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SOME SUFFICIENT CONDITIONS FOR UNIVALENCE OF CERTAIN FAMILIES OF INTEGRAL OPERATORS INVOLVING GENERALIZED BESSEL FUNCTIONS

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Abstract. The main object of this paper is to give sufficient conditions for certain families of integral operators, which are defined here by means of the normalized form of the generalized Bessel functions, to be univalent in the open unit disk. In particular cases, we find the corresponding simpler conditions for integral operators involving the Bessel function, the modified Bessel function and the spherical Bessel function.

1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and satisfy the usual normalization condition:

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z) \in \mathcal{A}$ which are univalent in $\mathbb{U}$. In the past two decades, many authors have determined various sufficient conditions for the univalence of various general families of integral operators as follows.

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The first family of integral operators, studied by Seenivasagan and Breaz (see [26]), is defined as follows (see also the recent investigations on this subject by Baricz and Frasin [7] and Srivastava *et al.* [27]):

$$\mathcal{F}_{\alpha_1,\cdots,\alpha_n,\beta}(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{j=1}^{n} \left(\frac{f_j(t)}{t}\right)^{1/\alpha_j} \mathrm{d}t\right]^{1/\beta}, \tag{1.1}$$

where each of the functions $f_j \quad (j = 1, \cdots, n)$ belongs to the class $\mathcal{A}$ and the parameters $\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \cdots, n)$ and $\beta \in \mathbb{C}$ are so constrained that the integral operators in (1.1) exist.

Remark 1. We note that, if $\alpha_j = \alpha \ (j = 1, \cdots, n)$, then the integral operator $\mathcal{F}_{\alpha_1,\cdots,\alpha_n,\beta}(z)$ reduces to the operator $\mathcal{F}_{\alpha,\beta}(z)$ which is related closely to some known integral operators investigated earlier in *Geometric Functions Theory* (see, for details, [28]). The operators $\mathcal{F}_{\alpha,\beta}(z)$ and $\mathcal{F}_{\alpha,\alpha}(z)$ were studied by Breaz and Breaz (see [12]) and Pescar (see [23]), respectively. Upon setting $\beta = 1$ and $\alpha = \beta = 1$ in $\mathcal{F}_{\alpha,\beta}(z)$, we can obtain the operators $\mathcal{F}_{\alpha,1}(z)$ and $\mathcal{F}_{1,1}(z)$ which were studied by Breaz and Breaz (see [11]) and Alexander (see [2]), respectively. Furthermore, in their special cases when

$$n = \beta = 1 \qquad \text{and} \qquad \alpha_j = \frac{1}{\alpha} \qquad (j = 1, \cdots, n),$$

the integral operators in (1.1) would obviously reduce to the operator $\mathcal{F}_{1/\alpha,1}(z)$ which was studied by Pescar and Owa (see [24]). In particular, for $\alpha \in [0, 1]$, a special case of the operator $\mathcal{F}_{1/\alpha,1}(z)$ was studied by Miller *et al.* (see [18]).

Remark 2. The second family of integral operators was introduced by Breaz and Breaz (see [13]) and it has the following form (see also a recent investigation on this subject by Breaz *et al.* [15]):

$$\mathcal{G}_{n,\gamma}(z) = \left[(n\gamma + 1) \int_0^z \prod_{j=1}^{n} [g_j(t)]^{\gamma} \, \mathrm{d}t\right]^{1/(n\gamma+1)}, \tag{1.2}$$

where the functions $g_j \in \mathcal{A} \quad (j = 1, \cdots, n)$ and the parameter $\gamma \in \mathbb{C}$ is so constrained that the integral operators in (1.2) exist. In particular, for $n = 1$, the integral operator $\mathcal{G}_{1,\gamma}(z)$ was studied by Moldoveanu and Pascu (see [19]).

Remark 3. The third family of integral operators was introduced by Breaz and Breaz (see [14]) and it has the following form:

$$\mathcal{H}_{\delta_1,\cdots,\delta_n,\mu}(z) = \left[\mu \int_0^z t^{\mu-1} \prod_{j=1}^{n} \left[h_j'(t)\right]^{\delta_j} \mathrm{d}t\right]^{1/\mu}, \tag{1.3}$$

where the functions $h_j \in \mathcal{A}$ $(j = 1, \cdots, n)$ and the parameters $\mu \in \mathbb{C}$ and $\delta_j \in \mathbb{C}$ $(j = 1, \cdots, n)$ are so constrained that the integral operators in (1.3) exist. In particular, for $\mu = 1$ in (1.3), the integral operator $\mathcal{H}_{\delta_1,\cdots,\delta_n,\mu}(z)$ reduces to the operator $\mathcal{H}_{\delta_1,\cdots,\delta_n}(z)$ which was studied by Breaz *et al.* (see [16]). We observe also that, for $n = \mu = 1$, the integral operator $\mathcal{H}(z)$ was introduced and studied by Pfaltzgraff (see [25]) and Kim and Merkes (see [17]).

Remark 4. The fourth family of integral operators was introduced by Pescar [22] as follows:

$$\mathcal{Q}_\lambda(z) = \left[\lambda \int_0^z t^{\lambda-1} \left(e^{q(t)}\right)^\lambda \mathrm{d}t\right]^{1/\lambda}, \tag{1.4}$$

where the function $q \in \mathcal{A}$ and the parameter $\lambda \in \mathbb{C}$ is so constrained that the integral operators in (1.4) exist.

Two of the most important and known univalence criteria for analytic functions defined in the open unit disk $\mathbb{U}$ were obtained by Ahlfors [1] and Becker [10] and by Becker (see [9]). Some extensions of these two univalence criteria were given by Pescar (see [21]) involving a parameter β (which, for $\beta = 1$, yields the Ahlfors-Becker univalence criterion) and by Pascu (see [20]) involving two parameters α and β (which, for $\beta = \alpha = 1$, yields Becker's univalence criterion). In our present investigation, we need these two univalence criteria which we recall here as Lemmas 1 and 2 below.

Lemma 1. (see [21]). *Let β and c be complex numbers such that*

$$\Re(\beta) > 0 \qquad \textit{and} \qquad |c| \leqq 1 \quad (c \neq -1).$$

If the function $f \in \mathcal{A}$ satisfies the following inequality:

$$\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zf''(z)}{\beta f'(z)} \right| \leqq 1 \qquad (z \in \mathbb{U}),$$

then the function F_β defined by

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1}\, f'(t)\mathrm{d}t\right)^{1/\beta} \tag{1.5}$$

is in the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{U}$.

Lemma 2. (see [20]). *If $f \in \mathcal{A}$ satisfies the following inequality*:

$$\left(\frac{1 - |z|^{2\Re(\alpha)}}{\Re(\alpha)}\right) \left|\frac{zf''(z)}{f'(z)}\right| \leqq 1 \qquad \big(z \in \mathbb{U};\ \Re(\alpha) > 0\big),$$

then, *for all $\beta \in \mathbb{C}$ such that $\Re(\beta) \geqq \Re(\alpha)$, the function F_β defined by (1.5) is in the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{U}$.*

Lemma 3 below is a consequence of the above-mentioned Becker's univalence criterion (see [9]) and the well-known Schwarz lemma.

Lemma 3. (see [22]). *Let the parameters $\lambda \in \mathbb{C}$ and $\theta \in \mathbb{R}$ be so constrained that*

$$\Re(\lambda) \geqq 1, \quad \theta > 1 \qquad \textit{and} \qquad 2\theta\,|\lambda| \leqq 3\sqrt{3}.$$

If the function $q \in \mathcal{A}$ satisfies the following inequality:

$$\left|zq'(z)\right| \leqq \theta \qquad (z \in \mathbb{U}),$$

then the function $\mathcal{Q}_\lambda : \mathbb{U} \to \mathbb{C}$, defined by

$$\mathcal{Q}_\lambda(z) = \left[\lambda \int_0^z t^{\lambda-1} \left(e^{q(t)}\right)^\lambda \mathrm{d}t\right]^{1/\lambda},$$

is in the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{U}$.

We next consider the following second-order linear homogeneous differential equation (see, for details, [30]):

$$\text{(1.6)} \quad z^2\omega''(z) + bz\omega'(z) + \left[dz^2 - \nu^2 + (1-b)\nu\right]\omega(z) = 0 \qquad (b, d, \nu \in \mathbb{C}).$$

A particular solution of the differential equation (1.6), which is denoted by $\omega_{\nu,b,d}(z)$, is called the generalized Bessel function of the first kind of order ν. In fact, we have the following familiar series representation for the function $\omega_{\nu,b,d}(z)$:

$$\text{(1.7)} \qquad \omega_{\nu,b,d}(z) = \sum_{n=0}^{\infty} \frac{(-d)^n}{n!\Gamma\left(\nu + n + \frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+\nu} \qquad (z \in \mathbb{C}),$$

where $\Gamma(z)$ stands for the Euler gamma function. The series in (1.7) permits us to study the Bessel, the modified Bessel and the spherical Bessel functions in a unified manner. Each of these particular cases of the function $\omega_{\nu,b,d}(z)$ is worthy of mention here.

- For $b = d = 1$ in (1.7), we obtain the familiar Bessel function $J_\nu(z)$ defined by (see [30]; see also [3])

$$\text{(1.8)} \qquad J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n+\nu} \qquad (z \in \mathbb{C}).$$

- For $b = -d = 1$ in (1.7), we obtain the modified Bessel function $I_\nu(z)$ defined by (see [30]; see also [3])

$$\text{(1.9)} \qquad I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n+\nu} \qquad (z \in \mathbb{C}).$$

- For $b-1=d=1$ in (1.7), we obtain the spherical Bessel function, which we choose to denote *here* by $K_\nu(z)$, defined by (see [3]; see also Remark 5 below)

$$K_\nu(z) := \sqrt{\frac{2}{z}}\, J_{\nu+\frac{1}{2}}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(\nu+n+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2n+\nu} \qquad (z \in \mathbb{C}). \tag{1.10}$$

Remark 5. The *spherical Bessel function* $j_n(z)$ *of the first kind* is usually defined by (see, for details, [30])

$$j_n(z) := \sqrt{\frac{\pi}{2z}}\, J_{n+\frac{1}{2}}(z) \qquad (n \in \mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\}).$$

On the other hand, in the theory of Bessel functions (see, for example, [30]), the notation $K_\nu(z)$ is usually meant for the *modified Bessel function of the third kind* (or the *Macdonald function*). Here, in our present investigation, we have found it to be convenient to use the same notation $K_\nu(z)$ which is defined here *markedly differently* by (1.10).

We now introduce the function $\varphi_{\nu,b,d}(z)$ defined, in terms of the generalized Bessel function $\omega_{\nu,b,d}(z)$, by

$$\varphi_{\nu,b,d}(z) = 2^\nu \Gamma\left(\nu + \frac{b+1}{2}\right) z^{1-\frac{\nu}{2}}\, \omega_{\nu,b,d}(\sqrt{z}). \tag{1.11}$$

By using the well-known Pochhammer symbol (or the *shifted* factorial) $(\lambda)_\mu$ defined, for $\lambda, \mu \in \mathbb{C}$ and in terms of the Euler Γ-function, by

$$(\lambda)_\mu := \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\mu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\mu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$, we obtain the following series representation for the function $\varphi_{\nu,b,d}(z)$ given by (1.11):

$$\varphi_{\nu,b,d}(z) = z + \sum_{n=1}^{\infty} \frac{(-d)^n}{4^n\,(\kappa)_n}\, \frac{z^{n+1}}{n!} \qquad \left(\kappa := \nu + \frac{b+1}{2} \notin \mathbb{Z}_0\right) \tag{1.12}$$

where

$$\mathbb{N} := \{1, 2, 3, \cdots\} \qquad \text{and} \qquad \mathbb{Z}_0 := \{0, -1, -2, \cdots\}.$$

For further results on this relative $\varphi_{\nu,b,d}(z)$ of the generalized Bessel function $\omega_{\nu,b,d}(z)$, we refer the reader to the recent papers (see, for example, [3-6, 8, 29]), where (among other things) some interesting functional inequalities, integral representations, extensions of some known trigonometric inequalities, and starlikeness, convexity and univalence of normalized analytic functions were established. In particular, Baricz and Frasin [7] investigated the univalence of some integral operators of the types given by (1.1), (1.2) and (1.4), which involve the normalized form of the ordinary Bessel function of the first kind. The main object of this paper is to give sufficient conditions for the families of integral operators of the types (1.1), (1.2), (1.3) and (1.4), which involve the normalized forms of the generalized Bessel functions of the first kind to be univalent in the open unit disk $\mathbb{U}$. We also extend and improve the aforementioned results of Baricz and Frasin [7]. At least in some cases, our main results are stronger than the results obtained in [7].

Recently, Baricz and Ponnusamy [8] proved the following lemma.

Lemma 4. (see [8]). *If the parameters* $\nu, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *are so constrained that*

$$\kappa > \max\left\{0, \frac{|d|}{4} - 1\right\},$$

then the function

$$\frac{\varphi_{\nu,b,d}(z)}{z} : \mathbb{U} \to \mathbb{C}$$

given by (1.12) *satisfies the following inequality*:

$$\frac{4\kappa(\kappa+1)-(2\kappa+1)\,|d|+\frac{1}{8}\,|d|^2}{\kappa\,[4(\kappa+1)-|d|]} \leqq \left|\frac{\varphi_{\nu,b,d}(z)}{z}\right| \leqq \frac{32\kappa^2-|d|}{8\kappa\,[4\kappa-|d|]} \quad (z\in\mathbb{U}). \tag{1.13}$$

Lemma 5. *If the parameters* $\nu, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *are so constrained that*

$$\kappa > \max\left\{0, \frac{|d|-2}{4}\right\},$$

then the function

$$\varphi_{\nu,b,d} : \mathbb{U} \to \mathbb{C}$$

defined by (1.12) *satisfies the following inequalities*:

$$\left|\varphi'_{\nu,b,d}(z) - \frac{\varphi_{\nu,b,d}(z)}{z}\right| \leqq \frac{(\kappa+1)\,|d|}{\kappa\,[4(\kappa+1)-|d|]} \quad (z \in \mathbb{U}), \tag{1.14}$$

$$\left|\frac{z\varphi'_{\nu,b,d}(z)}{\varphi_{\nu,b,d}(z)} - 1\right| \leqq \frac{8(\kappa+1)\,|d|}{32\kappa(\kappa+1) - 8(2\kappa+1)\,|d|+|d|^2} \quad (z \in \mathbb{U}), \tag{1.15}$$

$$\frac{4\kappa(\kappa+1)-(3\kappa+2)\,|d|}{\kappa\,[4(\kappa+1)-|d|]} \leqq \left|z\varphi'_{\nu,b,d}(z)\right| \leqq \frac{4\kappa(\kappa+1)+(\kappa+2)\,|d|}{\kappa\,[4(\kappa+1)-|d|]} \quad (z \in \mathbb{U}), \tag{1.16}$$

$$\left|z^2\varphi''_{\nu,b,d}(z)\right| \leqq \frac{|d|}{2\kappa}\frac{4(\kappa+1)+|d|}{4(\kappa+1)-|d|} \quad (z \in \mathbb{U}) \tag{1.17}$$

and

$$\left|\frac{z\varphi''_{\nu,b,d}(z)}{\varphi'_{\nu,b,d}(z)}\right| \leqq \frac{4(\kappa+1)\,|d|+|d|^2}{8\kappa(\kappa+1)-2(3\kappa+2)\,|d|} \quad (z\in\mathbb{U}). \tag{1.18}$$

Proof. We first prove the assertion (1.14) of Lemma 5. Indeed, by using the well-known triangle inequality:

$$|z_1+z_2| \leqq |z_1|+|z_2|$$

and the inequaliy:

$$n!(\kappa+1)_{n-1} \geqq n(\kappa+1)^{n-1} \qquad (n\in\mathbb{N}), \tag{1.19}$$

we have

$$\begin{aligned}\left|\varphi'_{\nu,b,d}(z)-\frac{\varphi_{\nu,b,d}(z)}{z}\right| &= \left|\sum_{n=1}^{\infty}\frac{n(-d)^n}{n!4^n(\kappa)_n}z^n\right| \leqq \sum_{n=1}^{\infty}\frac{n\,|d|^n}{n!4^n(\kappa)_n}\\ &= \frac{|d|}{4\kappa}\sum_{n=1}^{\infty}\frac{n\,|d|^{n-1}}{4^{n-1}n!(\kappa+1)_{n-1}} \leqq \frac{|d|}{4\kappa}\sum_{n=1}^{\infty}\left(\frac{|d|}{4(\kappa+1)}\right)^{n-1}\\ &= \frac{(\kappa+1)\,|d|}{\kappa\,[4(\kappa+1)-|d|]} \qquad \left(\kappa>\frac{|d|-2}{4}\right).\end{aligned}$$

Next, by combining the inequalities (1.13) with (1.14), we immediately see that the second assertion (1.15) of Lemma 5 holds true for all $z\in\mathbb{U}$ if

$$32\kappa(\kappa+1)-8(2\kappa+1)\,|d|+|d|^2>0.$$

In order to prove the assertion (1.16) of Lemma 5, we make use of the well-known triangle inequality and the following inequaliy:

$$2\cdot n!(\kappa+1)_{n-1} \geqq (n+1)(\kappa+1)^{n-1} \qquad (n\in\mathbb{N}). \tag{1.20}$$

We thus find that

$$\begin{aligned}\left|z\varphi'_{\nu,b,d}(z)\right| &= \left|z+\sum_{n=1}^{\infty}\frac{(n+1)(-d)^n}{n!4^n(\kappa)_n}z^{n+1}\right| \leqq 1+\sum_{n=1}^{\infty}\frac{(n+1)\,|d|^n}{n!4^n(\kappa)_n}\\ &= 1+\frac{|d|}{2\kappa}\sum_{n=1}^{\infty}\frac{(n+1)\,|d|^{n-1}}{4^{n-1}2n!(\kappa+1)_{n-1}} \leqq 1+\frac{|d|}{2\kappa}\sum_{n=1}^{\infty}\left(\frac{|d|}{4(\kappa+1)}\right)^{n-1}\\ &= \frac{4\kappa(\kappa+1)+(\kappa+2)\,|d|}{\kappa\,[4(\kappa+1)-|d|]} \qquad \left(\kappa>\frac{|d|-2}{4}\right),\end{aligned}$$

which is obviously positive if

$$4\kappa(\kappa+1)+(\kappa+2)\,|d|>0.$$

Similarly, by using the *reverse* triangle inequality:

$$|z_1-z_2| \geqq \big||z_1|-|z_2|\big|$$

and the inequality (1.20), we have

$$\begin{aligned}\left|z\varphi'_{\nu,b,d}(z)\right| &= \left|z+\sum_{n=1}^{\infty}\frac{(n+1)(-d)^n}{n!4^n(\kappa)_n}z^{n+1}\right| \geqq 1-\sum_{n=1}^{\infty}\frac{(n+1)\,|d|^n}{n!4^n(\kappa)_n}\\ &= 1-\frac{|d|}{2\kappa}\sum_{n=1}^{\infty}\frac{(n+1)\,|d|^{n-1}}{4^{n-1}2n!(\kappa+1)_{n-1}} \geqq 1-\frac{|d|}{2\kappa}\sum_{n=1}^{\infty}\left(\frac{|d|}{4(\kappa+1)}\right)^{n-1}\\ &= \frac{4\kappa(\kappa+1)-(3\kappa+2)\,|d|}{\kappa\,[4(\kappa+1)-|d|]} \qquad \left(\kappa>\frac{|d|-2}{4}\right),\end{aligned}$$

which is positive if

$$4\kappa(\kappa+1)-(3\kappa+2)\,|d|>0.$$

We now prove the assertion (1.17) of Lemma 5 by using again the triangle inequality and the following inequality:

$$4\cdot(n-1)!(\kappa+1)_{n-1} \geqq (n+1)(\kappa+1)^{n-1} \qquad (n\in\mathbb{N}\setminus\{1\}). \tag{1.21}$$

We thus have

$$\begin{aligned}\left|z^2\varphi''_{\nu,b,d}(z)\right| &= \left|\sum_{n=1}^{\infty}\frac{(n+1)n(-d)^n}{n!4^n(\kappa)_n}z^{n+1}\right| \leqq \sum_{n=1}^{\infty}\frac{(n+1)\,|d|^n}{(n-1)!4^n(\kappa)_n}\\ &= \frac{|d|}{\kappa}\left[\frac{1}{2}+\sum_{n=2}^{\infty}\left(\frac{n+1}{4\cdot(n-1)!}\right)\left(\frac{|d|^{n-1}}{4^{n-1}(\kappa+1)_{n-1}}\right)\right]\\ &\leqq \frac{|d|}{\kappa}\left[\frac{1}{2}+\sum_{n=2}^{\infty}\left(\frac{|d|}{4(\kappa+1)}\right)^{n-1}\right]\\ &= \frac{|d|}{2\kappa}\left(\frac{4(\kappa+1)+|d|}{4(\kappa+1)-|d|}\right) \qquad \left(\kappa>\frac{|d|-2}{4}\right).\end{aligned}$$

Finally, by combining the inequalities (1.16) and (1.17), we immediately deduce that (v) holds true for all $z\in\mathbb{U}$. Thus the proof of Lemma 5 is completed. ■

2. Univalence of Integral Operators Involving the Generalized Bessel Functions

Our first main result provides an application of Lemma 5 and contains sufficient univalence conditions for integral operators of the type (1.1) when the functions $f_j \quad (j=1,\cdots,n)$ are normalized forms of the generalized Bessel functions involving various parameters.

Theorem 1. *Let the parameters* $\nu_1, \cdots, \nu_n, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *be so constrained that*

$$\kappa_j > \frac{|d|-2}{4} \qquad \left(\kappa_j = \nu_j + \frac{b+1}{2} \quad (j = 1, \cdots, n)\right).$$

Consider the functions $\varphi_{\nu_j,b,d} : \mathbb{U} \to \mathbb{C}$ *defined by*

$$\varphi_{\nu_j,b,d}(z) = 2^{\nu_j}\, \Gamma\left(\nu_j + \frac{b+1}{2}\right) z^{1-\nu_j/2}\, \omega_{\nu_j,b,d}(\sqrt{z}). \tag{2.1}$$

Suppose also that

$\kappa = \min\{\kappa_1, \cdots, \kappa_n\},\ \Re(\beta) > 0,\ c \in \mathbb{C}\setminus\{-1\}$ *and* $\alpha_j \in \mathbb{C}\setminus\{0\}\ (j = 1, \cdots, n)$

and that these numbers satisfy the following inequality:

$$|c| + \frac{8(\kappa+1)\,|d|}{32\kappa(\kappa+1) - 8(2\kappa+1)\,|d| + |d|^2} \sum_{j=1}^{n} \frac{1}{|\beta\alpha_j|} \leqq 1.$$

Then the function $\mathcal{F}_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,\beta}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{F}_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,\beta}(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{j=1}^{n} \left(\frac{\varphi_{\nu_j,b,d}(t)}{t}\right)^{1/\alpha_j} \mathrm{d}t\right]^{1/\beta}, \tag{2.2}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Proof. We begin by setting $\beta = 1$ in (2.2) so that

$$\mathcal{F}_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,1}(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{\varphi_{\nu_j,b,d}(t)}{t}\right)^{1/\alpha_j} \mathrm{d}t.$$

First of all, we observe that, since $\varphi_{\nu_j,b,d} \in \mathcal{A}\ \ (j = 1, \cdots, n)$,

$$\varphi_{\nu_j,b,d}(0) = \varphi'_{\nu_j,b,d}(0) - 1 = 0.$$

Therefore, clearly, $\mathcal{F}_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,1} \in \mathcal{A}$, that is,

$$\mathcal{F}_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,1}(0) = \mathcal{F}'_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,1}(0) - 1 = 0.$$

On the other hand, it is easy to see that

$$\mathcal{F}'_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,1}(z) = \prod_{j=1}^{n} \left(\frac{\varphi_{\nu_j,b,d}(z)}{z}\right)^{1/\alpha_j} \tag{2.3}$$

and

$$\mathcal{F}''_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,1}(z) = \sum_{j=1}^{n} \frac{1}{\alpha_j} \left(\frac{\varphi_{\nu_j,b,d}(z)}{z} \right)^{(1-\alpha_j)/\alpha_j}$$

$$\cdot \left(\frac{z\varphi'_{\nu_j,b,d}(z) - \varphi_{\nu_j,b,d}(z)}{z^2} \right) \prod_{\substack{k=1 \\ (k\neq j)}}^{n} \left(\frac{\varphi_{\nu_k,b,d}(z)}{z} \right)^{1/\alpha_k}. \tag{2.4}$$

We thus find from (2.3) and (2.4) that

$$\frac{z\mathcal{F}''_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,1}(z)}{\mathcal{F}'_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,1}(z)} = \sum_{j=1}^{n} \frac{1}{\alpha_j} \left(\frac{z\varphi'_{\nu_j,b,d}(z)}{\varphi_{\nu_j,b,d}(z)} - 1 \right).$$

Now, by using the inequality (1.15) in Lemma 5 for each $\nu_j \quad (j = 1, \cdots, n)$, we obtain

$$\begin{aligned} \left| \frac{z\mathcal{F}''_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,1}(z)}{\mathcal{F}'_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,1}(z)} \right| &\leqq \sum_{j=1}^{n} \frac{1}{|\alpha_j|} \left| \frac{z\varphi'_{\nu_j,b,d}(z)}{\varphi_{\nu_j,b,d}(z)} - 1 \right| \\ &\leqq \sum_{j=1}^{n} \frac{1}{|\alpha_j|} \frac{8(\kappa_j+1)\,|d|}{32\kappa_j(\kappa_j+1) - 8(2\kappa_j+1)\,|d| + |d|^2} \\ &\leqq \sum_{j=1}^{n} \frac{1}{|\alpha_j|} \frac{8(\kappa+1)\,|d|}{32\kappa(\kappa+1) - 8(2\kappa+1)\,|d| + |d|^2} \end{aligned}$$

$$\left(z \in \mathbb{U};\ \kappa, \kappa_j := \nu_j + \frac{b+1}{2} > \frac{|d|-2}{4} \quad (j = 1, \cdots, n) \right),$$

where we have also used the fact that the function

$$\phi : \left(\frac{|d|-2}{4}, \infty \right) \to \mathbb{R},$$

defined by

$$\phi(x) = \frac{8(x+1)\,|d|}{32x(x+1) - 8(2x+1)\,|d| + |d|^2},$$

is decreasing and, consequently, we have

$$\frac{8(\kappa_j+1)\,|d|}{32\kappa_j(\kappa_j+1) - 8(2\kappa_j+1)\,|d| + |d|^2} \leqq \frac{8(\kappa+1)\,|d|}{32\kappa(\kappa+1) - 8(2\kappa+1)\,|d| + |d|^2}.$$

Finally, by using the triangle inequality and the assertion of Theorem 1, we get

$$\begin{aligned} &\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{z\mathcal{F}''_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,1}(z)}{\beta\mathcal{F}'_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,1}(z)} \right| \\ &\qquad \leqq |c| + \frac{8(\kappa+1)\,|d|}{32\kappa(\kappa+1) - 8(2\kappa+1)\,|d| + |d|^2} \sum_{j=1}^{n} \frac{1}{|\beta\alpha_j|} \leqq 1, \end{aligned}$$

which, in view of Lemma 1, implies that $\mathcal{F}_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,\beta} \in \mathcal{S}$. This evidently completes the proof of Theorem 1. ∎

Upon setting

$$\alpha_1 = \cdots = \alpha_n = \alpha$$

in Theorem 1, we immediately arrive at the following application of Theorem 1.

Corollary 1. *Let the parameters* $\nu_1, \cdots, \nu_n, b, c, d, \beta$ *and* $\kappa_j \quad (j = 1, \cdots, n)$ *be as in Theorem* 1. *Also let the functions* $\varphi_{\nu_j,b,d} : \mathbb{U} \to \mathbb{C}$ *be defined by* (2.1). *Suppose that*

$$\kappa = \min\{\kappa_1, \cdots, \kappa_n\} \quad \textit{and} \quad \alpha \in \mathbb{C} \setminus \{0\}$$

and that the following inequality holds true:

$$|c| + \frac{n}{|\beta\alpha|} \frac{8(\kappa+1)\,|d|}{32\kappa(\kappa+1) - 8(2\kappa+1)\,|d| + |d|^2} \leqq 1$$

Then the function $\mathcal{F}_{\nu_1,\cdots,\nu_n,b,d,\alpha,\beta}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{F}_{\nu_1,\cdots,\nu_n,b,d,\alpha,\beta}(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{j=1}^n \left(\frac{\varphi_{\nu_j,b,d}(t)}{t}\right)^{1/\alpha} \mathrm{d}t\right]^{1/\beta}, \tag{2.5}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Our second main result contains sufficient univalence conditions for an integral operator of the type (1.2) when the functions g_j are the normalized forms of the generalized Bessel functions with various parameters. The key tools in the proof are Lemma 2 and the inequality (1.15) of Lemma 5.

Theorem 2. *Let the parameters* $\nu_1, \cdots, \nu_n, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *be so constrained that*

$$\kappa_j > \frac{|d|-2}{4} \qquad \left(\kappa_j = \nu_j + \frac{b+1}{2} \ \ (j = 1, \cdots, n)\right).$$

Consider the functions $\varphi_{\nu_j,b,d} : \mathbb{U} \to \mathbb{C}$ *defined by* (2.1) *and let*

$$\kappa = \min\{\kappa_1, \cdots, \kappa_n\} \qquad \textit{and} \qquad \Re(\gamma) > 0.$$

Moreover, *suppose that the following inequality holds true*:

$$|\gamma| \leqq \frac{1}{n} \ \frac{32\kappa(\kappa+1) - 8(2\kappa+1)\,|d| + |d|^2}{8(\kappa+1)\,|d|} \ \Re(\gamma).$$

Then the function $\mathcal{G}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{G}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(z) = \left[(n\gamma+1) \int_0^z \prod_{j=1}^n \left(\varphi_{\nu_j,b,d}(t)\right)^{\gamma} \ \mathrm{d}t\right]^{1/(n\gamma+1)}, \tag{2.6}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Proof. Let us consider the function $\widetilde{\mathcal{G}}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(z):\mathbb{U}\to\mathbb{C}$ defined by

$$\widetilde{\mathcal{G}}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(z)=\int_0^z\prod_{j=1}^n\left(\frac{\varphi_{\nu_j,b,d}(t)}{t}\right)^\gamma \mathrm{d}t.$$

Observe that $\widetilde{\mathcal{G}}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}\in\mathcal{A}$, that is, that

$$\widetilde{\mathcal{G}}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(0)=\widetilde{\mathcal{G}}'_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(0)-1=0.$$

On the other hand, by using the assertion (1.15) of Lemma 5, the assertion of Theorem 2 and the fact that

$$\begin{aligned}&\frac{8(\kappa_j+1)\,|d|}{32\kappa_j(\kappa_j+1)-8(2\kappa_j+1)\,|d|+|d|^2}\\ &\qquad\leqq\frac{8(\kappa+1)\,|d|}{32\kappa(\kappa+1)-8(2\kappa+1)\,|d|+|d|^2}\qquad (j=1,\cdots,n),\end{aligned}$$

we have

$$\begin{aligned}&\frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\left|\frac{z\widetilde{\mathcal{G}}''_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(z)}{\widetilde{\mathcal{G}}'_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(z)}\right|\\ &\qquad\leqq\frac{|\gamma|}{\Re(\gamma)}\sum_{j=1}^n\left|\frac{z\varphi'_{\nu_j,b,d}(z)}{\varphi_{\nu_j,b,d}(z)}-1\right|\\ &\qquad\leqq\frac{n\,|\gamma|}{\Re(\gamma)}\frac{8(\kappa+1)\,|d|}{32\kappa(\kappa+1)-8(2\kappa+1)\,|d|+|d|^2}\leqq 1\qquad (z\in\mathbb{C}).\end{aligned}$$

Now, since $\Re(n\gamma+1)>\Re(\gamma)$ and the function $\mathcal{G}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}$ can be rewritten in the form:

$$\mathcal{G}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(z)=\left[(n\gamma+1)\int_0^z t^{n\gamma}\prod_{j=1}^n\left(\frac{\varphi_{\nu_j,b,d}(t)}{t}\right)^\gamma \mathrm{d}t\right]^{1/(n\gamma+1)},$$

Lemma 2 would imply that $\mathcal{G}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}\in\mathcal{S}$. This evidently completes the proof of Theorem 2. ∎

Choosing $n=1$ in Theorem 2, we have the following result.

Corollary 2. *Let the parameters* $\nu,b\in\mathbb{R}$ *and* $d\in\mathbb{C}$ *be so constrained that*

$$\kappa:=\nu+\frac{b+1}{2}>\frac{|d|-2}{4}.$$

Consider the function $\varphi_{\nu,b,d} : \mathbb{U} \to \mathbb{C}$ *defined by* (1.11). *Moreover, suppose that* $\Re(\gamma) > 0$ *and*

$$|\gamma| \leqq \frac{32\kappa(\kappa+1) - 8(2\kappa+1)\,|d| + |d|^2}{8(\kappa+1)\,|d|}\,\Re(\gamma).$$

Then the function $\mathcal{G}_{\nu,b,d,\gamma}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{G}_{\nu,b,d,\gamma}(z) = \left[(\gamma+1)\int_0^z (\varphi_{\nu,b,d}(t))^{\gamma}\ \mathrm{d}t\right]^{1/(\gamma+1)}, \tag{2.7}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

The following result contains another set of sufficient univalence conditions for an integral operator of the type (1.3) when the functions h_j are the normalized forms of the generalized Bessel functions involving various parameters. The key tools in the proof are Lemma 1 and the inequality (1.18) of Lemma 5.

Theorem 3. *Let the parameters* $\nu_1, \cdots, \nu_n, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *be so constrained that*

$$\kappa_j := \nu_j + \frac{b+1}{2} > \frac{|d|-2}{4} \qquad (j = 1, \cdots, n).$$

Consider the functions $\varphi_{\nu_j,b,d} : \mathbb{U} \to \mathbb{C}$ *defined by* (2.1). *Also let*

$$\kappa = \min\{\kappa_1, \cdots, \kappa_n\},\ \Re(\mu) > 0,\ c \in \mathbb{C} \setminus \{-1\} \quad \textit{and} \quad \delta_1, \cdots, \delta_n \in \mathbb{C}.$$

Moreover, suppose that the following inequality holds true:

$$|c| + \frac{4(\kappa+1)\,|d| + |d|^2}{8\kappa(\kappa+1) - 2(3\kappa+2)\,|d|}\sum_{j=1}^{n}\frac{|\delta_j|}{|\mu|} \leqq 1.$$

Then the function $\mathcal{H}_{\nu_1,\cdots,\nu_n,\delta_1,\cdots,\delta_n,b,d,\mu}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{H}_{\nu_1,\cdots,\nu_n,\delta_1,\cdots,\delta_n,b,d,\mu}(z) = \left[\mu\int_0^z t^{\mu-1}\prod_{j=1}^{n}\left(\varphi'_{\nu_j,b,d}(t)\right)^{\delta_j}\ \mathrm{d}t\right]^{1/\mu}, \tag{2.8}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Proof. Our demonstration of Theorem 3 is much akin to that of Theorem 1. Indeed, by considering the function $\mathcal{H}_{\nu_1,\cdots,\nu_n,\delta_1,\cdots,\delta_n,b,d}(z) : \mathbb{U} \to \mathbb{C}$ defined by

$$\mathcal{H}_{\nu_1,\cdots,\nu_n,\delta_1,\cdots,\delta_n,b,d}(z) = \int_0^z \prod_{j=1}^{n}\left(\varphi'_{\nu_j,b,d}(t)\right)^{\delta_j}\ \mathrm{d}t,$$

and using the inequality (1.18) of Lemma 5 for each $\nu_j \ (j=1,\cdots,n)$ in conjunction with the following easily derivable inequality:

$$\frac{4(\kappa_j+1)\,|d|+|d|^2}{8\kappa_j(\kappa_j+1)-2(3\kappa_j+2)\,|d|} \leqq \frac{4(\kappa+1)\,|d|+|d|^2}{8\kappa(\kappa+1)-2(3\kappa+2)\,|d|},$$

we are led eventually to the sufficient univalence conditions asserted by Theorem 3 by means of the triangle inequality and the assertion of Theorem 3. ■

Setting $n=1$ in Theorem 3, we immediately obtain the following result.

Corollary 3. *Let* $\nu, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *such that*

$$\kappa := \nu + \frac{b+1}{2} > \frac{|d|-2}{4}.$$

Consider the function $\varphi_{\nu,b,d} : \mathbb{U} \to \mathbb{C}$ *defined by* (1.11). *Also let* $\Re(\mu) > 0$, $c \in \mathbb{C} \setminus \{-1\}$ *and* $\delta \in \mathbb{C}$. *Moreover, suppose that the following inequality holds true*:

$$|c| + \left|\frac{\delta}{\mu}\right| \frac{4(\kappa+1)\,|d|+|d|^2}{8\kappa(\kappa+1)-2(3\kappa+2)\,|d|} \leqq 1.$$

Then the function $\mathcal{H}_{\nu,\delta,b,d,\mu}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{H}_{\nu,\delta,b,d,\mu}(z) = \left[\mu \int_0^z t^{\mu-1} \left(\varphi'_{\nu,b,d}(t)\right)^{\delta} \,\mathrm{d}t\right]^{1/\mu}, \tag{2.9}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Next, by applying Lemma 3 and the inequality (1.16) asserted by Lemma 5, we easily get the following result.

Theorem 4. *Let the parameters* $\nu, b \in \mathbb{R}$ *and* $d, \lambda \in \mathbb{C}$ *be so constrained that*

$$\kappa := \nu + \frac{b+1}{2} > \frac{|d|-2}{4}.$$

Consider the generalized Bessel function $\varphi_{\nu,b,d}$ *defined by* (1.11). *If* $\Re(\lambda) \geqq 1$ *and*

$$|\lambda| \leqq \frac{3\sqrt{3}\kappa\,[4(\kappa+1)-|d|]}{8\kappa(\kappa+1)+2(\kappa+2)\,|d|},$$

then the function $\mathcal{Q}_{\nu,b,d,\lambda} : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{Q}_{\nu,b,d,\lambda}(z) = \left[\lambda \int_0^z t^{\lambda-1} \left(e^{\varphi_{\nu,b,d}(t)}\right)^{\lambda} \,\mathrm{d}t\right]^{1/\lambda}, \tag{2.10}$$

is univalent in $\mathbb{U}$.

Taking into account the above results, we have the following particular cases.

2.1. Bessel Functions

Choosing $b = d = 1$, in (1.6) or (1.7), we obtain the Bessel function $J_\nu(z)$ of the first kind of order ν defined by (1.8). We observe also that

$$\mathcal{J}_{3/2}(z)=\frac{3\sin\sqrt{z}}{\sqrt{z}}-3\cos\sqrt{z},\quad \mathcal{J}_{1/2}(z)=\sqrt{z}\sin\sqrt{z}\quad\text{and}\quad \mathcal{J}_{-1/2}(z)=z\cos\sqrt{z}.$$

Corollary 4. *Let the function* $\mathcal{J}_\nu : \mathbb{U} \to \mathbb{C}$ *be defined by*

$$\mathcal{J}_\nu(z) = 2^\nu \Gamma(\nu+1)\, z^{1-\nu/2} J_\nu(\sqrt{z}).$$

Also let the following assertions hold true:

1. *Let* $\nu_1, \cdots, \nu_n > -1.25 \ (n \in \mathbb{N})$. *Consider the functions* $\mathcal{J}_{\nu_j} : \mathbb{U} \to \mathbb{C}$ *defined by*

$$\mathcal{J}_{\nu_j}(z) = 2^{\nu_j}\Gamma(\nu_j+1)\, z^{1-\nu_j/2} J_{\nu_j}(\sqrt{z}) \qquad (j = 1, \cdots, n). \tag{2.11}$$

Let $\nu = \min\{\nu_1, \cdots, \nu_n\}$ *and let the parameters* $\beta, c, \alpha_1, \cdots, \alpha_n$ *be as in Theorem* 1. *Moreover*, *suppose that these numbers satisfy the following inequality*:

$$|c| + \frac{\nu+2}{4\nu^2+10\nu+41/8}\sum_{j=1}^{n}\frac{1}{|\beta\alpha_j|} \leqq 1.$$

Then the function $\mathcal{F}_{\nu_1,\cdots,\nu_n,\alpha_1,\cdots,\alpha_n,\beta}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{F}_{\nu_1,\cdots,\nu_n,\alpha_1,\cdots,\alpha_n,\beta}(z) = \left[\beta\int_0^z t^{\beta-1}\prod_{j=1}^{n}\left(\frac{\mathcal{J}_{\nu_j}(t)}{t}\right)^{1/\alpha_j} \mathrm{d}t\right]^{1/\beta}, \tag{2.12}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$. *In the particular case when*

$$|c| + \frac{28}{233}\frac{1}{|\beta\alpha|} \leqq 1,$$

the function $\mathcal{F}_{3/2,\alpha,\beta}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{F}_{3/2,\alpha,\beta}(z) = \left[\beta\int_0^z t^{\beta-1}\left(\frac{3\sin\sqrt{t}}{t\sqrt{t}} - \frac{3\cos\sqrt{t}}{t}\right)^{1/\alpha} \mathrm{d}t\right]^{1/\beta},$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

2. *Let* $\nu_1, \cdots, \nu_n > -1.25 \ \ (n \in \mathbb{N})$ *and consider the normalized Bessel functions* $\mathcal{J}_{\nu_j} : \mathbb{U} \to \mathbb{C}$ *defined by* (2.11). *Also let* $\nu = \min\{\nu_1, \cdots, \nu_n\}$ *and* $\Re(\gamma) > 0$ *and suppose that these numbers satisfy the following inequality*:

$$|\gamma| \leqq \frac{1}{n}\frac{4\nu^2 + 10\nu + 41/8}{\nu + 2}\, \Re(\gamma).$$

Then the function $\mathcal{G}_{\nu_1,\cdots,\nu_n,n,\gamma}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{G}_{\nu_1,\cdots,\nu_n,n,\gamma}(z) = \left[(n\gamma+1)\int_0^z \prod_{j=1}^{n} \left(\mathcal{J}_{\nu_j}(t)\right)^{\gamma} \, \mathrm{d}t\right]^{1/(n\gamma+1)}, \tag{2.13}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$. *In the particular case when*

$$|\gamma| \leqq \frac{89}{20}\Re(\gamma),$$

the function $\mathcal{G}_{1/2,\gamma}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{G}_{1/2,\gamma}(z) = \left[(\gamma+1)\int_0^z \left(\sqrt{t}\sin\sqrt{t}\right)^{\gamma} \, \mathrm{d}t\right]^{1/(\gamma+1)},$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

3. *Let* $\nu_1, \cdots, \nu_n > -1.25 \ \ (n \in \mathbb{N})$ *and consider the normalized Bessel functions* $\mathcal{J}_{\nu_j} : \mathbb{U} \to \mathbb{C}$ *defined by* (2.11). *Let* $\nu = \min\{\nu_1, \cdots, \nu_n\}$ *and let the parameters* $\mu, c, \delta_1, \cdots, \delta_n$ *be as in Theorem* 3. *Moreover*, *suppose that these numbers satisfy the following inequality*:

$$|c| + \frac{4\nu + 9}{8\nu^2 + 18\nu + 6}\sum_{j=1}^{n}\frac{|\delta_j|}{|\mu|} \leqq 1$$

Then the function $\mathcal{H}_{\nu_1,\cdots,\nu_n,\delta_1,\cdots,\delta_n,\mu}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{H}_{\nu_1,\cdots,\nu_n,\delta_1,\cdots,\delta_n,\mu}(z) = \left[\mu\int_0^z t^{\mu-1}\prod_{j=1}^{n}\left(\mathcal{J}'_{\nu_j}(t)\right)^{\delta_j} \, \mathrm{d}t\right]^{1/\mu}, \tag{2.14}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$. *In particular*, *the function* $\mathcal{H}_{-1/2,\delta,\mu}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{H}_{-1/2,\delta,\mu}(z) = \left[\mu\int_0^z t^{\mu-1}\left(\cos\sqrt{t} - \frac{\sqrt{t}\sin\sqrt{t}}{2}\right)^{\delta} \, \mathrm{d}t\right]^{1/\mu},$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

4. *Let* $\lambda \in \mathbb{C}$ *and* $\nu > -1.25$ *and consider the normalized Bessel function* $\mathcal{J}_\nu(z)$ *given by* (1.8). *If* $\Re(\lambda) \geqq 1$ *and*

$$|\lambda| \leqq \frac{3\sqrt{3}(\nu+1)(4\nu+7)}{8\nu^2+26\nu+22},$$

then the function $\mathcal{Q}_{\nu,\lambda} : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{Q}_{\nu,\lambda}(z) = \left[\lambda \int_0^z t^{\lambda-1} \left(e^{\mathcal{J}_\nu(t)}\right)^\lambda \, \mathrm{d}t\right]^{1/\lambda}, \tag{2.15}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$. *In the particular case when* $|\lambda| \leqq 1.8959\cdots$, *the function* $\mathcal{Q}_{1/2,\lambda}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{Q}_{1/2,\lambda}(z) = \left[\lambda \int_0^z t^{\lambda-1} \left(e^{\sqrt{t}\sin\sqrt{t}}\right)^\lambda \, \mathrm{d}t\right]^{1/\lambda},$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Remark 6. Baricz and Frasin [7] proved that the following general integral operators:

$$\mathcal{F}_{\nu_1,\cdots,\nu_n,\alpha_1,\cdots,\alpha_n,\beta}(z), \quad \mathcal{G}_{\nu_1,\cdots,\nu_n,n,\gamma}(z) \qquad \text{and} \qquad \mathcal{Q}_{\nu,\lambda}(z)$$

defined by (2.12), (2.13) and (2.15), respectively, are actually univalent for all

$$\nu,\ \nu_1,\cdots,\nu_n > -0.690\,98\cdots.$$

From Corollary 4, we see that our results (*with* ν, $\nu_1,\cdots,\nu_n > -1.25$) are stronger than the Baricz-Frasin results for the same integral operators (see, for details, [7]).

2.2. Modified Bessel Functions

Taking $b = 1$ and $d = -1$ in (1.6) or (1.7), we obtain the modified Bessel function $I_\nu(z)$ of the first kind of order ν defined by (1.9). We observe also that

$$\mathcal{I}_{3/2}(z) = 3\cosh\sqrt{z} - \frac{3\sinh\sqrt{z}}{\sqrt{z}},$$
$$\mathcal{I}_{1/2}(z) = \sqrt{z}\sinh\sqrt{z} \qquad \text{and} \qquad \mathcal{I}_{-1/2}(z) = z\cosh\sqrt{z}.$$

Corollary 5. *Let the function* $\mathcal{I}_\nu : \mathbb{U} \to \mathbb{C}$ *be defined by*

$$\mathcal{I}_\nu(z) = 2^\nu \Gamma(\nu+1)\, z^{1-\nu/2} I_\nu(\sqrt{z}).$$

Also let the following assertions hold true:

1. *Let* $\nu_1,\cdots,\nu_n > -1.25$ $(n \in \mathbb{N})$. *Consider the functions* $\mathcal{I}_{\nu_j} : \mathbb{U} \to \mathbb{C}$ *defined by*

$$\mathcal{I}_{\nu_j}(z) = 2^{\nu_j}\Gamma(\nu_j+1)\, z^{1-\nu_j/2} I_{\nu_j}(\sqrt{z}) \qquad (j = 1, \cdots, n). \tag{2.16}$$

Let $\nu = \min\{\nu_1, \cdots, \nu_n\}$ *and let the parameters* $\beta, c, \alpha_1, \cdots, \alpha_n$ *be as in Theorem* 1. *Moreover, suppose that these numbers satisfy the following inequality*:

$$|c| + \frac{\nu+2}{4\nu^2+10\nu+41/8}\sum_{j=1}^{n}\frac{1}{|\beta\alpha_j|} \leqq 1.$$

Then the function $\mathcal{F}_{\nu_1,\cdots,\nu_n,\alpha_1,\cdots,\alpha_n,\beta}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{F}_{\nu_1,\cdots,\nu_n,\alpha_1,\cdots,\alpha_n,\beta}(z) = \left[\beta\int_0^z t^{\beta-1}\prod_{j=1}^{n}\left(\frac{\mathcal{I}_{\nu_j}(t)}{t}\right)^{1/\alpha_j} \mathrm{d}t\right]^{1/\beta}, \tag{2.17}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$. *In the particular case when*

$$|c| + \frac{28}{233}\frac{1}{|\beta\alpha|} \leqq 1,$$

the function $\mathcal{F}_{3/2,\alpha,\beta}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{F}_{3/2,\alpha,\beta}(z) = \left[\beta\int_0^z t^{\beta-1}\left(\frac{3\cosh\sqrt{t}}{t} - \frac{3\sinh\sqrt{t}}{t\sqrt{t}}\right)^{1/\alpha} \mathrm{d}t\right]^{1/\beta},$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

2. *Let* $\nu_1, \cdots, \nu_n > -1.25 \ (n \in \mathbb{N})$ *and consider the normalized modified Bessel functions* $\mathcal{I}_{\nu_j} : \mathbb{U} \to \mathbb{C}$ *defined by* (2.16). *Let* $\nu = \min\{\nu_1, \cdots, \nu_n\}$ *and* $\Re(\gamma) > 0$ *and suppose that these numbers satisfy the following inequality*:

$$|\gamma| \leqq \frac{1}{n}\left(\frac{4\nu^2+10\nu+41/8}{\nu+2}\right)\Re(\gamma).$$

Then the function $\mathcal{G}_{\nu_1,\cdots,\nu_n,n,\gamma}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{G}_{\nu_1,\cdots,\nu_n,n,\gamma}(z) = \left[(n\gamma+1)\int_0^z \prod_{j=1}^{n}\left(\mathcal{I}_{\nu_j}(t)\right)^{\gamma} \mathrm{d}t\right]^{1/(n\gamma+1)}, \tag{2.18}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$. *In the particular case when*

$$|\gamma| \leqq \frac{89}{20}\Re(\gamma),$$

the function $\mathcal{G}_{1/2,\gamma}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{G}_{1/2,\gamma}(z) = \left[(\gamma+1)\int_0^z \left(\sqrt{t}\sinh\sqrt{t}\right)^{\gamma}\,\mathrm{d}t\right]^{1/(\gamma+1)},$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

3. *Let* $\nu_1,\cdots,\nu_n > -1.25 \ \ (n \in \mathbb{N})$ *and consider the normalized modified Bessel functions* $\mathcal{I}_{\nu_j} : \mathbb{U} \to \mathbb{C}$ *defined by* (2.16). *Also let* $\nu = \min\{\nu_1,\cdots,\nu_n\}$ *and let the parameters* $\mu, c, \ \delta_1,\cdots,\delta_n$ *be as in Theorem* 3. *Suppose that these numbers satisfy the following inequality*:

$$|c| + \frac{4\nu+9}{8\nu^2+18\nu+6}\sum_{j=1}^{n}\frac{|\delta_j|}{|\mu|} \leqq 1.$$

Then the function $\mathcal{H}_{\nu_1,\cdots,\nu_n,\delta_1,\cdots,\delta_n,\mu}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{H}_{\nu_1,\cdots,\nu_n,\delta_1,\cdots,\delta_n,\mu}(z) = \left[\mu\int_0^z t^{\mu-1}\prod_{j=1}^{n}\left(\mathcal{I}'_{\nu_j}(t)\right)^{\delta_j}\,\mathrm{d}t\right]^{1/\mu},$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$. *In particular*, *the function* $\mathcal{H}_{-1/2,\delta,\mu}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{H}_{-1/2,\delta,\mu}(z) = \left[\mu\int_0^z t^{\mu-1}\left(\cosh\sqrt{t} + \frac{\sqrt{t}\sinh\sqrt{t}}{2}\right)^{\delta}\,\mathrm{d}t\right]^{1/(\gamma+1)},$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

4. *Let* $\lambda \in \mathbb{C}$ *and* $\nu > -1.25$ *and consider the normalized modified Bessel function* $\mathcal{I}_\nu(t)$ *given by* (1.9). *If* $\Re(\lambda) \geqq 1$ *and*

$$|\lambda| \leqq \frac{3\sqrt{3}(\nu+1)(4\nu+7)}{8\nu^2+26\nu+22},$$

then the function $\mathcal{Q}_{\nu,\lambda} : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{Q}_{\nu,\lambda}(z) = \left[\lambda\int_0^z t^{\lambda-1}\left(e^{\mathcal{I}_\nu(t)}\right)^{\lambda}\,\mathrm{d}t\right]^{1/\lambda},$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$. *In the particular case when* $|\lambda| \leqq 1.1809\cdots$, *the function* $\mathcal{Q}_{-1/2,\lambda}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{Q}_{-1/2,\lambda}(z) = \left[\lambda\int_0^z t^{\lambda-1}\left(e^{t\cosh\sqrt{t}}\right)^{\lambda}\,\mathrm{d}t\right]^{1/\lambda},$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

2.3. Spherical Bessel Functions

Upon setting $b = 2$ and $d = 1$ in (1.6) or (1.7), we obtain the spherical Bessel function $K_\nu(z)$ of the first kind of order ν defined *here* by (1.10).

Corollary 6. *Let* $\mathcal{K}_\nu : \mathbb{U} \to \mathbb{C}$ *be defined by*

$$\mathcal{K}_\nu(z) = 2^\nu \Gamma(\nu+1)\, z^{1-\nu/2} K_\nu(\sqrt{z}).$$

Also let the following assertions hold true:

1. *Let* $\nu_1, \cdots, \nu_n > -2.25 \ \ (n \in \mathbb{N})$. *Consider the functions* $\mathcal{K}_{\nu_j} : \mathbb{U} \to \mathbb{C}$ *defined by*

$$\mathcal{K}_{\nu_j}(z) = 2^{\nu_j} \Gamma(\nu_j+1)\, z^{1-\nu_j/2} K_{\nu_j}(\sqrt{z}) \qquad (j = 1, \cdots, n). \tag{2.19}$$

Let $\nu = \min\{\nu_1, \cdots, \nu_n\}$ *and let the parameters* $\beta, c, \alpha_1, \cdots, \alpha_n$ *be as in Theorem* 1. *Moreover*, *suppose that these numbers satisfy the following inequality*:

$$|c| + \frac{2\nu+5}{8\nu^2 + 28\nu + 89/4} \sum_{j=1}^{n} \frac{1}{|\beta\alpha_j|} \leqq 1.$$

Then the function $\mathcal{F}_{\nu_1,\cdots,\nu_n,\alpha_1,\cdots,\alpha_n,\beta}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{F}_{\nu_1,\cdots,\nu_n,\alpha_1,\cdots,\alpha_n,\beta}(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{j=1}^{n} \left(\frac{\mathcal{K}_{\nu_j}(t)}{t}\right)^{1/\alpha_j} \mathrm{d}t\right]^{1/\beta}, \tag{2.20}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

2. *Let* $\nu_1, \cdots, \nu_n > -2.25 \ \ (n \in \mathbb{N})$ *and consider the normalized spherical Bessel functions* $\mathcal{K}_{\nu_j} : \mathbb{U} \to \mathbb{C}$ *defined by* (2.19). *Let* $\nu = \min\{\nu_1, \cdots, \nu_n\}$ *and* $\Re(\gamma) > 0$ *and suppose that these numbers satisfy the following inequality*:

$$|\gamma| \leqq \frac{1}{n}\left(\frac{8\nu^2 + 28\nu + 89/48}{\nu+5}\right)\Re(\gamma).$$

Then the function $\mathcal{G}_{\nu_1,\cdots,\nu_n,n,\gamma}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{G}_{\nu_1,\cdots,\nu_n,n,\gamma}(z) = \left[(n\gamma+1) \int_0^z \prod_{j=1}^{n} \left(\mathcal{K}_{\nu_j}(t)\right)^{\gamma} \mathrm{d}t\right]^{1/(n\gamma+1)}, \tag{2.21}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

3. *Let* $\nu_1, \cdots, \nu_n > -2.25 \ \ (n \in \mathbb{N})$ *and consider the normalized spherical Bessel functions* $\mathcal{K}_{\nu_j} : \mathbb{U} \to \mathbb{C}$ *defined by* (2.19). *Also let* $\nu = \min\{\nu_1, \cdots, \nu_n\}$ *and let* *the parameters* $\mu, c, \delta_1, \cdots, \delta_n$ *be as in Theorem* 3. *Suppose that these numbers satisfy the following inequality*:

$$|c| + \frac{4\nu + 11}{8\nu^2 + 26\nu + 17} \sum_{j=1}^{n} \frac{|\delta_j|}{|\mu|} \leqq 1$$

Then the function $\mathcal{H}_{\nu_1,\cdots,\nu_n,\delta_1,\cdots,\delta_n,\mu}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{H}_{\nu_1,\cdots,\nu_n,\delta_1,\cdots,\delta_n,\mu}(z) = \left[\mu \int_0^z t^{\mu-1} \prod_{j=1}^{n} \left(\mathcal{K}'_{\nu_j}(t)\right)^{\delta_j} \,\mathrm{d}t\right]^{1/\mu}, \tag{2.22}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

4. *Let* $\lambda \in \mathbb{C}$ *and* $\nu > -2.25$ *and consider the normalized spherical Bessel function* $\mathcal{K}_\nu(t)$ *given by* (1.10). *If* $\Re(\lambda) \geqq 1$ *and*

$$|\lambda| \leqq \frac{3\sqrt{3}(2\nu + 3)(4\nu + 9)}{16\nu^2 + 68\nu + 74},$$

then the function $\mathcal{Q}_{\nu,\lambda} : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{Q}_{\nu,\lambda}(z) = \left[\lambda \int_0^z t^{\lambda-1} \left(e^{\mathcal{K}_\nu(t)}\right)^{\lambda} \,\mathrm{d}t\right]^{1/\lambda}, \tag{2.23}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

3. Univalence of Integral Operators for Sharp Bounds of the Generalized Bessel Functions

In our proof of Lemma 5, the key tools were the following inequalities [see also (1.19), (1.20) and (1.21)]:

$$n!(\kappa + 1)_{n-1} \geqq n(\kappa + 1)^{n-1} \qquad (\kappa > -1;\ n \in \mathbb{N}),$$

$$2 \cdot n!(\kappa + 1)_{n-1} \geqq (n + 1)(\kappa + 1)^{n-1} \qquad (\kappa > -1;\ n \in \mathbb{N})$$

and

$$4 \cdot (n - 1)!(\kappa + 1)_{n-1} \geqq (n + 1)(\kappa + 1)^{n-1} \qquad (\kappa > -1;\ n \in \mathbb{N} \setminus \{1\}).$$

Clearly, the above inequalities can be improved easily by using the following well-known inequality:

$$n! \geqq 2^{n-1} \qquad (\kappa > -1;\ n \in \mathbb{N}).$$

More precisely, we have

$$n!(\kappa + 1)_{n-1} \geqq [2(\kappa + 1)]^{n-1} \qquad (\kappa > -1;\ n \in \mathbb{N}) \tag{3.1}$$

and

$$2 \cdot (n-1)!(\kappa+1)_{n-1} \geqq [2(\kappa+1)]^{n-1} \qquad (\kappa > -1;\ n \in \mathbb{N} \setminus \{1\}). \tag{3.2}$$

The following lemma was proven by Baricz and Ponnusamy [8].

Lemma 6. (see [8]). *If the parameters* $\nu, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *are so constrained that*

$$\kappa > \max\left\{0, \frac{|d|}{8} - 1\right\},$$

then the function $\frac{\varphi_{\nu,b,d}(z)}{z} : \mathbb{U} \to \mathbb{C}$ *defined by* (1.12) *satisfies the following inequalities*:

$$\frac{8\kappa(\kappa+1) - (2\kappa+3)\,|d|}{\kappa\,[8(\kappa+1) - |d|]} \leqq \left|\frac{\varphi_{\nu,b,d}(z)}{z}\right| \leqq \frac{8\kappa + |d|}{8\kappa - |d|} \qquad (z \in \mathbb{U}) \tag{3.3}$$

and

$$\left|\left(\frac{\varphi_{\nu,b,d}(z)}{z}\right)'\right| \leqq \frac{|d|}{4\kappa}\left(\frac{8(\kappa+1) + |d|}{8(\kappa+1) - |d|}\right) \qquad (z \in \mathbb{U}). \tag{3.4}$$

Lemma 7. *If the parameters* $\nu, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *are so constrained that*

$$\kappa > \max\left\{0, \frac{|d|}{8} - 1\right\},$$

then the function $\varphi_{\nu,b,d} : \mathbb{U} \to \mathbb{C}$ *defined by* (1.12) *satisfies the following inequalities*:

$$\left|\varphi'_{\nu,b,d}(z) - \frac{\varphi_{\nu,b,d}(z)}{z}\right| \leqq \frac{|d|}{4\kappa}\left(\frac{8(\kappa+1) + |d|}{8(\kappa+1) - |d|}\right) \qquad (z \in \mathbb{U}), \tag{3.5}$$

$$\left|\frac{z\varphi'_{\nu,b,d}(z)}{\varphi_{\nu,b,d}(z)} - 1\right| \leqq \frac{8(\kappa+1)\,|d| + |d|^2}{32\kappa(\kappa+1) - 4(2\kappa+3)\,|d|} \qquad (z \in \mathbb{U}), \tag{3.6}$$

$$\begin{aligned} &\frac{32\kappa(\kappa+1) - 8(3\kappa+2)\,|d| - |d|^2}{4\kappa\,[8(\kappa+1) - |d|]} \\ &\qquad \leqq \left|z\varphi'_{\nu,b,d}(z)\right| \\ &\qquad \leqq \frac{32\kappa(\kappa+1) + 4(3\kappa+4)\,|d| + |d|^2}{4\kappa\,[8(\kappa+1) - |d|]} \qquad (z \in \mathbb{U}), \end{aligned} \tag{3.7}$$

$$\left|z^2\varphi''_{\nu,b,d}(z)\right| \leqq \frac{|d|}{2\kappa}\left(\frac{|d|^2}{8(\kappa+1)}\,\frac{8(\kappa+2)+|d|}{8(\kappa+2)-|d|} + \frac{8(\kappa+1)+|d|}{8(\kappa+1)-|d|}\right) \qquad (z \in \mathbb{U}) \tag{3.8}$$

and

$$\left|\frac{z\varphi''_{\nu,b,d}(z)}{\varphi'_{\nu,b,d}(z)}\right| \leqq \frac{|d|}{2} \tag{3.9}$$
$$\cdot\left(\frac{64(\kappa+1)^2\left[8(\kappa+2)-|d|\right]+128(\kappa+1)(\kappa+2)-\left[8(\kappa+2)+|d|\right]|d|^2}{2(\kappa+1)\left[8(\kappa+1)-|d|\right]\left\{16(\kappa+1)(2\kappa-|d|)-|d|\left(4\kappa+|d|\right)\right\}}\right)\quad (z\in\mathbb{U}).$$

Proof. By using the inequality (3.4) in Lemma 6, we obtain

$$\left|\varphi'_{\nu,b,d}(z)-\frac{\varphi_{\nu,b,d}(z)}{z}\right| = \left|z\left(\frac{\varphi_{\nu,b,d}(z)}{z}\right)'\right| \leqq \left|\left(\frac{\varphi_{\nu,b,d}(z)}{z}\right)'\right|$$
$$\leqq \frac{|d|}{4\kappa}\left(\frac{8(\kappa+1)+|d|}{8(\kappa+1)-|d|}\right) \qquad \left(z\in\mathbb{U};\ \kappa>\frac{|d|}{8}-1\right),$$

which proves the inequality (3.5).

Combining the inequality (3.5) of Lemma 7 with the inequality (3.3) of Lemma 6, we immediately arrive at the assertion (3.6) of Lemma 7.

By using the triangle inequality and the inequalies (3.1) and (3.2) above, we obtain

$$\begin{aligned}
&\left|z\varphi'_{\nu,b,d}(z)\right| \\
&\quad = \left|z+\sum_{n=1}^{\infty}\frac{(n+1)(-d)^n}{n!4^n(\kappa)_n}z^{n+1}\right| \leqq 1+\sum_{n=1}^{\infty}\frac{(n+1)\,|d|^n}{n!4^n(\kappa)_n} \\
&\quad = 1+\frac{|d|}{4\kappa}\left(1+2\sum_{n=2}^{\infty}\frac{|d|^{n-1}}{4^{n-1}2(n-1)!(\kappa+1)_{n-1}}+\sum_{n=1}^{\infty}\frac{|d|^{n-1}}{4^{n-1}n!(\kappa+1)_{n-1}}\right) \\
&\quad \leqq 1+\frac{|d|}{4\kappa}\left[1+2\sum_{n=2}^{\infty}\left(\frac{|d|}{8(\kappa+1)}\right)^{n-1}+\sum_{n=1}^{\infty}\left(\frac{|d|}{8(\kappa+1)}\right)^{n-1}\right] \\
&\quad = \frac{32\kappa(\kappa+1)+4(3\kappa+4)\,|d|+|d|^2}{4\kappa\left[8(\kappa+1)-|d|\right]} \qquad \left(z\in\mathbb{U};\ \kappa>\frac{|d|}{8}-1\right),
\end{aligned}$$

which yields the inequality (3.6). Similarly, by using the reverse triangle inequality in conjunction with the inequalities (3.1) and (3.2) above, we find that

$$\begin{aligned}
&\left|z\varphi'_{\nu,b,d}(z)\right| \\
&\quad = \left|z+\sum_{n=1}^{\infty}\frac{(n+1)(-d)^n}{n!4^n(\kappa)_n}z^{n+1}\right| \geqq 1-\sum_{n=1}^{\infty}\frac{(n+1)\,|d|^n}{n!4^n(\kappa)_n} \\
&\quad = 1-\frac{|d|}{4\kappa}\left[1+2\sum_{n=2}^{\infty}\frac{|d|^{n-1}}{4^{n-1}(n-1)!(\kappa+1)_{n-1}}+\sum_{n=1}^{\infty}\frac{|d|^{n-1}}{4^{n-1}n!(\kappa+1)_{n-1}}\right]
\end{aligned}$$

$$\begin{aligned}
&\geqq 1-\frac{|d|}{4\kappa}\left[1+2\sum_{n=2}^{\infty}\left(\frac{|d|}{8(\kappa+1)}\right)^{n-1}+\sum_{n=1}^{\infty}\left(\frac{|d|}{8(\kappa+1)}\right)^{n-1}\right]\\
&=\frac{32\kappa(\kappa+1)-8(3\kappa+2)\,|d|-|d|^2}{4\kappa\,[8(\kappa+1)-|d|]}>0 \quad \left(z\in\mathbb{U};\ \kappa>\frac{|d|}{8}-1\right),
\end{aligned}$$

provided that

$$32\kappa(\kappa+1)-8(3\kappa+2)\,|d|-|d|^2>0.$$

This proves the inequality (3.7).

From (1.12), we obtain the following derivative formulas (see [3, p. 161]):

$$\left(\frac{\varphi_{\nu,b,d}(z)}{z}\right)'=-\frac{d}{4\kappa}\ \frac{\varphi_{\nu+1,b,d}(z)}{z}$$

and

$$\left(\frac{\varphi_{\nu,b,d}(z)}{z}\right)''=\frac{d^2}{16\kappa(\kappa+1)}\ \frac{\varphi_{\nu+2,b,d}(z)}{z}.$$

Moreover, we can write

$$\left(\frac{\varphi_{\nu,b,d}(z)}{z}\right)''=\frac{\varphi''_{\nu,b,d}(z)}{z}-\frac{2}{z}\left(\frac{\varphi_{\nu,b,d}(z)}{z}\right)'$$

or, equivalently,

$$z^2\varphi''_{\nu,b,d}(z)=z^3\left(\frac{\varphi_{\nu,b,d}(z)}{z}\right)''+2z^2\left(\frac{\varphi_{\nu,b,d}(z)}{z}\right)'.$$

Now, from the last relation, (3.3) and (3.4), we find that

$$\begin{aligned}
\left|z^2\varphi''_{\nu,b,d}(z)\right| &= |z|^3\left|\left(\frac{\varphi_{\nu,b,d}(z)}{z}\right)''\right|+2\,|z|^2\left|\left(\frac{\varphi_{\nu,b,d}(z)}{z}\right)'\right|\\
&\leqq \frac{|d|^2}{16\kappa(\kappa+1)}\left|\frac{\varphi_{\nu+2,b,d}(z)}{z}\right|+2\left|\left(\frac{\varphi_{\nu,b,d}(z)}{z}\right)'\right|\\
&\leqq \frac{|d|^2}{16\kappa(\kappa+1)}\left(\frac{8(\kappa+2)+|d|}{8(\kappa+2)-|d|}\right)+\frac{|d|}{2\kappa}\left(\frac{8(\kappa+1)+|d|}{8(\kappa+1)-|d|}\right)\\
&= \frac{|d|}{2\kappa}\left(\frac{|d|^2}{8(\kappa+1)}\frac{8(\kappa+2)+|d|}{8(\kappa+2)-|d|}+\frac{8(\kappa+1)+|d|}{8(\kappa+1)-|d|}\right),
\end{aligned}$$

which proves the inequality (3.8).

Finally, by combining the inequalities (3.7) and (3.8) of Lemma 7, we immediately obtain the assertion (3.9) of Lemma 7. Our proof of Lemma 7 is thus completed. ■

Our first main result in this section is an application of Lemma 7 and contains sufficient conditions for integral operators of the type (2.2).

Theorem 5. *Let the parameters* $\nu_1, \cdots, \nu_n, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *be so constrained that*

$$\kappa_j := \nu_j + \frac{b+1}{2} > \frac{|d|}{8} - 1 \qquad (j = 1, \cdots, n).$$

Consider the functions $\varphi_{\nu_j,b,d} : \mathbb{U} \to \mathbb{C}$ *defined by* (2.1). *Also let*

$\kappa = \min\{\kappa_1, \cdots, \kappa_n\}$, $\Re(\beta) > 0$, $c \in \mathbb{C} \backslash \{-1\}$ *and* $\alpha_j \in \mathbb{C} \backslash \{0\}$ $(j = 1, \cdots, n)$.

Moreover, suppose that these numbers satisfy the following inequality:

$$|c| + \frac{8(\kappa+1)\,|d| + |d|^2}{32\kappa(\kappa+1) - 4(2\kappa+3)\,|d|} \sum_{j=1}^{n} \frac{1}{|\beta\alpha_j|} \leqq 1.$$

Then the function $\mathcal{F}_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,\beta}(z) : \mathbb{U} \to \mathbb{C}$, *defined by* (2.2), *is in the class* $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Proof. We begin by expressing the function $\mathcal{F}_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n}(z) : \mathbb{U} \to \mathbb{C}$ as follows:

$$\mathcal{F}_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n}(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{\varphi_{\nu_j,b,d}(t)}{t}\right)^{1/\alpha_j} \mathrm{d}t.$$

First of all, we observe that, since $\varphi_{\nu_j,b,d} \in \mathcal{A}$, that is,

$$\varphi_{\nu_j,b,d}(0) = \varphi'_{\nu_j,b,d}(0) - 1 = 0,$$

we have

$$\mathcal{F}_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n} \in \mathcal{A},$$

that is,

$$\mathcal{F}_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n}(0) = \mathcal{F}'_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n}(0) - 1 = 0.$$

On the other hand, it is easy to see that

$$\frac{z\mathcal{F}''_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n}(z)}{\mathcal{F}'_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n}(z)} = \sum_{j=1}^{n} \frac{1}{\alpha_j} \left(\frac{z\varphi'_{\nu_j,b,d}(z)}{\varphi_{\nu_j,b,d}(z)} - 1\right).$$

Thus, by using the inequality (3.6) of Lemma 7 for each ν_j $(j = 1, \cdots, n)$, we obtain

$$\begin{aligned}
\left|\frac{z\mathcal{F}''_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n}(z)}{\mathcal{F}'_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n}(z)}\right| &\leqq \sum_{j=1}^{n} \frac{1}{|\alpha_j|} \left|\frac{z\varphi'_{\nu_j,b,d}(z)}{\varphi_{\nu_j,b,d}(z)} - 1\right| \\
&\leqq \sum_{j=1}^{n} \frac{1}{|\alpha_j|} \left(\frac{8(\kappa_j+1)\,|d| + |d|^2}{32\kappa_j(\kappa_j+1) - 4(2\kappa_j+3)\,|d|}\right) \\
&\leqq \sum_{j=1}^{n} \frac{1}{|\alpha_j|} \left(\frac{8(\kappa+1)\,|d| + |d|^2}{32\kappa(\kappa+1) - 4(2\kappa+3)\,|d|}\right)
\end{aligned}$$

$$\left(z \in \mathbb{U};\ \kappa = \min\{\kappa_1, \cdots, \kappa_n\};\ \kappa_j := \nu_j + \frac{b+1}{2} > \frac{|d|}{8} - 1 \quad (j = 1, \cdots, n)\right).$$

Here we have used the fact that the function $\phi : \left(\frac{|d|}{8} - 1, \infty\right) \to \mathbb{R}$, defined by

$$\phi(x) = \frac{8(x+1)\,|d| + |d|^2}{32x(x+1) - 4(2x+3)\,|d|},$$

is decreasing and, consequently, that

$$\frac{8(\kappa_j+1)\,|d| + |d|^2}{32\kappa_j(\kappa_j+1) - 4(2\kappa_j+3)\,|d|} \leqq \frac{8(\kappa+1)\,|d| + |d|^2}{32\kappa(\kappa+1) - 4(2\kappa+3)\,|d|} \qquad (j = 1, \cdots, n).$$

Finally, by using the triangle inequality and the assertion of Theorem 5, we obtain

$$\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{z\mathcal{F}''_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n}(z)}{\beta\mathcal{F}'_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n}(z)} \right|$$
$$\leqq |c| + \frac{8(\kappa+1)\,|d| + |d|^2}{32\kappa(\kappa+1) - 4(2\kappa+3)\,|d|} \sum_{j=1}^{n} \frac{1}{|\beta\alpha_j|} \leqq 1,$$

which, in view of Lemma 1, implies that $\mathcal{F}_{\nu_1,\cdots,\nu_n,b,d,\alpha_1,\cdots,\alpha_n,\beta} \in \mathcal{S}$. This evidently completes the proof of Theorem 5. ∎

Upon setting

$$\alpha_1 = \cdots = \alpha_n = \alpha$$

in Theorem 5, we immediately arrive at the following application of Theorem 5.

Corollary 7. *Let the parameters* $\nu_1, \cdots, \nu_n, b, c, d, \beta$ *and* $\kappa_j \quad (j = 1, \cdots, n)$ *be prescribed as in Theorem* 5. *Also let*

$$\kappa = \min\{\kappa_1, \cdots, \kappa_n\} \qquad \textit{and} \qquad \alpha \in \mathbb{C} \setminus \{0\}.$$

Moreover, suppose that the functions $\varphi_{\nu_j,b,d} \in \mathcal{A}$ *are defined by* (2.1) *and the following inequality*:

$$|c| + \frac{n}{|\alpha\beta|} \left(\frac{8(\kappa+1)\,|d| + |d|^2}{32\kappa(\kappa+1) - 4(2\kappa+3)\,|d|} \right) \leqq 1$$

holds true. Then the function $\mathcal{F}_{\nu_1,\cdots,\nu_n,b,d,\alpha,\beta}(z) : \mathbb{U} \to \mathbb{C}$, *defined by* (2.5), *is in the class* $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Our second result in this section provides sufficient conditions for the integral operator in (2.6). The key tools in the proof are Lemma 2 and the inequality (3.6) of Lemma 7.

Theorem 6. *Let the parameters* $\nu_1, \cdots, \nu_n, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *be so constrained that*

$$\kappa_j := \nu_j + \frac{b+1}{2} > \frac{|d|}{8} - 1 \qquad (j = 1, \cdots, n).$$

Consider the functions $\varphi_{\nu_j,b,d} : \mathbb{U} \to \mathbb{C}$ *defined by* (2.1). *Also let*

$$\kappa = \min\{\kappa_1, \cdots, \kappa_n\} \qquad \textit{and} \qquad \Re(\gamma) > 0.$$

Moreover, suppose that these numbers satisfy the following inequality:

$$|\gamma| \leqq \frac{1}{n}\left(\frac{32\kappa(\kappa+1) - 4(2\kappa+3)\,|d|}{8(\kappa+1)\,|d| + |d|^2}\right)\Re(\gamma).$$

Then the function $\mathcal{G}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(z) : \mathbb{U} \to \mathbb{C}$, *defined by* (2.6), *is in the class* $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Proof. Let us consider the function $\widetilde{\mathcal{G}}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(z) : \mathbb{U} \to \mathbb{C}$ defined by

$$\widetilde{\mathcal{G}}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(z) = \int_0^z \prod_{j=1}^n \left(\frac{\varphi_{\nu_j,b,d}(t)}{t}\right)^{\gamma} \mathrm{d}t.$$

Observe that $\widetilde{\mathcal{G}}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma} \in \mathcal{A}$, that is, that

$$\widetilde{\mathcal{G}}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(0) = \widetilde{\mathcal{G}}'_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(0) - 1 = 0.$$

On the other hand, by using the inequality (3.6) of Lemma 7, the assertion of Theorem 6 and the fact that

$$\frac{8(\kappa_j+1)\,|d| + |d|^2}{32\kappa_j(\kappa_j+1) - 4(2\kappa_j+3)\,|d|} \leqq \frac{8(\kappa+1)\,|d| + |d|^2}{8\kappa(\kappa+1) - 4(2\kappa+3)\,|d|} \qquad (j = 1, \cdots, n),$$

we obtain

$$\begin{aligned}
\frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)} & \left|\frac{z\widetilde{\mathcal{G}}''_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(z)}{\widetilde{\mathcal{G}}'_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(z)}\right| \\
&\leqq \frac{|\gamma|}{\Re(\gamma)} \sum_{j=1}^n \left|\frac{z\varphi'_{\nu_j,b,d}(z)}{\varphi_{\nu_j,b,d}(z)} - 1\right| \\
&\leqq \frac{n\,|\gamma|}{\Re(\gamma)}\left(\frac{8(\kappa+1)\,|d| + |d|^2}{32\kappa(\kappa+1) - 4(2\kappa+3)\,|d|}\right) \leqq 1 \qquad (z \in \mathbb{U}).
\end{aligned}$$

Now, since

$$\Re(n\gamma + 1) > \Re(\gamma) \qquad (n \in \mathbb{N}),$$

the function $\mathcal{G}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}$ can be rewritten in the form:

$$\mathcal{G}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}(z) = \left[(n\gamma+1)\int_0^z t^{n\gamma}\prod_{j=1}^{n}\left(\frac{\varphi_{\nu_j,b,d}(t)}{t}\right)^{\gamma}\,\mathrm{d}t\right]^{1/(n\gamma+1)}$$

which, in view of Lemma 2, implies that $\mathcal{G}_{\nu_1,\cdots,\nu_n,b,d,n,\gamma}\in\mathcal{S}$. This evidently completes the proof of Theorem 6. ■

Choosing $n=1$ in Theorem 6, we have the following result.

Corollary 8. *Let the parameters* $\nu, b\in\mathbb{R}$ *and* $d\in\mathbb{C}$ *be so constrained that*

$$\kappa := \nu + \frac{b+1}{2} > \frac{|d|}{8} - 1.$$

Consider the function $\varphi_{\nu,b,d}:\mathbb{U}\to\mathbb{C}$ *defined by* (1.11). *Moreover, suppose that* $\Re(\gamma)>0$ *and*

$$|\gamma| \leqq \left(\frac{32\kappa(\kappa+1)-4(2\kappa+3)\,|d|}{8(\kappa+1)\,|d|+|d|^2}\right)\Re(\gamma).$$

Then the function $\mathcal{G}_{\nu,b,d,\gamma}(z):\mathbb{U}\to\mathbb{C}$, *defined by*

$$\mathcal{G}_{\nu,b,d,n,\gamma}(z) = \left[(\gamma+1)\int_0^z (\varphi_{\nu,b,d}(t))^{\gamma}\,\mathrm{d}t\right]^{1/(\gamma+1)},$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

The following result contains another set of sufficient conditions for integral operators of the type (2.8). The key tools in the proof are Lemma 1 and the inequality (3.9) of Lemma 7.

Theorem 7. *Let the parameters* $\nu_1,\cdots,\nu_n, b\in\mathbb{R}$ *and* $d\in\mathbb{C}$ *be so constrained that*

$$\kappa_j := \nu_j + \frac{b+1}{2} > \frac{|d|}{8} - 1 \qquad (j=1,\cdots,n).$$

Consider the functions $\varphi_{\nu_j,b,d}:\mathbb{U}\to\mathbb{C}$ *defined by* (2.1). *Also let*

$$\kappa=\min\{\kappa_1,\cdots,\kappa_n\},\ \ \Re(\mu)>0,\ \ c\in\mathbb{C}\backslash\{-1\}\quad \textit{and}\quad \delta_j\in\mathbb{C}\quad (j=1,\cdots,n).$$

Moreover, suppose that these numbers satisfy the following inequality:

$$\frac{|d|}{2}\left(\frac{64(\kappa+1)^2\,[8(\kappa+2)-|d|]+128(\kappa+1)(\kappa+2)-[8(\kappa+2)+|d|]\,|d|^2}{2(\kappa+1)\,[8(\kappa+1)-|d|]\,\{16(\kappa+1)(2\kappa-|d|)-|d|\,(4\kappa+|d|)\}}\right)\sum_{j=1}^{n}\frac{|\delta_j|}{\mu} \leqq 1-|c|\,.$$

Then the function $\mathcal{H}_{\nu_1,\cdots,\nu_n,\delta_1,\cdots,\delta_n,b,d,\mu}(z):\mathbb{U}\to\mathbb{C}$, *defined by* (2.8), *is in the class* $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Proof. The proof of Theorem 7 is much akin to that of Theorem 3, so we omit the details involved in this case. ■

By setting $n = 1$ in Theorem 7, we immediately obtain the following result.

Corollary 9. *Let the parameters* $\nu, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *be so constrained that*

$$\kappa := \nu + \frac{b+1}{2} > \frac{|d|}{8} - 1.$$

Consider the function $\varphi_{\nu,b,d} : \mathbb{U} \to \mathbb{C}$ *defined by* (1.11). *Also let*

$$\Re(\mu) > 0, \;\; c \in \mathbb{C} \setminus \{-1\} \qquad \textit{and} \qquad \delta \in \mathbb{C}.$$

Moreover, suppose that these numbers satisfy the following inequality:

$$\frac{|\delta d|}{2\,|\mu|}\left(\frac{64(\kappa+1)^2\,[8(\kappa+2)-|d|]+128(\kappa+1)(\kappa+2)-[8(\kappa+2)+|d|]\,|d|^2}{2(\kappa+1)\,[8(\kappa+1)-|d|]\,\{16(\kappa+1)(2\kappa-|d|)-|d|\,(4\kappa+|d|)\}}\right) \leqq 1-|c|\,.$$

Then the function $\mathcal{H}_{\nu,\delta,b,d,\mu}(z) : \mathbb{U} \to \mathbb{C}$, *defined by* (2.9), *is in the class* $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

By applying Lemma 3 and the inequality (3.7) of Lemma 7, we easily get the following result.

Theorem 8. *Let the parameters* $\nu, b \in \mathbb{R}$ *and* $d, \lambda \in \mathbb{C}$ *be so constrained that*

$$\kappa := \nu + \frac{b+1}{2} > \frac{|d|}{8} - 1.$$

Consider the generalized Bessel function $\varphi_{\,\nu,b,d}$ *defined by* (1.11). *If* $\Re(\lambda) \geqq 1$ *and*

$$|\lambda| \leqq \frac{6\sqrt{3}\kappa\,[8(\kappa+1)-|d|]}{32\kappa(\kappa+1)+4(3\kappa+4)\,|d|+|d|^2},$$

then the function $\mathcal{Q}_{\nu,b,d,\lambda} : \mathbb{U} \to \mathbb{C}$, *defined by* (2.10), *is in the class* $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Remark 7. In their special cases, Theorems 5 to 8 would readily yield the the corresonding results for the Bessel function $(b = d = 1)$, the modified Bessel function $(b = -d = 1)$ and the spherical Bessel function $(b-1=d=1)$ as asserted by Corollary 4, Corollary 5 and Corollary 6, respectively.

We now consider another set of inequalities (see [8, p. 12]):

(3.10) $(\kappa)_n > \kappa(\kappa+\alpha_0)^{n-1}$ and $n! > (1+\alpha_0)^{n-1}$ $(\kappa > 0;\ n \in \mathbb{N} \setminus \{1,2\})$, where

$$\alpha_0 \cong 1.302775637\cdots \tag{3.11}$$

is the greatest root of the following quadratic equation:

$$\alpha^2 + \alpha - 3 = 0. \tag{3.12}$$

Thus, by using the inequalities in (3.10) and the same steps as in the proofs of Lemma 5 and Lemma 7, we obtain some improved versions of Lemma 5 and Theorems 1 to 4.

Lemma 8. (see [8]). *If the parameters* $\nu, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *are so constrained that*

$$\kappa > \max\left\{0, \frac{|d|}{4(1+\alpha_0)} - \alpha_0\right\},$$

where α_0 *is given by* (3.11) *and* (3.12), *then the function*

$$\frac{\varphi_{\nu,b,d}(z)}{z} : \mathbb{U} \to \mathbb{C}$$

defined by (1.12) *satisfies the following inequalities*:

$$1 - \Phi(\kappa) < \left|\frac{\varphi_{\nu,b,d}(z)}{z}\right| \qquad (z \in \mathbb{U}) \tag{3.13}$$

and

$$\left|\left(\frac{\varphi_{\nu,b,d}(z)}{z}\right)'\right| < \frac{|d|}{4\kappa}\left[1 + \Phi(\kappa+1)\right] z \in \mathbb{U}, \tag{3.14}$$

where $\Phi(\kappa)$ *is defined by*

$$\Phi(\kappa) = -\frac{|d|^2}{16(1+\alpha_0)\kappa(\kappa+\alpha_0)} + \frac{|d|^2}{32\kappa(\kappa+1)} + \frac{|d|}{\kappa}\left(\frac{(1+\alpha_0)(\kappa+\alpha_0)}{4(1+\alpha_0)(\kappa+\alpha_0) - |d|}\right). \tag{3.15}$$

Lemma 9. *If the parameters* $\nu, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *are so constrained that*

$$\kappa > \max\left\{0, \frac{|d|}{4(1+\alpha_0)} - \alpha_0\right\},$$

then the function $\varphi_{\nu,b,d} : \mathbb{U} \to \mathbb{C}$ *defined by* (1.12) *satisfies the following inequalities*:

$$\left|\varphi'_{\nu,b,d}(z) - \frac{\varphi_{\nu,b,d}(z)}{z}\right| < \frac{|d|}{4\kappa}\left[1 + \Phi(\kappa+1)\right] \quad (z \in \mathbb{U}), \tag{3.16}$$

$$\left|\frac{z\varphi'_{\nu,b,d}(z)}{\varphi_{\nu,b,d}(z)} - 1\right| < \frac{|d|}{4\kappa}\left(\frac{1+\Phi(\kappa+1)}{1-\Phi(\kappa)}\right) \quad (z \in \mathbb{U}), \tag{3.17}$$

$$\frac{|d|}{4\kappa}\left[1-\Phi(\kappa+1)\right]+1-\Phi(\kappa) \leqq \left|z\varphi'_{\nu,b,d}(z)\right| \tag{3.18}$$

$$\leqq \frac{|d|}{4\kappa}\left[1+\Phi(\kappa+1)\right]+1+\Phi(\kappa) \quad (z \in \mathbb{U}), \tag{3.19}$$

$$\left|z^2\varphi''_{\nu,b,d}(z)\right| \leqq \frac{|d|}{2\kappa}\left(\frac{|d|}{8(\kappa+1)}\left[1+\Phi(\kappa+2)\right]+1+\Phi(\kappa+1)\right) \quad (z \in \mathbb{U}) \tag{3.20}$$

and

$$\left|\frac{z\varphi''_{\nu,b,d}(z)}{\varphi'_{\nu,b,d}(z)}\right| \leqq \frac{\frac{|d|}{2\kappa}\left(\frac{|d|}{8(\kappa+1)}\left[1+\Phi(\kappa+2)\right]+1+\Phi(\kappa+1)\right)}{\frac{|d|}{4\kappa}\left[1-\Phi(\kappa+1)\right]+1-\Phi(\kappa)} \quad (z \in \mathbb{U}), \tag{3.21}$$

where $\Phi(\kappa)$ *defined by* (3.15).

Proof. Our proof of Lemma 9 is very similar to that of Lemma 7. It is based upon the known inequalities in (3.10) and Lemma 8. We choose to omit the details involved. ■

By the aid of Lemmas 1 to 3 and Lemma 9, we prove the following resuts (Theorem 9 to 12). The proofs of Theorems 9 to 12 are very similar to the proofs of Theorems 1 to 4, respectively, so we omit the details involved in these cases.

Theorem 9. *Let the parameters* $\nu, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *be so constrained that*

$$\kappa := \nu + \frac{b+1}{2} > \frac{|d|}{4(1+\alpha_0)} - \alpha_0,$$

where α_0 *is given by* (3.11) *and* (3.12).
Consider the function $\varphi_{\nu,b,d} : \mathbb{U} \to \mathbb{C}$ *defined by* (1.11). *Also let*

$$\Re(\beta) > 0, \quad c \in \mathbb{C} \setminus \{-1\} \qquad \textit{and} \qquad \alpha \in \mathbb{C} \setminus \{0\}.$$

Moreover, suppose that these numbers satisfy the following inequality:

$$|c| + \frac{|d|}{4\kappa\,|\beta\alpha|}\left(\frac{1+\Phi(\kappa+1)}{1-\Phi(\kappa)}\right) \leqq 1.$$

where $\Phi(\kappa)$ *is defined by* (3.15). *Then the function* $\mathcal{F}_{\nu,b,d,\alpha,\beta}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{F}_{\nu,b,d,\alpha,\beta}(z) = \left[\beta \int_0^z t^{\beta-1}\left(\frac{\varphi_{\nu,b,d}(t)}{t}\right)^{1/\alpha} \mathrm{d}t\right]^{1/\beta},$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Theorem 10. *Let the parameters* $\nu, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *be so constrained that*

$$\kappa := \nu + \frac{b+1}{2} > \frac{|d|}{4(1+\alpha_0)} - \alpha_0,$$

where α_0 *is given by* (3.11) *and* (3.12). *Consider the function* $\varphi_{\nu,b,d} : \mathbb{U} \to \mathbb{C}$ *defined by* (1.11). *Also let* $\Re(\gamma) > 0$. *Moreover, suppose that these numbers satisfy the following inequality*:

$$|\gamma d| \leqq \left(\frac{4\kappa \, [1 - \Phi(\kappa)]}{1 + \Phi(\kappa+1)} \right) \Re(\gamma).$$

where $\Phi(\kappa)$ *is defined by* (3.15). *Then the function* $\mathcal{G}_{\nu,b,d,\gamma}(z) : \mathbb{U} \to \mathbb{C}$, *defined by* (2.7), *is in the class* $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Theorem 11. *Let the parameters* $\nu, b \in \mathbb{R}$ *and* $d \in \mathbb{C}$ *are so constrained that*

$$\kappa := \nu + \frac{b+1}{2} > \frac{|d|}{4(1+\alpha_0)} - \alpha_0,$$

where α_0 *is given by* (3.11) *and* (3.12). *Consider the function* $\varphi_{\nu,b,d} : \mathbb{U} \to \mathbb{C}$ *defined by* (1.11). *Also let*

$$\Re(\mu) > 0, \;\; c \in \mathbb{C} \setminus \{-1\} \qquad \textit{and} \qquad \delta \in \mathbb{C}.$$

Moreover, suppose that these numbers are satisfy the following inequality:

$$|c| + \left| \frac{\delta}{\mu} \right| \frac{\frac{|d|}{2\kappa} \left(\frac{|d|}{8(\kappa+1)} \, [1 + \Phi(\kappa+2)] + 1 + \Phi(\kappa+1) \right)}{\frac{|d|}{4\kappa} \, [1 - \Phi(\kappa+1)] + 1 - \Phi(\kappa)} \leqq 1,$$

where $\Phi(\kappa)$ *is defined by* (3.15). *Then the function* $\mathcal{H}_{\nu,\delta,b,d,\mu}(z) : \mathbb{U} \to \mathbb{C}$, *defined by* (2.9), *is in the class* $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

By applying Lemma 3 and the inequality (3.18) of Lemma 9, we arrive at the following result.

Theorem 12. *Let the parameters* $\nu, b \in \mathbb{R}$ *and* $d, \lambda \in \mathbb{C}$ *be so constrained that*

$$\kappa := \nu + \frac{b+1}{2} > \frac{|d|}{4(1+\alpha_0)} - \alpha_0,$$

where α_0 *is given by* (3.11) *and* (3.12). *Consider the generalized Bessel function* $\varphi_{\nu,b,d}$ *defined by* (1.11) . *If* $\Re(\lambda) \geqq 1$ *and*

$$|\lambda| \leqq \frac{6\sqrt{3}\kappa}{|d| \, [1 + \Phi(\kappa+1)] + 4\kappa \, [1 + \Phi(\kappa)]},$$

then the function $\mathcal{Q}_{\nu,b,d,\lambda} : \mathbb{U} \to \mathbb{C}$, *defined by* (2.10), *is in the class* $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Remark 8. By suitably specializing Theorems 9 to 12, we can obtain the corresponding results for the Bessel function $(b = d = 1)$, for the modified Bessel function $(b = -d = 1)$ and for the spherical Bessel function $(b - 1 = d = 1)$ as asserted by Corollary 4, Corollary 5 and Corollary 6, respectively.

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