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SOME TESTS AGAINST AGING BASED ON THE TOTAL TIME ON TEST TRANSFORM

by

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ABSTRACT

Let F be a life distribution with survival function $\bar{F} = 1 - F$ and finite mean $\mu = \int_0^\infty \bar{F}(x)dx$. The scaled total time on test transform $\varphi_F(t) = \int_0^{F^{-1}(t)} \bar{F}(x)dx/\mu$ was introduced by Barlow and Campo (1975) as a tool in the statistical analysis of life data. The properties IFR, IFRA, NBUE, DMRL and heavy-tailedness can be translated to properties of $\varphi_F(t)$. We discuss the previously known of these relationships and present some new results. Guided by properties of $\varphi_F(t)$ we suggest some test statistics for testing exponentiality against IFR, IFRA, NBUE, DMRL and heavy-tailedness, respectively. The asymptotic distributions of the statistics are derived and the asymptotic efficiencies of the tests are studied. The power for some of the tests is estimated by simulation for some alternatives when the sample size is $n = 10$ or $n = 20$.

1. Introduction and summary

In reliability theory the notion of aging plays an important role. Therefore several classes of life distributions have been introduced in order to model different aspects of aging. Among these are the classes of life distributions with the properties in the following definition.

DEFINITION 1.1 Let F be a life distribution (i.e. a distribution function with $F(0^-) = 0$) with survival function $\bar{F} = 1 - F$ and finite mean $\mu = \int_0^\infty \bar{F}(x)dx$ and let $S = \{t: \bar{F}(t) > 0\}$. Then F and $\bar{F}$ are said to be (or to have)

(i) *increasing failure rate (IFR)* if the conditional survival function

$$t \rightsquigarrow \frac{\bar{F}(x+t)}{\bar{F}(t)}$$

is a decreasing function on S when $x \geq 0$;

(ii) *increasing failure rate in average (IFRA)* if

$$t \rightsquigarrow \frac{-\ln \bar{F}(t)}{t}$$

is increasing on S;

(iii) *new better than used (NBU)* if

$$\bar{F}(x)\bar{F}(y) \geq \bar{F}(x+y)$$

for $x \geq 0$ and $y \geq 0$;

(iv) *new better than used in expectation (NBUE)* if

$$\bar{F}(x) \int_0^\infty \bar{F}(y)dy \geq \int^\infty \bar{F}(x+y)dy$$

for $x \geq 0$;

(v) *decreasing mean residual life (DMRL)* if

$$t \mapsto \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x)dx$$

is decreasing on S;

(vi) *harmonic new better than used in expectation (HNBUE)* if

$$\int_t^\infty \bar{F}(x)dx \leq \mu \exp(-t/\mu)$$

for $t \geq 0$. □

By reversing the inequalities and changing decreasing (increasing) to increasing (decreasing) we get the dual classes DFR, DFRA, NWU, NWUE, IMRL and HNWUE. Here D = decreasing, I = increasing and W = worse.

Between these classes of life distributions the implications in Figure 1.1 (but no other) hold.

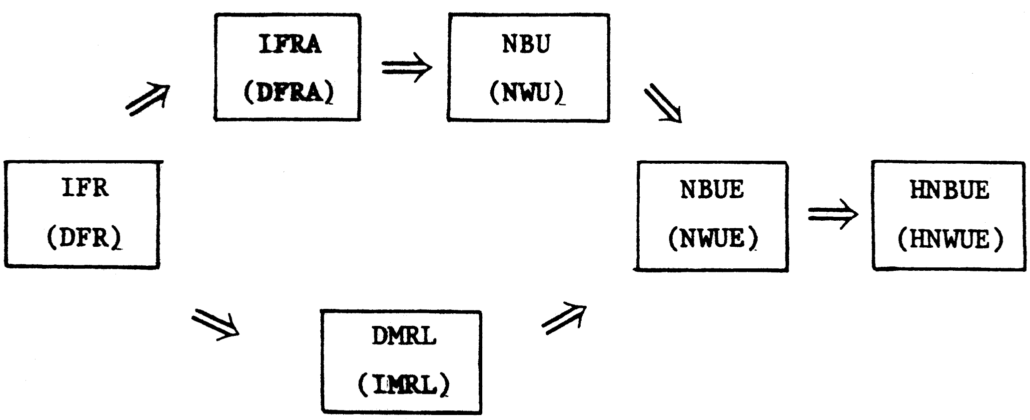


Figure 1.1 Implications between some classes of life distributions.

The IFR, IFRA, NBU, NBUE and DMRL classes of life distributions (with duals) and their properties were discussed e.g. by Bryson and Siddiqui (1969), Haines (1973) and Barlow and Proschan (1975). The HNBUE (HNWUE) class was introduced by Rolski (1975) and was further studied by Klefsjö (1977, 1980a, 1980b).

During recent years some tests have been suggested for testing

H_0: F is the exponential distribution against

(1.1) H_1: F is ∇ but not exponential,

where ∇ denotes IFR, IFRA, NBU, NBUE or DMRL. Such tests were proposed e.g. by Proschan and Pyke (1967), Barlow (1968) and Bickel and Doksum (1969) when ∇ = IFR, by Barlow and Campo (1975) and Bergman (1977) when ∇ = IFRA, by Hollander and Proschan (1972) and Koul (1977) when ∇ = NBU, by Hollander and Proschan (1975) and Koul (1978) when ∇ = NBUE and by Hollander and Proschan (1975) when ∇ = DMRL.

The scaled total time on test (TTT-) transform and the TTT-plot were introduced by Barlow and Campo (1975). These concepts have proved to be very useful in the statistical analysis of life data. In Section 2 we shall present the TTT-transform and the TTT-plot and give some of their properties. It is possible to translate some of the aging properties (i)-(vi) above (with duals) to properties of the scaled TTT-transform. Such correspondences will be discussed in Section 2. In Section 3 we shall use these properties of the scaled TTT-transform to get ideas for different test statistics for testing H_0 against H_1 when H_1 is of the type (1.1). We also present a test statistic for testing H_0 against heavy-tailedness. Two of the test statistics which we obtain have been derived before but from a different point of view. The asymptotic distributions of some of the test statistics are derived in Section 4 and the consistency of the tests is discussed in Section 5. Asymptotic efficiency results and some power estimates will be presented in Sections 6 and 7. In Section 8 the tests are illustrated on times between failures of air conditioning systems in jet airplanes.

2. The total time on test concept

2.1 The TTT-transform and the TTT-plot

We shall here briefly present the TTT-transform and the TTT-plot (TTT = Total Time on Test). For further details see e.g. Barlow and Campo (1975), Barlow (1979) and Bergman (1979).

Let F be a life distribution with finite mean μ. The *TTT-transform* H_F^{-1} of F is then defined by

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} \bar{F}(s)ds \quad \text{for } 0 \leq t \leq 1,$$

where

$$F^{-1}(t) = \inf\{x: F(x) \geq t\}.$$

There is a one-to-one correspondence between life distributions and their TTT-transforms.

Further, H_F^{-1} is continuous if and only if F is stricly increasing for $0 \leq x \leq F^{-1}(1)$ and H_F^{-1} is strictly increasing if and only if F is continuous.

Since the mean of F is given by

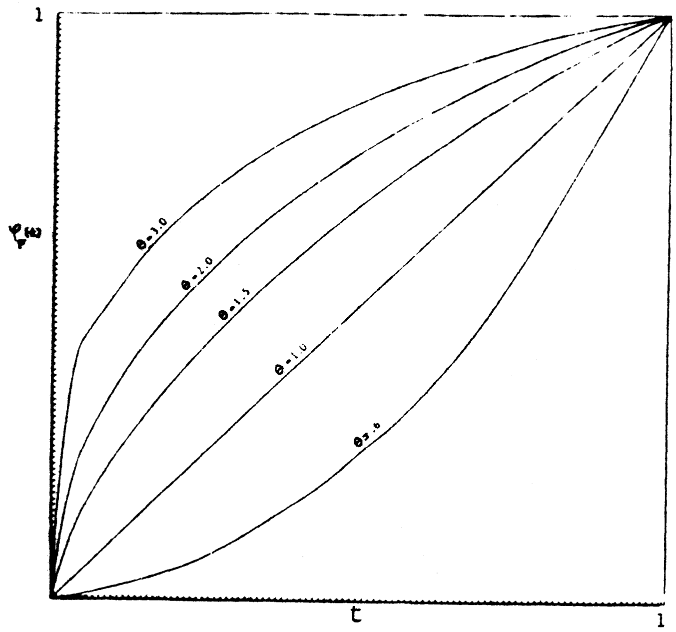
$$\mu = H_F^{-1}(1) = \int_0^{\infty} \bar{F}(s)ds,$$

the transform

$$(2.1) \qquad \varphi_F(t) = \frac{1}{\mu} \int_0^{F^{-1}(t)} \bar{F}(s)ds \quad \text{for } 0 \leq t \leq 1$$

is scale invariant and is named the *scaled TTT-transform*.

Figure 2.1 on p. 5 (from Barlow and Campo (1975)) illustrates the scaled TTT-transforms for some members of the family of Weibull distributions with $\bar{F}(x) = \exp(-(x/\alpha)^{\theta})$, $x \geq 0$. In particular we note that if F is the exponential distribution then the scaled TTT-transform is given by $\varphi_F(t) = t$, $0 \leq t \leq 1$. Further scaled TTT-transforms are presented by e.g. Barlow (1979), and Bergman (1979).



<u>Figure 2.1</u> Scaled TTT-transforms of some Weibull distributions with $\bar{F}(x) = \exp(-(x/\alpha)^{\theta})$.

Assume that $t(1) \leq t(2) \leq \ldots \leq t(n)$ is an ordered sample from a life distribution F (and let $t(0) = 0$). Further let

$$S_j = \sum_{k=1}^{j} (n-j+1)(t(k)-t(k-1)) \qquad \text{for } j = 1, 2, \ldots, n.$$

(and $S_0 = 0$) denote the total time on test at $t(j)$. Note that $S_n = \Sigma_{k=1}^{n} t(k)$.

A natural choice of estimator of the scaled TTT-transform is the *empirical scaled TTT-transform*

$$\varphi_n(t) = \frac{H_n^{-1}(t)}{H_n^{-1}(1)} \qquad \text{for } 0 \leq t \leq 1 ,$$

where

$$H_n^{-1}(t) = \int_0^{F_n^{-1}(t)} \bar{F}_n(s)ds \qquad \text{for } 0 \leq t \leq 1,$$

F_n is the empirical distribution function and $\bar{F}_n = 1 - F_n$. Calculations show that $H_n^{-1}(j/n) = S_j/n$ for $j = 0,1,\ldots,n$. By using

$$u_j = \frac{S_j}{S_n} \qquad \text{for } j = 0, 1, \ldots, n$$

we get that

$$\varphi_n(j/n) = u_j \qquad \text{for } j = 0, 1, \ldots, n.$$

The *TTT-plot* is now obtained by plotting u_j against j/n for $j =$ $= 0, 1, \ldots, n$ and then connecting the plotted points by straight lines as in Figure 2.2.

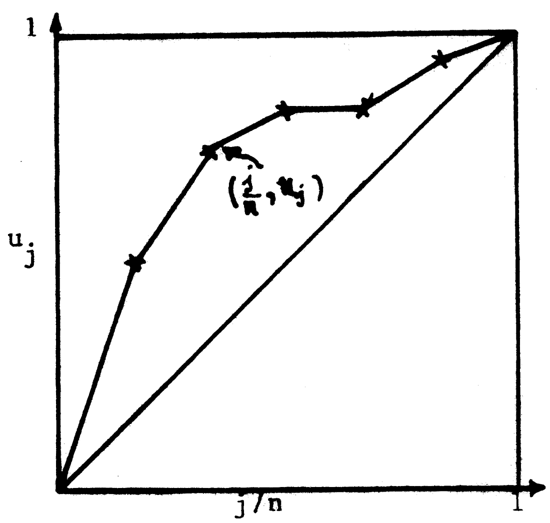


Figure 2.2 Example of a TTT-plot.

As a consequence of the Glivenko-Cantelli Lemma we get that if F is strictly increasing then

$$\varphi_n(j/n) \to \varphi_F(t)$$

with probability one and uniformly in [0,1] when $n \to \infty$ and $j/n \to t$ (see Barlow et al. (1972), p. 237). Because of this, Barlow and Campo (1975) suggested a comparison of the TTT-plot with graphs of scaled TTT-transforms for making model identification.

2.2 The TTT-transform and some aging properties

Some of the aging properties IFR, IFRA, NBU, NBUE and DMRL (with duals) can be translated to properties of the scaled TTT-transform. We shall here summarize the previously known of these relationships and give some comments on them. We shall also present some new results.

The following theorem in the IFR (DFR) case is due to Barlow and Campo (1975); see also Barlow (1979).

THEOREM 2.1 A life distribution F is IFR (DFR) if and only if the scaled TTT-transform $\varphi_F(t)$ is concave (convex) for $0 \leq t \leq 1$.

The proof is based on the fact that if F is absolutely continuous and strictly increasing then the derivative of $\varphi_F(t)$ is given by

$$\frac{d}{dt}\varphi_F(t) = \frac{1}{\mu r(F^{-1}(t))}$$

for almost all $t \in [0,1]$, where $r(x) = f(x)/\bar{F}(x)$ denotes the failure rate and $f(x)$ the density function.

Theorem 2.1 is illustrated in Figures 2.3 and 2.4.

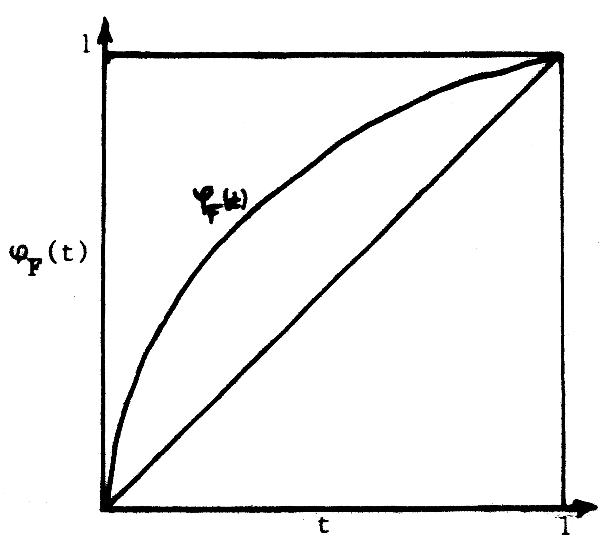


Figure 2.3 Scaled TTT-transform of an IFR life distribution.

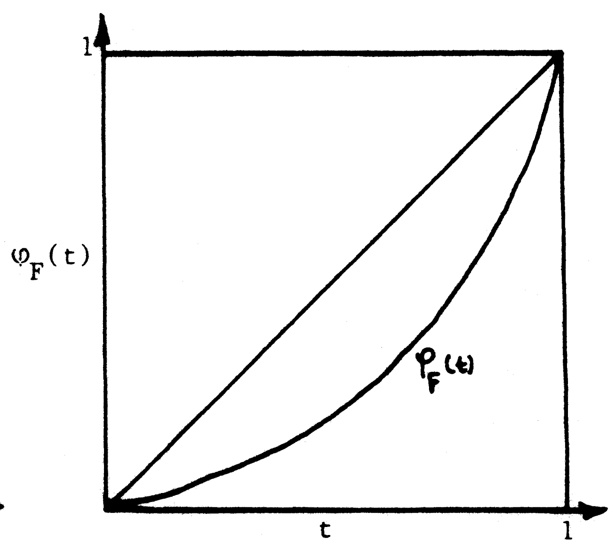


Figure 2.4 Scaled TTT-transform of a DFR life distribution.

Also the following theorem was given by Barlow and Campo (1975); see also Barlow (1979).

THEOREM 2.2 If F is a life distribution which is IFRA (DFRA) then $\varphi_F(t)/t$ is decreasing (increasing) for $0 < t < 1$.

Barlow and Campo (1975) proved the theorem when F is strictly increasing and absolutely continuous and then said that the general case follows from limiting arguments (which were not specified). Note that an IFRA life distribution is strictly increasing but not necessarily continuous and a DFRA life distribution is continuous but not necessarily strictly increasing.

Barlow (1979) also pointed out that there are life distributions which are not IFRA but for which $\varphi_F(t)/t$ is decreasing. An example of such a life distribution is

$$(2.2) \qquad F(x) = \begin{cases} 0 & \text{for } 0 \le x < 1/2 \\ 1 - \exp(-(c+x)) & \text{for } x \ge 1/2 , \end{cases}$$

where $c = \ln 2 - 1/2$ (see Barlow (1979)).

The NBU class of life distributions properly contains the IFRA class. Therefore it is natural to wonder if any implication holds between the NBU property and the property that $\varphi_F(t)/t$ is decreasing for $0 < t < 1$. The answer is no. It can be proved that F in (2.2) is not NBU (but is NBUE). Furthermore,

$$F(x) = 1 - \alpha^k \quad \text{for } k \le x < k+1 \text{ and } k = 0, 1, 2, \ldots$$

is NBU but $\varphi_F(t)/t$ is not decreasing.

In fact the property that $\varphi_F(t)/t$ is decreasing on $0 < t < 1$ is equivalent to that

(2.3) $$\frac{1}{F(x)} \int_0^x \bar{F}(s)ds \quad \text{is decreasing on } M = \{x: 0 < F(x) < 1\}.$$

The class of life distributions for which (2.3) holds has in fact emerged earlier in reliability theory. Marshall and Proschan (1972) studied replacement policies. They proved that under the age replacement policy (see e.g. Barlow and Proschan (1965), Chapter 4) with replacement age T the expectation of the length of time between succesive failures is decreasing in T if and only if (2.3) holds (see Marshall and Proschan (1972), Theorem 2.8). Marshall and Proschan (1972) also proved that (2.3) is true if F is IFR and that a life distribution for which (2.3) holds is NBUE.

In fact we have the following theorem.

<u>THEOREM 2.3</u> For a life distribution F the properties

(2.4) $$\frac{1}{F(x)} \int_0^x \bar{F}(s)ds \quad \text{is decreasing (increasing) on } M = \{x: 0 < F(x) < 1\}$$

and

(2.5) $$\varphi_F(t)/t \quad \text{is decreasing (increasing) for } 0 < t < 1$$

are equivalent.

<u>PROOF</u> Let us study the "decreasing" case. Then it can be seen that if any of (2.4) and (2.5) holds then F is strictly increasing on M. If F in addition is continuous the equivalence follows by substituting $t = F(x)$.

Now suppose that F is a life distribution with scaled TTT-transform $\varphi_F(t)$ for which (2.4) holds. Let $t_1 < t_2$ and $a_j = F^{-1}(t_j)$, $j = 1,2$. If $a_1 = a_2$ then $\varphi_F(t_1)/t_1 > \varphi_F(t_2)/t_2$. If $a_1 < a_2$ we can find two sequences $(y_j)_{j=1}^{\infty}$ and $(z_j)_{j=1}^{\infty}$ such that $(y_j)_{j=1}^{\infty}$ increases to a_2

and $(z_j)_{j=1}^{\infty}$ decreases to a_1 and $z_k < y_j$ for $j,k = 1,2,3,\ldots$.
From (2.4) we get that

$$\frac{1}{F(z_k)} \int_0^{z_k} \bar{F}(s)ds \geq \frac{1}{F(y_j)} \int_0^{y_j} \bar{F}(s)ds \qquad \text{for } j,k = 1,2,3,\ldots .$$

Letting $j \to \infty$ and $k \to \infty$ we obtain that $\varphi_F(t_1)/t_1 \geq \varphi_F(t_2)/t_2$. From this follows that (2.4) implies (2.5).

Now suppose that (2.5) holds. Let $x_1, x_2 \in M$ with $x_1 < x_2$ and let $t_j = F(x_j)$, $j = 1,2$. Since F is strictly increasing on M we get that $t_1 < t_2$ and $F^{-1}(t_j) = x_j$, $j = 1,2$. By substituting $t = F(x)$ in the inequality $\varphi_F(t_1)/t_1 \geq \varphi_F(t_2)/t_2$ it follows that $\int_0^{x_1} \bar{F}(s)ds/F(x_1) \geq$ $\geq \int_0^{x_2} \bar{F}(s)ds/F(x_2)$.

The proof in the "increasing" case is of similar character. Then F is continuous but not necessarily strictly increasing. □

In the NBUE (NWUE) case we have the following theorem.

<u>THEOREM 2.4</u> A life distribution is NBUE (NWUE) if and only if $\varphi_F(t) \geq (\leq)\ t$ for $0 \leq t \leq 1$.

The NBUE characterization was made by Bergman (1979). He used this characterization in connection with replacement policies. The statement in the NWUE case is new (as far as this author knows). The proof in the NWUE case is rather similar to that in the NBUE case. We want to point out here that the proof not only consist in reversing inequalities and also that the substitution $t = F(x)$ does not work in the general case since there are NBUE (NWUE) life distributions which are neither continuous nor strictly increasing.

Figure 2.5 illustrates the relations between the IFRA, the NBU and the NBUE classes and the class of life distributions for which $\varphi_F(t)/t$ is decreasing on $0 < t < 1$.

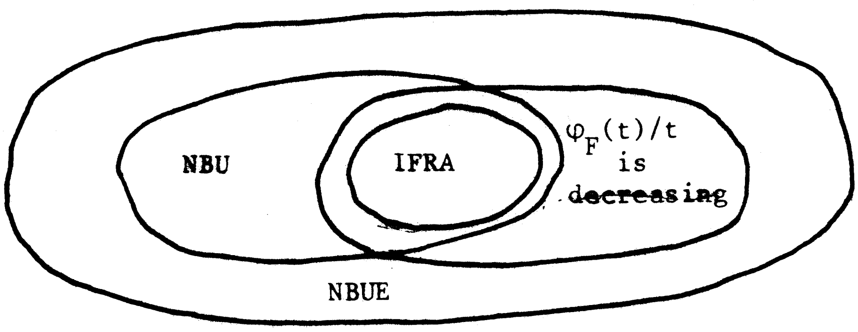


Figure 2.5 The relations between some classes of life distributions.

In the DMRL (IMRL) case we have the following characterization.

THEOREM 2.5 A life distribution F is DMRL (IMRL) if and only if $Q(t) = (1-\varphi_F(t))/(1-t)$ is decreasing (increasing) for $0 \leq t < 1$.

PROOF A life distribution which is DMRL (IMRL) is continuous (strictly increasing) but not necessarily strictly increasing (continuous). This is a consequence of the definition. If F is both continuous and strictly increasing the proof follows from the fact that

$$(2.6) \qquad \frac{1}{\bar{F}(x)} \int_x^{\infty} \bar{F}(s)ds = \frac{\mu(1 - \frac{1}{\mu}\int_0^x \bar{F}(s)ds)}{1 - F(x)}$$

by using the substitution $t = F(x)$. The proof in the general case is analogous to that of Theorem 2.3. □

The expressions "heavy-tailed" and "light-tailed" distributions are frequently used in the literature. Bryson (1974) gave a formal definition of the concept " heavy-tailed" distribution as one whose mean residual life $e_F(x) = \int_x^\infty \bar{F}(s)ds/\bar{F}(x)$ is an increasing function of x (i.e. F is IMRL) at least for sufficiently large values of x. However this definition has some disadvantages (see Vännman (1975)). Vännman (1975) instead proposed several (for absolutely continuous distributions) equivalent definitions of heavy-tailedness. One of these is the following.

DEFINITION 2.1 Assume that $L_2 = \lim_{x\to\infty} e_F(x)$ exists (finite or infinite). Then F is

(i) heavy-tailed if $L_2 = \infty$

(ii) light-tailed if $L_2 = 0$

(iii) a border-line case if $0 < L_2 < \infty$. □

REMARK If $\limsup_{x\to\infty} e_F(x) \neq \liminf_{x\to\infty} e_F(x)$ we cannot classify F by using $e_F(x)$.

From (2.6) we get that this definition of heavy-tailedness (light-tailedness) can be translated to the scaled TTT-transform.

THEOREM 2.6 A life distribution F is

(i) heavy-tailed if $\varphi_F'(1) = \infty$

(ii) light-tailed if $\varphi_F'(1) = 0$

(iii) a border-line case if $0 < \varphi_F'(1) < \infty$

REMARK With $\varphi_F'(1)$ we of course mean $\lim_{t\to 1-}\{(\varphi_F(t)-\varphi_F(1))/(t-1)\}$

Theorem 2.6 is illustrated in Figures 2.6 and 2.7.

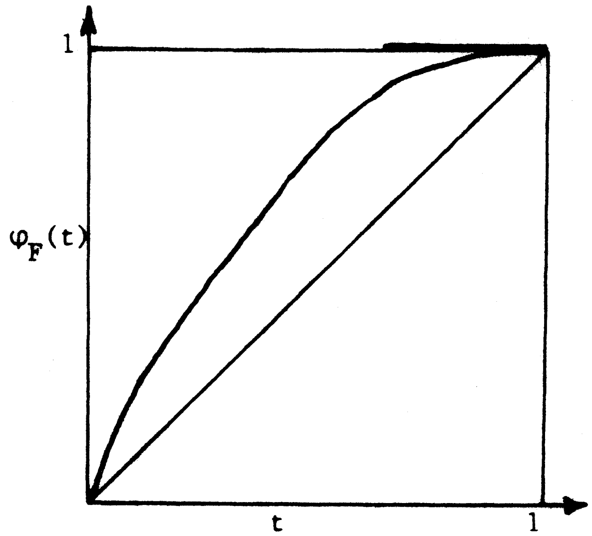


Figure 2.6 Scaled TTT-transform of a light-tailed life distribution.

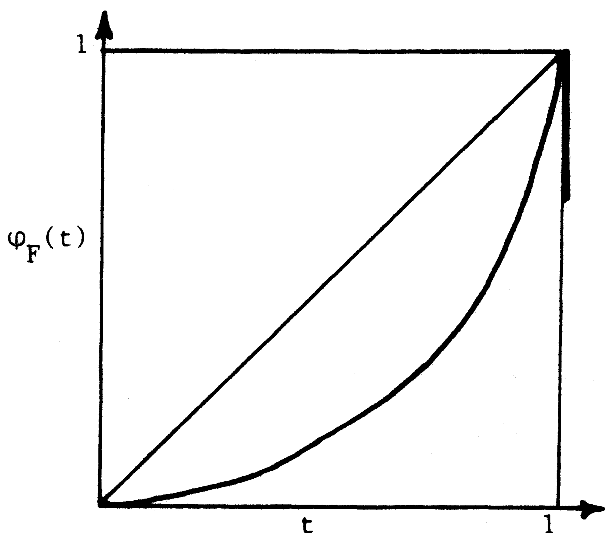


Figure 2.7 Scaled TTT-transform of a heavy-tailed life distribution.

Finally we want to mention the following facts.

No implication is known between the NBU (NWU) property and the TTT-transform.

For relationships between the HNBUE (HNWUE) property and the scaled TTT-transform, see Klefsjö (1980b).

For other interesting characterizations of the aging properties IFR, IFRA, NBU and NBUE, see Langberg, León and Proschan (1978). That paper also includes another proof of Theorem 2.1.

3. Test statistics based on the scaled TTT-transform

In this section we shall suggest some test statistics for testing exponentiality against different forms of aging.

Let $t(1) \leq t(2) \leq \ldots \leq t(n)$ denote an ordered sample from a life distribution F with finite mean $\mu = \int_0^\infty \bar{F}(x)dx$ (and let $t(0) = 0$). Further let

$$D_j = (n-j+1)(t(j)-t(j-1)) \quad \text{for } j = 1, 2, \ldots, n$$

denote the normalized spacings. Then $S_j = \Sigma_{k=1}^{j} D_k$, $j = 1, 2, \ldots, n$.

3.1 Against the IFR (DFR) alternative

To get ideas of how to form test statistics for testing exponentiality against the IFR (DFR) alternative we use that F is IFR (DFR) if and only if the scaled TTT-transform φ_F is concave (convex); cf. Theorem 2.1.

Now suppose that φ_F is concave. This means that linear interpolation between the points $(t_1,\varphi_F(t_1))$ and $(t_2,\varphi_F(t_2))$ always underestimates $\varphi_F(t)$ for $t_1 < t < t_2$. Therefore, with $\ell(t)$ according to Figure 3.1 on p. 15, we have $\ell(t) < \varphi_F(t)$. It is therefore reasonable to expect the TTT-plot based on a sample from an IFR life distribution to have the corresponding property. Thus we expect u_j to be larger than the value for $t = j/n$ on the line through the points $((j-1)/n,u_{j-1})$ and $((j+1)/n,u_{j+1})$; see Figure 3.2 on p. 15. This means that

$$(3.1) \qquad u_{j-1} + u_{j+1} - 2u_j < 0 \quad \text{for } j = 1, 2, \ldots, n-1.$$

A possible test statistic against the IFR alternative is therefore

$$A_1 = \sum_{j=1}^{n-1} (u_{j-1}+u_{j+1}-2u_j).$$

We expect a negative (positive) value of A_1 if F is IFR (DFR), but not exponential.

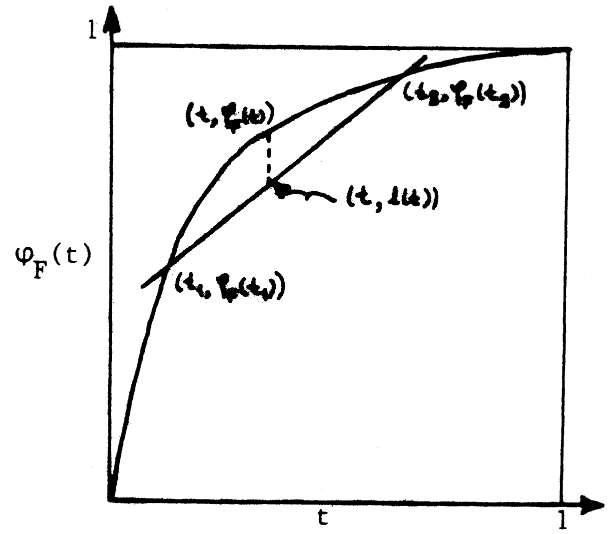


Figure 3.1

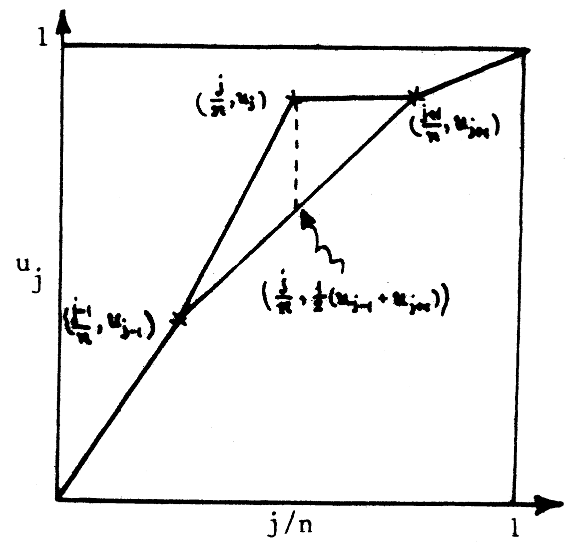


Figure 3.2

We get that

(3.2) $$A_1 = 1 - u_{n-1} - u_1 = (D_n - D_1)/S_n.$$

One advantage with A_1 is that its distribution under H_0 is easy to calculate. With the aid of the fact that $(u_1, u_2, \ldots u_{n-1})$ has the same distribution as an ordered sample of size $n-1$ from a uniform distribution over $[0,1]$ (cf. David (1970), p. 80) we get that

$$P(A_1 \leq x) = \begin{cases} \frac{1}{2}(1+x)^{n-1} & \text{for } -1 \leq x \leq 0 \\ 1 - \frac{1}{2}(1-x)^{n-1} & \text{for } 0 \leq x \leq 1. \end{cases}$$

Further, we get that

$$\lim_{n\to\infty} P(nA_1 \leq x) = \frac{1}{2}\int_{-\infty}^{x} e^{-|t|}dt.$$

This means that nA_1 asymptotically has a Laplace distribution. We note that the numerator of $A_1 = (D_n - D_1)/S_n$ is independent of $D_2, D_3, \ldots D_{n-1}$. A consequence of this fact is that a test based on A_1 is not consistent against the whole class of IFR (DFR) life distributions.

If φ_F is concave we not only expect (3.1) to hold. We also expect, for $j = 0, 1, \ldots, n-2$ and $k = 2, 3, \ldots, n-j$, that

(3.3) $\quad u_{j+\nu} > \ell((j+\nu)/n) \quad$ for $\nu = 1, 2, \ldots, k-1$

(see Figure 3.3).

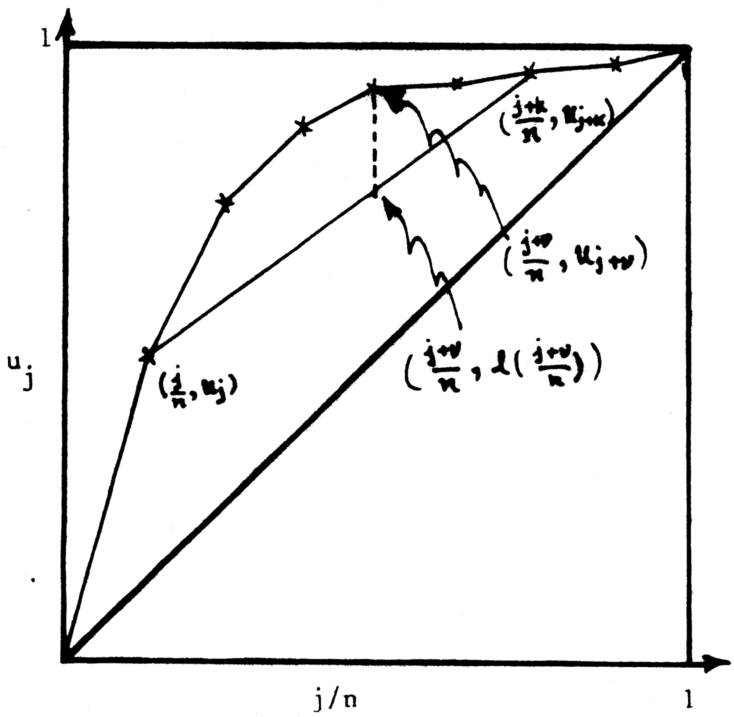


Figure 3.3

It is easily seen that (3.3) is equivalent to that

$$k(u_{j+\nu}-u_j) > \nu(u_{j+k}-u_j) \quad \text{for} \quad \nu = 1, 2, \ldots, k-1.$$

This gives the idea to study the test statistic

(3.4) $$A_2 = \sum_{j=0}^{n-2} \sum_{k=2}^{n-j} \sum_{\nu=1}^{k-1} \{k(u_{j+\nu}-u_j)-\nu(u_{j+k}-u_j)\} .$$

If F is IFR (DFR), but not exponential, we expect A_2 to be positive (negative).

After a good deal of calculation we get that

(3.5) $$A_2 = \sum_{j=1}^{n} \alpha_j D_j / S_n ,$$

where

(3.6) $$\alpha_j = \frac{1}{6}\{ (n+1)^3 j - 3(n+1)^2 j^2 + 2(n+1) j^3 \}.$$

The distribution of A_2 is more complicated than that of A_1. Since the normalized spacings $D_1, D_2, \ldots, D_n$ are independent and exponentially distributed under H_0 (see e.g. David (1970), p. 79) and $S_n = \Sigma_{j=1}^{n} D_j$

we get that

$$\left(\frac{D_1}{S_n}, \frac{D_2}{S_n}, \ldots, \frac{D_{n-1}}{S_n}\right)$$

has the density

$$f(x_1,x_2,\ldots,x_{n-1}) = (n-1)! \qquad \text{on } \Omega,$$

where $\Omega = \{(x_1,x_2,\ldots,x_{n-1}): x_j > 0,\ 1 \leq j \leq n-1,\ \Sigma_{j=1}^{n-1} x_j < 1\}$.

(cf. Hogg and Craig (1970),pp. 140-141). From this we can at least theoretically calculate the distribution of A_2. However, as we shall see in Section 4, A_2 is asymptotically normally distributed.

Before leaving the IFR case we note that the condition (3.1) can be written as

$$D_{j+1} < D_j \qquad \text{for } j = 1, 2, \ldots, n-1.$$

With

$$V_{ij} = \begin{cases} 1 & \text{if } D_j < D_i \\ 0 & \text{if } D_j \geq D_i \end{cases} \qquad \text{for } j = i+1, i+2, \ldots, n$$

and

$$(3.7) \qquad S = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} V_{ij}$$

we expect S to be large if F is IFR but not exponential . This statistic was proposed by Proschan and Pyke (1967).

3.2 Against the IFRA (DFRA) alternative

No condition is known of the scaled TTT-transform $\varphi_F(t)$ which is equivalent to the property that F is IFRA. However, from Theorem 2.2 it follows that "$\varphi_F(t)/t$ is decreasing for $0 < t < 1$" is a necessary (but not sufficient) condition for F to be IFRA.

If $\varphi_F(t)/t$ is decreasing we expect the corresponding to hold for the TTT-plot. This means that

$$(3.8) \qquad \frac{u_i}{i/n} > \frac{u_j}{j/n} \quad \text{for } j > i \text{ and } i = 1, 2, \ldots, n-1.$$

Multiplication by ij/n and summation over i and j gives the test statistic

$$(3.9) \qquad B = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (ju_i - iu_j).$$

We expect a positive (negative) value of B if F is IFRA (DFRA). Simplifying the expression in (3.9) we obtain that

$$(3.10) \qquad B = \sum_{j=1}^{n} \beta_j D_j / S_n ,$$

where

$$(3.11) \qquad \beta_j = \frac{1}{6}\{ 2j^3 - 3j^2 + j(1-3n-3n^2) + 2n + 3n^2 + n^3 \}.$$

We note that B has the same form as A_2. Hence what was said about the distribution of A_2 is true for B as well.

3.3 Against the NBUE (NWUE) alternative

We know from Theorem 2.4 that a life distribution F is NBUE (NWUE) if and only if $\varphi_F(t) \geq (\leq)\ t$ for $0 \leq t \leq 1$. This gives rise to the test statistic

$$C = \sum_{j=1}^{n} \{ u_j - \frac{j}{n} \}$$

for testing against the NBUE (NWUE) alternative. Direct calculations give

$$(3.12) \qquad C = \sum_{j=1}^{n-1} u_j - \frac{n-1}{2} .$$

If F is NBUE (NWUE), but not exponential, we expect C to be positive (negative).

We note that

$$C = V - \frac{n-1}{2} = nK^*,$$

where $V = \Sigma_{j=1}^{n-1} u_j$ is the cumulative TTT-statistic (see e.g. Barlow et al. (1972), Chapter 6) proposed for testing against "F is IFR (DFR)" and

$$(3.13) \qquad K^* = \sum_{j=1}^{n} (\frac{3}{2} - \frac{2j}{n} + \frac{1}{2n}) t(j)/S_n$$

was introduced by Hollander and Proschan (1975) for testing against the NBUE (NWUE) alternative. Hence, C, V and K* are equivalent test statistics

3.4 Against the DMRL (IMRL) alternative

By using Theorem 2.5 and the same idea as earlier we expect that if F is DMRL (IMRL) then

$$\frac{1 - u_j}{1 - j/n} < (>) \frac{1 - u_i}{1 - i/n} \qquad \text{for } j > i \text{ and } i = 0, 1, \ldots, n-1.$$

After multiplication by $(n-i)(n-j)/n$ and summation we get the test statistic

$$(3.14) \qquad M_1 = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \{(n-j)(1-u_i) - (n-i)(1-u_j)\}.$$

If F is DMRL (IMRL), but not exponential, we expect M_1 to be positive (negative).

We note that M_1 is proportional to the test statistic V^* proposed by Hollander and Proschan (1975) for testing against DMRL (IMRL).

If

$$Q(t) = \frac{1 - \varphi_F(t)}{1 - t}$$

has a derivative then $Q(t)$ is decreasing if and only if $Q'(t) \leq 0$. Since

$$Q'(t) = \frac{(1-\varphi_F(t)) - (1-t)\varphi_F'(t)}{(1-t)^2}$$

we expect $(1-\varphi_F(t)) - (1-t)\varphi_F'(t)$ to be negative if F is DMRL. From this we get the idea of studying

$$(1-u_j) - (1-\tfrac{j}{n}) \frac{u_{j+1} - u_{j-1}}{2/n} \qquad \text{for } j = 1, 2, \ldots, n-1,$$

where $n(u_{j+1} - u_{j-1})/2$ is an approximation of the derivative. A possible test statistic against the DMRL (IMRL) alternative is therefore

$$M_2 = \sum_{j=1}^{n-1} \{2(1-u_j) - (n-j)(u_{j+1}-u_{j-1})\} .$$

Substituting $u_j = \Sigma_{\nu=1}^{j} D_\nu/S_n$ we get that

$$(3.15) \qquad M_2 = \sum_{j=1}^{n} \gamma_j D_j/S_n ,$$

where $\gamma_1 = 1-n$ and $\gamma_j = 4j-3-2n$ for $2 \leq j \leq n$. This is another test statistic of the same form as A_2 and B (and M_1).

We remark that there is a relation between the test statistics M_2 and C. As is easily seen

$$M_2 = nu_1 - 4C - 1.$$

3.5 Against heavy-tailedness (light-tailedness)

Theorem 2.6 gives the idea of using the "derivative" at $t = 1$ of the TTT-plot

$$(3.16) \qquad E = \frac{u_n - u_{n-1}}{\frac{n}{n} - \frac{n-1}{n}} = n(1-u_{n-1})$$

as a test statistic against heavy-tailedness (light-tailedness).

If F is heavy-tailed (light-tailed) we expect E to be large and positive (small and positive).

An advantage with E is that it has a simple distribution under H_0. Since

$$P(u_{n-1} \leq x) = x^{n-1} \qquad \text{for} \quad 0 \leq x \leq 1$$

we have

$$P(E \leq x) = 1 - (1 - \frac{x}{n})^{n-1} \qquad \text{for} \quad 0 \leq x \leq n.$$

From this it follows that E is asymptotically exponentially distributed with mean 1 under H_0. We also observe that E (or equivalently u_{n-1}) has another advantage as a test statistic. It can be seen from the TTT-plot if H_0 can be rejected by comparing u_{n-1} with the rejection bound. If we use u_{n-1}^{n-1}, which is uniformly distributed under H_0, the rejection bound is even independent of the sample size.

Other test statistics for testing against heavy-tailedness (light-tailedness) are

$$(3.17) \qquad T_2 = t(r)/t(m)$$

$$(3.18) \qquad T_3 = t(r)/\left(\frac{1}{m}\sum_{j=1}^{m} t(j)\right)$$

and

$$(3.19) \qquad T_4 = \left(\frac{1}{k-s}\sum_{j=s+1}^{k} t(j)\right)\Big/\left(\frac{1}{r}\sum_{j=1}^{r} t(j)\right)$$

where $r < m$ and $r < s < k$ are chosen in a suitable way (which depends on F). These statistics were proposed and studied by Vännman (1975).

4. The asymptotic distributions

First let us summarize our new test statistics.

Against IFR:

$$A_1 = (D_n - D_1)/S_n \quad \text{(see (3.2))}$$

$$A_2 = \sum_{j=1}^{n} \alpha_j D_j / S_n \quad \text{(see (3.5))},$$

$$\text{with } \alpha_j = \frac{1}{6}\{(n+1)^3 j - 3(n+1)^2 j^2 + 2(n+1) j^3\}.$$

Against IFRA:

$$B = \sum_{j=1}^{n} \beta_j D_j / S_n \quad \text{(see (3.10))},$$

$$\text{with } \beta_j = \frac{1}{6}(n^3 + 3n^2 - 3n^2 j - 3nj - 3j^2 + j + 2n + 2j^3).$$

Against DMRL:

$$M_2 = \sum_{j=1}^{n} \gamma_j D_j / S_n \quad \text{(see (3.15))},$$

$$\text{with } \gamma_1 = 1-n,\ \gamma_j = 4j-3-2n,\ 2 \le j \le n.$$

Against heavy-tailedness:

$$E = n(1-u_{n-1}) \quad \text{(see (3.16))}.$$

We note that they are all scale invariant. Here A_2, B and M_2 are asymptotically normally distributed under rather general assumptions on F. Let us consider A_2.

By using that $D_j = (n-j+1)(t(j)-t(j-1))$ we can write the numerator of A_2 as

$$\sum_{j=1}^{n} \alpha_j D_j = \sum_{j=1}^{n} \alpha_j' t(j) .$$

Here

$$\alpha_j' = \frac{1}{6}(8nj^3-15n^2j^2+8n^3j-n^4) + p(n^a j^b), \tag{4.1}$$

where $p(n^a j^b)$ denotes a sum of terms of the form $n^a j^b$ with $a+b \leq 3$. These terms can be ignored without disturbing the asymptotic properties of A_2. This means that if

$$J_A(u) = \frac{1}{6}(8u^3-15u^2+8u-1)$$

then A_2/n^4 and $\Sigma_{j=1}^{n} J_A(j/n)t(j)/S_n$ has the same asymptotic properties.

With

$$\mu(J_A,F) = \int_0^\infty x\, J_A(F(x))dF(x) \tag{4.2}$$

$$\sigma^2(J_A,F) = \int_0^\infty \int_0^\infty J_A^*(F(x))J_A^*(F(y))\{F(\min(x,y))-F(x)F(y)\}dxdy \tag{4.3}$$

$$J_A^*(u) = J_A(u) - \frac{\overline{\mu(J_A,F)}}{\mu} \tag{4.4}$$

and

$$\mu = \int_0^\infty \bar{F}(x)dx$$

it can be proved, by using Slutsky's Theorem and Theorems 2 and 3 in Stigler (1974), that

(4.5) $$\mathcal{L}\left(\frac{\sqrt{n}\left(\frac{A_2}{n^4}-\frac{\mu(J_A,F)}{\mu}\right)}{\sigma(J_A,F)/\mu}\right) \to N(0,1) \quad \text{when} \quad n \to \infty$$

if $\int_0^\infty x^2 dF(x) < \infty$ and $\sigma^2(J_A,F) > 0$ (cf. Hollander and Proschan (1980)). Under H_0 we have $F(x) = F_0(x) = 1 - \exp(-\lambda x)$, $x \geq 0$. Straightforward calculations show that

$$\mu(J_A,F_0) = 0$$

and

$$\frac{\sigma^2(J_A,F_0)}{\mu^2(F_0)} = \frac{1}{7560}.$$

Consequently, under H_0 we get that

(4.6) $$\mathcal{L}(A_2\sqrt{7560/n^7}) \to N(0,1) \quad \text{when} \quad n \to \infty.$$

In the same manner we also have

(4.7) $$\mathcal{L}\left(\frac{\sqrt{n}\left(\frac{B}{n^3}-\frac{\mu(J_B,F)}{\mu}\right)}{\sigma(J_B,F)/\mu}\right) \to N(0,1) \quad \text{when} \quad n \to \infty,$$

where

$$J_B(u) = \tfrac{1}{6}(4-6u-6u^2+8u^3).$$

Under H_0 we get that

$$\mathcal{L}(B\sqrt{210/n^5}) \to N(0,1) \quad \text{when} \quad n \to \infty.$$

For M_2 we have

$$M_2 = \frac{\sum_{j=1}^{n} \delta_j t(j)}{\sum_{j=1}^{n} t(j)} + \frac{(n^2-n+1)t(1)}{\sum_{j=1}^{n} t(j)},$$

where

$$\delta_j = 8j - 6n - 3 \quad \text{for } j = 1, 2, \ldots, n.$$

From this it follows that if

(4.8) $\sqrt{n}\, t(1) \to 0$ in probability when $n \to \infty$

then

$$(4.9) \quad \mathcal{L}\left(\frac{\sqrt{n}\left(\frac{M_2}{n} - \frac{\mu(J_M,F)}{\mu}\right)}{\sigma(J_M,F)/\mu}\right) \to N(0,1) \quad \text{when } n \to \infty,$$

where

$$J_M(u) = 8u - 6.$$

The condition (4.8) can be written as

$$\lim_{n\to\infty} \bar{F}^n(x/\sqrt{n}) = 0 \quad \text{for } x > 0$$

and holds e.g. for $\bar{F}(x) = \exp(-(x/\alpha)^\theta)$, $x \geq 0$, when $\theta < 2$. In particular we get under H_0 that

$$\mathcal{L}\left(M_2\sqrt{3n/((n-1)(4n-5))}\right) \to N(0,1) \quad \text{when } n \to \infty.$$

We also point out that the asymptotic distribution under H_0 for A_2, B and M_2 can be derived by using Liapounov's Theorem (see e.g. Chung (1974), p. 185), Slutsky's Theorem and rather straightforward calculations.

5. Consistency

In this section we shall prove that the tests based on the asymptotically normally distributed statistics A_2, B and M_2 (if (4.8) holds) are consistent against the class of continuous IFR, IFRA and DMRL life distributions, respectively. For M_2 this follows from the fact that (if (4.8) holds) it is asymptotically equivalent to C (cf. p. 20) which is consistent against the larger class of continuous NBUE life distributions (see Hollander and Proschan (1975)). We concentrate on A_2 and B . Since these statistics are asymptotically normally distributed with $\mu(J_A,F_0) = \mu(J_B,F_0) = 0$, where F_0 denotes the exponential distribution, the consistency follows if $\mu(J_A,F) > 0$ when F is IFR (but not exponential) and $\mu(J_B,F) > 0$ when F is IFRA (but not exponential). Let

$$\Psi_A = \iiint_{\Omega_A} \left[\frac{\varphi(t+u)-\varphi(t)}{u} - \frac{\varphi(t+s)-\varphi(t)}{s}\right] su\, ds\, dt\, du\,,$$

where

$$\Omega_A = \{(s,t,u):\ 0\le u\le s,\ 0\le s\le 1-t,\ 0\le u\le 1\}$$

and

$$\Psi_B = \iint_{\Omega_B} \left[\frac{\varphi(t)}{t} - \frac{\varphi(s)}{s}\right] st\, ds\, dt\,,$$

where

$$\Omega_B = \{(s,t):\ 0\le s\le t,\ 0\le t\le 1\}.$$

If F is IFR (IFRA) but not exponential it follows that $\Psi_A > 0$ $(\Psi_B > 0)$. Hence the proof is complete if we can show that $\Psi_A = \mu(J_A,F)/\mu$ and $\Psi_B = \mu(J_B,F)/\mu$. A straightforward integration by parts gives that

$$\Psi_A = \frac{1}{6}\int_0^1 \varphi(u)du + \int_0^1 u\varphi(u)du - \int_0^1 u^2\varphi(u)du$$

and

$$\Psi_B = -\frac{1}{6}\int_0^1 \varphi(u)\,du + \int_0^1 u^2\varphi(u)\,du.$$

By using the fact that , for a continuous life distribution F,

$$\frac{1}{\mu}\int_0^\infty \bar{F}^{k+1}(x)\,dx = k\int_0^1 (1-u)^{k-1}\varphi(u)\,du \qquad \text{for} \quad k \geq 1$$

the statement now follows.

We can here give the reason why we got the test statistics V^* and K^* of Hollander and Proschan (1975) by considering the TTT-plot.

Our test statistic M_1 is proportional to

$$\sum_{i=0}^{n-1}\sum_{j=i+1}^{n}\left\{\frac{1-u_i}{1-\frac{i}{n}} - \frac{1-u_j}{1-\frac{i}{n}}\right\}\left(1-\frac{i}{n}\right)\left(1-\frac{j}{n}\right)\frac{1}{n}\frac{1}{n}$$

which is the sample analogue of

$$\iint_{0<s<t<1}\left[\frac{1-\varphi(s)}{1-s} - \frac{1-\varphi(t)}{1-t}\right](1-s)(1-t)\,ds\,dt.$$

With $s = F(x)$ and $t = F(y)$ this integral can be written

$$(5.1) \qquad \iint_{x<y}\left\{\frac{\int_x^\infty \bar{F}(u)\,du}{\bar{F}(x)} - \frac{\int_y^\infty \bar{F}(u)\,du}{\bar{F}(y)}\right\}\bar{F}(x)\bar{F}(y)\,dF(x)\,dF(y)$$

and V^* is the sample analogue of (5.1). For C and K^* the situation is analogous.

6. The asymptotic efficiency

When comparing consistent tests of a simple hypothesis $\theta = \theta_0$ against an alternative hypothesis $\theta > \theta_0$ (say) different measures of asymptotic efficiencies can be applied (see e.g. Rao (1965),pp. 390-396). When the test statistic T is asymptotically normally distributed with mean $\mu(\theta)$ and variance $\sigma^2(\theta)/n$ the most frequently used measure when testing $\theta = \theta_0$ against a sequence of alternative hypothesis $\theta = \theta_0 + cn^{-1/2}$

is the Pitman efficiency given by

(6.1) $E_F(T) = (\mu'(\theta_0))^2/\sigma^2(\theta_0).$

Here θ_0 corresponds to the exponential distribution. By using (4.3)-(4.8) we can now calculate the Pitman efficiency in the form

(6.2) $E_F(T) = \{\mu'(J_T,F)_{\theta=\theta_0}\}^2/\{\sigma^2(J_T,F)_{\theta=\theta_0}\}$

for $T = A_2$, B, M_2 and V^*.

We have calculated $E_F(T)$ for linear failure rate, Makeham, Pareto, Weibull and gamma alternatives given respectively by

$F_1(x) = 1 - \exp(-(x + \frac{1}{2}\theta x^2))$ for $x \geq 0,\ \theta \geq 0,$

$F_2(x) = 1 - \exp(-(x+\theta(x+e^{-x}-1))$ for $x \geq 0,\ \theta \geq 0,$

$F_3(x) = 1 - (1+\theta x)^{-1/\theta}$ for $x \geq 0,\ \theta \geq 0,$

$F_4(x) = 1 - \exp(-x^{\theta})$ for $x \geq 0,\ \theta > 0,$

$F_5(x) = \frac{1}{\Gamma(\theta)} \int_0^x t^{\theta-1} e^{-t} dt$ for $x \geq 0,\ \theta > 0.$

Here F_1, F_2, F_4 (for $\theta \geq 1$) and F_5 (for $\theta \geq 1$) are IFR and F_2, F_4 (for $\theta \leq 1$) and F_5 (for $\theta \leq 1$) are DFR.

For F_1, F_2 and F_3 we get H_0 for $\theta = \theta_0 = 0$ and for F_4 and F_5 when $\theta = \theta_0 = 1$.

Table 6.1 on p. 29 shows for each one of F_1-F_5 the asymptotic relative efficiencies of the test statistics compaired to the one with the largest Pitman efficiency value. The last column gives in each case the largest Pitman efficiency value of the included statistics.

TABLE 6.1 The Pitman asymptotic relative efficiency.

		T				
		A_2	B	M_2	V*	E_{max}
F	F_1	0.44	0.31	0.91	1.00	0.820
	F_2	0.70	0.70	1.00	0.70	0.083
	F_3	0.44	0.31	0.91	1.00	0.820
	F_4	0.51	0.87	1.00	0.49	1.441
	F_5	0.39	1.00	0.90	0.28	0.498

As a further comparison we mention that for K* (see p.19) , S (see (3.7) p. 17) and $W = \Sigma_{j=1}^{n} j\ell n(1-R_j/(n+1))$, where R_j is the range of D_j (see Bickel and Doksum (1969)), we have for all distributions F that

$$E_F(S) = \frac{3}{4} E_F(W) = \frac{3}{4} E_F(K^*) = \frac{3}{4} E_F(M_2).$$

(see Bickel and Doksum (1969)). It may also be of interest to know that for the test statistic T_4 (see p. 22) with optimal choices of $1 \leq r < s < k \leq n$ we have $E_{F_3}(T_4) = 0.96$ and $E_{F_4}(T_4) = 1.50$ (see Vännman (1975)).

The perhaps surprisingly small values of $E_{F_2}(T)$ in Table 6.1 depend on the fact that the failure rate of the Makeham distribution is equal to $1 + \theta(1-\exp(-t))$, which very soon becomes approximately constant.

7. A small sample study

In practice we often have small samples. Therefore an investigation of the different test statistics when the sample size is small is desirable. We have made a minor study with the different test statistics (cf. p. 22).

Against

IFR: A_1, A_2 ;

IFRA: B ;

NBUE: K* ;

DMRL: V* ;

heavy-tailedness: E.

However, before doing this study we need the lower and upper percentile points in the different cases. For A_1 and E they are easily calculated exactly. For V* and K* they are tabulated in Hollander and Proschan (1975) and Barlow et al. (1972), respectively. For A_2 and B we carried out a simulation study for n = 10 and n = 20 with 20000 replications each. See Table 7.1 and Table 7.2.

TABLE 7.1 Critical values of the IFR-statistic $A_2\sqrt{7560/n^7}$

	Lower tail			Upper tail		
	1%	5%	10%	1%	5%	10%
n = 10	-3.228	-2.395	-1.922	3.279	2.393	1.951
n = 20	-2.812	-1.968	-1.573	2.803	1.966	1.578

TABLE 7.2 Critical values of the IFRA-statistic $B\sqrt{210/n^5}$

	Lower tail			Upper tail		
	1%	5%	10%	1%	5%	10%
n = 10	-2.156	-1.671	-1.376	2.906	1.976	1.463
n = 20	-2.132	-1.625	-1.323	2.690	1.850	1.416

We have simulated the power of tests with significance level $\alpha = 0.05$ for some Weibull, Pareto and gamma alternatives (cf. p. 28), both for $n = 10$ and $n = 20$. We also included the life distribution

(7.1) $$F(t) = (1-e^{-3t})(1-e^{-7t}) \quad \text{for } t \geq 0$$

which is IFRA but not IFR (see Barlow and Proschan (1975), p. 83). The power estimates are based on 2000 simulations each. For an interesting comparison in the $n = 20$ case we also included the test statistic T_4 (see (3.19)), studied by Vännman (1975), in the following forms

(7.2) $$T_4W = t(20)/\sum_{j=1}^{7} t(j)$$

and

(7.3) $$T_4P = t(20)/\sum_{j=1}^{14} t(j).$$

The choices of the parameters k, s and r (to 20, 19 and 7(14), respectively) are done in such a manner that T_4W and T_4P have as large asymptotic efficiency as possible against the Weibull and Pareto alternative, respectively (see Vännman (1975)).

The lower and upper percentile points for T_4W and T_4P were simulated with 20000 replications each (see Tables 7.3 and 7.4).

The power estimates are given in Tables 7.5 and 7.6 on p.32.

TABLE 7.3 Critical values of the test statistic T_4W in (7.2)

Lower tail			Upper tail		
1%	5%	10%	1%	5%	10%
0.809	1.072	1.263	4.518	5.618	8.138

TABLE 7.4 Critical values of the test statistic T_4P in (7.3)

Lower tail			Upper tail		
1%	5%	10%	1%	5%	10%
0.229	0.275	0.309	0.834	0.985	1.318

TABLE 7.5 Power estimates based on 2000 samples of size $n = 20$ with significance level $\alpha = 0.05$.

	A_1	A_2	B	V*	K*	E	T_4W	T_4P
Weibull $\theta = 1.5$	0.44	0.37	0.62	0.30	0.65	0.08	0.63	0.50
Weibull $\theta = 2.0$	0.78	0.75	0.97	0.59	0.98	0.15	0.98	0.93
Weibull $\theta = 0.8$	0.18	0.27	0.33	0.27	0.35	0.16	0.35	0.29
F acc to (7.1)	0.35	0.12	0.39	0.08	0.31	0.07	0.31	0.19
Pareto $\theta = 1/2$	0.37	0.42	0.33	0.55	0.56	0.36	0.52	0.54
Gamma $\theta = 2.0$	0.53	0.30	0.65	0.21	0.62	0.08	0.60	0.41

TABLE 7.6 Power estimates based on 2000 samples of size $n = 10$ with significance level $\alpha = 0.05$.

	A_1	A_2	B	V*	K*	E
Weibull $\theta = 1.5$	0.30	0.24	0.40	0.20	0.37	0.09
Weibull $\theta = 2.0$	0.61	0.51	0.76	0.39	0.77	0.13
Weibull $\theta = 0.8$	0.15	0.17	0.20	0.17	0.21	0.12
F acc to (7.1)	0.21	0.10	0.23	0.10	0.19	0.07
Pareto $\theta = 1/2$	0.26	0.26	0.23	0.31	0.34	0.24
Gamma $\theta = 2.0$	0.33	0.17	0.37	0.15	0.34	0.09

Tables 7.5 and 7.6 show some interesting facts. We observe that of the two IFR statistics A_1 and A_2 the simpler A_1 has larger power values than A_2 in most cases. Further we note that the power values of A_1 are smaller than the corresponding values of B and K*. Observe however that K* is a test statistic against the larger class of NBUE distributions. We also note that the IFRA statistic B has power values quite comparable to the values of K*. For the IFRA distribution F defined by (7.1) the power values of B are even larger than those of K*. We observe that the test statistic E against heavy-tailedness has unpleasant small power values in all cases except the Pareto case. We also note that T_4W has larger power values than T_4P in all the Weibull cases but smaller power values when F is the Pareto distibution. This is of course expected.

8. An illustration

Barlow et al. (1972) studied times between air conditioner failures on selected aircrafts (see Barlow et al. (1972), pp. 269-271 and Proschan (1963)). By using the cumulative TTT-statistic $V = \Sigma_{k=1}^{n-1} u_k$ (cf. p. 19) Barlow et al. (1972) tested

H_0: F is the exponential distribution

against

H_1: F is DFR (but not exponential).

Table 8.1 (from Proschan (1963)) shows the failure data from five different aircrafts. When the data from a single airplane is tested H_0 is rejected at the significance level $\alpha = 0.05$ only for plane 7915.

TABLE 8.1 Intervals between failures of air conditioning equipments on aircrafts (from Proschan (1963)).

Plane					
7907	7908		7915	7916	8044
194	413	201	359	50	487
15	14	118	9	254	18
41	58	34	12	5	100
29	37	31	270	283	7
33	100	18	603	35	98
181	65	18	3	12	5
	9	67	104		85
	169	57	2		91
	447	62	438		43
	184	7			230
	36	22			3
		34			130

As an illustration we computed the values of the different test statistics studied in this report by using 20 values, randomly chosen, from the failure times of plane 7908 (the times 413, 118 and 18 were excluded). The result is summarized in Table 8.2 on p. 35. There is a good agreement with the result obtained by Barlow et al. (1972). The only exception is that E indicates a heavy-tailed distribution.

It is well-known that a mixture of distributions , all of which are DFR (including the exponential distribution), itself is DFR (see Barlow and Proschan (1975), p. 104). Therefore, if we pool together failure times from different aircrafts and regard this as a sample from a life distribution F we expect H_0 to be rejected.

As an illustration we chose the first five values in Table 8.1 from each of the planes 7907, 7908, 7916 and 8044 (we did not use plane 7915 since the failure intervals from this plane alone were shown by Barlow et al. (1972) to come from a DFR life distribution). Table 8.3 on p. 36 shows the values of the different statistics and the corresponding rejection areas. As we can see, H_0 is rejected at the significance level $\alpha = 0.05$ if we use A_2, B, V^* or K^*. H_0 is not rejected if we use A_1 or E. This may depend on the fact that these statistics are more sensitive to extreme values. Furthermore, Tables 7.5 and 7.6 on p. 32 indicate that A_1 has less power against DFR life distributions than A_2, B, V^* and K^*.

TABLE 8.2 Values of the test statistics for failure times from plane 7908.

	Values of the test statistics	Rejection area	
		$\alpha = 5\%$	$\alpha = 10\%$
A_1	0.064	> 0.114	> 0.081
$A_2\sqrt{7560/n^7}$	-0.944	< -1.968	< -1.573
$B\sqrt{210/n^5}$	-0.007	< -1.625	< -1.323
$V^*\sqrt{210n}$	-1.796	< -1.865	< -1.434
K^*	-0.063	< -0.104	< -0.081
E	0.047	< 0.050	< 0.100

TABLE 8.3 Values of the test statistics for five failure times from each of the planes 7907, 7908, 7916 and 8044.

	Values of the test statistics	Rejection area	
		α = 5%	α = 10%
A_1	-0.011	> 0.114	> 0.081
$A_2\sqrt{7560/n^7}$	-2.431	< -1.968	< -1.573
$B\sqrt{210/n^5}$	-1.484	< -1.625	< -1.323
$V^*\sqrt{210n}$	-1.973	< -1.865	< -1.434
K^*	-0.111	< -0.104	< -0.081
E	0.533	< 0.050	< 0.100

As a further illustration we have included the TTT-plots based on the two samples. See Figures 8.1 and 8.2.

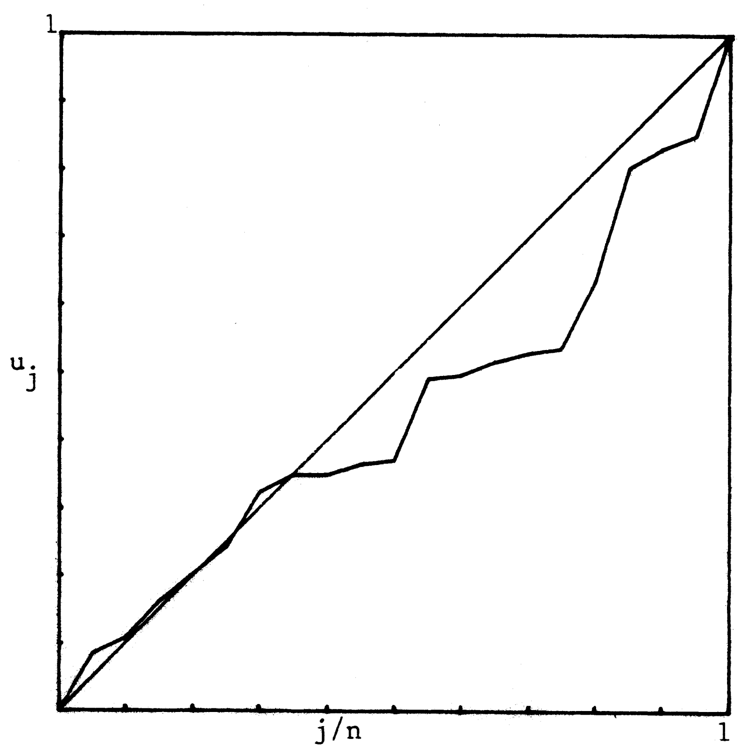


Figure 8.1 TTT-plot based on 20 failure times from plane 7908 (see Table 8.1).

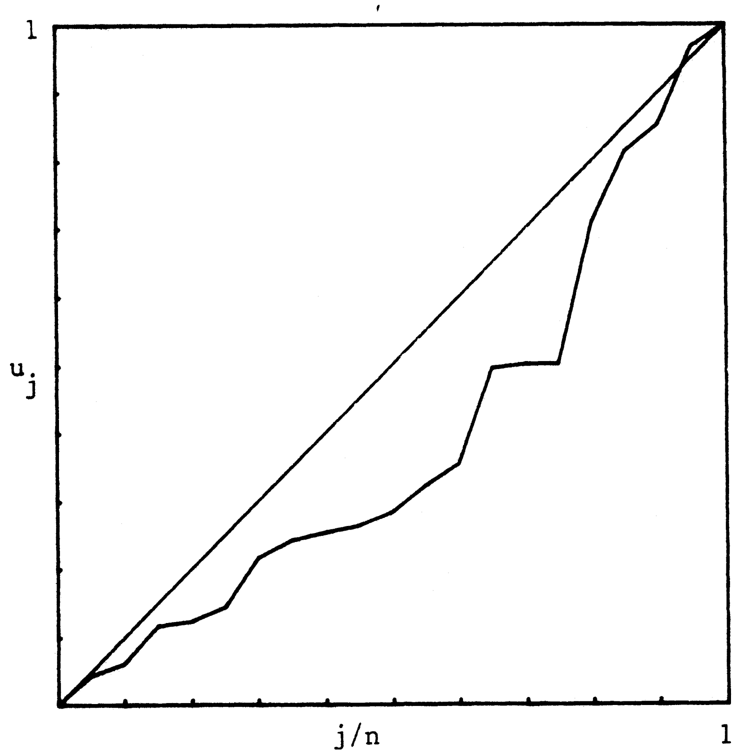


Figure 8.2 TTT-plot based on five failure times from each of the planes 7907, 7908, 7916 and 8044 (see Table 8.1).

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