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SOME τ-EXTENSIONS OF LAURICELLA FUNCTIONS OF SEVERAL VARIABLES

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Abstract. Motivated mainly by certain interesting extensions of the τ-hypergeometric function defined by Virchenko *et al.* [11] and some τ-Appell's function introduced by Al-Shammery and Kalla [1], we introduce here the τ-Lauricella functions $F_A^{(n),\tau_1,\ldots,\tau_n}$, $F_B^{(n),\tau_1,\ldots,\tau_n}$ and $F_D^{(n),\tau_1,\ldots,\tau_n}$ and the confluent forms $\Phi_2^{(n),\tau_1,\ldots,\tau_n}$ and $\Phi_D^{(n),\tau_1,\ldots,\tau_n}$ of n variables. We then systematically investigate their various integral representations of each of these τ-Lauricella functions including their generating functions. Various (known or new) special cases and consequences of the results presented here are also considered.

1. Introduction, definitions and preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}^-$ and $\mathbb{C}$ denote the sets of positive integers, negative integers and complex numbers, respectively,

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \qquad \text{and} \qquad \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}.$$

The classical Gauss's hypergeometric function (see, *e.g.*, [7, 9]), is defined by

$$(1) \qquad {}_2F_1\left[\begin{matrix} a,\, b; \\ c; \end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\frac{z^n}{n!} = {}_2F_1(a,\, b;\, c;\, z),$$

$$(|z| < 1\,;\, \Re(c) > \Re(b) > 0).$$

Here, and in what follows, $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ denotes the Pochhammer symbol (or the shifted factorial) which is defined (in general) by

$$(2) \quad (\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}$$

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it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [9, p. 21 *et seq.*]), $\mathbb{N}$ being (as above) the set of positive integers.

The generalized hypergeometric function [7, p. 487, Eq. (15)] with p numerator and q denominator is defined as follows:

$$(3) \qquad {}_pF_q\left[\begin{array}{c}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{array} z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\,\frac{z^n}{n!}.$$

An interesting generalization of the generalized hypergeometric function ${}_pF_q$ is due to Wright ${}_p\Psi_q$(see, *e.g.*, [12, 13]):

$$(4) \qquad \begin{aligned}{}_p\Psi_q(z) &= {}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),(a_2,A_2),\ldots,(a_p,A_p);\\ (b_1,B_1),(b_2,B_2),\ldots,(b_p,B_q);\end{array} z\right]\\ &=\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+A_jk)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\,\frac{z^k}{k!},\end{aligned}$$

where the coefficients $A_i(i=1,\ldots,p)$ and $B_j(j=1,\ldots,q)$ are positive real numbers such that $1+\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i\geq 0$.

This generalized form of the hypergeometric function has been investigated by M. Dotsenko [3] and V. Malovichko [6]. One of the interesting special case considered in [3], has the form:

$$(5) \qquad {}_2R_1^{\omega,\mu}(z)= {}_2R_1\,(a,\,b;\,c;\,\omega,\,\mu;\,x)=\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}(a)_n\,\frac{\Gamma(b+\frac{\omega}{\mu}n)}{\Gamma(c+\frac{\omega}{\mu}n)}\,\frac{z^n}{n!},$$

$$(|z|<1\,;\,\Re(c)>\Re(b)>0)\,.$$

Further, Virchenko *et al.* [11, p. 90, Eq. (5)] have considered the following τ-hypergeometric function by letting $\frac{\omega}{\mu}=\tau$ as:

$$(6) \qquad {}_2R_1^{\tau}(z)= {}_2R_1\,(a,\,b;\,c;\,\tau;\,x)=\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}(a)_n\,\frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\,\frac{z^n}{n!},$$

$$(\tau>0,\ |z|<1\,;\,\Re(c)>\Re(b)>0)\,.$$

The τ-confluent hypergeometric function is introduced by N. Virchenko [10] as:

$$(7) \qquad {}_1\Phi_1^{\tau}(z)= {}_1\Phi_1^{\tau}\,(b;\,c;\,;\,x)=\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\,\frac{z^n}{n!},$$

$$(\tau>0,\ \Re(c)>\Re(b)>0)\,.$$

The special case when $\tau=1$, (6) and (7) yields the familiar Gauss's hypergeometric and confluent hypergeometric function [7].

Moreover, Al-Shammery and Kalla [1] introduced and studied various properties of some τ-Appell's hypergeometric functions as:

$$(8) \qquad F_1^{\tau_1,\tau_2}[\alpha,\beta_1,\beta_2;\gamma;x_1,x_2]$$

$$:= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{m_1,,m_2=0}^{\infty} \frac{\Gamma(\alpha+\tau_1 m_1+\tau_2 m_2)(\beta_1)_{m_1}(\beta_2)_{m_2}}{\Gamma(\gamma+\tau_1 m_1+\tau_2 m_2)} \frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!},$$

$$\big(\tau_1,\tau_2>0;\ \max\{\mid x_1\mid,\mid x_2\mid\}<1\big).$$

(9) $F_2^{\tau_1,\tau_2}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2]$

$$:= \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\beta_1)\Gamma(\beta_2)} \sum_{m_1,m_2=0}^{\infty} \frac{(\alpha)_{m_1+m_2}\Gamma(\beta_1+\tau_1 m_1)\Gamma(\beta_2+\tau_2 m_2)}{\Gamma(\gamma_1+\tau_1 m_1)\Gamma(\gamma_2+\tau_2 m_2)} \frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!},$$

$$\big(\tau_1,\tau_2>0;\ |x_1|+|x_2|<1\big).$$

(10)

$F_3^{\tau_1,\tau_2}[\alpha_1,\alpha_2,\beta_1,\beta_2;\gamma;x_1,x_2]$

$$:= \frac{\Gamma(\gamma)}{\Gamma(\beta_1)\Gamma(\beta_2)} \sum_{m_1,m_2=0}^{\infty} \frac{(\alpha_1)_{m_1}(\alpha_2)_{m_2}\Gamma(\beta_1+\tau_1 m_1)\Gamma(\beta_2+\tau_2 m_2)}{\Gamma(\gamma+\tau_1 m_1+\tau_2 m_2)} \frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!},$$

$$\big(\tau_1,\tau_2>0;\ \max\{\mid x_1\mid,\mid x_2\mid\}<1\big).$$

In a sequel to the aforementioned work by Virchenko *et al.* [11], Al-Shammery and Kalla [1] and L. Galué [4] introduced the some τ-Appell hypergeometric functions $F_1^{\tau_1,\tau_2}$, $F_2^{\tau_1,\tau_2}$ and $F_3^{\tau_1,\tau_2}$ and some τ-Humbert's functions in two variables, respectively and investigated their various properties including integral representations. Motivated essentially by the demonstrated potential for applications of these τ-hypergeometric functions ${}_2R_1^{\tau}(z)$ and ${}_1\Phi_1^{\tau}(z)$, and the τ-Appell hypergeometric functions $F_1^{\tau_1,\tau_2}$, $F_2^{\tau_1,\tau_2}$ and $F_3^{\tau_1,\tau_2}$ in many diverse areas of mathematical, physical, engineering and statistical sciences (see, for details, [1, 4, 5, 11] and the references cited therein), we aim here at systematically investigating the family of some τ-Lauricella's functions $F_A^{(n),\tau_1,\ldots,\tau_n}$, $F_B^{(n),\tau_1,\ldots,\tau_n}$ and $F_D^{(n),\tau_1,\ldots,\tau_n}$ of n variables. For each of these τ-multivariate hypergeometric functions, we derive various integral representations of Eulerian and Laplace type. We then obtain generating functions for these τ-Lauricella hypergeometric functions. Some interesting special cases (known or new) of our main results are also pointed out.

2. The τ-Lauricella functions of n variables

In this section, we consider the extensions of three Lauricella hypergeometric functions $F_A^{(n)}$, $F_D^{(n)}$ and $F_D^{(n)}$ of n variables in terms of additional parameters $\tau_1,\ldots,\tau_n$ as follows: For α, $\beta_1,\ldots,\ \beta_n\in\mathbb{C}$ and $\gamma_1,\ldots,\gamma_n\in\mathbb{C}\setminus\mathbb{Z}_0^-$, we have

(11) $F_A^{(n),\tau_1,\ldots,\tau_n}[\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n]$

$$:= \frac{\Gamma(\gamma_1)\cdots\Gamma(\gamma_n)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}\Gamma(\beta_1+\tau_1 m_1)\cdots\Gamma(\beta_n+\tau_n m_n)}{\Gamma(\gamma_1+\tau_1 m_1)\cdots\Gamma(\gamma_n+\tau_n m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!},$$

$$\big(\tau_1, \ldots, \tau_n > 0;\ |x_1| + \cdots + |x_n| < 1\big),$$

(12)

$$F_B^{(n),\tau_1,\ldots,\tau_n}[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n]$$
$$:= \frac{\Gamma(\gamma)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}\Gamma(\beta_1+\tau_1 m_1)\cdots\Gamma(\beta_n+\tau_n m_n)}{\Gamma(\gamma+\tau_1 m_1+\cdots+\tau_n m_n)} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!},$$

$$\big(\tau_1, \ldots, \tau_n > 0;\ \max\{|\ x_1\ |, \ldots, |\ x_n\ |\} < 1\big),$$

(13)

$$F_D^{(n),\tau_1,\ldots,\tau_n}[\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n]$$
$$:= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{\Gamma(\alpha+\tau_1 m_1+\cdots+\tau_n m_n)(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{\Gamma(\gamma+\tau_1 m_1+\cdots+\tau_n m_n)} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!},$$

$$\big(\tau_1, \ldots, \tau_n > 0;\ \max\{|\ x_1\ |, \ldots, |\ x_n\ |\} < 1\big).$$

The confluent forms of $F_B^{(n),\tau_1,\ldots,\tau_n}$ and $F_D^{(n),\tau_1,\ldots,\tau_n}$ are given as:

(14)

$$\Phi_2^{(n),\tau_1,\ldots,\tau_n}[\beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n]$$
$$:= \frac{\Gamma(\gamma)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{\Gamma(\beta_1+\tau_1 m_1)\cdots\Gamma(\beta_n+\tau_n m_n)}{\Gamma(\gamma+\tau_1 m_1+\cdots+\tau_n m_n)} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!},$$

$$\big(\tau_1, \ldots, \tau_n > 0;\ |\ x_1\ | < \infty, \ldots, |\ x_n\ | < \infty\big)$$

and

(15)

$$\Phi_D^{(n),\tau_1,\ldots,\tau_n}[\alpha, \beta_1, \ldots, \beta_{n-1}; \gamma; x_1, \ldots, x_n]$$
$$:= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{\Gamma(\alpha+\tau_1 m_1+\cdots+\tau_n m_n)(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{\Gamma(\gamma+\tau_1 m_1+\cdots+\tau_n m_n)} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!},$$

$$\big(\tau_1, \ldots, \tau_n > 0;\ \max\{|\ x_1\ |, \ldots, |\ x_n\ |\} < 1\big),$$

where we have use the *interdependence formula*:

(16)

$$\Phi_2^{(n),\tau_1,\ldots,\tau_n}[\beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n]$$
$$:= \lim_{\min\{|\alpha_1|,\ldots,|\alpha_n|\}\to\infty} F_B^{(n),\tau_1,\ldots,\tau_n}[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n; \gamma; \tfrac{x_1}{\alpha_1}, \ldots, \tfrac{x_n}{\alpha_n}]$$

and

(17)

$$\Phi_D^{(n),\tau_1,\ldots,\tau_n}[\alpha, \beta_1, \ldots, \beta_{n-1}; \gamma; x_1, \ldots, x_n]$$
$$:= \lim_{|\beta_n|\to\infty} F_D^{(n),\tau_1,\ldots,\tau_n}[\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_{n-1}, \frac{x_n}{\beta_n}]$$

respectively.

Remark 1. The special case of (11), (12) and (13) when $n = 2$ are seen to yield the known extension of the τ-Appell's hypergeometric functions (9), (10) and (8) as $F_2^{\tau_1,\tau_2}$, $F_3^{\tau_1,\tau_2}$ and $F_1^{\tau_1,\tau_2}$, respectively.

Also for $n = 2$ in (14) and (15), yields the confluent forms of τ-Appell's hypergeometric functions [1].

3. Integral representations of τ-Lauricella functions

In this section, we present certain integral representations of the τ-Lauricella hypergeometric functions $F_A^{(n),\tau_1,\ldots,\tau_n}$ $F_B^{(n),\tau_1,\ldots,\tau_n}$ and $F_D^{(n),\tau_1,\ldots,\tau_n}$ of n variables. We also obtain the integral representations of the confluent form of τ-Lauricella hypergeometric functions $\Phi_2^{(n),\tau_1,\ldots,\tau_n}$ and $\Phi_D^{(n),\tau_1,\ldots,\tau_n}$.

Theorem 1. *The following integral representation for* $F_A^{(n)}$ *in* (11) *holds true*:

$$\begin{aligned} (18)\qquad & F_A^{(n),\tau_1,\ldots,\tau_n}[\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n] \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-t}\,t^{\alpha-1}\;{}_1\Phi_1^{\tau_1}\left[\begin{matrix}\beta_1;\\ \gamma_1;\end{matrix}\; x_1 t\right]\cdots\;{}_1\Phi_1^{\tau_n}\left[\begin{matrix}\beta_n;\\ \gamma_n;\end{matrix}\; x_n t\right]dt \\ & \big(\tau_1,\ldots,\tau_n>0;\ \Re(x_1+\cdots+x_n)<1,\ \Re(\alpha)>0\big). \end{aligned}$$

Proof. Using the integral representation of the Pochhammer symbol

$$(\alpha)_{m_1+\cdots+m_n}$$

and the definition of τ-confluent hypergeometric function (7) in (11), we are led to the desired result (18) asserted by Theorem 1. □

Theorem 2. *The following n-tuple integral representation for* $F_A^{(n)}$ *in* (11) *holds true*:

$$\begin{aligned} (19)\qquad & F_A^{(n),\tau_1,\ldots,\tau_n}[\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n] \\ &= \frac{1}{B(\beta_1,\gamma_1-\beta_1)\cdots B(\beta_n,\gamma_n-\beta_n)} \\ &\quad\cdot\int_0^1\cdots\int_0^1 t_1^{\beta_1-1}\cdots t_n^{\beta_n-1}(1-t_1)^{\gamma_1-\beta_1-1}\cdots(1-t_n)^{\gamma_n-\beta_n-1} \\ &\quad\cdot\ (1-x_1t_1^{\tau_1}-\cdots-x_nt_n^{\tau_n})^{-\alpha}\,dt_1\cdots dt_n \\ & \big(\tau_1,\ldots,\tau_n>0;\ \Re(\gamma_j)>\Re(\beta_j)>0\ \ (j=1,\ldots,n)\big). \end{aligned}$$

Proof. Considering the following elementary identity involving the Beta function $B(\beta,\gamma)$:

$$(20)\qquad \frac{(\beta)_\nu}{(\gamma)_\nu}=\frac{B(\beta+\nu,\gamma-\beta)}{B(\beta,\gamma-\beta)}=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta+\nu-1}(1-t)^{\gamma-\beta-1}\,dt$$

$$\big(\Re(\gamma)>\Re(\beta)>\max\{0,-\Re(\nu)\}\big)$$

in (11) and using the elementary series identity [9, p. 52, Eq. 1.6(3)]:

$$\sum_{m_1,\ldots,m_n=0}^{\infty} \Omega\left(m_1+\cdots+m_n\right) \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} = \sum_{m=0}^{\infty} \Omega(m) \frac{(x_1+\cdots+x_m)^m}{m!}, \tag{21}$$

we get the desired multiple integral representation (19) asserted by Theorem 2. □

Theorem 3. *The following n-tuple integral representation for $F_B^{(n)}$ in* (12) *holds true*:

$$\begin{aligned} &F_B^{(n),\tau_1,\ldots,\tau_n}[\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n] \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma-\beta_1-\cdots-\beta_n)} \\ &\quad\cdot\int\cdots\int t_1^{\beta_1-1}\cdots t_n^{\beta_n-1}(1-t_1-\cdots-t_n)^{\gamma-\beta_1-\cdots-\beta_n-1} \\ &\quad\cdot\ (1-x_1t_1^{\tau_1})^{-\alpha_1}\cdots(1-x_nt_n^{\tau_n})^{-\alpha_n}\,dt_1\,\cdots\,dt_n \end{aligned} \tag{22}$$

$$\big(t_j \geqq 0,\ t_1+\cdots+t_n \leqq 1,\ \tau_j>0;\ \Re(\beta_j)>0\ \ (j=1,\ldots,n),\ \Re(\gamma-\beta_1-\cdots-\beta_n)>0\big).$$

Proof. Using the following Eulerian elementary integral [9, p. 275, Eq. (3)]:

$$\int\cdots\int t_1^{\beta_1-1}\cdots t_n^{\beta_n-1}(1-t_1-\cdots-t_n)^{\gamma-\beta_1-\cdots-\beta_n-1}\,dt_1\,\cdots\,dt_n = \frac{\Gamma(\gamma)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\beta_1+\cdots+\beta_n+\gamma)} \tag{23}$$

$$\big(t_j \geqq 0,\ t_1+\cdots+t_n \leqq 1;\ \Re(\beta_j)>0\ \ (j=1,\ldots,n),\ \Re(\gamma)>0\big)$$

in (12), we get the desired multiple integral representation (22) asserted by Theorem 3. □

Theorem 4. *The following n-tuple integral representation for $F_B^{(n)}$ in* (12) *holds true*:

$$\begin{aligned} &F_B^{(n),\tau_1,\ldots,\tau_n}[\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n] \\ &= \frac{1}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}\int_0^\infty\cdots\int_0^\infty e^{-t_1-\cdots-t_n}\, t_1^{\alpha_1-1}\cdots t_n^{\alpha_n-1}\,\Phi_2^{(n),\tau_1,\ldots,\tau_n} \\ &\quad\cdot[\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n]\,dt_1\,\cdots\,dt_n \end{aligned} \tag{24}$$

$$\big(\tau_1,\ldots,\tau_n>0,\ \min\{\Re(\alpha_1),\ldots,\Re(\alpha_n)\}>0\big).$$

Proof. Using the integral representation of Pochhammer symbols

$$(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}$$

in the definition (12) and using the definition of confluent form of τ-Lauricella function (14), we are led to the desired n-tuple integral representation (24) asserted by Theorem 4. □

Theorem 5. *The following integral representation for $F_D^{(n)}$ in* (13) *holds true*:

$$\begin{aligned}(25)\qquad & F_D^{(n),\tau_1,\ldots,\tau_n}[\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n] \\ &= \frac{1}{B(\beta_1,\gamma_1-\beta_1)\cdots B(\beta_n,\gamma_n-\beta_n)} \\ &\quad \cdot\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-x_1t^{\tau_1})^{-\beta_1}\cdots(1-x_nt^{\tau_n})^{-\beta_n}dt \\ &\big(\tau_1,\ldots,\tau_n>0,\ \max\{|\arg(1-x_1)|,\ldots,|\arg(1-x_n)|\}<\pi; \\ &\quad \Re(\gamma_j)>\Re(\beta_j)>0\ \ (j=1,\ldots,n)\big).\end{aligned}$$

Proof. Using the elementary identity (20) involving the Beta function $B(\alpha,\gamma)$ in (13) and employing the generalized binomial expansion

$$(1-zt)^{-\beta}=\sum_{n=0}^{\infty}(\beta)_n\,\frac{(zt)^n}{n!},$$

we get the desired multiple integral representation (25) asserted by Theorem 5. □

Theorem 6. *The following integral representation for $F_D^{(n)}$ in* (13) *holds true*:

$$\begin{aligned}(26)\qquad & F_D^{(n),\tau_1,\ldots,\tau_n}[\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n] \\ &= \frac{1}{\Gamma(\beta_n)}\int_0^\infty e^{-t}t^{\beta_n-1}\ \Phi_D^{(n),\tau_1,\ldots,\tau_n}[\alpha,\beta_1,\ldots,\beta_{n-1};\gamma;x_1t,\ldots,x_nt]dt \\ &\qquad\big(\tau_1,\ldots,\tau_n>0,\ \Re(x_n)<1;\ \Re(\beta_n)>0\big).\end{aligned}$$

Proof. Using the integral representation of the Pochhammer symbol $(\beta_n)_{m_n}$ in (13) and using the definition of confluent form of τ-Lauricella Functions (15), we are led to the desired result. □

Theorem 7. *The following n-tuple integral representation for $\Phi_2^{(n)}$ in* (14) *holds true*:

$$\begin{aligned}(27)\qquad & \Phi_2^{(n),\tau_1,\ldots,\tau_n}[\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n] \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma-\beta_1-\cdots-\beta_n)} \\ &\quad\cdot\int\cdots\int t_1^{\beta_1-1}\cdots t_n^{\beta_n-1}(1-t_1-\cdots-t_n)^{\gamma-\beta_1-\cdots-\beta_n-1}\end{aligned}$$

$$\cdot\ e^{-x_1 t^{\tau_1}+\cdots+e^{-x_n t^{\tau_n}}}\ dt_1 \cdots dt_n$$

$$\big(t_j \geqq 0,\ t_1+\cdots+t_n \leqq 1,\ \tau_j > 0;$$
$$\Re(\beta_j) > 0\ \ (j=1,\ldots,n),\ \Re(\gamma-\beta_1-\cdots-\beta_n) > 0\big).$$

Proof. Using the Eulerian elementary integral (23) in (14), and employing the expansion

$$e^z = \sum_{n=0}^{\infty} \frac{(z)^n}{n!}$$

we get the desired multiple integral representation (27) asserted by Theorem 7. □

Theorem 8. *The following integral representation for* $\Phi_D^{(n)}$ *in* (15) *holds true*:

$$\Phi_D^{(n),\tau_1,\ldots,\tau_n}[\alpha,\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n] \tag{28}$$
$$= \frac{1}{B(\beta_1,\gamma_1-\beta_1)\cdots B(\beta_n,\gamma_{n-1}-\beta_{n-1})}$$
$$\cdot \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-x_1 t^{\tau_1})^{-\beta_1}\cdots(1-x_{n-1}t^{\tau_{n-1}})^{-\beta_{n-1}} e^{-x_n t^{\tau_n}}\, dt$$
$$\big(\tau_1,\ldots,\tau_n > 0,\ \max\{|\arg(1-x_1)|,\ldots,|\arg(1-x_n)|\} < \pi;$$
$$\Re(\gamma_j) > \Re(\beta_j) > 0\ \ (j=1,\ldots,n)\big).$$

Proof. Using the elementary identity (20) involving the Beta function $B(\alpha,\gamma)$ in (13) and employing the binomial expansion

$$(1-z)^{-\beta} = \sum_{n=0}^{\infty} (\beta)_n \frac{(z)^n}{n!} \quad \text{and} \quad e^z = \sum_{n=0}^{\infty} \frac{(z)^n}{n!}$$

we get the desired multiple integral representation (28) asserted by Theorem 8. □

In particular, when $n = 2$ in the integral representations (26), (18) and (24), we can deduce presumably new integral representations for the τ-Appell's hypergeometric functions $F_1^{\tau_1,\tau_2}$, $F_2^{\tau_1,\tau_2}$ and $F_3^{\tau_1,\tau_2}$, respectively which are asserted by Corollary 1 below.

Corollary 1. *Each of the following integral representations for* τ*-Appell's hypergeometric functions hold true*:

$$F_1^{\tau_1,\tau_2}[\alpha,\beta_1,\beta_2;\gamma;x_1,x_2] \tag{29}$$
$$= \frac{1}{\Gamma(\beta_2)} \int_0^{\infty} e^{-t} t^{\beta_2-1}\ \Phi_1^{\tau_1,\tau_2}[\alpha,\beta_1;\gamma;x_1 t,x_2 t]dt$$
$$\big(\tau_1,\tau_2 > 0,\ \Re(x_2) < 1;\ \Re(\beta_2) > 0\big),$$

$$F_2^{\tau_1,\tau_2}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2] \tag{30}$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t}\, t^{\alpha-1}\, {}_1\Phi_1^{\tau_1} \left[\begin{matrix} \beta_1; \\ \gamma_1; \end{matrix} \; x_1 t \right] {}_1\Phi_1^{\tau_2} \left[\begin{matrix} \beta_2; \\ \gamma_2; \end{matrix} \; x_2 t \right] dt$$

$$\big(\tau_1, \tau_2 > 0;\ \Re(x_1 + x_2) < 1,\ \Re(\alpha) > 0\big)$$

and

(31) $$F_3^{\tau_1,\tau_2}[\alpha_1, \alpha_2, \beta_1, \beta_2; \gamma; x_1, x_2] = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty \int_0^\infty e^{-t_1-t_2}\, t_1^{\alpha_1-1} t_2^{\alpha_2-1}\, \Phi_2^{\tau_1,\tau_2}[\beta_1, \beta_2; \gamma; x_1, x_2]\, dt_1\, dt_2$$

$$\big(\tau_1, \tau_2 > 0,\ \min\{\Re(\alpha_1), \Re(\alpha_2)\} > 0\big).$$

Remark 2. The special cases of (19), (24) and (25), when $n = 2$ are seen to yield the known integral representations of the τ-Appell's hypergeometric functions [1].

Also for $n = 2$ in (27) and (28), yields the confluent forms of τ-Appell's hypergeometric functions [1].

Remark 3. It is obviously the case when $\tau_1 = \cdots = \tau_n = 1$ in (19), (24) and (25) reduces to the known integral representations for the Lauricella functions (see, for details, [8]).

4. Generating functions of τ-Lauricella functions

In this section, we derive linear and bilinear generating relations for the τ-hypergeometric functions of one, two and n variables. Further, we obtain some special cases as corollaries.

Theorem 9. *Each of the following generating functions for τ-Lauricella functions $F_A^{(n),\tau_1,\ldots,\tau_n}$ in* (11) *holds true*:

(32) $$\sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!}\, F_A^{(n),\tau_1,\ldots,\tau_n}[\lambda + k, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n]\, t^k = (1-t)^{-\lambda}\, F_A^{(n),\tau_1,\ldots,\tau_n} \left(\lambda, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; \frac{x_1}{1-t}, \ldots, \frac{x_n}{1-t} \right)$$

$$(\tau_1, \ldots, \tau_n > 0,\ \lambda \in \mathbb{C} \text{ and } |t| < 1),$$

(33) $$\sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!}\, F_A^{(n),\tau_1,\ldots,\tau_n}[-k, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n]\, t^k = (1-t)^{-\lambda}\, F_A^{(n),\tau_1,\ldots,\tau_n} \left(\lambda, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; \frac{x_1 t}{t-1}, \ldots, \frac{x_n t}{t-1} \right)$$

$$(\tau_1, \ldots, \tau_n > 0,\ \lambda \in \mathbb{C} \text{ and } |t| < 1)$$

and

(34) $$\sum_{k=0}^{\infty} F_A^{(n),\tau_1,\ldots,\tau_n}[-k, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n]\, \frac{t^k}{k!}$$

$$= e^t \; {}_1\Phi_1^{\tau_1}\left[\begin{array}{c} \beta_1; \\ \gamma_1; \end{array} \; -x_1 t\right] \cdots \; {}_1\Phi_1^{\tau_n}\left[\begin{array}{c} \beta_n; \\ \gamma_n; \end{array} \; -x_n t\right]$$

$$(\tau_1, \ldots, \tau_n > 0, \; \lambda \in \mathbb{C} \; and \; |t| < 1).$$

Proof. Using definition (11) in left-hand side, we have

$$\sum_{k=0}^{\infty} (\lambda)_k \; F_A^{(n),\tau_1,\ldots,\tau_n}[\lambda+k, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n]$$

$$= \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} \left(\frac{\Gamma(\gamma_1)\cdots\Gamma(\gamma_n)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\lambda+k)_{m_1+\cdots+m_n}\Gamma(\beta_1+\tau_1 m_1)\cdots\Gamma(\beta_n+\tau_n m_n)}{\Gamma(\gamma_1+\tau_1 m_1)\cdots\Gamma(\gamma_n+\tau_n m_n)} \, \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \right) t^k$$

$$= \frac{\Gamma(\gamma_1)\cdots\Gamma(\gamma_n)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\lambda)_{m_1+\cdots+m_n}\Gamma(\beta_1+\tau_1 m_1)\cdots\Gamma(\beta_n+\tau_n m_n)}{\Gamma(\gamma_1+\tau_1 m_1)\cdots\Gamma(\gamma_n+\tau_n m_n)} \, \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \left(\sum_{k=0}^{\infty} (\lambda+m_1+\cdots+m_n)_k \frac{t^k}{k!} \right)$$

where we have reversal the order of summation and using the identity

$$(\lambda)_k(\lambda+k)_{m_1+\cdots+m_n} = (\lambda)_{m_1+\cdots+m_n}(\lambda+m_1+\cdots+m_n)_k,$$

the binomial theorem

$$(1-t)^{-\lambda-m_1-\cdots-m_n} = \sum_{k=0}^{\infty} \frac{(\lambda+m_1+\cdots+m_n)_k}{k!} t^k \qquad (|t|<1),$$

and identification of the series over k from (11) as

$$F_A^{(n),\tau_1,\ldots,\tau_n}\left(\lambda, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; \frac{x_1}{1-t}, \ldots, \frac{x_n}{1-t}\right)$$

then leads to the assertion in Theorem 9.

A similar procedure yields the result (33).

Next, by virtue of the following limit formula:

$$\lim_{|\lambda|\to\infty} \left\{ (\lambda)_n \left(\frac{z}{\lambda}\right)^n \right\} = z^n \quad (n \in \mathbb{N}_0)$$

when t is replaced by $\frac{t}{\lambda}$ and $|\lambda| \to \infty$ in (33), we get the desired exponential generating function asserted by (34). □

In particular, when $n=2$ and $n=1$ in (32), (33) and (34), we can deduce *presumably new* generating functions for the τ-Appell hypergeometric function $F_2^{\tau_1,\tau_2}$ and τ-Gauss's hypergeometric function ${}_2R_1^{\tau}(z)$, respectively which are asserted by Corollaries 2 and 3 below. We state here the resulting generating functions without proof.

Corollary 2. *The following generating function for*

$$F_2^{\tau_1,\tau_2}(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2)$$

in (9) *holds true*:

(35) $$\sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} \; F_2^{\tau_1,\tau_2}(\lambda+k, \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2) \; t^k$$

$$= (1-t)^{-\lambda}\, F_2^{\tau_1,\tau_2}\left(\lambda, \beta_1, \beta_2; \gamma_1, \gamma_2; \frac{x_1}{1-t}, \frac{x_2}{1-t}\right)$$

$$(\tau_1, \tau_2 > 0,\ \lambda \in \mathbb{C} \text{ and } |t| < 1),$$

(36)
$$\sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!}\, F_2^{\tau_1,\tau_2}(-k, \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2)\, t^k$$

$$= (1-t)^{-\lambda}\, F_2^{\tau_1,\tau_2}\left(\lambda, \beta_1, \beta_2; \gamma_1, \gamma_2; \frac{x_1 t}{t-1}, \frac{x_2 t}{t-1}\right)$$

$$(\tau_1, \tau_2 > 0,\ \lambda \in \mathbb{C} \text{ and } |t| < 1)$$

and

(37)
$$\sum_{k=0}^{\infty} F_2^{\tau_1,\tau_2}(-k, \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2)\, \frac{t^k}{k!}$$

$$= e^t\ {}_1\Phi_1^{\tau_1}\left[\begin{matrix} \beta_1; \\ \gamma_1; \end{matrix}\ -x_1 t\right]\ {}_1\Phi_1^{\tau_2}\left[\begin{matrix} \beta_2; \\ \gamma_2; \end{matrix}\ -x_2 t\right]$$

$$(\tau_1, \tau_2 > 0,\ \lambda \in \mathbb{C} \text{ and } |t| < 1).$$

Corollary 3. *The following generating function for* ${}_2R_1\,(\alpha,\ \beta;\ \gamma;\ \tau;\ z)$ *defined by* (6) *holds true*:

(38)
$$\sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!}\ {}_2R_1\,(\lambda + k,\ \beta;\ \gamma;\ \tau;\ z)\, t^k = (1-t)^{-\lambda}\ {}_2R_1\left(\lambda,\ \beta;\ \gamma;\ \tau;\ \frac{z}{1-t}\right)$$

$$(\lambda \in \mathbb{C},\ \tau > 0 \text{ and } |t| < 1),$$

(39)
$$\sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!}\ {}_2R_1\,(-k,\ \beta;\ \gamma;\ \tau;\ z)\, t^k = (1-t)^{-\lambda}\ {}_2R_1\left(\lambda,\ \beta;\ \gamma;\ \tau;\ \frac{zt}{t-1}\right)$$

$$(\lambda \in \mathbb{C},\ \tau > 0 \text{ and } |t| < 1)$$

and

(40)
$$\sum_{k=0}^{\infty} {}_2R_1\,(-k,\ \beta;\ \gamma;\ \tau;\ z)\, \frac{t^k}{k!} = e^t\ {}_1\Phi_1^{\tau}\left[\begin{matrix} \beta; \\ \gamma; \end{matrix}\ -zt\right]$$

$$(\lambda \in \mathbb{C},\ \tau > 0 \text{ and } |t| < 1).$$

Remark 4. It is obviously the case when $\tau_1 = \cdots = \tau_n = 1$ in (32), (33) and (34), reduces to the known generating functions for the Lauricella functions (see, for details, [9]).

Theorem 10. *Each of the following generating functions for* τ*-Lauricella functions in* (12) *and* (13) *holds true*:

(41)
$$\sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!}\, F_B^{(n),\tau_1,\ldots,\tau_n}[\lambda + k, \alpha_2, \ldots, \alpha_n, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n]\, t^k$$

$$= (1-t)^{-\lambda}\ F_B^{(n),\tau_1,\ldots,\tau_n}\left(\lambda, \alpha_2, \ldots, \alpha_n, \beta_1, \ldots, \beta_n; \gamma; \frac{x_1}{1-t}, x_2, \ldots, x_n\right)$$

$$(\tau_1, \ldots, \tau_n > 0,\ \lambda \in \mathbb{C} \textit{ and } |t| < 1)$$

and

$$\sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!}\ F_D^{(n),\tau_1,\ldots,\tau_n}[\alpha, \lambda + k, \beta_2, \ldots, \beta_n; \gamma; x_1, \ldots, x_n]\ t^k \tag{42}$$

$$= (1-t)^{-\lambda}\ F_D^{(n),\tau_1,\ldots,\tau_n}\left(\alpha, \lambda, \beta_2, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; \frac{x_1}{1-t}, x_2, \ldots, x_n\right)$$

$$(\tau_1, \ldots, \tau_n > 0,\ \lambda \in \mathbb{C} \textit{ and } |t| < 1).$$

Proof. The proof of Theorem 10 is similar to that of Theorem 9. □

In particular, when $n = 2$ in (42) and (41), we can deduce *presumably new* generating functions for the τ-Appell hypergeometric function $F_1^{\tau_1,\tau_2}$ and $F_3^{\tau_1,\tau_2}$, respectively which are asserted by Corollary 4 below. We state here the resulting generating functions without proof.

Corollary 4. *Each of the following generating functions for τ-Appell functions in* (8) *and* (10) *holds true*:

$$\sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!}\ F_1^{\tau_1,\tau_2}[\alpha, \lambda + k, \beta_2; \gamma; x_1, x_2]\ t^k \tag{43}$$

$$= (1-t)^{-\lambda}\ F_1^{\tau_1,\tau_2}\left(\alpha, \lambda, \beta_2; \gamma_1, \gamma_2; \frac{x_1}{1-t}, x_2\right)$$

$$(\tau_1, \tau_2 > 0,\ \lambda \in \mathbb{C} \textit{ and } |t| < 1)$$

and

$$\sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!}\ F_3^{\tau_1,\tau_2}[\lambda + k, \alpha_2, \beta_1, \beta_2; \gamma; x_1, x_2]\ t^k \tag{44}$$

$$= (1-t)^{-\lambda}\ F_3^{\tau_1,\tau_2}\left(\lambda, \alpha_2, \beta_1, \beta_2; \gamma; \frac{x_1}{1-t}, x_2\right)$$

$$(\tau_1, \tau_2 > 0,\ \lambda \in \mathbb{C} \textit{ and } |t| < 1).$$

Theorem 11. *The following bilinear generating function for ${}_2R_1^{\tau}(z)$ defined by* (6) *holds true*:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_2R_1^{\tau}\left(\lambda + n, \alpha; \beta; x\right)_2 R_1^{\tau}\left(-n, \gamma; \delta; y\right)\ t^n \tag{45}$$

$$= (1-t)^{-\lambda}\ F_2^{\tau_1,\tau_2}\left(\lambda, \alpha, \gamma; \beta, \delta; \frac{x}{1-t}, -\frac{yt}{1-t}\right).$$

Proof. The proof of Theorem 11 is similar to that of Theorem 9. □

5. Concluding remarks and observations

In our present investigation, with the help of additional parameters $\tau_1, \ldots, \tau_n$, we have introduced and studied some τ-Lauricella functions of n variables. As a special case, when $n = 2$, we obtain their τ-Appell functions of two variables [1]. Further, we give various integral representations of these extended functions and generating relations for these τ-Lauricella functions. Various (known or new) special cases results presented here are also considered. The special cases of the results obtained in this paper when $\tau_1 = \cdots = \tau_n = 1$, would reduces to the corresponding known results for the Lauricella functions (see, for details, [2, 8, 9]). The expressions of the integrals, which we have evaluated in this paper, are new and generalize the results in the existing literature (see [2, 8, 9]).

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