SOME STRONG CONVERGENCE RESULTS OF RANDOM ITERATIVE ALGORITHMS WITH ERRORS IN BANACH SPACES

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ABSTRACT. In this paper, we study the strong convergence and stability of a new two step random iterative scheme with errors for accretive Lipschitzian mapping in real Banach spaces. The new iterative scheme is more acceptable because of much better convergence rate and less restrictions on parameters as compared to random Ishikawa iterative scheme with errors. We support our analytic proofs by providing numerical examples. Applications of random iterative schemes with errors to variational inequality are also given. Our results improve and establish random generalization of results obtained by Chang [4], Zhang [31] and many others.

1. Introduction and preliminaries

The machinery of random fixed point theory provides a convenient way of modelling many problems arising in non-linear analysis, probability theory and for solution of random equations in applied sciences. With the recent rapid developments in random fixed point theory, there has been a renewed interest in random iterative schemes [5, 6, 7, 22, 23, 24, 26]. In linear spaces, Mann and Ishikawa iterative schemes are two general iterative schemes which have been successfully applied to fixed point problems [1, 2, 13, 14]. In recent, many stability and convergence results of iterative schemes have been established, using Lipschitz accretive (or pseudo-contractive) mapping in Banach spaces [4, 8, 31]. Since in deterministic case the consideration of error terms is an important part of any iterative scheme, therefore motivated by the work of Ćirić [11, 12, 13, 14, 15], we introduce a two step random iterative scheme with errors and prove that the iterative scheme is stable with respect to T with Lipschitz condition where T is an accretive mapping in arbitrary real Banach space.

Let X be a real separable Banach space and let J denote the normalized duality pairing from X to 2^{X^*} given by

$$J(x) = \{f \in X^* : \langle x, f \rangle = \|x\|\,\|f\|, \|f\| = \|x\|\}, \quad x \in X,$$

where X^* denote the dual space of X and $\langle \cdot, \cdot \rangle$ denote the generalized duality pairing between X and X^*.

Suppose (Ω, Σ) denotes a measurable space consisting of a set Ω and sigma algebra Σ of subsets of Ω and C, a nonempty subset of X. Then random Mann iterative scheme with errors is defined as follows:

$$x_{n+1}(w) = (1 - \alpha_n)x_n(w) + \alpha_n T(w, x_n(w)) + u_n(w),$$

(1.1)   for each $w \in \Omega$, $n \geq 0$,

where $0 \leq \alpha_n \leq 1$, $x_0 : \Omega \to C$, an arbitrary measurable mapping and $\{u_n(w)\}$ is a sequence of measurable mappings from Ω to C.

Also, random Ishikawa iterative scheme with errors is defined as follows:

$$x_{n+1}(w) = (1 - \alpha_n)x_n(w) + \alpha_n T(w, y_n(w)) + u_n(w),$$
$$y_n(w) = (1 - \beta_n)x_n(w) + \beta_n T(w, x_n(w)) + v_n(w),$$

(1.2)   for each $w \in \Omega$, $n \geq 0$,

where $0 \leq \alpha_n, \beta_n \leq 1$, $x_0 : \Omega \to C$, an arbitrary measurable mapping and $\{u_n(w)\}$, $\{v_n(w)\}$ are sequences of measurable mappings from Ω to C.

Obviously $\{x_n(w)\}$ and $\{y_n(w)\}$ are sequences of mappings from Ω into C.

Also, we consider the following two step random iterative scheme with errors $\{x_n(w)\}$ defined by

$$x_{n+1}(w) = (1 - \alpha_n)y_n(w) + \alpha_n T(w, y_n(w)) + u_n(w),$$
$$y_n(w) = (1 - \beta_n)x_n(w) + \beta_n T(w, x_n(w)) + v_n(w),$$

(1.3)   for each $w \in \Omega$, $n \geq 0$,

where $\{u_n(w)\}$, $\{v_n(w)\}$ are sequences of measurable mappings from Ω to C, $0 \leq \alpha_n, \beta_n \leq 1$ and $x_0 : \Omega \to C$, an arbitrary measurable mapping.

Remark 1. Putting $\beta_n = 0$, $v_n = 0$ in (1.3) and (1.2), we get random Mann iterative scheme with errors.

Now we give some definitions and lemmas, which will be used in the proof of our main results.

Definition 1.1. A mapping $g : \Omega \to C$ is said to be measurable if $g^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of X.

Definition 1.2. A function $F : \Omega \times C \to C$ is said to be a random operator if $F(\cdot, x) : \Omega \to C$ is measurable for every $x \in C$.

Definition 1.3. A measurable mapping $p : \Omega \to C$ is said to be random fixed point of the random operator $F : \Omega \times C \to C$, if $F(w, p(w)) = p(w)$ for all $w \in \Omega$.

Definition 1.4. A random operator $F : \Omega \times C \to C$ is said to be continuous if for fixed $w \in \Omega$, $F(w, \cdot) : C \to C$ is continuous.

In the sequel, I denotes the identity operator on X, $D(T)$ and $R(T)$ denote the domain and the range of T, respectively.

Definition 1.5. Let $T : \Omega \times X \to X$ be a mapping. Then

    (i) T is said to be Lipschitizian, if for any $x, y \in X$ and $w \in \Omega$, there exists $L > 0$ such that

(1.4) $$\|T(w, x) - T(w, y)\| \leq L\|x - y\|.$$

    (ii) T is said to be nonexpansive, if for any $x, y \in X$ and $w \in \Omega$,

(1.5) $$\|T(w, x) - T(w, y)\| \leq \|x - y\|.$$

    (iii) $T : \Omega \times X \to X$ is pseudo-contractive [8] if and only if for all $x, y \in X, w \in \Omega$ and for all $r > 0$ the following inequality holds:

(1.6) $$\|x - y\| \leq \|(1 + r)(x - y) - r(T(w, x) - T(w, y))\|$$

or equivalently if and only if for all $x, y \in X$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2$$

    (iv) T is said to be accretive [8], if and only if for all $x, y \in X$ and for all $r > 0$ the following inequality holds:

(1.7) $$\|x - y\| \leq \|x - y + r(T(w, x) - T(w, y))\|$$

or equivalently if and only if for all $x, y \in X$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \geq 0.$$

    (v) If T is accretive and $R(I + \lambda T) = X$ for any $\lambda > 0$, then T is called m-accretive [20, 32].

Accretive mappings are connected with nonexpansive mappings. It is well known that if T is accretive [10], then $(I + T)^{-1}$ is a nonexpansive single-valued mapping from $R(I + \lambda T)$ to $D(T)$. The interest in accretive mappings also stems from the following facts:

    (a) If T is accretive, then solutions of the equation $Tx = 0$ correspond to the equilibrium points of some evolution systems [29].

    (b) Many physical problems arising in applied mathematics can be modelled in terms of initial value problem of the form:

$$\frac{dx}{dt} = -Tx, x(0) = x_0, \quad \text{where } T \text{ is an accretive mapping.}$$

    (c) Their connection with the well-known class of pseudo-contractive mappings (T is pseudo-contractive if and only if $I - T$ is accretive).

Suppose that X is a real reflexive Banach space, $T, A : X \to X$, $g : X \to X^*$ are three mappings, and $\varphi : X^* \to R \cup \{\infty\}$ is a function with a Gateaux differential $\partial\varphi$. Then u is a solution of a variational inequality if for any given $f \in X$, there exists a $u \in X$ such that

$$g(u) \in D(\partial\varphi),$$

(1.8) $\quad \langle Tu - Au - f, v - g(u) \rangle \geq \varphi(g(u)) - \varphi(v) \quad \text{for all } v \in X^*.$

Lemma 1.6 ([4]). *Suppose X is a real reflexive Banach space, $\partial\varphi \circ g : X \to 2^X$ is a mapping, then the following conclusions are equivalent*:

  (i) *$x^* \in X$ is a solution of variational inclusion problem (1.8)*;

  (ii) *$x^* \in X$ is a fixed point of the mapping $S : X \to 2^X$*;

$$S(x) = f - (Tx - Ax + \partial\varphi(g(x))) + x;$$

  (iii) *$x^* \in X$ is a solution of the equation $f = Tx - Ax + \partial\varphi(g(x))$.*

Lemma 1.7 ([20]). *Suppose X is an arbitrary real Banach space, $T : D(T) \subset X \to X$ is accretive and continuous, and $D(T) = X$. Then T is m-accretive.*

Lemma 1.8 ([32]). *Suppose X is an arbitrary real Banach space, $T : D(T) \subset X \to X$ is an m-accretive mapping. Then the equation $x + Tx = f$ has a unique solution in $D(T)$ for any $f \in X$.*

Lemma 1.9. *Let $x_n(w)$ be a sequence of real numbers satisfying the following inequality*:

$$x_{n+1} \leq \delta x_n + \sigma_n, \quad n \geq 1,$$

where $x_n \geq 0$, $\sigma_n \geq 0$ and $\lim\limits_{n\to\infty} \sigma_n = 0$, $0 \leq \delta < 1$. Then $x_n \to 0$ as $n \to \infty$.

Definition 1.10 ([1]). Let $T : \Omega \times C \to C$ be a random operator, where C is a nonempty closed convex subset of a real separable Banach space X. Let $x_0 : \Omega \to C$ be any measurable mapping. The sequence $\{x_{n+1}(w)\}$ of measurable mappings from Ω to C, for $n = 0, 1, 2, \ldots$ generated by the certain random iterative scheme involving a random operator T is denoted by $\{T, x_n(w)\}$ for each $w \in \Omega$. Suppose that $x_n(w) \to p(w)$ as $n \to \infty$ for each $w \in \Omega$, where $p \in RF(T)$. Let $\{p_n(w)\}$ be any arbitrary sequence of measurable mappings from Ω to C. Define the sequence of measurable mappings $k_n : \Omega \to R$ by $k_n(w) = d(p_n(w), \{T, p_n(w)\})$. If for each $w \in \Omega$, $k_n(w) \to 0$ as $n \to \infty$ implies $p_n(w) \to p(w)$ as $n \to \infty$ for each $w \in \Omega$, then the random iterative scheme is said to be stable with respect to the random operator T.

2. Convergence and stability results

In this section, we establish the convergence and stability results of revised two step random iterative scheme with errors (1.3) and random Ishikawa iterative scheme with errors in real Banach spaces.

Theorem 2.1. *Let X be a real Banach space, $T : \Omega \times X \to X$ be a Lipschitzian random mapping with a Lipschitz constant $L \geq 1$, such that $(-T)$ is accretive. Let $\{x_n(w)\}$ be the random iterative scheme with errors defined by (1.3), with the following restrictions*:

    (i) $0 < \alpha < \alpha_n - L^2(1 + L)\alpha_n^2 - \beta_n(L - 1) < 1 \ (n \geq 0)$.

    (ii) $\lim\limits_{n \to \infty} u_n(w) = 0, \ \lim\limits_{n \to \infty} v_n(w) = 0$.

Then

    (I) *the sequence $\{x_n(w)\}$ converges strongly to a unique fixed point $p(w)$ of T.*

    (II) *the sequence $\{x_n(w)\}$ is stable. Moreover, $\lim\limits_{n \to \infty} p_n(w) = p(w)$ implies $\lim\limits_{n \to \infty} k_n(w) = 0$.*

Proof. (I) From (1.3), we have

$$(x_{n+1}(w) - p(w)) - \alpha_n(T(w, x_{n+1}(w)) - T(w, p(w)))$$

$$(2.1) \quad = (1 - \alpha_n)(y_n(w) - p(w)) - \alpha_n(T(w, x_{n+1}(w)) - T(w, y_n(w)) + u_n(w).$$

Since $(-T)$ is accretive and Lipschitzian mapping, so using (2.1) and (1.7), we get

$$\|x_{n+1}(w) - p(w)\|$$
$$\leq \|x_{n+1}(w) - p(w) - \alpha_n(T(w, x_{n+1}(w)) - T(w, p(w)))\|$$
$$= \|(1 - \alpha_n)(y_n(w) - p(w)) - \alpha_n(T(w, x_{n+1}(w)) - T(w, y_n(w))) + u_n(w)\|$$
$$(2.2) \quad \leq (1 - \alpha_n)\|y_n(w) - p(w)\| + \alpha_n\|T(w, y_n(w)) - T(w, x_{n+1}(w))\| + \|u_n(w)\|.$$

Now, using Lipschitz condition on T, (1.3) implies

$$\|(T(w, x_{n+1}(w)) - T(w, y_n(w)))\|$$
$$\leq L\|x_{n+1}(w) - y_n(w)\|$$
$$\leq L\alpha_n\|y_n(w) - T(w, y_n(w))\| + L\|u_n(w)\|$$
$$\leq L\alpha_n\|y_n(w) - p(w)\| + L\alpha_n\|T(w, y_n(w)) - p(w)\| + L\|u_n(w)\|$$
$$(2.3) \quad = L\alpha_n(1 + L)\|y_n(w) - p(w)\| + L\|u_n(w)\|.$$

Also, from (1.3), we have the following estimate:

$$\|y_n(w) - p(w)\|$$
$$\leq (1 - \beta_n)\|x_n(w) - p(w)\| + \beta_n\|T(w, x_n(w)) - p(w)\| + \|v_n(w)\|$$
$$\leq (1 - \beta_n)\|x_n(w) - p(w)\| + \beta_n L\|x_n(w) - p(w)\| + \|v_n(w)\|$$
$$(2.4) \quad = [1 + \beta_n(L - 1)]\|x_n(w) - p(w)\| + \|v_n(w)\|.$$

Using inequalities (2.2)-(2.4), we arrive at

$$\|x_{n+1}(w) - p(w)\|$$
$$\leq (1 - \alpha_n)[1 + \beta_n(L - 1)]\|x_n(w) - p(w)\| + (1 - \alpha_n)\|v_n(w)\|$$

$$+ \alpha_n^2 L(1+L)[1+\beta_n(L-1)]\|x_n(w)-p(w)\| + (1+\alpha_n L)\|u_n(w)\|$$
$$+ \alpha_n^2 L(1+L)\|v_n(w)\|$$
$$\leq [1+\beta_n(L-1)][1-\alpha_n+\alpha_n^2 L(1+L)]\|x_n(w)-p(w)\| + (1+L)\|u_n(w)\|$$
$$+ [1+L(1+L)]\|v_n(w)\|$$
$$\leq [1-\{\alpha_n-\alpha_n^2 L^2(1+L)-\beta_n(L-1)\}]\|x_n(w)-p(w)\|+(1+L)\|u_n(w)\|$$
$$+ [1+L(1+L)]\|v_n(w)\|$$

$$(2.5) \quad \leq [1-\alpha]\|x_n(w)-p(w)\| + (1+L)\|u_n(w)\| + [1+L(1+L)]\|v_n(w)\|.$$

Now, put $[1-\alpha]=\delta$ and $[1+L(1+L)]\|v_n(w)\| + (1+L)\|u_n(w)\| = \sigma_n$.
    Then (2.5) reduces to

$$\|x_{n+1}(w)-p(w)\| \leq \delta\|x_n(w)-p(w)\| + \sigma_n.$$

Therefore, using conditions (i)-(ii) and Lemma 1.9, above inequality yields $\lim_{n\to\infty}\|x_{n+1}(w)-p(w)\| = 0$, that is $\{x_n(w)\}$ defined by (1.3) converges strongly to a random fixed point $p(w)$ of T.
    (II) Suppose that $\{p_n(w)\} \subset X$, is an arbitrary sequence,

$$k_n(w) = \|p_{n+1}(w) - (1-\alpha_n)q_n(w) - \alpha_n T(w,q_n(w)) - u_n(w)\|,$$

where

$$q_n(w) = (1-\beta_n)p_n(w) + \beta_n T(w,p_n(w)) + v_n(w) \quad \text{and} \quad \lim_{n\to\infty} k_n(w) = 0.$$

Then

$$\|p_{n+1}(w) - T(w,p(w))\|$$
$$= \|p_{n+1}(w) - (1-\alpha_n)q_n(w) - \alpha_n T(w,q_n(w)) - u_n(w)\|$$
$$+ \|(1-\alpha_n)q_n(w) + \alpha_n T(w,q_n(w)) + u_n(w) - T(w,p(w))\|$$
$$(2.6) \qquad = k_n(w) + \|r_n - T(w,p(w))\|,$$

where

$$(2.7) \qquad r_n = (1-\alpha_n)q_n(w) + \alpha_n T(w,q_n(w)) + u_n(w).$$

Then using (2.7), we have

$$(r_n(w)-p(w)) - \alpha_n(T(w,r_n) - T(w,p(w)))$$
$$= (1-\alpha_n)(q_n(w)-p(w)) - \alpha_n(T(w,r_n(w)) - T(w,q_n(w))) + u_n(w)$$

which further implies

$$\|r_n(w)-p(w)\|$$
$$\leq \|r_n(w)-p(w) - \alpha_n(T(w,r_n(w)) - T(w,p(w)))\|$$
$$= \|(1-\alpha_n)(q_n(w)-p(w)) - \alpha_n(T(w,r_n(w)) - T(w,q_n(w))) + u_n(w)\|$$
$$(2.8) \quad \leq (1-\alpha_n)\|(q_n(w)-p(w))\| + \alpha_n\|(T(w,r_n(w)) - T(w,q_n(w)))\| + \|u_n(w)\|.$$

Now, similar to (2.3) and (2.4), we have the following estimates:

(2.9)
$$\|(T(w, r_n(w)) - T(w, q_n(w)))\| \le L\alpha_n(1+L)\|q_n(w) - p(w)\| + L\|u_n(w)\|,$$

(2.10)
$$\|q_n(w) - p(w)\| \le [1 + \beta_n(L-1)]\|(p_n(w) - p(w))\| + \|v_n(w)\|.$$

Using estimates (2.8)-(2.10), we arrive at

$$\|r_n(w) - p(w)\|$$
$$\le [1 - \{\alpha_n - \alpha_n^2 L^2(1+L) - \beta_n(L-1)\}]\|(p_n(w) - p(w))\|$$
(2.11)
$$+ (1+L)\|u_n(w)\| + [1 + L(1+L)]\|v_n(w)\|.$$

Substituting (2.11) in (2.6), we obtain

$$\|p_{n+1}(w) - T(w, p(w))\|$$
$$\le k_n(w) + [1 - \{\alpha_n - \alpha_n^2 L^2(1+L) - \beta_n(L-1)\}]\|(p_n(w) - p(w))\|$$
(2.12)
$$+ (1+L)\|u_n(w)\| + [1 + L(1+L)]\|v_n(w)\|.$$

Hence again using Lemma 1.9, together with conditions (i)-(ii), (2.12) yields $\lim\limits_{n\to\infty} p_n(w) = p(w)$.

Therefore, the iteration (1.3) is T-stable.

Further, let $\lim\limits_{n\to\infty} p_n(w) = p(w)$, then using (2.11), we have

$$k_n(w)$$
$$= \|p_{n+1}(w) - (1 - \alpha_n)q_n(w) - \alpha_n T(w, q_n(w)) - u_n(w)\|$$
$$= \|p_{n+1}(w) - r_n(w)\|$$
$$\le \|p_n(w) - p(w)\| + \|r_n(w) - p(w)\|$$
$$\le \|p_n(w) - p(w)\| + [1 - \{\alpha_n - \alpha_n^2 L^2(1+L) - \beta_n(L-1)\}]\|(p_n(w) - p(w))\|$$
$$+ (1+L)\|u_n(w)\| + [1 + L(1+L)]\|v_n(w)\|,$$

which implies $\lim\limits_{n\to\infty} k_n(w) = 0$. This completes the proof of Theorem 2.1. $\quad\square$

Putting $\beta_n = 0$, in Theorem 2.1, we have the following obvious corollary:

Corollary 2.2. *Let X be a real Banach space, $T : \Omega \times X \to X$ be a Lipschitzian random mapping with a Lipschitz constant $L \ge 1$, such that $(-T)$ is accretive. Let $\{x_n(w)\}$ be the random Mann iterative scheme with errors defined by (1.1) with the following conditions:*

(i) $0 < \alpha < \alpha_n - L^2(1+L)\alpha_n^2 < 1$ $(n \ge 0)$.
(ii) $\lim\limits_{n\to\infty} u_n(w) = 0$.

Then

(I) *the sequence $\{x_n(w)\}$ converges strongly to a unique fixed point $p(w)$ of T.*

(II) *the sequence $\{x_n(w)\}$ is stable. Moreover,* $\lim\limits_{n\to\infty} p_n(w) = p(w)$ *implies* $\lim\limits_{n\to\infty} k_n(w) = 0$.

Theorem 2.3. *Let X be a real Banach space, $T : \Omega \times X \to X$ be Lipschitzian random mapping with a Lipschitz constant $L \geq 1$, such that $(-T)$ is accretive. Let $\{x_n(w)\}$ be the random Ishikawa iterative scheme with errors defined by (1.2) with the following restrictions:*

(i) $0 < \alpha < \alpha_n - \alpha_n L(1 + L)(\alpha_n + \beta_n) - L(L^2 - 1)\alpha_n\beta_n < 1$ $(n \geq 0)$.

(ii) $\lim\limits_{n\to\infty} u_n(w) = 0$, $\lim\limits_{n\to\infty} v_n(w) = 0$.

Then

(I) *the sequence $\{x_n(w)\}$ converges strongly to a unique fixed point $p(w)$ of T.*

(II) *the sequence $\{x_n(w)\}$ is stable. Moreover,* $\lim\limits_{n\to\infty} p_n(w) = p(w)$ *implies* $\lim\limits_{n\to\infty} k_n(w) = 0$.

Proof. Using (1.2), we have

$$(x_{n+1}(w) - p(w)) - \alpha_n(T(w, x_{n+1}) - T(w, p(w)))$$
$$(2.13) \quad = (1-\alpha_n)(x_n(w)-p(w))-\alpha_n(T(w, x_{n+1}(w))-T(w, y_n(w)))+u_n(w).$$

Using (2.13) and (1.7), we get

$$\|x_{n+1}(w)-p(w)\|$$
$$\leq \|x_{n+1}(w)-p(w)-\alpha_n(T(w, x_{n+1}(w))-T(w, p(w)))\|$$
$$= \|(1-\alpha_n)(x_n(w)-p(w))-\alpha_n(T(w, x_{n+1}(w))-T(w, y_n(w)))+u_n(w)\|$$
$$(2.14) \quad \leq (1-\alpha_n)\|x_n(w)-p(w)\|+\alpha_n\|T(w, x_{n+1}(w))-T(w, y_n(w))\|+\|u_n(w)\|.$$

As T is a Lipschitz mapping with constant L, so we have the following estimates:

$$\|T(w, x_{n+1}(w)) - T(w, y_n(w))\|$$
$$\leq L\|x_{n+1}(w) - y_n(w)\|$$
$$(2.15) \quad \leq L[(1-\alpha_n)\|x_n(w)-y_n(w)\|+\alpha_n\|T(w, y_n(w))-y_n(w)\|+\|u_n(w)\|],$$

$$\|T(w, (y_n(w))) - y_n(w)\|$$
$$\leq (1 + L)\|y_n(w) - p(w)\|$$
$$\leq (1 + L)[(1 - \beta_n)\|x_n(w) - p(w)\| + \beta_n\|T(w, (x_n(w))) - x_n(w)\|$$
$$+ \|v_n(w)\|]$$
$$(2.16) \quad \leq (1 + L)[1 + (L - 1)\beta_n]\|x_n(w) - p(w)\| + (1 + L)\|v_n(w)\|$$

and

$$\|x_n(w) - y_n(w)\| \leq \beta_n\|x_n(w) - T(w, (x_n(w)))\| + \|v_n(w)\|$$
$$(2.17) \qquad\qquad\qquad \leq (1 + L)\beta_n\|x_n(w) - p(w)\| + \|v_n(w)\|.$$

Using (2.16) and (2.17), (2.15) yields

$$\|(T(w, x_{n+1}(w)) - T(w, y_n(w)))\|$$
$$\leq \{L(1+L)(1-\alpha_n)\beta_n + L(1+L)[1+(L-1)\beta_n]\alpha_n\}\|x_n(w) - p(w)\|$$
$$+ L(1 + L\alpha_n)\|v_n(w)\| + L\|u_n(w)\|$$
$$\leq [L(1+L)(\alpha_n+\beta_n) + L(L^2-1)\alpha_n\beta_n]\|x_n(w) - p(w)\|$$
$$(2.18) \quad + L(1+L)\|v_n(w)\| + L\|u_n(w)\|.$$

Substituting (2.18) into (2.14), we arrive at

$$\|x_{n+1}(w) - p(w)\|$$
$$\leq [1 - \{\alpha_n - \alpha_n L(1+L)(\alpha_n+\beta_n) - L(L^2-1)\alpha_n\beta_n\}]\|(x_n(w) - p(w))\|$$
$$+ L(1+L)\alpha_n\|v_n(w)\| + (1 + L\alpha_n)\|u_n(w)\|$$
$$\leq [1 - \{\alpha_n - \alpha_n L(1+L)(\alpha_n+\beta_n) - L(L^2-1)\alpha_n\beta_n\}]\|(x_n(w) - p(w))\|$$
$$+ L(1+L)\|v_n(w)\| + (1+L)\|u_n(w)\|$$
$$(2.19) \quad \leq [1-\alpha]\|(x_n(w) - p(w))\| + L(1+L)\|v_n(w)\| + (1+L)\|u_n(w)\|.$$

Now, put $[1-\alpha] = \delta$ and $L(1+L)\|v_n(w)\| + (1+L)\|u_n(w)\| = \sigma_n$.

Then (2.19) reduces to

$$\|x_{n+1}(w) - p(w)\| \leq \delta\|x_n(w) - p(w)\| + \sigma_n.$$

Therefore, using conditions (i)-(ii) and Lemma 1.9, above inequality yields $\lim_{n\to\infty} \|x_{n+1}(w) - p(w)\| = 0$, that is $\{x_n(w)\}$ defined by (1.2) converges strongly to a random fixed point $p(w)$ of T.

(II) The proof of this part can hold on the same lines as in the proof of part (II) in Theorem 2.1. $\square$

Now, we demonstrate the following example to prove the validity of our results.

Example 2.4. Let $\Omega = [0, 2]$ and Σ be the sigma algebra of Lebesgue's measurable subsets of Ω. Take $X = R$ and define random operator T from $\Omega \times X$ to X as $T(w, x) = w - x$. Then the measurable mapping $\xi : \Omega \to X$ defined by $\xi(w) = \frac{w}{2}$, for every $w \in \Omega$, serve as a random fixed point of T. It is easy to see that the operator T is a Lipschitz random operator with Lipschitz constant $L = 1$ such that $(-T)$ is accretive and $\alpha_n = \frac{1}{(1+L)^3}$, $\beta_n = \frac{1}{(1+L)^6}$, $\|u_n\| = \|v_n\| = \frac{1}{(n+1)}$ satisfies all the conditions (i)-(ii) given in Theorem 2.1. and Theorem 2.3.

Remark 2.5. New random iterative scheme is more acceptable as compared to random Ishikawa iterative scheme with errors due to following reasons:

(1) In deterministic case, for accretive mappings new two step iterative with errors has better convergence rate as compared to Ishikawa iterative scheme with errors.

(2) For convergence, weak control conditions on parameters are required in new two step random iterative with errors as compared to random Ishikawa iterative scheme with errors.

Proof. (1) Let $T(x) = 1 - x$, $L = 1$, $\alpha_n = \frac{1}{(1+L)^3}$, $\beta_n = \frac{1}{(1+L)^6}$, $\|u_n\| = \|v_n\| = \frac{1}{(n+1)}$. Then taking initial approximation $x_0 = 1$, convergence of new two step and Ishikawa iterative schemes with errors to the fixed point 0.5 of operator T is shown in the following table. From table, it is obvious that new two step iterative scheme with errors has much better convergence rate as compared to Ishikawa iterative scheme with errors.

Number of iterations	New two step iterative scheme with errors		Ishikawa iterative scheme with errors	
N	Tx_n	x_{n+1}	Tx_n	x_{n+1}
0	0	2.83203	0	1.9707
1	−1.83203	2.19437	−0.970703	1.93049
2	−1.19437	1.73106	−0.930489	1.89137
3	−0.731063	1.39444	−0.891374	1.85333
4	−0.394444	1.14987	−0.853328	1.81632
5	−0.14987	0.972171	−0.816323	1.78033
6	0.0278291	0.843062	−0.78033	1.74532
7	0.156938	0.749256	−0.745321	1.71127
8	0.250744	0.6811	−0.711269	1.67815
9	0.3189	0.63158	−0.678149	1.64593
10	0.36842	0.595601	0	1.9707
⋮	⋮	⋮	⋮	⋮
40	0.499991	0.500007	0.00118339	0.985177
41	0.499993	0.500005	0.0148229	0.971911
42	0.499995	0.500003	0.0280895	0.959007
43	0.499997	0.500003	0.0409933	0.946456
44	0.499997	0.500002	0.0535442	0.934248
45	0.499998	0.500001	0.065752	0.922374
46	0.499999	0.500001	0.077626	0.910825
47	0.499999	0.500001	0.0891753	0.899591
48	0.499999	0.500001	0.100409	0.888665
49	0.499999	0.5	0.111335	0.878037
50	**0.5**	**0.5**	0.121963	0.8677
⋮	⋮	⋮	⋮	⋮
528	**0.5**	**0.5**	0.499999	0.500001
529	**0.5**	**0.5**	0.499999	0.500001
530	**0.5**	**0.5**	0.499999	0.500001
531	**0.5**	**0.5**	0.499999	0.500001
532	**0.5**	**0.5**	0.499999	0.500001
533	**0.5**	**0.5**	0.499999	0.500001
534	**0.5**	**0.5**	0.499999	0.500001
535	**0.5**	**0.5**	0.499999	0.500001
536	**0.5**	**0.5**	0.499999	0.500001
537	**0.5**	**0.5**	0.499999	0.500001
538	**0.5**	**0.5**	0.499999	0.5
539	**0.5**	**0.5**	**0.5**	**0.5**
⋮	⋮	⋮	⋮	⋮

(2) If we take $L = 1$, $\alpha_n = \frac{1}{4L(1+L)+L}$, $\beta_n = \frac{1}{4L(1+L)}$, then both conditions $0 < \alpha_n - L^2(1+L)\alpha_n^2 - \beta_n(L-1) < 1$ and $0 < \alpha_n - \alpha_n L(1+L)(\alpha_n + \beta_n) - L(L^2-1)\alpha_n\beta_n < 1$, are satisfied.

But if we take $L = 1$, $\alpha_n = \frac{1}{(1+L)^3}$ and $\beta_n = \frac{1}{(1+L)}$, then $0 < \alpha_n - L^2(1 + L)\alpha_n^2 - \beta_n(L - 1) < 1$ is satisfied but $0 < \alpha_n - \alpha_n L(1 + L)(\alpha_n + \beta_n) - L(L^2 - 1)\alpha_n\beta_n < 1$, is not satisfied.

So,
$$0 < \alpha_n - \alpha_n L(1 + L)(\alpha_n + \beta_n) - L(L^2 - 1)\alpha_n\beta_n < 1,$$
is stronger condition than
$$0 < \alpha_n - L^2(1 + L)\alpha_n^2 - \beta_n(L - 1) < 1.$$

3. Applications

In this section, we apply the random iterative schemes with errors to find solution of nonlinear random variational inclusion problem.

Theorem 3.1. *Let $T, A : \Omega \times X \to X$, $g : \Omega \times X \to X^*$ are three random operators on real reflexive Banach space X and $\varphi : X^* \to R \cup \{\infty\}$, a function with continuous Gateaux differential $\partial\varphi$, such that $T - A + \partial\varphi \circ g - I : \Omega \times X \to X$ is a Lipschitzian accretive random operator with a Lipschitz constant $L \geq 1$. Define a random operator $S : \Omega \times X \to X$ by $S(w, x) = f - (T(w, x) - A(w, x) + \partial\varphi(g(w, x))) + x(w)$, where $f \in X$ is any given point. For any given $x_0(w) \in X$, let $\{x_n(w)\}$ be the random iterative scheme with errors defined by*

$$x_{n+1}(w) = (1 - \alpha_n)y_n(w) + \alpha_n S(w, y_n(w)) + u_n(w),$$
$$y_n(w) = (1 - \beta_n)x_n(w) + \beta_n S(w, x_n(w)) + v_n(w),$$

(3.1) $$\text{for each } w \in \Omega, n \geq 0,$$

where $\{u_n(w)\}$, $\{v_n(w)\}$ are measurable sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

(i) $0 < \alpha < \alpha_n - L^{*2}(1 + L^*)\alpha_n^2 - \beta_n(L^* - 1) < 1$, $L^* = 1 + L$.
(ii) $\lim_{n \to \infty} u_n(w) = 0$, $\lim_{n \to \infty} v_n(w) = 0$.

Then the iterative scheme (3.1) converges to the unique solution $x^*(w) \in X$ of the following nonlinear variational inclusion problem

$$g(w, u) \in D(\partial\varphi),$$
$$\langle T(w, u) - A(w, u) - f, v - g(w, u)\rangle \geq \varphi(g(w, u)) - \varphi(v),$$

(3.2) $$\text{for all } v \in X^*.$$

Proof. We shall complete the proof in two steps. In the first step, we show that nonlinear variational inclusion problem (3.2) has a unique solution $x^* \in X$. In the second step, we show that iterative scheme (3.1) converges to the unique solution.

Step 1. As $T - A + \partial\varphi \circ g - I$ is a Lipschitzian accretive mapping, so by Lemma 1.7, $T - A + \partial\varphi \circ g - I$ is m-accretive. Hence by Lemma 1.8, for any $f \in X$, the equation

$$f = T(w, x) - A(w, x) + \partial\varphi(g(w, x)) - I(w, x) + x(w)$$

has a unique solution $x^*(w) \in X$. Then, using Lemma 1.6, $x^*(w) \in X$ will be the solution of nonlinear variational inclusion problem (3.2) and it is fixed point of S.

Step 2. Now since $T - A + \partial\varphi \circ g - I : \Omega \times X \to X$ is a Lipschitzian accretive operator with a Lipschitz constant $L \geq 1$, so $S : \Omega \times X \to X$ is also a Lipschitzian mapping with Lipschitz constant $L^* = 1 + L$, such that $(-S)$ is an accretive operator. Now, replacing T by S in (1.3), L by L^* in condition (i) of Theorem 2.1 and following the same steps as in the proof of Theorem 2.1, it is easy to see that the random iterative scheme (3.1) converges to the unique solution $x^* \in X$ of nonlinear variational inclusion problem (3.2).   □

Letting $\varphi \equiv 0$, $u_n(w) = v_n(w) = 0$, in Theorem 3.1, we can obtain the following theorem.

Theorem 3.2. *Let $T, A : \Omega \times X \to X$, $g : \Omega \times X \to X^*$ are three random operators on real reflexive Banach space X, such that $T - A - I : \Omega \times X \to X$ is a Lipschitzian accretive operator with a Lipschitz constant $L \geq 1$. Define a random operator $S : \Omega \times X \to X$ by $S(w, x) = f - (T(w, x) - A(w, x)) + x(w)$, where $f \in X$ is any given point. For any given $x_0(w) \in X$, let $\{x_n(w)\}$ be the random iterative scheme defined by*

$$x_{n+1}(w) = (1 - \alpha_n)y_n(w) + \alpha_n S(w, y_n(w)),$$

(3.3) $y_n(w) = (1 - \beta_n)x_n(w) + \beta_n S(w, x_n(w))$ *for each $w \in \Omega, n \geq 0$,*

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:
   (i) $0 < \alpha < \alpha_n - L^{*2}(1 + L^*)\alpha_n^2 - \beta_n(L^* - 1) < 1$,
Then the iterative scheme (3.3) converges to the unique solution $x^* \in X$ of nonlinear variational inequality

$$\langle T(w, u) - A(w, u) - f, v - g(w, u) \rangle \geq 0 \quad \text{for all } v \in X^*.$$

Theorem 3.3. *Let $T, A : \Omega \times X \to X$, $g : \Omega \times X \to X^*$ are three random operators on real reflexive Banach space X and $\varphi : X^* \to R \cup \{\infty\}$, a function with continuous Gateaux differential $\partial\varphi$, such that $T - A + \partial\varphi \circ g - I : \Omega \times X \to X$ is a Lipschitzian accretive operator with a Lipschitz constant $L \geq 1$. Define an random operator $S : \Omega \times X \to X$ by $S(w, x) = f - (T(w, x) - A(w, x) + \partial\varphi(g(w, x))) + x(w)$, where $f \in X$ is any given point. For any given $x_0(w) \in X$, let $\{x_n(w)\}$ be the random Ishikawa iterative scheme with errors defined by*

$$x_{n+1}(w) = (1 - \alpha_n)x_n(w) + \alpha_n S(w, y_n(w)) + u_n(w),$$

$$y_n(w) = (1 - \beta_n)x_n(w) + \beta_n S(w, x_n(w)) + v_n(w),$$

(3.4)   *for each $w \in \Omega, n \geq 0$,*

where $\{u_n(w)\}$, $\{v_n(w)\}$ are measurable sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$, satisfying the following conditions:
   (i) $0 < \alpha < \alpha_n - \alpha_n L^*(1 + L^*)(\alpha_n + \beta_n) - L^*(L^2 - 1)\alpha_n\beta_n < 1$.
   (ii) $\lim\limits_{n \to \infty} u_n(w) = 0$, $\lim\limits_{n \to \infty} v_n(w) = 0$.

Then the iterative scheme (3.4) *converges to the unique solution* $x^* \in X$ *of nonlinear variational inclusion problem* (3.2).

Proof. Using Theorem 2.3 and following the same arguments as in the proof of Theorem 3.1, it is easy to show the random Ishikawa iterative scheme with errors (3.4) converges to the unique solution $x^* \in X$ of nonlinear variational inclusion problem (3.2). □

Theorem 3.4. *Let* $T, A : \Omega \times X \to X$, $g : \Omega \times X \to X^*$ *are three random operators on real reflexive Banach space* X *and* $\varphi : X^* \to R \cup \{\infty\}$, *a function with continuous Gateaux differential* $\partial \varphi$, *such that* $T - A + \partial\varphi \circ g - I : \Omega \times X \to X$ *is a Lipschitzian accretive operator with a Lipschitz constant* $L \geq 1$. *Define a random operator* $S : \Omega \times X \to X$ *by* $S(w, x) = f - (T(w, x) - A(w, x) + \partial\varphi(g(w, x))) + x(w)$, *where* $f \in X$ *is any given point. For any given* $x_0(w) \in X$, *let* $\{x_n(w)\}$ *be the random Mann iterative scheme with errors defined by*

$$x_{n+1}(w) = (1 - \alpha_n)x_n(w) + \alpha_n S(w, x_n(w)) + u_n(w),$$
(3.5) $$\text{for each } w \in \Omega, \ n \geq 0,$$

where $\{u_n(w)\}$ *is a measurable sequence in* X *and* $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *satisfying following conditions*:

(i) $0 < \alpha < \alpha_n - L^2(1 + L)\alpha_n^2 < 1$.
(ii) $\lim\limits_{n \to \infty} u_n(w) = 0$.

Then the iterative scheme (3.5) *converges to the unique solution* $x^* \in X$ *of nonlinear variational inclusion problem* (3.2).

Remark 3.5. Results involving random Ishikawa and Mann iterative schemes to solve variational inclusion problem (3.2) or variational inequality can be proved as special cases of Theorems 3.1-3.4.

Remark 3.6. Our results are generalization, improvement and extension of some of the well known results in the following sense:

1. Theorems 3.1 and 3.3 are randomization of Theorem 3.1 of Chang [4] as well as Theorem 2.1 of Zhang [31], using new convergence technique and weak restrictive condition on parameters. In fact in Theorems 3.1 and 3.3, α_n and β_n need not converge to zero as in Theorem 3.1 of Chang [4] and Theorem 2.1 of Zhang [31].
2. Theorem 3.3 holds in reflexive real Banach spaces, whereas Theorem 3.1 of Chang [4] has been proved in uniformly smooth Banach spaces.
3. In Theorems 3.1 and 3.3, unlike in Theorem 3.1 of Chang [4] and Theorem 2.1 of Zhang [31],the boundedness of range of S or Sx_n and Sy_n, is not required.
4. The Ishikawa iterative scheme has been replaced with more general random Ishikawa iterative scheme with errors and more acceptable new two step random iterative scheme with errors.

5. Stability of more acceptable new two step random iterative scheme with errors has been proved in Theorem 2.1.

6. Theorem 3.1 of Chang [4], generalizes and improves the results in [16, 17, 18, 21, 27, 28, 30], so Theorems 3.1 and 3.3 extend and establish random generalization of the work of [16, 17, 18, 21, 27, 28, 30].

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