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Some Subclasses of q-Analytic Starlike Functions

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Abstract

We define two new subclasses, $\mathcal{S}_q^*(\alpha)$ and $\mathcal{TS}_q^*(\alpha)$, of analytic univalent functions. We obtain a sufficient condition for analytic univalent functions to be in $\mathcal{S}_q^*(\alpha)$ and we prove that this condition is also necessary for the functions in the class $\mathcal{TS}_q^*(\alpha)$. We also obtain extreme points, distortion bounds, covering result, convex combination and convolution properties for the functions in the class $\mathcal{TS}_q^*(\alpha)$.

1 Introduction

Let $\mathcal{S}$ be the class of functions $\mathfrak{f}$ which are analytic and univalent in the open unit disc $\mathcal{U} = \{\mathfrak{z} \in \mathbb{C} : |\mathfrak{z}| < 1\}$ given by

$$\mathfrak{f}(\mathfrak{z}) = \mathfrak{z} + \sum_{\mathfrak{n}=2}^{\infty} a_{\mathfrak{n}} \mathfrak{z}^{\mathfrak{n}}. \tag{1}$$

Let $\mathcal{T}$ denote the class of functions $\mathfrak{f}$ in $\mathcal{S}$ of the form

$$\mathfrak{f}(\mathfrak{z}) = \mathfrak{z} - \sum_{\mathfrak{n}=2}^{\infty} |a_{\mathfrak{n}}| \mathfrak{z}^{\mathfrak{n}}. \tag{2}$$

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We recollect here the q-difference operator that was used in geometric function theory and in several areas of science. We give basic definitions and properties about the q-difference operator that are used in this study (for details see [1] and [3]). For $0 < q < 1$, we defined the q-integer $[\mathrm{h}]_q$ by

$$[\mathrm{h}]_q = \frac{1-q^{\mathrm{h}}}{1-q}, \quad (\mathrm{h} = 1, 2, 3, ...).$$

Notice that if $q \to 1^-$ then $[\mathrm{h}]_q \to \mathrm{h}$.

In [4], İsmail et al. used q- calculus, in the theory of analytic univalent functions by defining a class of complex-valued functions that are analytic on the open unit disk $\mathcal{U}$ with the normalizations $\mathfrak{f}(0) = 0$, $\mathfrak{f}'(0) = 1$, and $|\mathfrak{f}(q z)| \leq |\mathfrak{f}(z)|$ on $\mathcal{U}$ for every q, $q \in (0, 1)$. The q-difference operator of analytic functions $\mathfrak{f}$ given by (1) are by definition, given as follows (see [3])

$$\mathcal{D}_q\mathfrak{f}(z) = \begin{cases} \frac{\mathfrak{f}(z)-\mathfrak{f}(qz)}{(1-q)z} & ;\ z \neq 0 \\ \mathfrak{f}'(0) & ;\ z = 0 \end{cases}.$$

Thus, for the function $\mathfrak{f}$ of the form (1), we have

$$\mathcal{D}_q\mathfrak{f}(z) = 1 + \sum_{\mathrm{h}=2}^{\infty} [\mathrm{h}]_q\, a_{\mathrm{h}} z^{\mathrm{h}-1} \tag{3}$$

and

$$\begin{aligned} \mathcal{D}_q\left(\mathcal{D}_q\mathfrak{f}(z)\right) &= \sum_{\mathrm{h}=2}^{\infty} [\mathrm{h}]_q\, [\mathrm{h}-1]_q\, a_{\mathrm{h}} z^{\mathrm{h}-2} \\ &= \sum_{\mathrm{h}=2}^{\infty} [\mathrm{h}]_q \left(\frac{[\mathrm{h}]_q - 1}{q}\right) a_{\mathrm{h}} z^{\mathrm{h}-2}. \end{aligned} \tag{4}$$

Denote by $\mathcal{S}_q^*(\alpha)$ the subclass of $\mathcal{S}$ consisting of functions $\mathfrak{f}$ of the form (1) that satisfy the condition

$$Re\left(\alpha\frac{z^2\mathcal{D}_q\left(\mathcal{D}_q\mathfrak{f}(z)\right)}{\mathfrak{f}(z)} + \frac{z\mathcal{D}_q\mathfrak{f}(z)}{\mathfrak{f}(z)}\right) > 0, \tag{5}$$

where $\alpha \geq 0$. Finally, we let $\mathcal{TS}_q^*(\alpha) \equiv \mathcal{S}_q^*(\alpha) \cap \mathcal{T}$.

By suitably specializing the parameters, the classes $\mathcal{S}_q^*(\alpha)$ reduces to the various subclasses of analytic univalent functions. Such as,

(i) $\mathcal{S}_q^*(\alpha) = H(\alpha)$ for $q \to 1^-$ (see [5]),

(ii) $\mathcal{S}_q^*(\alpha) = S^*$ for $\alpha = 0$ and $q \to 1^-$ (see [2]),

(iii) $\mathcal{S}_q^*(\alpha) = S_q^*$ for $\alpha = 0$ (see [4]).

In addition to these, [6], [7], [8] references can be consulted for studies in this field. In this paper, we first obtain a sufficient condition for analytic univalent functions $\mathfrak{f}$ given by (1) to be in $\mathcal{S}_q^*(\alpha)$ and then we prove that this condition is also necessary for the functions in the class $\mathcal{TS}_q^*(\alpha)$. We also obtain extreme points, distortion bounds, covering result, convex combination and convolution properties for the functions in the class $\mathcal{TS}_q^*(\alpha)$.

2 Main Results

Theorem 1. *Let a function* $\mathfrak{f}$ *given by* (1) *and satisfies*

$$\sum_{\mathrm{h}=1}^{\infty} \left(\alpha \, [\mathrm{h}]_q^2 + (q-\alpha) \, [\mathrm{h}]_q \right) |a_{\mathrm{h}}| \leq 2q, \tag{6}$$

where $0 \leq \alpha$ *and then* $\mathfrak{f} \in \mathcal{S}_q^*(\alpha)$.

Proof. In order to prove that $\mathfrak{f} \in \mathcal{S}_q^*(\alpha)$, we shall show that for $0 \leq \alpha$, $Re\left(\frac{\alpha z^2 \mathcal{D}_q(\mathcal{D}_q\mathfrak{f}(z)) + z\mathcal{D}_q\mathfrak{f}(z)}{\mathfrak{f}(z)}\right) > 0$. We know that $Re(w) \geq 0$ if and only if $|1+w| \geq |1-w|$. So, it suffices to show that

$$\left|\alpha z^2 \mathcal{D}_q \left(\mathcal{D}_q \mathfrak{f}(z)\right) + z\mathcal{D}_q\mathfrak{f}(z) + \mathfrak{f}(z)\right| - \left|\alpha z^2 \mathcal{D}_q \left(\mathcal{D}_q \mathfrak{f}(z)\right) + z\mathcal{D}_q\mathfrak{f}(z) - \mathfrak{f}(z)\right| \geq 0.$$

Now,

$$\left|\alpha z^2\mathcal{D}_q\left(\mathcal{D}_q f(z)\right)+z\mathcal{D}_q f(z)+f(z)\right|-\left|\alpha z^2\mathcal{D}_q\left(\mathcal{D}_q f(z)\right)+z\mathcal{D}_q f(z)-f(z)\right|$$

$$\begin{aligned}
&= \left|2z+\sum_{\mathrm{h}=2}^{\infty}\left(\alpha\,[\mathrm{h}]_q\,[\mathrm{h}-1]_q+[\mathrm{h}]_q+1\right)a_{\mathrm{h}}z^{\mathrm{h}}\right| \\
&\quad -\left|\sum_{\mathrm{h}=2}^{\infty}\left(\alpha\,[\mathrm{h}]_q\,[\mathrm{h}-1]_q+[\mathrm{h}]_q-1\right)a_{\mathrm{h}}z^{\mathrm{h}}\right| \\
&\geq 2|z|-\sum_{\mathrm{h}=2}^{\infty}\left(\alpha\frac{[\mathrm{h}]_q^2}{q}+(q-\alpha)\frac{[\mathrm{h}]_q}{q}+1\right)|a_{\mathrm{h}}||z|^{\mathrm{h}} \\
&\quad -\sum_{\mathrm{h}=2}^{\infty}\left(\alpha\frac{[\mathrm{h}]_q^2}{q}+(q-\alpha)\frac{[\mathrm{h}]_q}{q}-1\right)|a_{\mathrm{h}}||z|^{\mathrm{h}} \\
&\geq \frac{2|z|}{q}\left(q-\sum_{\mathrm{h}=2}^{\infty}\left(\alpha\,[\mathrm{h}]_q^2+(q-\alpha)\,[\mathrm{h}]_q\right)|a_{\mathrm{h}}||z|^{\mathrm{h}-1}\right) \\
&\geq 0, \quad \text{in view of (6).}
\end{aligned}$$

This completes the proof. □

Theorem 2. *Let* f *be given by* (2). *Then* $f\in\mathcal{TS}_q^*(\alpha)$ *if and only if* f *satisfies the condition* (6) *holds.*

Proof. Since $\mathcal{TS}_q^*(\alpha)\subset\mathcal{S}_q^*(\alpha)$ therefore, 'if' part of the Theorem 2 can be proved from Theorem 1, thus we prove 'only if' part of Theorem 2. Let $f\in\mathcal{TS}_q^*(\alpha)$, then

$$Re\left(\frac{\alpha z^2\mathcal{D}_q\left(\mathcal{D}_q f(z)\right)+z\mathcal{D}_q f(z)}{f(z)}\right)>0.$$

equivalently,

$$\left|\frac{-\sum_{\mathrm{h}=2}^{\infty}\frac{1}{q}\left(\alpha\,[\mathrm{h}]_q^2+(q-\alpha)\,[\mathrm{h}]_q-q\right)|a_{\mathrm{h}}|z^{\mathrm{h}-1}}{2-\sum_{\mathrm{h}=2}^{\infty}\frac{1}{q}\left(\alpha\,[\mathrm{h}]_q^2+(q-\alpha)\,[\mathrm{h}]_q+q\right)|a_{\mathrm{h}}|z^{\mathrm{h}-1}}\right|<1.$$

We note that since the above condition holds for all values of z, $|z| = r < 1$. Choosing the values of z on positive real axis, where $0 \leq z = r < 1$, we obtain

$$\frac{\sum_{\mathrm{n}=2}^{\infty} \frac{1}{q}\left(\alpha\,[\mathrm{n}]_q^2 + (q-\alpha)\,[\mathrm{n}]_q - q\right)|a_{\mathrm{n}}|r^{\mathrm{n}-1}}{2 - \sum_{\mathrm{n}=2}^{\infty} \frac{1}{q}\left(\alpha\,[\mathrm{n}]_q^2 + (q-\alpha)\,[\mathrm{n}]_q + q\right)|a_{\mathrm{n}}|r^{\mathrm{n}-1}} < 1. \tag{7}$$

If the condition (6) doesn't hold then,the condition (7) doesn't hold for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0,1)$ for which the quotient (7) is greater than 1. This contradicts the required condition for $f \in \mathcal{TS}_q^*(\alpha)$ and so the proof is complete. The result is sharp for functions given by

$$f_{\mathrm{n}}(z) = z - \frac{q}{[\mathrm{n}]_q\left(\alpha\left([\mathrm{n}]_q - 1\right) + q\right)} z^{\mathrm{n}}, \quad \mathrm{n} \geq 2.$$

□

Corollary 3. *If* $f \in \mathcal{TS}_q^*(\alpha)$ *then*

$$|a_{\mathrm{n}}| \leq \frac{q}{[\mathrm{n}]_q\left(\alpha\left([\mathrm{n}]_q - 1\right) + q\right)}, \quad \mathrm{n} \geq 2.$$

Theorem 4. *Let* $f \in \mathcal{TS}_q^*(\alpha)$. *Then, for* $|z| = r < 1$, *we have*

$$r - \frac{1}{(1+\alpha)(1+q)} r^2 \leq |f(z)| \leq r + \frac{1}{(1+\alpha)(1+q)} r^2.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f \in \mathcal{TS}_q^*(\alpha)$. Taking the absolute value of f we have

$$\begin{aligned} |f(z)| &\leq r + \sum_{\mathrm{n}=2}^{\infty} |a_{\mathrm{n}}| r^{\mathrm{n}} \\ &\leq r + \frac{r^2}{(1+\alpha)(1+q)} \sum_{\mathrm{n}=2}^{\infty} \frac{1}{q}\left(\alpha\,[\mathrm{n}]_q^2 + (q-\alpha)\,[\mathrm{n}]_q\right)|a_{\mathrm{n}}| \\ &< r + \frac{1}{(1+\alpha)(1+q)} r^2. \end{aligned}$$

We note that result in Theorem 4 is sharp for the following function,

$$\mathfrak{f}_2(z) = z - \frac{1}{(1+\alpha)(1+q)} z^2$$

at $z = \pm r$. □

The following covering result follows from the left hand inequality in Theorem 4.

Corollary 5. *Let* $\mathfrak{f}$ *of the form* (2) *be so that* $\mathcal{TS}_q^*(\alpha)$*. Then*

$$\left\{ w : |w| < \frac{q+\alpha(1+q)}{(1+\alpha)(1+q)} \right\} \subset \mathfrak{f}(\mathcal{U}).$$

Here, we consider extreme points for functions $\mathfrak{f} \in \mathcal{TS}_q^*(\alpha)$.

Theorem 6. *Let* $\mathfrak{f}_1(z) = z$ *and* $\mathfrak{f}_{\mathrm{h}}(z) = z - \frac{q}{[\mathrm{h}]_q\left(\alpha\left([\mathrm{h}]_q-1\right)+q\right)} z^{\mathrm{h}}$ *for* $\mathrm{h} \geq 2$*. Then* $\mathfrak{f} \in \mathcal{TS}_q^*(\alpha)$ *if and only if it can be expressed in the form* $\mathfrak{f}(z) = \sum_{\mathrm{h}=1}^{\infty} \delta_{\mathrm{h}} \mathfrak{f}_{\mathrm{h}}(z)$*, where* $\delta_{\mathrm{h}} \geq 0$ *and* $\sum_{\mathrm{h}=1}^{\infty} \delta_{\mathrm{h}} = 1$*. In particular, the extreme points of* $\mathcal{TS}_q^*(\alpha)$ *are* $\{\mathfrak{f}_{\mathrm{h}}\}$*.*

Proof. We first assume

$$\begin{aligned} \mathfrak{f}(z) &= \sum_{\mathrm{h}=1}^{\infty} \delta_{\mathrm{h}} \mathfrak{f}_{\mathrm{h}}(z) \\ &= z - \sum_{\mathrm{h}=2}^{\infty} \delta_{\mathrm{h}} \frac{q}{[\mathrm{h}]_q\left(\alpha\left([\mathrm{h}]_q-1\right)+q\right)} z^{\mathrm{h}}. \end{aligned}$$

Next, since

$$\sum_{\mathrm{h}=2}^{\infty} \delta_{\mathrm{h}} \frac{q}{[\mathrm{h}]_q\left(\alpha\left([\mathrm{h}]_q-1\right)+q\right)} \frac{[\mathrm{h}]_q\left(\alpha\left([\mathrm{h}]_q-1\right)+q\right)}{q}$$

$$= \sum_{\mathrm{h}=2}^{\infty} \delta_{\mathrm{h}} = 1 - \delta_1 \leq 1,$$

therefore by Theorem 2, $\mathfrak{f} \in \mathcal{TS}_q^*(\alpha)$.

Conversely, suppose $\mathfrak{f} \in \mathcal{TS}_q^*(\alpha)$. Since

$$|a_{\mathfrak{n}}| \leq \frac{q}{[\mathfrak{n}]_q\left(\alpha\left([\mathfrak{n}]_q - 1\right) + q\right)}, \quad \mathfrak{n} \geq 2,$$

we may set

$$\delta_{\mathfrak{n}} = \frac{[\mathfrak{n}]_q\left(\alpha\left([\mathfrak{n}]_q - 1\right) + q\right)}{q}|a_{\mathfrak{n}}|, \ \mathfrak{n} \geq 2 \text{ and } \delta_1 = 1 - \sum_{\mathfrak{n}=2}^{\infty} \delta_{\mathfrak{n}}.$$

Then,

$$\begin{aligned}
\sum_{\mathfrak{n}=1}^{\infty} \delta_{\mathfrak{n}} \mathfrak{f}_{\mathfrak{n}}(z) &= \delta_1 \mathfrak{f}_1(z) + \sum_{\mathfrak{n}=2}^{\infty} \delta_{\mathfrak{n}} \mathfrak{f}_{\mathfrak{n}}(z) \\
&= \delta_1 z + \sum_{\mathfrak{n}=2}^{\infty} \delta_{\mathfrak{n}} \left(z - \frac{q}{[\mathfrak{n}]_q\left(\alpha\left([\mathfrak{n}]_q - 1\right) + q\right)} z^{\mathfrak{n}} \right) \\
&= \sum_{\mathfrak{n}=1}^{\infty} \delta_{\mathfrak{n}} z - \sum_{\mathfrak{n}=2}^{\infty} \delta_{\mathfrak{n}} \frac{q}{[\mathfrak{n}]_q\left(\alpha\left([\mathfrak{n}]_q - 1\right) + q\right)} z^{\mathfrak{n}} \\
&= z - \sum_{\mathfrak{n}=2}^{\infty} |a_{\mathfrak{n}}| z^{\mathfrak{n}} = \mathfrak{f}(z).
\end{aligned}$$

Hence the proof is complete. □

For analytic functions

$$\mathfrak{f}_{\mathfrak{r}}(z) = z - \sum_{\mathfrak{n}=2}^{\infty} |a_{\mathfrak{n},\mathfrak{r}}| z^{\mathfrak{n}} \quad (\mathfrak{r} = 1, 2),$$

the convolution of $\mathfrak{f}_{\mathfrak{r}}$ is given by

$$(\mathfrak{f}_1 * \mathfrak{f}_2)(z) = \mathfrak{f}_1(z) * \mathfrak{f}_2(z) = z - \sum_{\mathfrak{n}=2}^{\infty} |a_{\mathfrak{n},1}||a_{\mathfrak{n},2}| z^{\mathfrak{n}}.$$

Using this definition, we show that the class $\mathcal{TS}_q^*(\alpha)$ is closed under convolution.

Theorem 7. *Let* $\mathfrak{f}_1 \in \mathcal{TS}_q^*(\alpha_1)$ *and* $\mathfrak{f}_2 \in \mathcal{TS}_q^*(\alpha_2)$, *where* $0 \leq \alpha_1 \leq \alpha_2$. *Then* $\mathfrak{f}_1 * \mathfrak{f}_2 \in \mathcal{TS}_q^*(\alpha_2) \subset \mathcal{TS}_q^*(\alpha_1)$.

Proof. We wish to show that $\mathfrak{f}_1 * \mathfrak{f}_2$ satisfies the coefficient condition (6). For $\mathfrak{f}_1 \in \mathcal{TS}_q^*(\alpha_1)$, we note that $|a_{\mathfrak{n},1}| \leq 1$. Now, for the coefficients of convolution function $\mathfrak{f}_1 * \mathfrak{f}_2$, we can write

$$\begin{aligned}
&\sum_{\mathfrak{n}=1}^{\infty} \frac{[\mathfrak{n}]_q \left(\alpha_1 \left([\mathfrak{n}]_q - 1\right) + q\right)}{q} |a_{\mathfrak{n},1}||a_{\mathfrak{n},2}| \\
&\leq \sum_{\mathfrak{n}=1}^{\infty} \frac{[\mathfrak{n}]_q \left(\alpha_1 \left([\mathfrak{n}]_q - 1\right) + q\right)}{q} |a_{\mathfrak{n},2}| \\
&\quad \leq \sum_{\mathfrak{n}=1}^{\infty} \frac{[\mathfrak{n}]_q \left(\alpha_2 \left([\mathfrak{n}]_q - 1\right) + q\right)}{q} |a_{\mathfrak{n},2}| \leq 2,
\end{aligned}$$

since $0 \leq \alpha_1 \leq \alpha_2$ and $\mathfrak{f}_2 \in \mathcal{TS}_q^*(\alpha_2)$. Therefore $\mathfrak{f}_1 * \mathfrak{f}_2 \in \mathcal{TS}_q^*(\alpha_2) \subset \mathcal{TS}_q^*(\alpha_1)$. □

Now, we show that $\mathcal{TS}_q^*(\alpha)$ is closed under convex combinations of its members.

Theorem 8. *The class* $\mathcal{TS}_q^*(\alpha)$ *is closed under convex combinations.*

Proof. For $\mathfrak{r} = 1, 2, 3, ...$, let $\mathfrak{f}_{\mathfrak{r}} \in \mathcal{TS}_q^*(\alpha)$, where $f_{\mathfrak{r}}$ is given by

$$\mathfrak{f}_{\mathfrak{r}}(\mathfrak{z}) = \mathfrak{z} - \sum_{\mathfrak{n}=2}^{\infty} |a_{\mathfrak{n}_{\mathfrak{r}}}| \mathfrak{z}^{\mathfrak{n}}.$$

Then by (6), we obtain

$$\sum_{\mathfrak{n}=1}^{\infty} \frac{[\mathfrak{n}]_q \left(\alpha \left([\mathfrak{n}]_q - 1\right) + q\right)}{q} |a_{\mathfrak{n}_{\mathfrak{r}}}| \leq 2. \tag{8}$$

For $\sum_{\mathfrak{r}=1}^{\infty} \zeta_{\mathfrak{r}} = 1$, $0 \leq \zeta_{\mathfrak{r}} \leq 1$, the convex combinations of $\mathfrak{f}_{\mathfrak{r}}$ may be written as

$$\sum_{\mathfrak{t}=1}^{\infty}\zeta_{\mathfrak{t}}\mathfrak{f}_{\mathfrak{t}}(\mathfrak{z})=\mathfrak{z}-\sum_{\mathfrak{n}=2}^{\infty}\left(\sum_{\mathfrak{t}=1}^{\infty}\zeta_{\mathfrak{t}}|a_{\mathfrak{n}_{\mathfrak{t}}}|\right)\mathfrak{z}^{\mathfrak{n}}. \tag{9}$$

Then by (8),

$$\begin{aligned}
&\sum_{\mathfrak{n}=1}^{\infty}\frac{[\mathfrak{n}]_q\left(\alpha\left([\mathfrak{n}]_q-1\right)+q\right)}{q}\left(\sum_{\mathfrak{t}=1}^{\infty}\zeta_{\mathfrak{t}}|a_{\mathfrak{n}_{\mathfrak{t}}}|\right)\\
=\ &\sum_{\mathfrak{t}=1}^{\infty}\zeta_{\mathfrak{t}}\sum_{\mathfrak{n}=1}^{\infty}\frac{[\mathfrak{n}]_q\left(\alpha\left([\mathfrak{n}]_q-1\right)+q\right)}{q}|a_{\mathfrak{n}_{\mathfrak{t}}}|\\
\leq\ &2\sum_{\mathfrak{t}=1}^{\infty}\zeta_{\mathfrak{t}}=2.
\end{aligned}$$

Thus, we get

$$\sum_{\mathfrak{t}=1}^{\infty}\zeta_{\mathfrak{t}}\mathfrak{f}_{\mathfrak{t}}(\mathfrak{z})\in\mathcal{TS}_q^*(\alpha).$$

□

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