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SUIPR Report No. 383

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ABSTRACT

In this paper, the so-called triggered emissions of whistlers are considered. The effects of a nonlinear interaction between a whistler wave and the particles resonating with it are considered in some detail. The interaction gives rise to two closely related effects, namely a change in the amplification or absorption by the plasma, and a phase-bunching of the resonant and near-resonant particles, giving rise to a current. The current due to the phase-bunched particles has roughly the same structure (frequency and wave number) as the wave causing it, and will therefore give rise to emission of a new whistler wave, acting like an antenna. The possibility of having a self-sustained process of phase-bunching and emission is investigated. The structure of a region where this process can take place is proposed, and the corresponding conditions that have to be met are stated.

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†On leave from the University of Bergen, Norway.

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# 1. INTRODUCTION

Since the discovery of the so-called triggered emissions (TE) of whistlers in the magnetosphere, a number of suggestions and theories have been put forward to explain the phenomenon (see Helliwell [1,2,3] and the list of references given there). The complexity of the problem can be appreciated from the following list of characteristics, none of which seems to be irrelevant: (a) TE are a non-stationary, or transient phenomenon, (b) the inhomogeneity along the direction of propagation plays an important role, in particular for the frequency variation of the TE (see [1,2]), and (c) TE are basically due to some nonlinear process. This is readily deduced from the fact that the frequency of the TE can differ considerably from that of the triggering signal.

As is fairly evident from the list above, a complete mathematical treatment, taking into account all the relevant features, is a hopeless task. The most recent theories that have been put forward [1,4,5], have concentrated their effort on finding the nonlinear mechanism underlying the TE . There seems to be general agreement that the mechanism responsible is a nonlinear wave-particle interaction. The nonlinear effects available can crudely be categorized into (a) wave-wave interactions, and (b) wave-particle interactions. The first category consists of interactions between three or more waves, resulting in a redistribution of the wave energies. These interactions and their feasibility to produce TE have been considered by Harker and Crawford [6]. A typical result from their calculation is that a typical length scale, $\ell_1$ , for the transfer of energy from a pump wave to two other waves is $1/\ell_1 \sim kB_w/B_0$ , where $k$ is the wave number; $B_w$ is the magnetic field of the pumping whistler wave, and $B_0$ is the earth's magnetic field.

The second category consists of the nonlinear interaction between a finite amplitude wave train, and the particles that happen to be in resonance with it. The resonant particles are the ones that see the electric and magnetic field vectors of the wave rotate at the local gyrofrequency. Thus the doppler-shifted frequency, $\omega - \underset{\sim}{v}\cdot\underset{\sim}{k}$ , seen by these particles equals the gyrofrequency, $\Omega$ , where $\omega$ is the wave frequency, and $\underset{\sim}{v}$ is the particle velocity. Taking the whistler wave to propagate along the static magnetic field, and recollecting that $\omega < \Omega$ for the whistler mode, one has the resonance condition $\omega + v_{\|}k = \Omega$ , which we may write as

$$v_{\|} = V_0 \equiv \frac{\Omega - \omega}{k} \quad , \tag{1}$$

where $v_{\|}$ is the velocity component along the static magnetic field. It should be noted that the resonant particles move in the opposite direction to the wave.

The interaction between the wave and the particles resonating with it, results in a distortion of the particle velocity distribution function near the resonance velocity. This in turn leads to a change in the absorption or amplification properties. It also leads to a certain amount of phase bunching of the resonant particles. A typical length scale, $\ell_2$ , for these processes to take place is of order $1/\ell_2 \sim k(B_w/B_0)^{1/2}$ . Inserting $B_0 \sim 10^{-2}$ G , and $B_w \sim 10^{-7} - 10^{-8}$ G one arrives at $1/k\ell_2 \sim 3 \times 10^{-3} - 10^{-3}$ . The corresponding figures for the wave-wave interactions are $1/k\ell_1 \sim 10^{-5} - 10^{-6}$ , which are too small to give any significant effect. For the wave-particle interaction, however, the length $\ell_2$ is sufficiently small to be of interest in the present problem.

In this paper we will work roughly along the lines suggested by Helliwell [1], trying to calculate the effect of the wave-particle interaction, and investigating the possibility of having a self-sustained process giving rise to the observed emission. The plan of the paper is as follows.

In Section 2, the interaction between an electron and a whistler wave in the magnetosphere is considered. In particular, we investigate the effect of the inhomogeneity of the earth's magnetic field.

In Section 3, the collective interaction of the resonant particles with the wave is investigated. First, the linear case is considered, for the sake of completeness. This case has been dealt with previously by Liemohn [7], and Lee et al. [8]. Then the nonlinear damping or growth resulting from the nonlinear interaction between the whistler wave and the resonant, and near-resonant particles, is investigated.

Recently this has also been investigated by Palmadesso et al. [9] using a slightly different approach. An application of our findings is made to some experimental observations published by McNeill [10].

In Section 4, the phase-bunching of the resonant electrons due to the whistler wave is investigated, and it is shown that this effect is closely related to the nonlinear damping (or growth) effect. This effect was suggested by Helliwell [1] to be responsible for the observed emissions.

In Section 5 the emission from the phase-bunched particles is discussed. The possibility of having a self-sustained process, where the emitted wave phase-bunches new particles, which subsequently give new emission, is discussed. The structure of a region where this self-sustained process might take place is described, and the corresponding requirement on the frequency variation of the emitted whistler is discussed.

## 2. NONLINEAR INTERACTION BETWEEN AN ELECTRON AND A WHISTLER WAVE IN THE MAGNETOSPHERE

Consider an electron of charge $e$, and mass $m$, moving in the earth's magnetic field $\underset{\sim}{B}$, in the presence of a circularly-polarized whistler mode (WM) signal propagating along $\underset{\sim}{B}$. Let $\underset{\sim}{B}_w = B_w \{\cos(\omega t - \int kdz), \sin(\omega t - \int kdz), 0\}$ be the magnetic field vector of the WM wave, where $z$ is the coordinate along the static magnetic field, $\underset{\sim}{B}$, and $k$ is the wave number. Its relation to the frequency, $\omega$, is given by the cold plasma dispersion relation

$$k = \frac{\omega_p}{c}\left(\frac{\omega}{\Omega - \omega}\right)^{1/2} \tag{2}$$

Here $c$ is the speed of light, $\omega_p$ and $\Omega$ are the electron plasma frequency and gyrofrequency, respectively. The fact that $(\omega_p/\Omega)^2 \gg 1$ in the domain of interest for the present paper has been taken into account to reduce the dispersion relation to this form.

The equations of motion are

$$\dot{v}_{\parallel} = v_{\perp} a \sin\Psi + \frac{v_{\perp}^2}{2B}\frac{\partial B}{\partial z}, \quad \dot{v}_{\perp} = -(v_{\parallel} + \omega/k) a \sin\Psi - \frac{v_{\parallel}v_{\perp}}{2B}\frac{\partial B}{\partial z},$$

$$\dot{\varphi} = \Omega - [(v_{\parallel} + \omega/k)/v_{\perp}] a \cos\Psi, \tag{3}$$

where the velocity of the electron is $\underset{\sim}{v} = \{v_{\perp}\cos\varphi, v_{\perp}\sin\varphi, -v_{\parallel}\}$, and $\Psi = \varphi + \int kdz - \omega t + \pi$ is the angle between $-\underset{\sim}{B}_w$ and $v_{\perp}$. The last term in each of the first two equations comes from the lowest order guiding centre approximation for an electron in an inhomogeneous magnetic field. The quantity $a = eB_w/mc$ is the gyrofrequency corresponding to the wave magnetic field.

Neglecting terms of relative order of magnitude $(a/\Omega)^{1/2} (\ll 1)$, the following equations are readily deduced from Eq. (3),

$$\frac{d}{dt}[v_{\parallel}^2 + v_{\perp}^2] = -\frac{\omega}{k} v_{\perp} a \sin\Psi,$$

(Continued)

$$\frac{d}{dt}\left[\left(v_{\|}+\frac{\omega}{k}\right)^2+v_\perp^2\right] = \frac{\omega}{k}\frac{v_\perp^2}{2\Omega}\frac{\partial\Omega}{\partial z} - 2v_{\|}\frac{\omega}{k}\left(v_{\|}+\frac{\omega}{k}\right)\frac{1}{k}\frac{\partial k}{\partial z} \quad ,$$

$$\ddot{\Psi} + \bar{\omega}^2 \sin\Psi = k\left[\dot{V}_0 - (v_\perp^2/2\Omega)\frac{\partial\Omega}{\partial z}\right] + \dot{k}(V_0 - v_{\|}) \quad , \quad (4)$$

where $\bar{\omega}^2 = a\, kv_\perp$ . For particles near resonance, Eq. (4) reduces to

$$\frac{d}{dt}\left[v_{\|}^2 + v_\perp^2\right] = -\frac{\omega}{k} v_\perp\, a \sin\Psi \quad , \quad (5a)$$

$$\frac{d}{dt}\left[\left(v_{\|}+\frac{\omega}{k}\right)^2+v_\perp^2\right] = \frac{\omega}{k}\left(\frac{\Omega\omega}{k^2}+\frac{v_\perp^2}{2}\right)\frac{1}{\Omega}\frac{\partial\Omega}{\partial z} \quad , \quad (5b)$$

$$\ddot{\Psi} + \bar{\omega}^2 \sin\Psi = k\left[\dot{V}_0 - (v_\perp^2/2\Omega)\frac{\partial\Omega}{\partial z}\right] \quad , \quad (5c)$$

where in Eq. (5b) the so-called gyrofrequency model for the variation of the electron density (see Helliwell [3]) has been used. In this model, the density is proportional to the static magnetic field.

## 2.1 Homogeneous Case

Near the equator, where the variation in the earth's magnetic field is very small, it makes sense to neglect the right-hand sides of Eqs. (5b) and (5c). These two equations are then easily integrated once to give the following integrals of motion

$$u^2 \equiv \left(v_{\|}+\frac{\omega}{k}\right)^2 + v_\perp^2 \quad , \quad \chi^2 \equiv (v_{\|} - V_0)^2 - 2\frac{a}{k}v_\perp \cos\Psi \quad . \quad (6)$$

From these two integrals, it is readily seen that electrons near resonance can be phase-trapped by the wave, in the sense that $\Psi$ is confined to a region inside the interval $(-\pi, \pi)$ . This is demonstrated by the phase diagram of Fig. 1, where curves of constant $u$ and $\chi$ are plotted in a $v_{\|} - \Psi$ plane. Later it will be found more convenient to use the coordinate $z$ as independent variable instead of time $t$ .

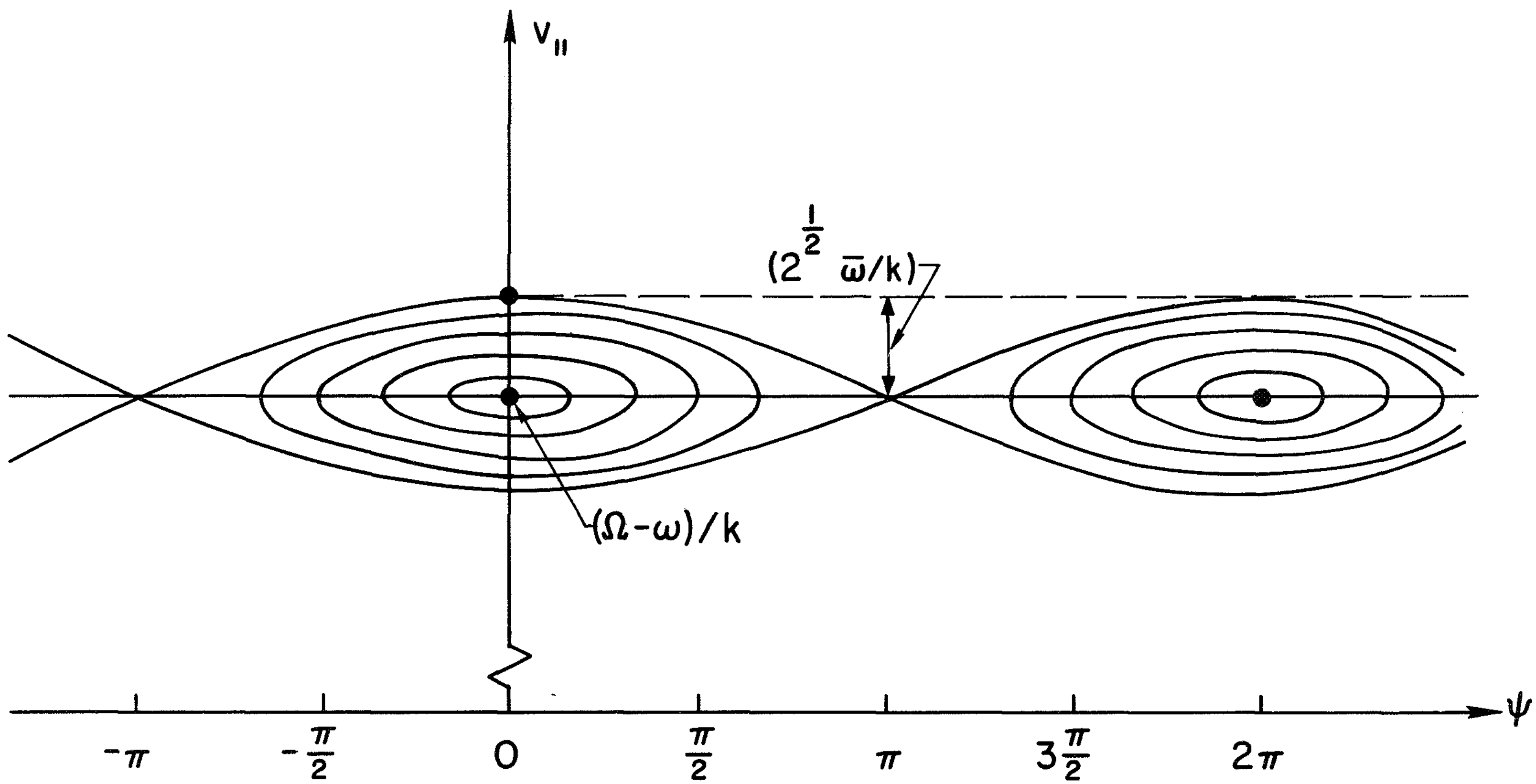


Fig. 1. Phase diagram.

Making this transformation of variable, and again neglecting terms of relative order of magnitude $(a/\Omega)^{1/2}$, one obtains from Eq. (5c)

$$\frac{d^2\Psi}{dz^2} + \kappa^2 \sin\Psi = 0 \quad , \tag{7}$$

where $\kappa^2 = \bar{\omega}^2/v_{\|}^2$. For resonant and near-resonant particles, $\kappa^2 = k^2\bar{\omega}^2/(\Omega - \omega)^2$.

Equation (7) is analogous to the equation of motion of a pendulum suspended in a gravity field, and shows that the relative phase angle $\Psi$ between $\underset{\sim}{v}_{\perp}$ and $-\underset{\sim}{B}_w$ oscillates around the equilibrium position $\Psi = 0$. The period for small deviations from this equilibrium is $2\pi/\kappa$. The solution of Eq. (7) in terms of elliptic functions is given in the Appendix.

## 2.2 Inhomogeneous Case

When the particle moves farther away from the equator, the inhomogeneity of the earth's magnetic field can no longer be neglected, and the full equations (5a)-(5c) must be considered.

The "pendulum" equation (5c) now has an additional force, corresponding to the right-hand side of the equation, which is proportional to the difference in acceleration between an imaginary particle moving with the resonance velocity $V_0$, and an electron in the absence of a wave field. As long as the magnitude of this force does not exceed $\bar{\omega}^2$, the electron may still be phase-trapped in the wave. Thus trapping is only possible if the condition

$$\bar{\omega}^2 > k \left| \dot{V}_0 - \frac{v_{\perp}^2}{2\Omega}\frac{\partial\Omega}{\partial z} \right| \tag{8}$$

is satisfied. Using the gyrofrequency model for the density variation, the condition above takes the form

$$\bar{\omega}^2 > \left(1 + \frac{\omega}{2\Omega} + \frac{\Omega-\omega}{2\Omega}\tan^2\alpha\right)\left|V_0\frac{\partial\Omega}{\partial z}\right| \quad , \tag{9}$$

where $\alpha$ is the pitch angle of the particle velocity $(\tan\alpha = v_{\perp}/v_{\|})$.

In the following, it is shown that condition (9) can only be satisfied over a relatively small region around the equator. For this region the formula

$$\Omega = \Omega_0 \left[ 1 + \frac{9}{2} \left(\frac{z}{R}\right)^2 \right] \quad , \tag{10}$$

gives a good approximation to the dipole field, where $R$ is the distance from the center of the earth. Using this formula, together with the inequality (9), one obtains

$$\frac{z}{R} < \left(\frac{2\pi a R}{9\Omega_0 \lambda}\right) \tan\alpha \left(1 + \frac{\omega}{2\Omega} + \frac{\Omega-\omega}{2\Omega} \tan^2\alpha\right)^{-1} \tag{11}$$

Using data relevant to TE experiments [1,2]; $R \sim 3R_0$ , where $R_0$ is the radius of the earth, $\lambda \sim 2$ km , $B \sim 1.2 \times 10^{-2}$ G , and $\omega \sim \Omega_0/2$ , and assuming $\alpha \sim 45^o$ and $B_w \sim 10^{-7} - 10^{-8}$ G , one obtains $\ell_c \sim 0.24\ R_0 - 0.024\ R_0$ for the length of the interaction region. The corresponding length for the phase oscillation becomes $\ell_B \sim 0.06\ R_0 - 0.2\ R_0$ , where $\ell_B = 2\pi/\kappa$ . Comparing the figures for $\ell_c$ and $\ell_B$ it will be seen that a field strength of $10^{-8}$ G is too small to give any nonlinear effects. For fields of about $10^{-7}$ G , $\ell_c$ and $\ell_B$ become of the same order of magnitude. At $10^{-7}$ G , we have $\ell_c \sim 4\ell_B$ . The case where $\ell_c$ and $\ell_B$ are of the same order of magnitude will be referred to as the low field case. When $B_w \sim 10^{-6}$ G , one finds that $\ell_c \sim 10^2\ \ell_B$ . The situation when $\ell_c \gg \ell_B$ is henceforth referred to as the high field case.

In the high field case, the length scale of the phase oscillations becomes much smaller than the scale length of the inhomogeneity. It is therefore appropriate to look for an adiabatic invariant corresponding to the phase oscillations. The existence of such an adiabatic invariant has already been pointed out by Laird and Knox [11], who did not, however, calculate it in any detail. A calculation can be done rather easily, following an approach similar to theirs.

One defines a Hamiltonian given by

$$H = \frac{m_e}{2} (v_{\parallel} - V_0)^2 + m_e \frac{\bar{\omega}^2}{k^2} (1 - \cos \Psi) \tag{12}$$

From this Hamiltonian, defining the momentum $p = m_e(v_{\parallel} - V_0)$, one obtains the equations of motion

$$\dot{z} = v_{\parallel} - V_0 = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = \frac{\partial H}{\partial z} \tag{13}$$

Classical mechanics then asserts the existence of an adiabatic invariant, $J$, given by [12]

$$J = \oint p dz = - \frac{m_e}{k} \oint (v_{\parallel} - V_0) \, d\Psi \quad , \tag{14}$$

when the parameters $(V_0, k, \bar{\omega})$ are varying slowly. The integrals above, which are evaluated along a closed loop in phase space for $H = \text{const.}$, are readily calculated to be

$$J = 8 \frac{H}{\bar{\omega}} \left[ \frac{E(\eta) - (1-\eta^2) K(\eta)}{\eta^2} \right] \quad , \tag{15}$$

where $E(\eta)$ and $K(\eta)$ are the complete elliptic integrals of the second and first kind respectively, and $\eta$ is defined in the Appendix to be

$$\eta = \left( \frac{1}{2} + \frac{\chi^2 k^2}{4\bar{\omega}^2} \right)^{1/2} \quad , \tag{16}$$

and $\chi^2$ is given by Eq. (6). The particles are trapped only when $\eta < 1$. For particles trapped near the 'bottom of the well' $(\eta = 0)$, one obtains the following expression for $J$, using asymptotic expressions for $E$ and $K$,

$$J = 2\pi \frac{H}{\bar{\omega}} \left( 1 + \frac{\eta^2}{8} + \ldots \right) \tag{17}$$

The first term of this expression is the adiabatic invariant of a harmonic oscillator of frequency $\bar{\omega}$ and energy $H$, which is in good agreement with what one would expect for those particles.

## 2.3 Energy Exchange

The energy exchange between the wave and a trapped particle depends on the relative phase angle $\Psi$, as can be seen from (5a). In the homogeneous case, $\Psi$ oscillates around $\Psi = 0$ where the energy exchange is zero. Thus the electron energy will oscillate with a period of the order of $2\pi/\bar{\omega}$.

In the inhomogeneous case, however, $\Psi = 0$ is no longer an equilibrium value for the phase angle. The equilibrium value, $\Psi_0$, for which $\ddot{\Psi} = 0$ is now given by

$$\sin \Psi_0 = \frac{k}{\bar{\omega}^2} \left[ \dot{V}_0 - (v_\perp^2/2\Omega) \frac{\partial\Omega}{\partial z} \right] \tag{18}$$

If the right-hand side of Eq. (18) varies sufficiently slowly, the relative phase angle of a trapped electron will oscillate around this new equilibrium value for $\Psi$, with a period that for small oscillations is slightly longer than in the homogeneous case, namely $2\pi/\bar{\omega}(\cos \Psi_0)^{1/2}$. For electrons staying near this new equilibrium position, which can be considered as the centre of gravity of the distribution of trapped electrons, the rate of energy transfer can be obtained from Eqs. (5a) and (18) to be

$$\frac{d}{dt} [v_\parallel^2 + v_\perp^2] = - \frac{\omega}{k} \left[ \dot{V}_0 - (v_\perp^2/2\Omega) \frac{\partial\Omega}{\partial z} \right] \quad . \tag{19}$$

Equation (19) now shows an increase of energy when $\dot{V}_0 - (v_\perp^2/2\Omega)\, \partial\Omega/\partial z < 0$, i.e., when the particle travels towards the equator, and a corresponding decrease when the particle moves away from the equator.

As these results only concern particles very near exact resonance, no conclusion can be drawn for the overall damping or amplification of a WM wave.

## 3. WHISTLER AMPLIFICATION (OR DAMPING)

In this section, whistler amplification and damping in the magnetosphere are considered. Liemohn [7], and Lee et al. [8], have dealt with this question in considerable detail within the limits of linear theory. For the sake of completeness, this linear theory will be considered briefly here.

### 3.1 Linear Theory

In the following, a similar approach to that in Ref. [7] is followed. We shall, however, consider a class of distribution functions that is somewhat different from those considered by Liemohn. The particle distribution functions that he deals with are apparently not equilibrium distributions, in the sense that they depend on the kinetic energy, $E$ , and the magnetic moment, $\mu$ , as well as the coordinate along a field line. Now an equilibrium distribution does not depend on the coordinate along a field line, and in the following we shall choose a class of functions meeting that requirement.

It is assumed that the electron population consists mainly of a low temperature thermal distribution, $f_0$ , plus a small nonthermal tail of energetic electrons, $\delta f_0$

$$f = f_0 + \delta f_0 \quad . \tag{20}$$

Linearizing the Vlasov equation, and using Maxwell's equations assuming solutions varying as $\exp[i(\omega t - \int k dz) - \int k_i dz]$ , where again the z-axis is along the magnetic field, one arrives at the following expression for $k_i$ for the WM (see Ref. [7])

$$\frac{k_i}{k} = \frac{1}{2}\left(\frac{\pi\omega_p}{ck}\right)^2 \int_0^\infty [\delta f_0]_{v_\parallel = V_0} v_\perp^2 dv_\perp \quad , \tag{21}$$

where $[\delta f_0]$ has been written for $[v_\perp \partial/\partial v_\parallel - (v_\parallel + \omega/k)\,\partial/\partial v_\perp]\,\delta f_0$ , and advantage has been taken of the assumption that $\delta f_0 \ll f_0$ , which implies $k_i \ll k$ . Further, we have used the fact that the parameters are not varying significantly over a wavelength, i.e., $k \gg (\partial B/\partial z)/B$ .

Assuming now the simple class of nonthermal distribution functions

$$\delta f_0 = \text{Const} \times \frac{\mu^n}{(E + E_0)^m} \quad , \qquad (22)$$

where $\mu = m_e v_\perp^2/2B$ is the magnetic moment, $E$ is the kinetic energy, and $E_0$ is a constant related to the mean kinetic energy of the nonthermal distribution. In terms of $v_\perp$ and $v_\parallel$, Eq. (22) can be written as

$$\delta f_0 = A \frac{(\beta v_\perp^2)^n}{(v_\perp^2 + v_\parallel^2 + v_0^2)^m} \quad , \qquad (23)$$

where $A$ is a constant chosen such that the integral $\int \delta f_0 \, d\underset{\sim}{v}$, evaluated at the equator, is equal to the ratio between the densities of the nonthermal and the thermal electrons. The quantity $\beta = \Omega_0/\Omega$, where $\Omega_0$ is the gyrofrequency at the equator, at the field line considered.

Substituting Eq. (23) into Eq. (21) one obtains

$$\frac{k_i}{k} = S(m,n) \left(\frac{\omega}{\Omega_0}\right)^{-n-N} \frac{\left(\frac{n+1}{n} - \frac{\Omega}{\omega}\right)}{\left(\frac{\Omega}{\omega}\right)^{n-N} \left(\frac{\Omega}{\omega} - 1\right)^{3N}} \quad , \qquad (24)$$

where $N = (m - n - 2)/2$, and $S(m,n)$ is a positive dimensionless constant depending on $n$, $m$, $A$, and the values of $\omega_p$ and $\Omega$ at the equator. For lower altitudes, where $\Omega \gg \omega$, one has

$$\frac{k_i}{k} \approx - S(m,n) \, \omega^{-n-N} \left(\frac{\Omega}{\omega}\right)^{3-m} \qquad (25)$$

It is seen from Eq. (24) that amplification occurs when

$$\frac{\Omega}{\omega} > \frac{n+1}{n} \qquad (26)$$

When $\Omega_0/\omega$ is somewhat smaller than $1 + 1/n$, Eq. (24) implies that the region around the equator will absorb WM wave energy.

The variation of $k_i/[k_0 S(m,n)]$ , where $k_0$ is the wave number at the equator, has been plotted in Fig. 2 as a function of geometric latitude for two sets of values of $m$ and $n$ .

From the previous section, it is evident that nonlinear wave-particle interaction with a monochromatic wave can only take place in a relatively narrow region around the magnetic equator. For a certain frequency band, $\omega > \Omega_0 n/(n+1)$ , the linear theory predicts a transition from amplification to damping in the equatorial region. Consequently, for this frequency band, any change that the nonlinear wave-particle interaction might produce in the damping or growth of these waves may be significant. In the following, therefore, the nonlinear correction to the damping (or growth) near the equator will be investigated.

### 3.2 Nonlinear Theory

In the following, the effect of the nonlinear wave-particle interaction on the damping (or growth) of the WM wave is considered. The effect on one electron was discussed in Section 2, and it was pointed out that particles near or at the resonant velocity, $V_0$ , were 'phase-trapped' in the WM wave. Due to this interaction, the electron velocity distribution is changing near the resonant velocity, and consequently the rate of absorption (or amplification) of the WM wave changes.

The region A (see Fig. 3b) where the nonlinear interaction can take place has a length $\ell_c$ that was estimated in Section 2 . According to the figures given in that section, A is a fairly narrow region around the equatorial plane of the field line considered. It appears from the previous discussion that two rather different cases should be taken into account, namely the low field case, where $\ell_B \sim \ell_c$ , and the high field case, where $\ell_c \gg \ell_B$ . In the following, the low field case is considered.

In this case $\ell_B$ and $\ell_c$ are both smaller than the lengths of wave trains corresponding to a Morse dot [0.25 $R_0$ ($\sim$ 50 ms)] and a dash [0.75 $R_0$ ($\sim$ 150 ms)] . Further, it is possible to show that the acceleration of the wave and the particles due to the inhomogeneity of the field is not of much importance within most of the region A , and can be neglected. In this case, therefore, it makes sense to use a homogeneous model (see Fig. 3a)

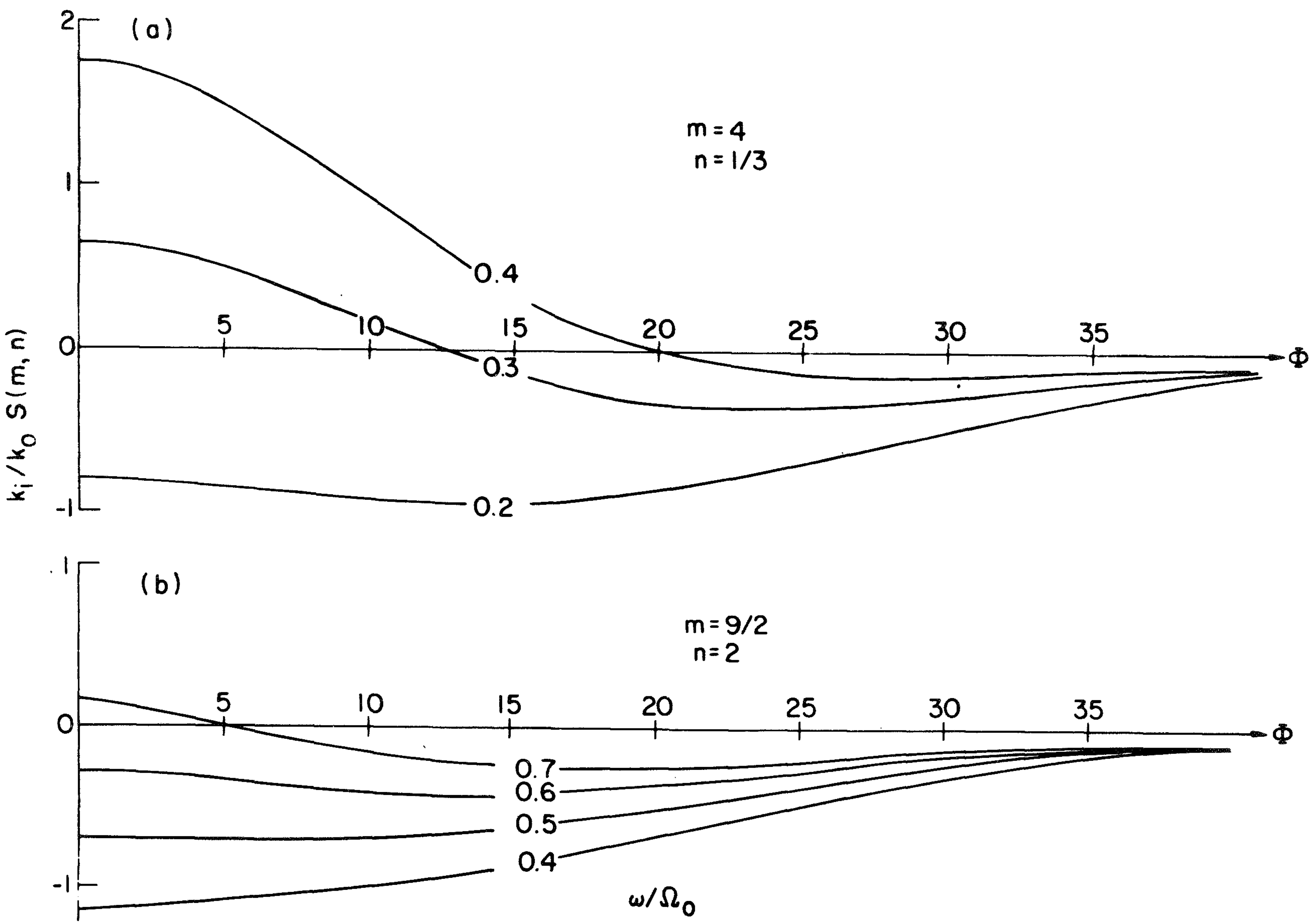


Fig. 2. Linear damping as a function of geomagnetic latitude.

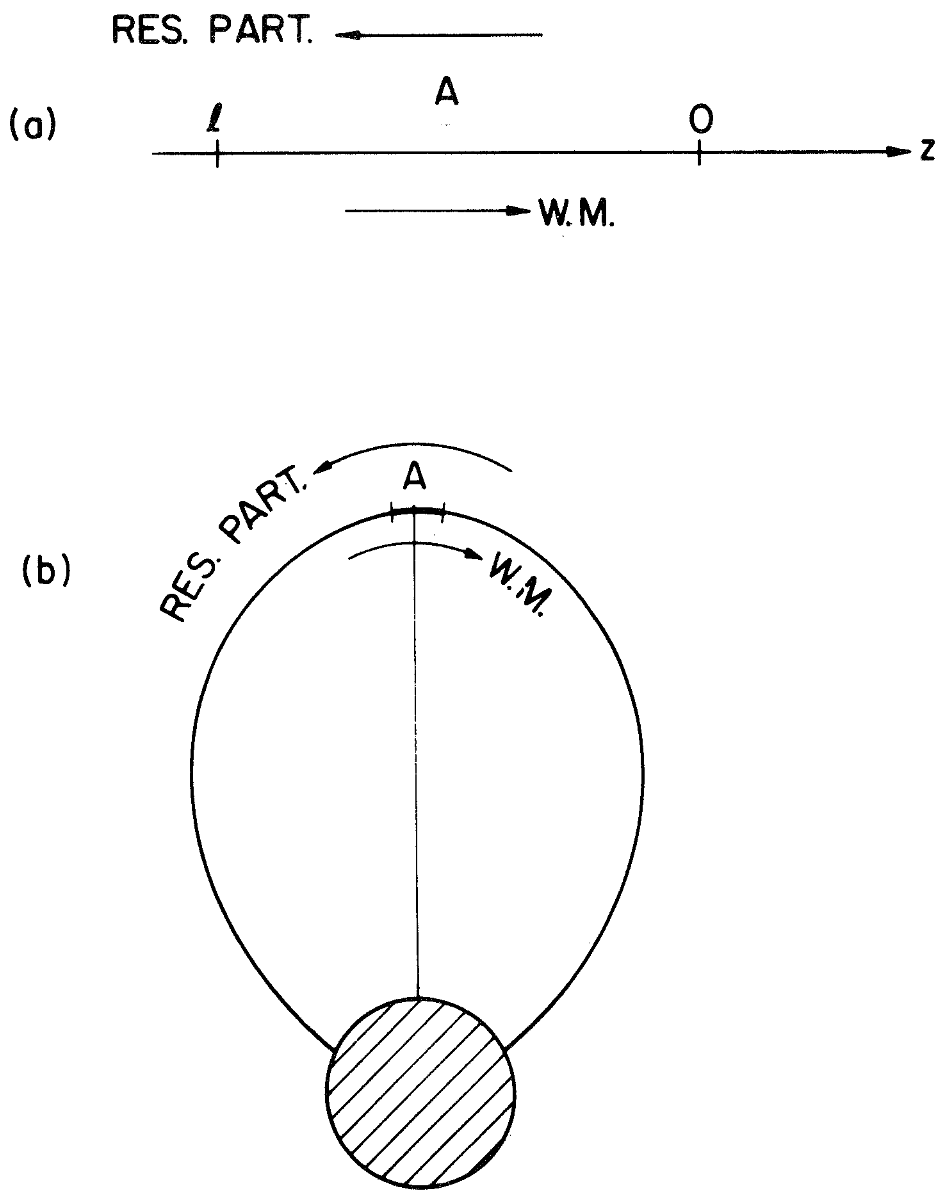


Fig. 3. Diagram of interaction region.

where the wave train occupies the interaction region, moving from the left to the right, and the particles are streaming in from the right $(z > 0)$ . In this model, the electrons to the right of $A$ are unperturbed, and their distribution function can be taken as that given by Eqs. (20) and (23). As the electrons are streaming through $A$ , those at or near resonance, belonging of course to the nonthermal tail of the distribution, are interacting with the WM wave. The electron velocity distribution function will consequently be changing through $A$ , for velocities near $V_0 = (\Omega-\omega)/k$ .

Introducing the variable $w = v_{\parallel} - V_0$ , the unperturbed nonthermal distribution function can be considered a function of $u$ [given by Eq. (6)] and $w$ only, and can be expanded near $w = 0$ in the form

$$\delta f_0(u,w) = \delta f_0(u,0) + \left.\frac{\partial \delta f_0}{\partial w}\right|_0 w + \ldots \tag{27}$$

Because $\delta f_0$ can be assumed to vary only insignificantly for small values of $w$ , the first two terms in the expansion of Eq. (27) will give sufficient accuracy to this region. For the perturbed distribution (for $z < 0$ ), an expansion like Eq. (27) does not make sense because the WM wave will impose a fine structure on the particle distribution for small $w$ .

To evaluate the perturbed distribution, $\delta f$ , Liouville's theorem is applied, which states that

$$\delta f(\underset{\sim}{v},z) = \delta f_0[u,w_0(\underset{\sim}{v},z)] \quad , \tag{28}$$

where $w_0$ is the unperturbed value of $w$ for $z > 0$ , and advantage has been taken of the fact that $u$ is an integral of the motion. Using Eqs. (27) and (28), one obtains

$$\delta f(\underset{\sim}{v},z) = \delta f_0(u,0) + \left.\frac{\partial \delta f_0}{\partial w}\right|_0 w_0(\underset{\sim}{v},z) \tag{29}$$

To obtain the nonlinear growth (or damping), a method similar to that applied by O'Neill [13] to the problem of nonlinear damping of Langmuir waves, may be applied. This method is not self-consistent, in the sense that the perturbed particle trajectories are calculated on the assumption that the wave amplitude is constant, which it is not, of course, in the presence of amplification or damping. Although crude, the method has shown good qualitative agreement with experiment [14], and will be applied with some confidence in the following.

Invoking the conservation of energy, and assuming a steady state, we have

$$\frac{\partial T}{\partial z} + \frac{\partial}{\partial z}\left[\frac{1}{8\pi}\left(E_w^2 + B_w^2\right)\right] = 0 \quad , \tag{30}$$

where $T$ represents the kinetic energy of the particles. Defining the coefficient of damping (or growth), $k_i$, as $k_i = -(\partial B_w/\partial z)/B_w$, and taking advantage of the fact that the equivalent permittivity $\epsilon \gg 1$ for the case of interest here (because $\omega_p \gg \Omega$), we obtain

$$k_i = \frac{\omega_p^2}{2c^2a^2}\frac{\partial}{\partial z}\int v^2\,\delta f\,d\underset{\sim}{v} \tag{31}$$

To evaluate the above integral, the expression for $\delta f$ [Eq. (29)] is applied, together with the expressions for $w_0$ [Eqs. (A.5) and (A.6)] derived in the Appendix. Applying the result $w_0(w,\Psi) = -w_0(-w,-\Psi)$, which is most readily deduced from Fig. 1, one obtains

$$\frac{k_i}{k} = \frac{\omega_p^2(\Omega-\omega)}{c^2k^2a^2}\frac{\partial}{\partial z}\int \left.\frac{\partial \delta f_0}{\partial w}\right|_0 w\,w_0\,d\underset{\sim}{v} \tag{32}$$

The integral in Eq. (32) is conveniently broken up into two parts: one for the trapped particles, and one for the nearly trapped particles. For the trapped particles, the variables of integration $\eta$, $v_\perp$, and $\xi$ (see the Appendix) are introduced, giving the term

$$\frac{\partial}{\partial z}\left\{\frac{8\omega_p^2(\Omega-\omega)}{c^2k^4a}\int_0^\infty v_\perp^2 \left.\frac{\partial\delta f_0}{\partial w}\right|_0 dv_\perp \int_0^1 \eta^3 d\eta \int_{-\pi}^{\pi}\frac{\varpi\,\cos(\xi/2)\,\mathrm{cn}[F(\xi/2,\eta)-\kappa z,\eta]\,d\xi}{[1-\eta^2\sin(\xi/2)]^{1/2}}\right\} . \tag{33}$$

For the nearly trapped particles, the variables of integration $\eta$ , $v_\perp$ ; and $\Psi$ are more convenient. One obtains

$$\frac{\partial}{\partial z}\left\{\frac{8\omega_p^2(\Omega-\omega)}{c^2k^4a}\int_0^\infty v_\perp^2 \left.\frac{\partial\delta f_0}{\partial w}\right|_0 dv_\perp \int_1^\infty \eta^2 d\eta \int_{-\pi}^{\pi} \bar{\omega}\mathrm{dn}[F(\Psi/2,1/\eta) - \eta\kappa z, 1/\eta]d\Psi\right\} . \tag{34}$$

Adding the expressions (33) and (34); differentiating, and making the substitutions $\eta \to 1/\eta$ and $\Psi \to \xi$ in Eq. (34), gives

$$\frac{k_i}{k} = \frac{\pi^2\omega_p^2}{2c^2k^2}\int_0^\infty v_\perp^3 \left.\frac{\partial\delta f_0}{\partial w}\right|_0 dv_\perp \int_0^1 d\eta \int_{-\pi}^{\pi} H(\eta,\xi,\kappa z)\, d\xi$$

$$H = \frac{16}{\pi^2}\left\{\eta^3 \frac{\cos(\xi/2)\,\mathrm{sn}[F-\kappa z,\eta]\mathrm{dn}[F-\kappa z,\eta]}{[1-\eta^2\sin^2(\xi/2)]^{1/2}} + \frac{1}{\eta^3}\,\mathrm{sn}\left[F-\frac{\kappa}{\eta}z,\eta\right]\mathrm{cn}\left[F-\frac{\kappa}{\eta},\eta\right]\right\} , \tag{35}$$

where $F = F(\xi/2,\eta)$ .

Applying asymptotic expressions for the Jacobi elliptic function, and performing the $\eta$ and $\xi$ integrations, one obtains

$$\lim_{z\to 0^-}\int_0^1 d\eta \int_{-\pi}^{\pi} H\, d\xi = 1 , \tag{36}$$

and the expression left for $k_i$ is readily identified as the rate of damping (or growth), $k_i^L$ , derived from the linear theory, noting that

$$v_\perp \left.\frac{\partial\delta f_0}{\partial w}\right|_0 = [\delta f_0]_{v_\parallel = V_0} \tag{37}$$

The dependence of H on $v_\perp$ only enters through $\kappa$ , which can be written in terms of the pitch angle, $\alpha$ , as

$$\kappa = k\,[a\tan\alpha/(\Omega-\omega)]^{1/2} \tag{38}$$

Finally one obtains $k_i$ in the form

$$k_i = k_i^L \int_0^{\pi/2} g(\alpha) \left\{ \int_0^1 d\eta \int_{-\pi}^{\pi} H \, d\xi \right\} d\alpha \quad , \tag{39}$$

where the form of $g(\alpha)$ is found from Eq. (23) to be const. $\times (\cos\alpha)^{2(m-n)-3} \times (\sin\alpha)^{2n+1}$, and the constant is chosen such that

$$\int_0^{\pi/2} g(\alpha) \, d\alpha = 1 \tag{40}$$

The result of a numerical integration of Eq. (39) for $m = 4$ and $n = 1/2$ is given in Fig. 4(a), which displays $k_i/k_i^L$ as a function of the argument $-kz[a/(\Omega-\omega)]^{1/2}$. The interaction region is on the negative z-axis starting at $z = 0$.

### 3.3 Discussion

It is clear from the previous two sections, that along most of the path of the WM wave, except in a fairly narrow region around the equator, the question of amplification or absorption is adequately answered by the linear theory. This theory predicts for a rather wide class of electron distribution functions, that the main amplification or absorption (see Fig. 2) occurs on the upper part of the field line $(-25^\circ < \phi < 25^\circ)$. For frequencies $\omega$ such that

$$\omega > \frac{n}{n+1} \Omega_0 \quad ,$$

the WM wave is damped on the top of the field line. In the nonlinear interaction region A, however, the damping is changed as indicated in Fig. 4(a), except for the front region of the wave train which moves through A according to the linear theory.

So far, it has been assumed that the electron velocity distribution to the right of A is undisturbed. This seems to be a reasonable assumption when one considers wave trains of short duration. If, however, a steady state WM wave ('key-down') is applied, one must take into account that the electrons interacting with the WM wave are reflected in the mirror field of the earth, and after one complete bounce period again enter the interaction region from the right. If these particles have

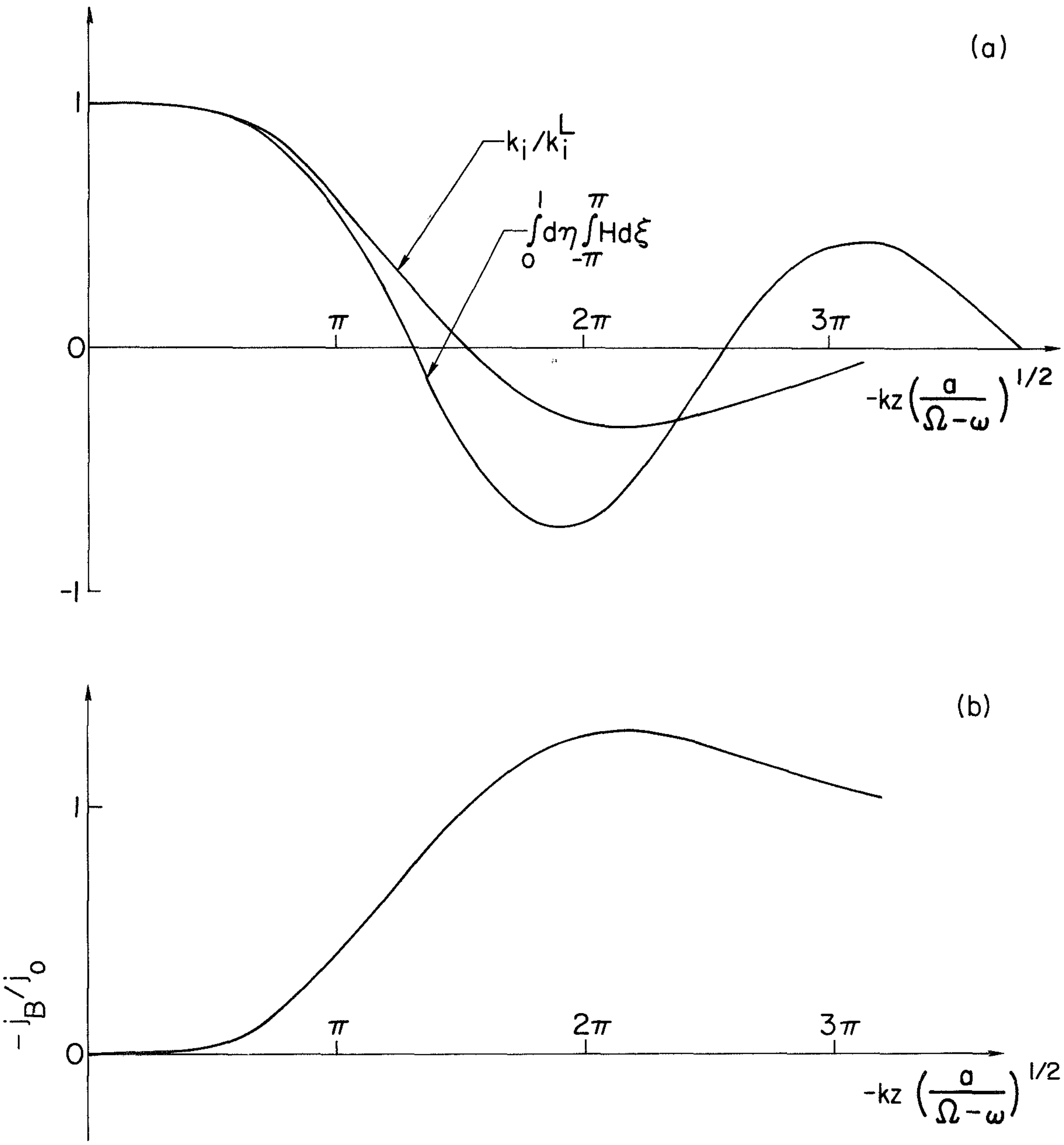


Fig. 4. Nonlinear damping and current due to phase-bunching.

not in the meantime been scattered by magnetospheric disturbances other than the WM wave, they will interact with the wave again repeatedly. The phase angle relative to the WM wave on successive arrivals at A will be different, however. This will eventually lead to a complete 'stirring' of the electron velocity distribution near the resonance velocity, resulting in a new equilibrium where the WM wave and the plasma do not exchange energy. In the homogeneous case, this corresponds to an equilibrium where the distribution function, $\delta f$, is a function of $u$ [given by Eq. (6)] only, in the neighborhood of the resonance velocity $[(\partial f/\partial w)|_0 = 0]$. For such a distribution function, the damping (or amplification) vanishes.

An experimental result supporting these conclusions has been reported by McNeill [10]. He received signals from NPG Seattle on 18.6 kHz at Lower Hutt, New Zealand, corresponding to whistler mode propagation to an altitude of 2.4 $R_0$. The transmitter had a one-hour cycle of 50 min Morse transmission followed by a 5 min silent period ('key-up') and finally a 5 min key-down transmission.

It was observed that after a key-down period the Morse signals sometimes came through with a power of the order of 10 dB above the power of the Morse signals immediately preceding the key-up, key-down periods. A typical decay time for this enhancement was estimated by McNeill to be approximately 50 min.

Now, a key-down period of 5 min should allow for something like $10^2$ passages through A by a resonant electron, and thus provide time enough for a new equilibrium to be approached. The increase in power of a transmitted signal due to the new equilibrium is readily estimated to be $\exp[2\ell_c k_i^L]$, assuming that the linear theory predicts absorption at the equator for the frequency supplied, so an increase of 10 dB would correspond to $\ell_c k_i^L \sim 1/2$, which appears to be a reasonable figure.

## 4. NONLINEAR PHASE-BUNCHING

In the following, the so-called nonlinear phase-bunching will be considered. This effect has been discussed by Helliwell [1] as a possible source of TE , using a simplified model for the phase-trapped electrons.

It is instructive to start with Helliwell's model, which consists of electrons in exact resonance with a WM wave. Upon entering the wave train, the particles are evenly distributed in phase. This is shown in Fig. 5(a), where the resonant particles are distributed along the $\Psi$-axis in the phase diagram between $-\pi$ and $\pi$ . They all lie within the trapping region, and are all in exact resonance $(w = 0)$ . After having moved through the wave train for a quarter of a bunching period, $2\pi/\overline{\omega}$ , the distribution of the same particles in phase space is shown in Fig. 5(b). It will be seen that nearly all the particles are now located in a small interval around $\Psi = 0$ . Thus, the particles that upon entering the wave train were evenly distributed in phase have now become phase-bunched.

Consider next the more realistic model sketched in Fig. 5(c). The electrons are again distributed evenly in phase. Instead of consisting of only one beam with $w = 0$ , they are now given a distribution, $f(w)$ , whose level lines are drawn as straight lines parallel to the $\Psi$-axis, indicating that $f(w)$ has a gradient. After a quarter of a bunching period the phase diagram looks like that sketched in Fig. 5(d). As will be seen from this figure, the phase-bunching is still there, even if it is not as pronounced as in the previous case [Fig. 5(b)]. The bunching is now roughly proportional to the gradient in $f(w)$ . If there were no gradient in $f(w)$ initially the bunching would disappear altogether.

The phase-bunching, and the nonlinear damping are of course closely related. To see the relation, it should be pointed out that if a population of trapped and nearly trapped electrons has become phase-bunched, this implies that their distribution function, $\delta f$ [see Eq. (29)], has become a periodic function of $\Psi = \varphi + kz - \omega t$ . This fact again implies

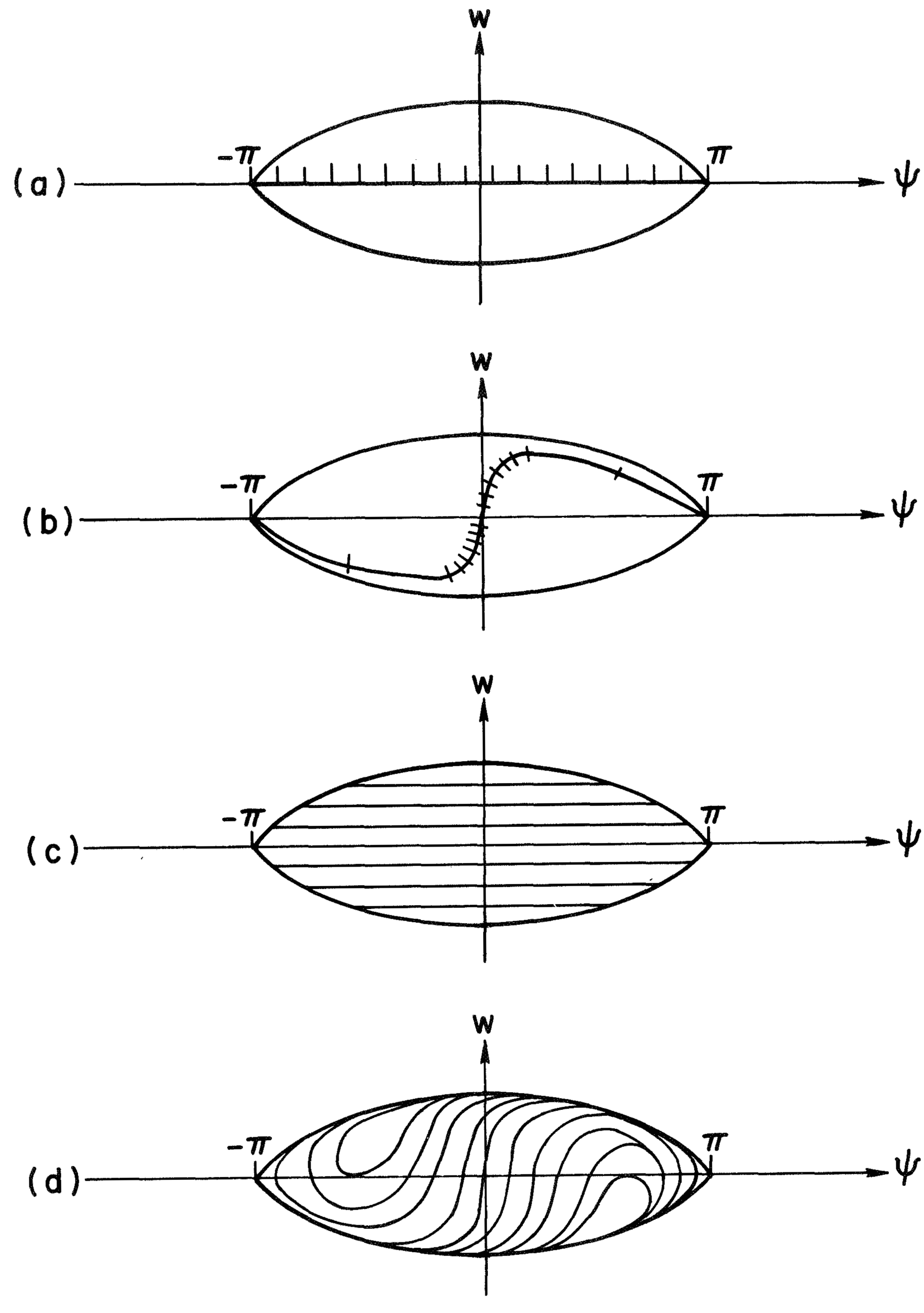


Fig. 5. Phase diagram illustrating phase-bunching.

that the particles near and at resonance give rise to a current $\underset{\sim}{j}$ , given by

$$\underset{\sim}{j} = e \int \underset{\sim}{v}\, \delta f \, d\underset{\sim}{v} = e \int \underset{\sim}{v}\, w_0(\underset{\sim}{v}, z) \left. \frac{\partial \delta f_0}{\partial w} \right|_0 d\underset{\sim}{v} \quad , \tag{41}$$

where Eq. (29) has been used, and advantage has been taken of the fact that the integral of the first term in Eq. (29) vanishes because $\delta f_0(u,0)$ is not a function of $\varphi$ . Introducing Eqs. (A.5) and (A.6) in Eq. (41), and integrating, one finally obtains

$$\underset{\sim}{j} = - A\,[- \sin(\omega t - kz) \,,\, \cos(\omega t - kz) \,,\, 0] \quad , \tag{42}$$

where

$$A = \frac{16\, e n_0 a}{k} \int_0^\infty v_\perp^3 \left. \frac{\partial \delta f_0}{\partial w} \right|_0 dv_\perp \int_0^1 d\eta \int_{-\pi}^{\pi} G(\eta, z, \xi)\, d\xi \quad ,$$

and

$$G(\eta,z,\xi) = \sin(\xi/2) \left\{ \eta^3 \, \mathrm{cn}\, [F(\xi/2,\eta) - \kappa z, \eta] + \frac{1}{\eta^3} \frac{\mathrm{dn}\, [F(\xi/2,\eta) - (\kappa z/\eta), \eta] \cos(\xi/2)}{[1 - \eta^2 \sin^2(\xi/2)]^{1/2}} \right\} \quad .$$

In the absence of resonant particles, the WM wave will propagate with a constant amplitude, i.e., without amplification or absorption. In that case, the current due to the wave is simply

$$(c/4\pi)\, \nabla \times \underset{\sim}{B}_w = (kc/4\pi)\, \underset{\sim}{B}_w \quad , \tag{43}$$

where the displacement current, being of relative order of magnitude $(\Omega/\omega_p)^2 \ll 1$ , has been neglected. In the presence of the resonant and near-resonant particles the total current density in the plasma is given by

$$(kc/4\pi)\, \underset{\sim}{B}_w + \underset{\sim}{j} \quad , \tag{44}$$

where $\underset{\sim}{j}$ is given by Eqs. (41) and (42). One of Maxwell's equations now gives

$$\nabla \times \underset{\sim}{B}_w = k\underset{\sim}{B}_w + \frac{4\pi}{c}\underset{\sim}{j} \quad , \tag{45}$$

where the displacement current has again been neglected. Taking into consideration that $B_w = |\underset{\sim}{B}_w|$ is now a slowly-varying function of $z$, one obtains

$$\underset{\sim}{j} = -\frac{c}{4\pi} k_i B_w \left[-\sin(\omega t - kz) , \cos(\omega t - kz) , 0\right] \quad . \tag{46}$$

Comparing Eqs. (42) and (46) one obtains

$$A = (c/4\pi)\, k_i B_w \tag{47}$$

One can consequently use the expression for $A$ given by Eq. (42) to evaluate $k_i$, as an alternative method.

To obtain the part of $\underset{\sim}{j}$ which results from the nonlinear bunching, the linear limit $\underset{\sim}{j}_0$ of $\underset{\sim}{j}$ must be subtracted. Denoting the current due to the nonlinear bunching as $\underset{\sim}{j}_B$, one has

$$\underset{\sim}{j}_B = \underset{\sim}{j} - \underset{\sim}{j}_0 = (c/4\pi)\,(k_i - k_i^L)\, B_w \left[-\sin(\omega t - kz) , \cos(\omega t - kz) , 0\right] \quad . \tag{48}$$

In Fig. 4(b) the quantity $-j_B/j_0$ resulting from the computed value of $k_i$, is displayed as a function of the argument $-kz[a/(\Omega-\omega)]^{1/2}$.

Although the preceding calculations apply only to the low field case, it is to be expected that they will also give a good indication of what is happening in the high field case. In this case, $\ell_c \gg \ell_B$, implying that when the wave train occupies most of the interaction region, the resonant particles traveling through it will experience several oscillations in the relative phase angle $\Psi$. Due to variation of the frequency of oscillation with position in the trapping region, one expects a 'smearing out' of the distribution to occur. This will eventually result in the establishment of a new equilibrium, where the wave no longer interacts with the nonthermal electron velocity distribution.

Thus, the electron distribution upon leaving the wave train no longer gives rise to amplification or damping, nor does it carry a current due to phase-bunching. There are two exceptions, however, which should be pointed out. They occur when the distance from the front of the wave train to the entrance of the interaction region, or the distance from the exit of the interaction region to the rear end of the wave train, is approximately $2\pi/\kappa$ . In these two cases, phase-bunching will occur.

It has been suggested by Sudan and Ott [5] that, due to the acceleration effect, new particles will constantly be caught by the wave, thus giving rise to a cumulative effect which should enhance the number of particles taking part in the phase-bunching. This, however, does not appear to be correct. Considering a phase diagram of the type given in Fig. 1, and recalling from elementary mechanics that the motion on such a phase plane is 'incompressible', it appears that the only way of absorbing more particles into the trapping region, is to expand this region. Such an expansion admittedly takes place as the wave travels towards the equator, but the maximum area of the trapping region occurs at the equator, where one has essentially homogeneity. Thus, it appears that the acceleration effect decreases rather than increases the number of trapped particles.

Over a part of the interaction region where acceleration can be considered constant, Eq. (5c) can be integrated if we consider $\bar{\omega}^2$ and the right-hand side of the equation as constant. The corresponding phase diagram is sketched in Fig. 6, illustrating the fact that there is no flux of particles into the trapping region.

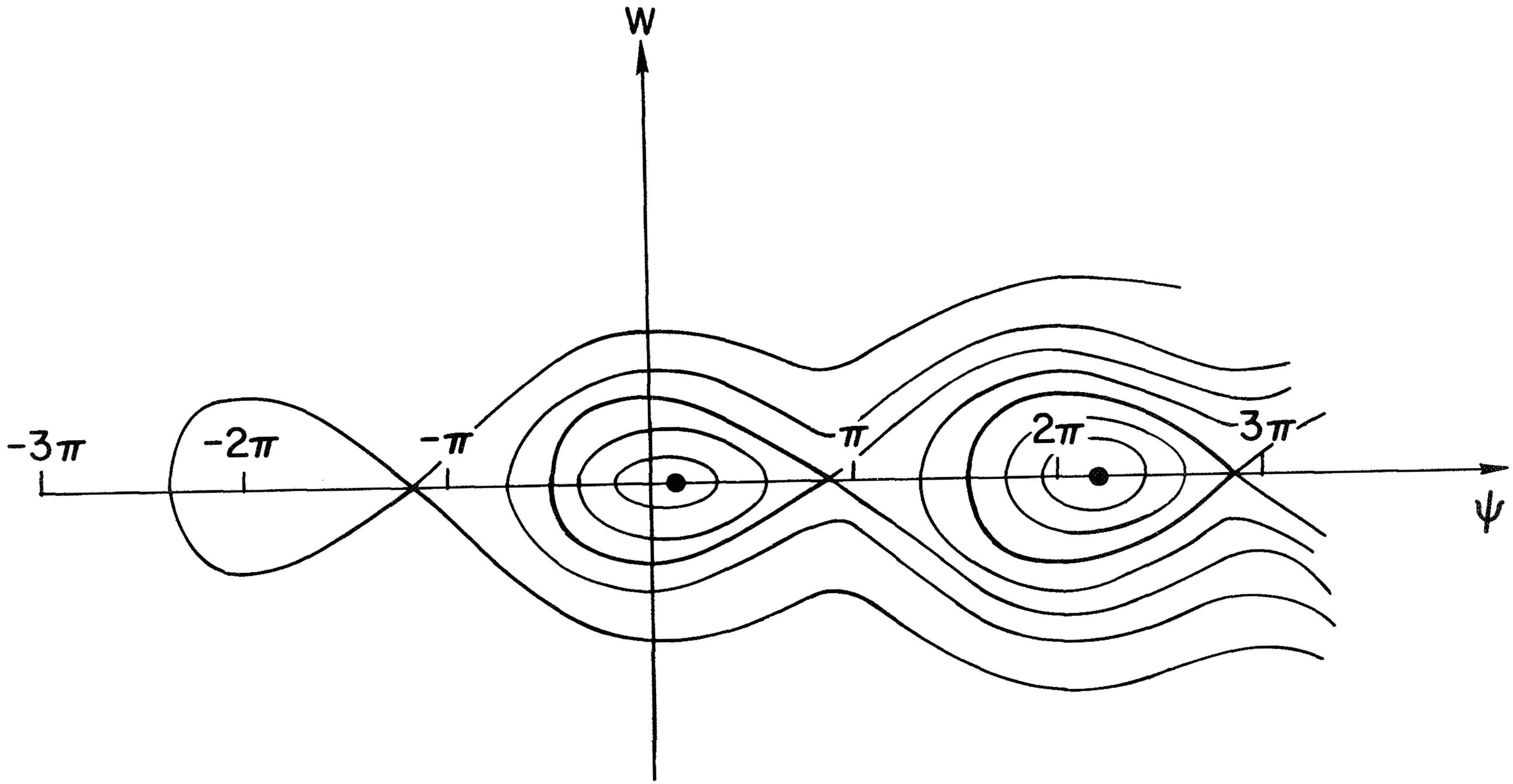


Fig. 6. Phase diagram for electrons in an accelerated wave.

## 5. EMISSION

It appears from the previous section that when the resonant particles interact with a WM wave train over a length approximately given by $2\pi/\kappa$ before they leave it, they carry a current $j_B$ . This current has a structure, i.e., a frequency $\omega$ and a wave number $k$ . In the neighborhood of the rear end of the wave train, $\omega$ and $k$ have the same values as the frequency and wave number of the wave producing the current (henceforth referred to as the 'triggering' wave).

As the particles get farther away from the triggering wave $\omega$ , $k$ , and the velocity $v_{\|}$ of the particles, are all varying slowly due to the inhomogeneity of the earth's magnetic field. Also, a certain amount of phase-mixing takes place, due to the fact that the current-carrying particles have a certain width of distribution in $v_{\perp}$ and $v_{\|}$ . A characteristic length for the phase-mixing to take place can be estimated to be $2\pi/\kappa$ , in the absence of any phase-correlating effects.

Clearly, the current due to the phase-bunched particles will give rise to the emission of a new WM wave. In fact the resonant particles act like an antenna. The gain of this antenna will depend on its length, or rather its 'coherence length', $\ell$ , i.e., the maximum length over which a part of an emitted wave can stay in resonance with the current-carrying particles.

A rough estimate of the amplitude of the emitted wave, $B_E$ , at the front end of the antenna can be found using the wave equation, and Eq. (48) for the current density. We have

$$B_E \sim k_i \ell B_W \quad , \tag{49}$$

where the value of $k_i$ at the end of the triggering wave train is used.

If no phase-correlating effects are present, a reasonable upper bound on $\ell$ seems to be $2\pi/\kappa$ . The emitted wave itself, however, will have a certain phase-correlating effect. In order to explain the observed emissions in experiments with artificially triggered emission, it appears to be necessary to produce a self-sustained process in which the emitted WM phase-bunches resonant particles which subsequently produce more emission. In the following, this possibility is considered.

Let $E$ denote the region in which the self-sustained process of phase-bunching and emission takes place. $E$ is not necessarily a stationary domain. It may move relative to the equator. It will be assumed that the linear theory predicts amplification within $E$ for the frequency band emitted. Consider the model of $E$ sketched in Fig. 7. It is convenient to subdivide it into two sections which have been denoted the phase-bunching section, and the antenna section. Through the first section, the phase-bunching is growing, together with a change in the amplification rate, $k_i$, which is shown in the lower curve of Fig. 7. The amplitude of the emission shown in Fig. 7 is growing from the rear end of $E$. The variation in amplitude in the antenna section depends rather critically upon the value of $k_i$ there. If $k_i$ varies as indicated with the broken curve in Fig. 7, the gain from the antenna will be seriously reduced. If, on the other hand, $k_i$ varies as indicated by the solid curve, the growth in the antenna section will be approximately linear, as indicated. The reason why $k_i$ is shown to vary insignificantly over the antenna section is that the amplitude of the emitted signal is still so small there that the bunching length is greater than the 'length' of the antenna, $\ell_A$. When, however, the emitted wave reaches the phase-bunching section, the value of $k_i$ increases (see also Fig. 4 to compare the variation of bunching with the variation of $k_i$), and the growth from there on is exponential.

The effect that saturates the growth across the region $E$ has already been indicated. If the amplitude gets too high, the bunching occurs earlier, together with a transition of $k_i$ from amplification to damping. Thus, in the antenna section, the emission becomes seriously limited by damping, and the total growth is reduced. A rough estimate of the growth through the region $E$ gives the following approximate condition which must be satisfied

$$k_i^L \ell_A \exp\left[\int k_i \, dz\right] \sim 1 \quad , \tag{50}$$

where the integral is taken through the whole region $E$. Both in Eqs. (49) and (50) it has been assumed that there is little or no damping across the antenna section.

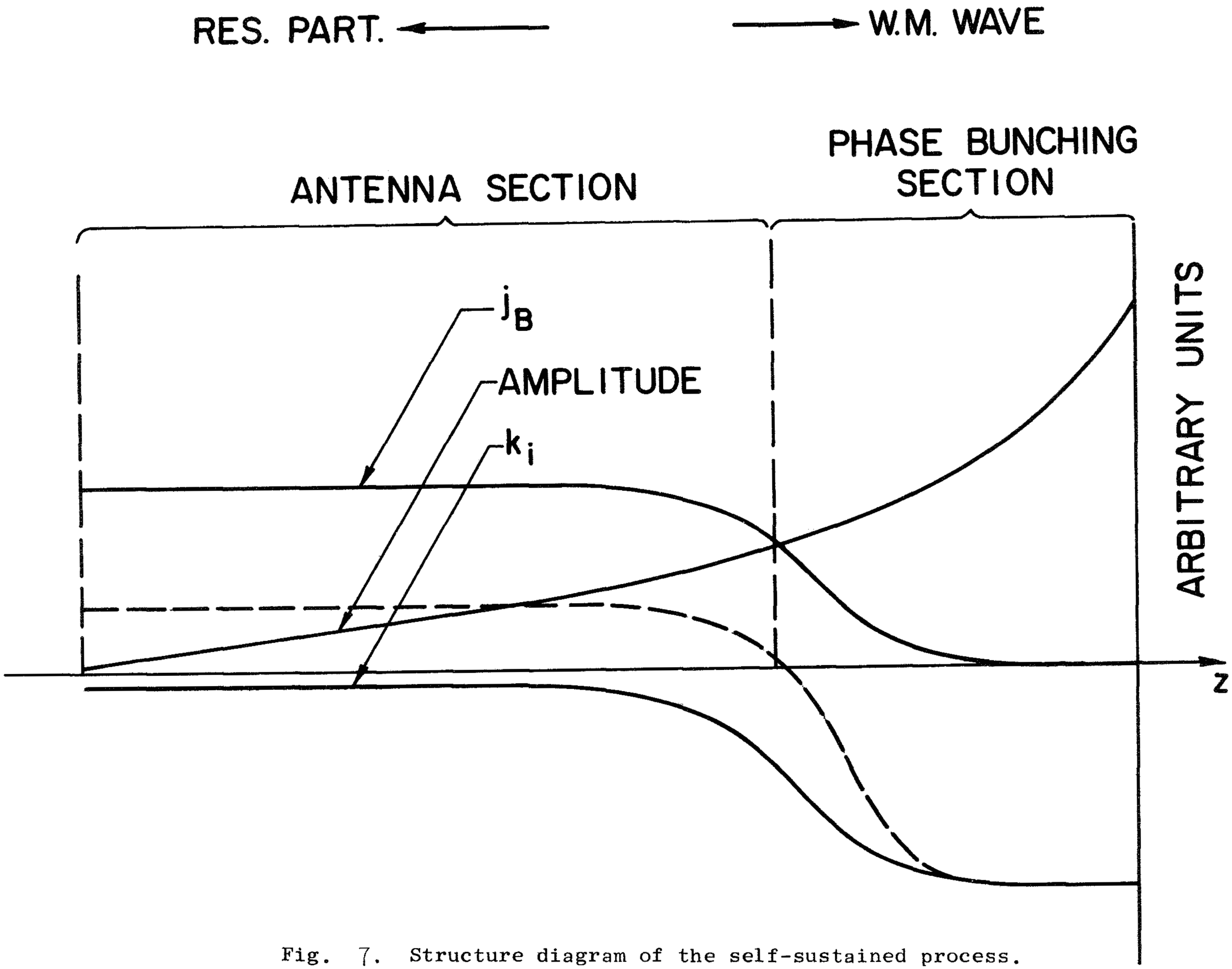


Fig. 7. Structure diagram of the self-sustained process.

So far it has been implicitly assumed that the emitted wave and the resonant particles stay in resonance throughout E . If this is not so, one is not going to get any significant gain from the antenna, nor can the bunching be of any importance as already discussed in Section 2. The requirement that the emitted wave should stay in resonance with the resonant particles throughout an extended region of space, has been pointed out and discussed by previous authors (see Helliwell [1,2]). Analytically, it can be stated by the equations

$$\Omega - \omega - kv_{\parallel} = 0 \quad , \quad \frac{d}{dt}(\Omega - \omega - kv_{\parallel}) = 0 \quad , \tag{51}$$

where $d/dt$ denotes the rate of change experienced by a resonant particle.

It should be pointed out that it is not enough to have Eq. (51) satisfied for one particular resonant particle throughout E . Equation (51) must be satisfied for all resonant particles within E . It should also be noted that Eq. (51) implies that the right-hand side of Eq. (5c) vanishes. Thus, a nonlinear interaction of the type discussed in Sections 3 and 4 can take place even if the inhomogeneity of the region E would not allow these processes to take place for a wave of constant frequency. Since now the frequency $\omega$ can vary, an additional degree of freedom is introduced such that the condition for trapping is no longer as restrictive as that given in Eq. (8). Taking the variation of $\omega$ into account Eq. (51) leads to the equation

$$\frac{d\omega}{dt} \equiv \frac{\partial\omega}{\partial t} - V_0 \frac{\partial\omega}{\partial z} = A \frac{\partial\Omega}{\partial z} \quad , \tag{52}$$

where

$$A = V_0 \left(1 + \frac{\omega}{2\Omega} + \frac{\Omega - \omega}{2\Omega} \tan^2 \alpha\right) \Big/ (1 + V_0/v_g) \quad ,$$

and $v_g(\omega)$ is the group velocity corresponding to the frequency $\omega$ .

In addition to Eq. (52) one has the equation

$$\frac{\partial\omega}{\partial t} + v_g(\omega) \frac{\partial\omega}{\partial z} = 0 \quad , \tag{53}$$

which states that as one moves along with the group velocity, $v_g(\omega)$, one sees a constant frequency $\omega$. The equation (52) and (53) appear to be incompatible in the general case. The variation of $v_g$ as a function of $z$ and $\omega$, can be considered small within the region E. If it is neglected altogether, the solution of Eq. (53) is

$$\omega = \omega(\tau) \quad , \tag{54}$$

where $\tau = t - z/v_g$.

A dependence of $\omega$ on $\tau$ alone does not generally satisfy Eq. (52). There is one notable exception, however, namely the case where the variation of $A(\partial\Omega/\partial z)$ vanishes, or is very small. In that case, an approximate solution to Eqs. (52) and (53) is given by

$$\omega = \alpha\tau \quad , \tag{55}$$

where $\alpha = A(\partial\Omega/\partial z)/(1 + V_0/v_g)$, and represents the rate of increase (or decrease) of the frequency, and is assumed to be a small quantity. If $\alpha$ were a constant, Eq. (55) would provide an exact solution to Eqs. (52) and (53). If this is not so, Eq. (55) will satisfy Eqs. (52) and (53) except for terms of order $\alpha^2$. Equation (55) predicts a linear increase or decrease depending on whether the resonant particles are streaming towards the equator or away from it, respectively. The rate of increase of $\omega$ is proportional to the distance of E from the equator. If E is moving, a change in the rate of change of $\omega$ results.

## 6. DISCUSSION

In this paper, we have investigated some of the basic physical effects that are believed to be responsible for the observed emissions of whistlers in the magnetosphere. First, we considered the effects of a nonlinear interaction between a WM wave and the particles resonating with it. It was pointed out that the interaction has two closely related effects, namely a variation of the amplification (or absorption) of the WM wave, and a phase-bunching of the resonant and near-resonant particles. The phase-bunching gives rise to a current which, in the absence of inhomogeneities, has the same structure (frequency and wave number) as the wave causing it. In the presence of inhomogeneities, the frequency and wave number of the current will vary slowly. The structured current, acting like an antenna, will in turn cause emission of a new whistler mode wave traveling in the same direction, and with nearly the same frequency and wave number (inhomogeneous case) as the wave bunching the particles.

The possibility of having a self-sustained process was then investigated, and the structure of a region, E , in which such a process can take place was proposed. We arrived at two conditions for the existence of such a region, namely Eq. (50) and Eq. (51), which have to be satisfied throughout E . It appears that these conditions can be satisfied if the amplification predicted by the linear theory is sufficiently strong. This question is currently under consideration for typical magnetospheric conditions. In the proposed model, the initial energy needed for the process to take place comes from the triggering wave. The main energy feeding the process, however, comes from the anisotropy in the distribution of the hot electrons.

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## APPENDIX: SOLUTION OF THE PENDULUM EQUATION

In the following, the integration of the equation of motion for an electron interacting with a WM wave is indicated. In Section 2, the equation of motion (7) for the angle $\Psi = \varphi + kz - \omega t + \pi$ , was derived. Using the coordinate $z$ as the independent variable, one obtains:

$$\frac{d^2\Psi}{dz^2} + \kappa^2 \sin\Psi = 0 \quad , \tag{A.1}$$

where $\kappa^2 = \bar{\omega}^2/v_{\|}^2$ . For trapped, and nearly trapped particles, $\kappa^2$ will be approximated by $\bar{\omega}^2 k^2/(\Omega-\omega)^2$ . Equation (A.1) is now readily integrated once to produce:

$$\left(\frac{1}{2}\frac{d\Psi}{dz}\right)^2 = \kappa^2\eta^2 \left(1 - \frac{1}{\eta^2}\sin^2\frac{\Psi}{2}\right) \quad , \tag{A.2}$$

where $\eta = [1/2 + \chi^2 k^2/(4\bar{\omega}^2)]^{1/2}$ , and $\chi^2$ is given by Eq. (6) in Section 2.

For $1 < \eta^2$ , Eq. (A.2) is readily integrated to give

$$F(\Psi_0/2 \, , \, 1/\eta) = F(\Psi/2 \, , \, 1/\eta) - \eta\kappa z \quad , \quad 1 < \eta^2 \quad , \tag{A.3}$$

where $F$ is an elliptic integral of the first kind.

For $\eta^2 < 1$ , the substitution $\sin(\Psi/2) = \eta \sin(\xi/2)$ makes it possible to integrate Eq. (A.2) to give

$$F(\xi_0/2, \eta) = F(\xi/2, \eta) - \kappa z \quad , \quad \eta^2 < 1 \tag{A.4}$$

We shall need the quantity $w_0$ , which is the unperturbed value of $w$ , where $w$ is defined by:

$$w \equiv v_{\|} - V_0 = \pm\, 2\,(\bar{\omega}/k)\,[\eta^2 - \sin^2(\Psi/2)]^{1/2} \quad .$$

For $\eta^2 < 1$ one obtains:

$$w_0 = \pm\, 2\,(\bar{\omega}/k)\,\eta\,\mathrm{cn}\,[F(\xi/2,\eta) - \kappa z, \eta] \quad , \quad \eta^2 < 1 \quad , \tag{A.5}$$

and for $\eta^2 > 1$ one obtains:

$$w_0 = \pm 2 (\bar{\omega}/k) \eta \, \mathrm{dn} [F(\Psi/2 , 1/\eta) - \eta \kappa z , 1/\eta] \quad , \quad \eta^2 > 1 \quad , \qquad (A.6)$$

where cn and dn are two of the Jacobi elliptic functions. Here only the '+' sign is used in the determination of the square root in Eq. (A.6). Thus $w_0/w > 0$ for $1 < \eta^2$, as it should be [dn $(x , 1/\eta)$ is positive for all $x$ ].

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