}} \Omega^{\dot{B}}_{\dot{D}} \wedge \theta^{A\dot{D}}, \tag{20}$$

$$\begin{aligned} 2D\varphi^A \wedge S_{AB} \wedge D\chi^B &\equiv d[\varphi^A S_{AB} \wedge D\chi^B - D\varphi^A \wedge S_{AB}\chi^B] - 2\varphi^A \chi^B S_{AM} \wedge \Omega^M_B \\ &\equiv d[\varphi^A S_{AB} \wedge D\chi^B - D\varphi^A \wedge S_{AB}\chi^B] - \frac{i}{4} R \varphi^A \chi_A \eta, \end{aligned} \tag{21}$$

where φ^A and χ^A are any two-component Weyl spinors, S_{AB} is the self dual spinorial 2-form $S_{AB} := \frac{1}{2}\theta_A{}^{\dot{D}} \wedge \theta_{B\dot{D}}$ and Ω_{AB} is the spinorial curvature.

Four dimensional Dirac spinors and matrices can be expressed via Weyl spinors with the help of the relations:

$$\overline{\psi} = (\chi_A \quad \varphi^{\dot{A}}), \qquad \psi = \begin{pmatrix} \varphi^A \\ \chi_{\dot{A}} \end{pmatrix}, \qquad \gamma^a = \sqrt{2} \begin{pmatrix} 0 & \sigma^{aA\dot{B}} \\ \sigma^a_{B\dot{A}} & 0 \end{pmatrix}, \tag{22}$$

where $\sigma^a{}_{A\dot{B}}$ satisfies the identity $\sigma^a{}_{A\dot{B}} \sigma_{aC\dot{D}} = \varepsilon_{AC} \varepsilon_{\dot{B}\dot{D}}$.

Identities (19)–(21) are complex, but with the help of (22) one can easily conclude that eq (21) is the imaginary part of the (18) and the analog of (18) without γ is the real part of (21). Similarly, eq (7) is the real part of (19).

We also note that such identities are not restricted only to the Riemann or Riemann-Cartan connection. In 4 dimensions we may, for example, replace $\omega^{\alpha\beta}$ and $\Omega^{\alpha\beta}$ with their self dual combinations: $A^{\alpha\beta} := \frac{1}{2}(\omega^{\alpha\beta} + \frac{1}{2} i \epsilon^{\alpha\beta}{}_{\mu\nu} \omega^{\mu\nu})$ and $F^{\alpha\beta} := \frac{1}{2}(\Omega^{\alpha\beta} + \frac{1}{2} i \epsilon^{\alpha\beta}{}_{\mu\nu} \Omega^{\mu\nu})$ (for an application see Nester, Tung and Zhang 1993). Moreover, similar identities apply to Yang-Mills connections, for example

$$2D\overline{\psi} \wedge D\psi \equiv d(\overline{\psi} D\psi - (D\overline{\psi})\psi) - 2F^p \overline{\psi} T_p \psi, \tag{23}$$

where $D\psi := d\psi + A^p T_p \psi$ and $F^p := dA^p + \epsilon^p{}_{qr} A^q \wedge A^r$ is the Yang-Mills field strength.

We have focused on identities linear in curvature. However, identities of the type discussed here need not be so restricted; we may choose, for example, $A = \Omega^{ab}\gamma_{ab}$ in eq (5) to obtain an identity involving quadratic curvature terms. Clearly there are many other possibilities. We are not yet aware of any applications for such higher order identities.

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