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SOME SUFFICIENT CONDITIONS FOR THE COMPARABILITY OF TWO DIFFERENTIAL OPERATORS

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ABSTRACT. The comparison of differential operators is a problem of the theory of partial differential operators with constant coefficients. This problem up to now doesn't have a complete solution. It was formulated in the sixties by Lars Hörmander in his monograph "The Analysis of Linear Partial Differential Operators". Many facts of the theory of partial differential equations can be formulated by using the concept of pre-order relation over the set of differential operators, however it is too complicated to check the comparability condition of two differential operators. In this paper we get some sufficient conditions for the comparability of two differential operators. [1]

Introduction Let $POL_{\mathbb{C}}(n,m)$ be the set of all polynomials in n variables with complex coefficients of degree m, and $POL_{\mathbb{R}}(n,m)$ be the set of all polynomials in n variables with real coefficients of degree m, and $P(\zeta) = \sum_\alpha C_\alpha \zeta^\alpha$ any such polynomial , where α is multi-index, that is an n-tuple $(\alpha_1, \cdots, \alpha_n)$ of non-negative
integers.

Definition 1. The function $\tilde{P}(\zeta) = \sqrt{\sum_\alpha |\partial^\alpha P(\zeta)|^2}$ of the polynomial $P(\zeta)$ is called Hörmander's function.

Definition 2. If $P(D)$ and $Q(D)$ are differential operators such that $\frac{\tilde{Q}(\zeta)}{\tilde{P}(\zeta)} < C$, $\zeta \in \mathbb{R}^n$, we shall say that Q is weaker than P and write $Q < P$, or that P is stronger than Q and write $Q < P$. If $P < Q < P$, the operators are called equally strong.

The space $POL_{\mathbb{C}}(n,m)$ has the dimension $\nu = \frac{(m+n)!}{m!n!}$. We denote by τ_ζ the linear translation operator by a vector $\zeta \in \mathbb{C}^n$.

The following formula holds [see[3] chapter 2]:

$$\|\tau_\zeta P\|_o = \sqrt{\sum_\alpha \frac{1}{\alpha!} \left|(\overline{\partial}^\alpha P)(\zeta)\right|^2}$$

Definition 3. For each $P(\zeta) \in POL_{\mathbb{C}}(n,m)$, we will denote by $span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P))$ the linear span of translation $\{P(\zeta + x) : x \in \mathbb{R}^n\}$.

Definition 4. The polynomial $P(\zeta)$ is called regular if $\dim\left[span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P))\right] = \frac{(m+p)!}{(m!)(p)!}$,
$p = \dim \Lambda'_{\mathbb{R}^n}(P)$, where $\Lambda_{\mathbb{R}^n}(P) = \{\zeta \in \mathbb{R}^n : P(\zeta + x) = P(x)$ for all $x \in \mathbb{R}^n\}$is the $\mathbb{R}-$manifold of linearity of the polynomial P and $\dim \Lambda'_{\mathbb{R}^n}(P)$ is some space in $\mathbb{R}^n$ such that $\mathbb{R}^n = \Lambda_{\mathbb{R}^n}(P) \oplus \Lambda'_{\mathbb{R}^n}(P)$.

We should point out that each element $\sum\limits_{i=1}^{l} t_i P(\zeta+\eta_i)$ of the space $span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P))$ can be

written in the form

$$\sum_{i=1}^{l} t_i P(\zeta + \eta_i) = \sum_{|\alpha|\leq m} [\frac{(\overline{\partial}^{\alpha} P)(\zeta)}{\alpha!}(\sum_{i=1}^{l} t_i \eta_i^{\alpha})],$$

where $\eta_i, i = 1, ..., l$ are arbitrary vectors from $\mathbb{R}^n$and $t_i, i = 1, ...l$ are arbitrary complex numbers.

We note that the coefficients $\sum\limits_{i=1}^{l} t_i \eta_i^{\alpha}, |\alpha| \leq m$ can be given any values which are symmetric over the indices $\alpha_1, ..., \alpha_n$ that corresponds to the choice of l, t_i and η_i, where $\alpha = (\alpha_1, ..., \alpha_n)$.

The Main Results. We fix on $Diff_{\mathbb{C}}(n, m) \times POL_{\mathbb{C}}(n, m)$ the bilinear form

$$Diff_{\mathbb{C}}(n, m) \times POL_{\mathbb{C}}(n, m) \ni (R(\overline{\partial}), P(\zeta)) \to (R(\overline{\partial})P)(0) \in \mathbb{C} \qquad (1)$$

where $Diff_{\mathbb{C}}(n, m)$ is the set of all partial differential operators with constant complex coefficients. It is obvious that the bilinear form (1) reduces the spaces $Diff_{\mathbb{C}}(n, m)$ and $POL_{\mathbb{C}}(n, m)$ to duality. Thus the space $POL^*_{\mathbb{C}}(n, m)$ (the dual of $POL_{\mathbb{C}}(n, m)$) can be identified with $Diff_{\mathbb{C}}(n, m)$, and the dual of $Diff_{\mathbb{C}}(n, m)$ can be identified with $POL_{\mathbb{C}}(n, m)$.

Lemma 1. Each linear continuous functional $f$on the space $span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P))$ can be written in the following form $f(S) = (R_f(\overline{\partial})S)(0)$, where $S(\zeta) \in span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P))$, $R_f(\overline{\partial}) \in Diff_{\mathbb{C}}(n, m)$.

Proof. By Hahn-Banach theorem, each functional $f$in the space $span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P))$ (for each $P(\zeta) \in POL_{\mathbb{C}}(n, m)$, we will denote by $span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P))$ the linear *span* of translations $\{P(\zeta + x)\}$, $x \in \mathbb{R}^n$) can be extended to a linear continuous functional on the space $POL_{\mathbb{C}}(n, m)$.

It is clear that the functional $R_f(\overline{\partial})$ from the above lemma is not uniquely defined.

Let $\{(\overline{\partial}^{\alpha} P)(\zeta)\}\alpha \in \mathbb{Z}_P$ be a basis of $span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P))$. Then each vector $(\zeta + \eta) \in span_{\mathbb{C}}$ can be written in the following form

$$S(\zeta, \eta) = \sum_{\alpha \in \mathbb{Z}_P} P_{\alpha}(\eta)(\overline{\partial}^{\alpha} P)(\zeta)$$

where the polynomials $P_\alpha(\eta) \in POL_{\mathbb{C}}(n,m)$ are defined uniquely. The definition of the linear *span* of translations at once implies that

$$span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P)) \subseteq \{ \sum_{\alpha \in \mathbb{Z}_P} C_\alpha P_\alpha(\zeta); C_\alpha \in \mathbb{C}, \alpha \in \mathbb{Z}_P \}.$$

This implies that

$$span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P)) = \{ \sum_{\alpha \in \mathbb{Z}_P} C_\alpha P_\alpha(\zeta); C_\alpha \in \mathbb{C}, \alpha \in \mathbb{Z}_P \}$$

and the vectors $\{P_\alpha(\zeta)\}_{\alpha \in \mathbb{Z}_P}$ are linearly independent since $\dim_{\mathbb{C}}(span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P)))$ equals the value of multi-indices from $\mathbb{Z}_P$.

In what follows f_ζ denotes $f \in POL^*_{\mathbb{C}}(n,m)$ acting on ζ variable.

Lemma 2. For each polynomial $P(\zeta) \in POL_{\mathbb{C}}(n,m)$ the following equality holds

$$\begin{aligned} Span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P)) &= \{f_\zeta(\tau_\eta(P)) : f \in POL^*_{\mathbb{C}}(n,m)\} \\ &= \{f_\eta(\tau_\zeta(P)) : f \in POL_{\mathbb{C}}(n,m)\}. \end{aligned}$$

Proof. The proof is an immediate consequence of the fact that, for any complex numbers $C_\alpha \in \mathbb{C}, \alpha \in \mathbb{Z}_P$ we can determine a linear continuous functional $f \in POL^*_{\mathbb{C}}(n,m)$ such that $f((\overline{\partial}^\alpha P)(\zeta)) = C_\alpha, \alpha \in \mathbb{Z}_P$.

Corollary 1. For each polynomial $P(\zeta) \in POL_{\mathbb{C}}(n,m)$ the following equality holds

$$Span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P)) = \{(R(\overline{\partial})P)(\zeta) : R(\overline{\partial}) \in Diff_{\mathbb{C}}(n,m)\}.$$

The proof of this Corollary follows from Lemma 2 taking into account that $Diff_{\mathbb{C}}(n,m)$ can be identified by means of coupling (1).

Definition 5. The collection of differential operators $R_0(\overline{\partial}), \cdots, R_\rho(\overline{\partial})$ is called P-linearly independent if the equality

$$\sum_{i=0}^{\rho} C_i(R_i(\overline{\partial})P)(\zeta) \equiv 0, \qquad \forall \zeta \in \mathbb{R}^n$$

holds if and only if $C_0 = \cdots = C_\rho = 0$.

Lemma 3. Any collection of $\rho + 1 = \dim_{\mathbb{C}}(span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(0))$ operators $R_0(\overline{\partial}), \cdots, R_\rho(\overline{\partial})$ which are P-linearly independent over $\mathbb{C}$ forms a basis of some subspace $Vect'_D(P)$ which is complementary to the vector subspace

$$Vect_D(P) = \{R(\overline{\partial}) \in Diff_{\mathbb{C}}(n,m) : (R(\overline{\partial})P)(\zeta) \equiv 0, \zeta \in \mathbb{R}^n\}$$

of the space $Diff_{\mathbb{C}}(n,m)$. Thus

$$Diff_{\mathbb{C}}(n,m) = Vect_D(P) \oplus Vect'_D(P)$$

Proof. Let $Vect_D(P)$ be spanned by $R_0(\overline{\partial}), \cdots, R_\rho(\overline{\partial})$. We shall prove that $Vect_D(P) \cap Vect'_D(P) = \{\theta\}$ where θ is a finite subset of vectors. In fact, if $\sum_{i=0}^{\rho} C_i R_i(\overline{\partial}) \in Vect_D(P)$, then $\sum_{i=0}^{\rho} C_i(R_i(\overline{\partial})P)(\zeta) \equiv 0$ from where $C_0 = C_1 = \cdots = C_\rho = 0$. On the other hand, if there exist an operator $R(\overline{\partial}) \in Diff_{\mathbb{C}}(n,m)$ such that for any $C_0, \cdots, C_\rho$ we have $R(\overline{\partial}) - \sum_{i=0}^{\rho} C_i R_i(\overline{\partial}) \notin Vect_D(P)$, then $(R(\overline{\partial})P)(\zeta)$ does not belong to the linear span $\sum_{i=0}^{\rho} C_i (R_i(\overline{\partial})P)(\zeta)$, since we can assume that $R(\overline{\partial}) \notin Vect_D(P)$. But, $(R(\overline{\partial})P)(\zeta) \in span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P))$ and the vectors $(R_i(\overline{\partial})P)(\zeta)$ form a basis of $span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P))$. Thus for some collection $\overline{C}_0, \cdots, \overline{C}_\rho$ we should have

$$\left(R(\overline{\partial}) - \sum_{i=0}^{\rho} \overline{C}_i R_i(\overline{\partial})\right) P(\zeta) \equiv 0$$

which means that

$$R(\overline{\partial}) - \sum_{i=0}^{\rho} \overline{C}_i R_i(\overline{\partial}) \in Vect_D(P).$$

We introduce the following notations
$Wect_v = \{Q(\zeta) \in POL_{\mathbb{C}}(n,m) : (R(\overline{\partial})Q)(\zeta) \equiv 0, \forall \zeta \in \mathbb{R}^n$ and $\forall R(\overline{\partial}) \in Vect_D(P)\}$
and $Wect_v^0 = \{Q(\zeta) \in POL_{\mathbb{C}}(n,m) : (R(\overline{\partial})Q)(0) \equiv 0, \text{for } \forall R(\overline{\partial}) \in Vect_D(P)\}$
It is clear that $span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P)) \subseteq Wect_v \subseteq Wect_v^0$

Theorem. For each polynomial $P(\zeta) \in POL_{\mathbb{C}}(n,m)$, the following equality holds

$$Span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P)) = Wect_v = Wect_v^0.$$

Proof. We shall prove that the spaces $Vect'_D(P)$ and $Wect_v^0$ are in duality. For this we consider the standard duality $Vect'_D(P) \times Wect_v^0 \ni (R(\overline{\partial}), S(\zeta)) \to (R(\overline{\partial})S)(0)$. We need to establish the separability of this bilinear form with respect to both variables.

We assume that the vector $R_0(\overline{\partial}) \neq \theta$ is fixed. If there does not exist a vector $S_0(\zeta)$ such that $(R_0(\overline{\partial})S_0)(0) \neq 0$, then

$$(R_0(\overline{\partial})S)(0) = 0 \tag{2}$$

for all $S(\zeta) \in Wect_v^0$. We assume

$$S_x(\zeta) = P(\zeta + x) \in Span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P)) \subseteq Wect_v^0,$$

and get, by definition (2),

$$(R_0(\overline{\partial})S_x)(0) = R_0(\overline{\partial})P(x) \equiv 0, \forall x \in \mathbb{R}^n,$$

Whence

$$R_0(\partial) \in Vect_D(P) \cap Vect'_D(P) = \{\Theta\}.$$

Consequently, $R_0(\overline{\partial}) = \theta$, which contradicts the choice of $R_0(\overline{\partial})$. This means that there exists a vector $S_0(\zeta) \in Wect_v^0, S_0(\zeta) \neq 0$ such that $(R_0(\overline{\partial})S_0)(0) \neq 0$.

On the other hand let $S_0(\zeta) \not\equiv 0$ and $S_0(\zeta) \in Wect_v^0$, we need to prove the existence of $R_0(\overline{\partial}) \in Vect'_D(P)$ such that

$$(R_0(\overline{\partial})S_0)(0) \neq 0$$

If such operator $R_0(\overline{\partial})$ does not exist, we will have $(R(\overline{\partial})S_0)(0) = 0$ for each $R(\overline{\partial})$ since $S_0(\zeta) \in Wect_v^0$. But then $S_0(\zeta) \equiv 0, \forall \zeta \in \mathbb{R}^n$ which contradict the choice of $S_0(\zeta) \not\equiv O$.

Consequently the spaces $Vect'_D$ and $Wect_v^0$ are in duality and the bilinear form which reduces them into duality is separable with respect to both variables. From where implies that

$$\dim_{\mathbb{C}}[Wect_v^0] = \dim_{\mathbb{C}}[Vect_D(P)] = \rho + 1.$$

References

[1] Lars Hörmander, *The analysis of Linear Partial Differential Operators I, Springer Verlag, Berlin, 1983*

[2] Lars Hörmander, *The analysis of Linear Partial Differential Operators II, Springer Verlag, Berlin, 1983*

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