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SOME THEOREMS ON GRAPH CONGRUENCES (*)

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Abstract. — *We prove a theorem on graph congruences. This theorem is the key step for the characterization of syntactic semigroups of languages of dot-depth at most one.*

Résumé. — *On démontre un théorème sur les congruences de graphe. Ce théorème est utilisé de façon cruciale dans la caractérisation des langages de hauteur* 1 *dans la hiérarchie de Brzozowski.*

1. INTRODUCTION

In proving the correspondence between certain varieties of languages and semigroups, one of the key steps is a theorem on directed graphs, more precisely on graph congruences. The first theorem of this kind, appeared originally in [1] in the proof of the correspondence between locally testable languages and locally idempotent and commutative semigroups, though it was not formulated as a separate result on graphs. The treatment of this result as a theorem on graph congruence is due to Eilenberg [2, pp. 222-228].

Let m be an integer, $m \geqq 1$ and let ${}_m\sim$ relate any coterminal paths which traverse the same set of m-tuples of edges. In [4] Simon has proved that the family of all $\mathcal{J}$-trivial congruences of finite index corresponds to the family of congruences covered by ${}_m\sim$ for some m, when the underlying graph consists of one vertex (Simon's result was not formulated as a theorem on graphs). In the paper, we show that this is not true, when the underlying graph has more vertices than one. We prove (Theorem 2) that the family of graph congruences covered

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by ${}_m\sim$ for some m, corresponds to the family of all dot $(\mathcal{J})$-trivial graph congruences of finite index, where dot$(\mathcal{J})$ is the concatenation closure of the Green relation $\mathcal{J}$, or equivalently, it is the smallest congruence covered by $\mathcal{J}$. This result is used in [3] for the characterization of syntactic semigroups of languages with dot-depth at most one.

2. PRELIMINARIES

Let A be a non-empty, finite set, called alphabet. The cardinality of A will be denoted by $|A|$. A^+ (respectively, A^*) is the free semigroup (respectively, free monoid) generated by A. Elements of A^* are called words. The empty word in A^* is denoted by λ (identity of A^*). The concatenation of two words $x, y \in A^*$ is denoted by xy. The length of a word x is denoted by $|x|$.

Let $\sim$ be an equivalence relation on A^*. For $x \in A^*$ $[x]_\sim$ means the equivalence class of $\sim$ containing x. An equivalence relation on A^* is a congruence iff for $x, y \in A^*$, $x \sim y$ implies $uxv \sim uyv$ for all $u, v \in A^*$.

For terminology related to graphs we follow Eilenberg's monograph [2].

A directed graph G consists of two sets, an alphabet A and the set of vertices V along with two functions: $\alpha, \omega : A \to V$. Elements of A are also called edges in this case.

Two letters (or edges) $a, b \in A$ are called consecutive if $a\omega = b\alpha$. Let $D \subset A^2$ be the set of all words ab such that a and b are non-consecutive. Then the set of all paths of G is:

$$P = A^+ - A^* D A^*.$$

Functions α, ω can be extended to $\alpha, \omega : P \to V$ in the following way: if $x = a_1 a_2 \ldots a_n \in P$, then $x\alpha = a_1\alpha$, $x\omega = a_n\omega$, where $a_1, a_2, \ldots, a_n \in A$, $n \geqq 1$. For each vertex $v \in V$ we adjoint to P a trivial path 1_v; $1_v\alpha = 1_v\omega = v$.

A path x is called a loop, if $x\alpha = x\omega$. We say that two paths x and y are consecutive if $x\omega = y\alpha$. In this case the concatenation xy is again a path. Two paths x and y are coterminal, if $x\alpha = y\alpha$ and $x\omega = y\omega$.

For any two binary relations $\sim_1$ and $\sim_2$ on P we say that $\sim_1$ is greater than $\sim_2$ (or $\sim_1$ is covered by $\sim_2$), we write $\sim_1 \supseteqq \sim_2$, if for any $x, y \in$ P $x \sim_2 y$ implies $x \sim_1 y$.

An equivalence relation $\sim$ on P is called a graph congruence if it satisfies the following conditions:

(i) if $x \sim y$, then x and y are coterminal;

(ii) if $x \sim y$ and $w \sim z$, and x, w are consecutive, then $xw \sim yz$.

In this paper we shall deal only with graph congruences of finite index. Now we define three basic families of graph congruences which we investigate:

(i) Let $(a_1, a_2, \ldots, a_m) \in A \times A \times \ldots \times A$ (m-times), $m \geqq 1$. We shall write $(a_1, a_2, \ldots, a_m) \in x$ and say that $(a_1, a_2, \ldots, a_m)$ appears in x, $x \in A^*$, if $x = x_0 a_1 x_1 a_2 x_2 \ldots a_m x_m$ for some $x_0, x_1, \ldots, x_m \in A^*$.

For each integer m, $m \geqq 1$ and for each $x \in A^*$ define:

$$x \tau_m = \{(a_1, a_2, \ldots, a_n) \mid m \geqq n \geqq 1, (a_1, a_2, \ldots, a_n) \in x\}.$$

Instead of τ_1 we simply write τ. Now, we can define a graph congruence ${}_m\!\sim$ on P as follows: for $x, y \in P$.

$x \,{}_m\!\sim y$ iff x and y are coterminal and $x \tau_m = y \tau_m$. By convention we set $1_v \tau_m = \emptyset$ for any vertex v, $v \in V$. It is easily verified that ${}_m\!\sim$ is a graph congruence of finite index on P.

(ii) For any n, $n \geqq 1$, let us define a binary relation ${}_n\!-$ on P, in the following way: for $x, y \in P$:

$$x \,{}_n\!- y \text{ iff } x = x' x_1 x_2 \ldots x_n x'' \quad \text{and} \quad y = x' y_1 y_2 \ldots y_n x'',$$

for some $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that $x_i \tau = y_j \tau$, $i, j = 1, 2, \ldots, n$, and $x_1 x_2 \ldots x_n$, $y_1 y_2 \ldots y_n$ are coterminal paths.

Define ${}_n\!=$ to be the reflexive and transitive closure of ${}_n\!-$.

Equivalently, ${}_n\!=$ is the smallest graph congruence on P satisfying the condition: $x_1 x_2 \ldots x_n \,{}_n\!= y_1 y_2 \ldots y_n$ whenever $x_i \tau = y_j \tau$, $i, j = 1, 2, \ldots, n$.

(iii) For any n, $n \geqq 1$, let us define a binary relation ${}_n\!\approx$ on P as follows: for $x, y \in P$, $x \,{}_n\!\approx y$ iff:

$$x = x' x_1 x_2 \ldots x_n u_1 u_2 \ldots u_n x'',$$
$$y = x' y_1 y_2 \ldots y_n w_1 w_2 \ldots w_n x'',$$

for some:

$$x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_n, y_1, y_2, \ldots, y_n, w_1, w_2, \ldots, w_n,$$

such that:

$$x_i \tau = y_j \tau, \qquad u_i \tau = w_j \tau \qquad \text{for} \quad i, j = 1, 2, \ldots, n$$

and:

$$x_1 x_2 \ldots x_n u_1 u_2 \ldots u_n \qquad \text{and} \qquad y_1 y_2 \ldots y_n w_1 w_2 \ldots w_n$$

are coterminal paths.

Define ${}_n\simeq$ to be the reflexive and transitive closure of ${}_n\approx$.

Equivalently, ${}_n\simeq$ is the smallest graph congruence on P satisfying the condition: $x_1 x_2 \ldots x_n u_1 u_2 \ldots u_n \;{}_n\simeq\; y_1 y_2 \ldots y_n w_1 w_2 \ldots w_n$, whenever $x_i \tau = y_j \tau$ and $u_i \tau = w_j \tau$ for $i, j = 1, 2, \ldots, n$.

NOTATION: Let $A_1, A_2, \ldots, A_h \subseteqq A, h \geqq 1$. Then $(A_1, A_2, \ldots, A_h)$ will denote the set of k-tuples:

$$(a_1^1, a_1^2, \ldots, a_1^{k_1}, a_2^1, a_2^2, \ldots, a_2^{k_2}, \ldots, a_h^1, a_h^2, \ldots, a_h^{k_h})$$

such that:

$$\{a_i^1, a_i^2, \ldots, a_i^{k_i}\} = A_i, \qquad |A_i| = k_i,$$

$$k = \sum_{i=1}^{h} |A_i|, \qquad i = 1, 2, \ldots, h.$$

If $A_1 = A_2 = \ldots = A_h$ we denote this set by (A_1^h). By $(A_1, A_2, \ldots, A_h) \in x$ for $x \in A^*$, we mean that there is at least one k-tuple from the set $(A_1, A_2, \ldots, A_h)$ which appears in x.

Let $\sim$ be any graph congruence on P. We adapt here Green relation $\mathscr{J}$ for graph congruences. For $x, y \in P$:

$x \mathscr{J} y$ iff there are paths z_1, z_2, z_3 and z_4 such that $z_1 x z_2 \sim y$ and $z_3 y z_4 \sim x$.

However, we will also need the concatenation closure of $\mathscr{J}$, denoted by $\operatorname{dot}(\mathscr{J})$, and defined as follows: for $x, y \in P$:

$x \operatorname{dot}(\mathscr{J})\, y$ iff for some n, $n \geqq 1$, $x = x_1 x_2 \ldots x_n$, $y = y_1 y_2 \ldots y_n$ and $x_i \mathscr{J} y_i$ for $i = 1, 2, \ldots, n$.

We will say that a graph congruence $\sim$ on P is $\mathscr{J}$ (or $\operatorname{dot}(\mathscr{J})$)-trivial if for any coterminal paths $x \mathscr{J} y$ (or $x \operatorname{dot}(\mathscr{J})\, y$, respectively) implies $x \sim y$.

3. RESULTS

The aim of this paper is to show that the family of $\mathscr{J}$-trivial graph congruence does not correspond to the family of graph congruences covered by the graph congruence ${}_m\sim$ for some $m \geqq 1$, when the number of vertices of the underlying graph is greater than 1. In opposite, Simon [4] has proved that this is the case, if the underlying graph has exactly one vertex. The following example is suggested by our results. Let $V = \{1, 2\}$, $A = \{a, b, c, d\}$ and $a\alpha = c\alpha = b\omega = d\omega = 1$, $a\omega = c\omega = b\alpha = d\alpha = 2$. Define the congruence $\sim$ by its congruence classes:

$$\{1_1\}, \{1_2\}, a(ba)^*, (ab)^+, (ab)^+\ c(dc)^*, (ab)^+\ (cd)^+, b(ab)^*,$$
$$(ba)^+, bc(dc)^*, b(cd)^+, c(dc)^*, (cd)^+, d(cd)^*, (dc)^+$$

and four classes containing all other paths, according to the coterminality.

This congruence is $\mathcal{J}$-trivial, but $(ab)^+ (cd)^+ \not\sim (ab)^+ ad(cd)^+$. Thus for any m we have $\sim \not< {}_m\sim$. Of course, if we consider that vertices 1 and 2 represent the same vertex, then our congruence comes to be not $\mathcal{J}$-trivial. In fact, we have $a(ba)^* \mathcal{J} (ab)^+$, however, in the case of two vertices 1 and 2, these classes are not coterminal.

THEOREM 1: *For any graph congruence of finite index on P the following are equivalent:*

(*a*) $\sim$ *is* $\mathcal{J}$*-trivial;*

(*b*) *there exists an integer n,* $n \geqq 1$*, such that for all loops u, v about the same vertex:*

$$u(vu)^n \sim (vu)^n \sim (vu)^n v;$$

(*c*) *there exists an integer m,* $m \geqq 1$*, such that* $\sim < {}_m=$.

THEOREM 2: *For any graph congruence of finite index on P the following are equivalent:*

(*a*) $\sim$ *is dot* ($\mathcal{J}$)*-trivial;*

(*b*) *there exists an integer n,* $n \geqq 1$*, such that for all loops* $u_1 u_2$ *and* $v_1 v_2$ *about the same vertex, where paths* u_1 *and* v_1 *are coterminal:*

$$(u_1 u_2)^n u_1 v_2 (v_1 v_2)^n \sim (u_1 u_2)^n (v_1 v_2)^n;$$

(*c*) *there exists an integer n,* $n \geqq 1$*, such that* $\sim <_n \simeq$;

(*d*) *there exists an integer m,* $m \geqq 1$*, such that* $\sim <_m \sim$.

4. PROOF OF THEOREM 1

(*a*) $\Rightarrow$ (*c*): Let $\sim$ be a $\mathcal{J}$-trivial congruence of finite index on *P*. From the definition of congruence ${}_m=$ it follows that it is sufficient to show that $x_1 x_2 \ldots x_m \sim y_1 y_2 \ldots y_m$ whenever $x_1 x_2 \ldots x_m$ and $y_1 y_2 \ldots y_m$ are coterminal, and $x_i \tau = y_j \tau$ $(i, j = 1, 2, \ldots, m)$ for some *m*. Since $\sim$ is $\mathcal{J}$-trivial, it is sufficient to show that $x_1 x_2 \ldots x_m \mathcal{J} y_1 y_2 \ldots y_m$ whenever $x_i \tau = y_j \tau$ for some *m*. We prove this by the following:

LEMMA 3: *Let* $\sim$ *be a* $\mathcal{J}$*-trivial congruence of finite index on P. Then for* $m \geqq 2$ (*index* $\sim +1$):

$$x_1 x_2 \ldots x_m \mathcal{J} y_1 y_2 \ldots y_m,$$

whenever $x_i \tau = y_j \tau$ *for* $i, j = 1, 2, \ldots, m$.

Proof: We may assume that $m=2(\text{index} \sim +1)$. Since $x_i\tau=y_j\tau$ for any i and j, then for any k, $k=1, 2, \ldots, m/2$ we may write $x_{2k}=x'_{2k}x''_{2k}$ for x'_{2k} such that all paths $x_1 x_2 \ldots x_{2k-1} x'_{2k} (k=1, 2, \ldots, m/2)$ are coterminal. By the choice of m, there exist k_1 and k_2, $1 \leqq k_1 < k_2 \leqq m/2$ such that:

$$(0) \qquad x_1 x_2 \ldots x_{2k_1-1} x'_{2k_1} \sim x_1 x_2 \ldots x_{2k_2-1} x'_{2k_2}.$$

We claim that for any path z such that $z\alpha=(x_1 x_2 \ldots x_{2k_2-1} x'_{2k_2})\omega$ and $z\tau \subseteqq x_i\tau$ we have:

$$(1) \qquad x_1 x_2 \ldots x_{2k_1-1} x'_{2k_1} \mathcal{J}\, x_1 x_2 \ldots x_{2k_2-1} x'_{2k_2} z.$$

We apply the induction on the length of z. If $|z|=0$, (1) follows by (0). Let $z=wr$, $|w| \geqq 0$ and $r \in x_i\tau$. Now, $k_2 > k_1$ implies that:

$$x_1 x_2 \ldots x_{2k_2-1} x'_{2k_2} wr = x_1 x_2 \ldots x_{2k_1-1} x'_{2k_1} urvwr,$$

for some $u, v \in P$ such that $x_1 x_2 \ldots x_{2k_2-1} x'_{2k_2} = x_1 x_2 \ldots x_{2k_1-1} x'_{2k_1} urv$. Evidently, paths $x_1 x_2 \ldots x_{2k_2-1} x'_{2k_2} w$ and $x_1 x_2 \ldots x_{2k_1-1} x'_{2k_1} u$ are coterminal. By the induction assumption:

$$x_1 x_2 \ldots x_{2k_2-1} x'_{2k_2} w \,\mathcal{J}\, x_1 x_2 \ldots x_{2k_1-1} x'_{2k_1}.$$

Hence:

$$x_1 x_2 \ldots x_{2k_2-1} x'_{2k_2} w \,\mathcal{J}\, x_1 x_2 \ldots x_{2k_1-1} x'_{2k_1} u.$$

Since $\sim$ is a $\mathcal{J}$-trivial graph congruence, we have:

$$x_1 x_2 \ldots x_{2k_2-1} x'_{2k_2} wr \sim x_1 x_2 \ldots x_{2k_1-1} x'_{2k_1} ur.$$

Consequently, by (0):

$$x_1 x_2 \ldots x_{2k_2-1} x'_{2k_2} wr \,\mathcal{J}\, x_1 x_2 \ldots x_{2k_1-1} x'_{2k_1}.$$

Thus the claim holds.

By this claim $x_1 x_2 \ldots x_{2k_1-1} x'_{2k_1} \mathcal{J}\, x_1 x_2 \ldots x_m$. Since $x_i\tau=y_j\tau$, we can find $u, v \in P$ such that $x_m=uv$ and $u\omega=y_1\alpha$. Hence by the claim, $x_1 x_2 \ldots x_{m-1} u\, y_1 y_2 \ldots y_m \mathcal{J}\, x_1 x_2 \ldots x_m$. By symmetry, $y_1 y_2 \ldots y_{m-1} u_1 x_1 x_2 \ldots x_m \mathcal{J}\, y_1 y_2 \ldots y_m$ for some u_1 such that $y_m=u_1 v_1$ and $u_1\omega=x_1\alpha$. Thus $x_1 x_2 \ldots x_m \mathcal{J}\, y_1 y_2 \ldots y_m$. □

$(c) \Rightarrow (e)$: Congruence ${}_m{=}$ satisfies (b) for $n=m$. Hence, also $\sim$ satisfies (b).

$(b) \Rightarrow (a)$: Let $x \mathcal{J} y$ and let x, y be coterminal paths. By the definition $x \sim z_1 y z_2$ and $y \sim z_3 x z_4$ for some loops z_1, z_3 about the same vertex and for some loops z_2, z_4 about the same vertex. Since $\sim$ is a graph congruence, then $x \sim (z_1 z_3)^n x (z_4 z_2)^n$. Consequently, (*b*) implies $x \sim z_3 x z_4 \sim y$. □

5. PROOF OF THEOREM 2

(*a*) ⇒ (*c*): Let $\sim$ be a dot$(\mathcal{J})$-trivial congruence of finite index on P. From the definition of congruence ${}_n\simeq$ it follows that it is sufficient to show that there is n, $n \geqq 1$ such that:

$$x_1 x_2 \ldots x_n u_1 u_2 \ldots u_n \sim y_1 y_2 \ldots y_n w_1 w_2 \ldots w_n,$$

whenever $x_1 x_2 \ldots x_n u_1 u_2 \ldots u_n$ and $y_1 y_2 \ldots y_n w_1 w_2 \ldots w_n$ are coterminal paths and $x_i \tau = y_j \tau$, $u_i \tau = w_j \tau$ $(i, j = 1, 2, \ldots, n)$. Since $\sim$ is dot $(\mathcal{J})$-trivial, it follows that $\sim$ is also $\mathcal{J}$-trivial. By Lemma 1 for $n \geqq 2$ (index $\sim +1$) $x_1 x_2 \ldots x_n \mathcal{J} y_1 y_2 \ldots y_n$ and $u_1 u_2 \ldots u_n \mathcal{J} w_1 w_2 \ldots w_n$. Hence, by the definition of dot $(\mathcal{J})$-triviality:

$$x_1 x_2 \ldots x_n u_1 u_2 \ldots u_n \sim y_1 y_2 \ldots y_n w_1 w_2 \ldots w_n. \quad \square$$

(*c*) ⇒ (*d*): We will prove that for each n there exists an integer m, $m \geqq 1$, such that ${}_n\simeq\, < {}_m\sim$. We claim that it is sufficient to set:

$$m = m(n, k) = n^2 \sum_{j=1}^{k} \sum_{i=1}^{j} i = \frac{n^2}{6} k(k+1)(k+2),$$

where $k = |A|$. The proof is by induction on k.

$k = 1$

Then for $m = m(n, 1) = n^2$, $x\,{}_m\sim y$ implies $|x|, |y| \geqq n^2$ or $x = y$. Consequently, $x\,{}_n\simeq y$.

General induction assumption

If $|A| = k \geqq 1$, ${}_n\simeq\, < {}_m\sim$ for $m = m(n, k)$.

Now, let $|A| = k+1$, $k \geqq 1$, and let $x\,{}_m\sim y$ for $m = m(n, k+1)$. For x we define a unique factorization of x as follows: $x = x_1 x_2 \ldots x_p x_{p+1}$, where for $i = 1, 2, \ldots, p$, $p \geqq 0$, x_i is the shortest prefix of $x_i x_{i+1} \ldots x_p x_{p+1}$ such that $x_i \tau = A$, and $x_{p+1} \tau \subsetneqq A$. If $p \geqq n$, then $m = m(n, k+1) > n(k+1)$ implies that the similar factorization of y, namely $y_1 y_2 \ldots y_r y_{r+1}$, must be such that $r \geqq n$. Hence, by the definition, $x\,{}_n\simeq y$.

Assume $p < n$. Then $m = m(n, k+1) > n(k+1)$ implies that $r = p$. Let us define $m(n, k+1, p) = m(n, k) + p.n. \sum_{i=1}^{k+1} i$. Evidently, $m(n, k+1) = m(n, k+1, n)$. We prove that if the above factorizations of x and y are $x_1 x_2 \dots x_p x_{p+1}$ and $y_1 y_2 \dots y_p y_{p+1}$ respectively, $0 \leqq p < n$, then for $m = m(n, k+1, p)$ $x\,{}_m\!\sim y$ implies $x\,{}_n\!\simeq y$. We apply the induction on p.

$p = 0$

It follows that $x\tau = y\tau \nsubseteq A$. Since $m = m(n, k+1, 0) = m(n, k)$, then by the general induction assumption $x\,{}_m\!\sim y$ implies $x\,{}_n\!\simeq y$.

Induction assumption for p

If $x = x_1 x_2 \dots x_p x_{p+1}$ and $y = y_1 y_2 \dots y_p y_{p+1}$ are factorizations as above for some p, $0 \leqq p < n-1$ then for $m = m(n, k+1, p)$ $x\,{}_m\!\sim y$ implies $x\,{}_n\!\simeq y$.

Let $r = p+1$ and let $x = x_1 x_2 \dots x_r x_{r+1}$ and $y = y_1 y_2 \dots y_r y_{r+1}$ be the factorization as above. Assume $x\,{}_m\!\sim y$ for $m = m(n, k+1, r)$.

Consider $x_r x_{r+1}$. Let a be the last letter of x_r. One can write $x_r x_{r+1} = x'' x'$, where x' is the shortest suffix of $x_r x_{r+1}$ such that $x' \tau = A$. Let b be the first letter of x'. There are two cases which we investigate separately:

(1) $|x_r| = |x''| + 1$ i.e. $a = b$. Then $x_r x_{r+1} = zat$ for some $z, t \in A^*$ and $a \notin z\tau \cup t\tau$;

(2) if $|x_r| > |x''| + 1$, then $x_r x_{r+1} = zbwat$ for some $z, w, t \in A^*$, $a \neq b$ and $a \notin (zbw)\tau$, $b \notin (wat)\tau$.

(1) In this case, $m = m(n, k+1, r) > (r-1)(k+1) + 2$ and $x\,{}_m\!\sim y$ imply that $y_r y_{r+1} = uav$ for some $u, v \in A^*$ such that $a \notin u\tau \cup v\tau$. Also, by the same argument $u\tau = z\tau \nsubseteq A$ and $t\tau = v\tau \nsubseteq A$.

Hence, $x = x_1 x_2 \dots x_{r-1} zat$ and $y = y_1 y_2 \dots y_{r-1} uav$. Since $a \notin z\tau \cup t\tau$, then for $q = m(n, k+1, r) - (r-1)(k+1) - 1$, $(A^{r-1}, a, a_1, a_2, \dots, a_q) \underline{\in} x (\underline{\in} y)$ iff $(a_1, a_2, \dots, a_q) \underline{\in} t (\underline{\in} v$, respectively). Hence $x\,{}_m\!\sim y$ implies $t\,{}_q\!\sim v$. Since $q > m(n, k)$, the by the general induction assumption $t\,{}_n\!\simeq v$.

Similarly, $(a_1, a_2, \dots, a_q, a) \underline{\in} x (\underline{\in} y)$ iff $(a_1, a_2, \dots, a_q) \underline{\in} x_1 x_2 \dots x_{r-1} z$ $(\underline{\in} y_1 y_2 \dots y_{r-1} u$, respectively). Hence, $x\,{}_m\!\sim y$ implies $x_1 x_2 \dots x_{r-1} z$ ${}_q\!\sim y_1 y_2 \dots y_{r-1} u$ for $q = m(n, k+1, r) - 1$. Consequently, by the induction assumption for $p = r-1$, we obtain $x_1 x_2 \dots x_{r-1} z\,{}_n\!\simeq y_1 y_2 \dots y_{r-1} u$.

Altogether, since ${}_n\!\simeq$ is a graph congruence, we have $x\,{}_n\!\simeq y$.

(2) As in (1), $m = m(n, k+1, r) > (r-1)(k+1) + 2$ and $x\,{}_m\!\sim y$ imply that $y_r y_{r+1} = ubsav$ for some $u, s, v \in A^*$ such that $a \notin (ubs)\tau$ and $b \notin (sav)\tau$.

In this part of the proof we shall use certain special factorizations defined as follows: for $z \in A^+$ let $z = z_1 z_2 \dots z_l$ $(l \geqq 1)$ be a factorization of z such that for

$i=1, 2, \ldots, l z_i$ is the shortest prefix of $z_i z_{i+1} \ldots z_l$ such that $z_i \tau = (z_i z_{i+1} \ldots z_l)\tau \neq \emptyset$. Of course, $z_i \tau \supseteq z_{i+1}\tau$. Such factorization always exists and it is unique. For $z=\lambda$ we assume that $l=0$. Now, for $z \in A^+$ and for an integer n, $n \geqq 1$, we define the left n-factorization of z as follows:

(i) If for some j, $z_j \tau = z_{j+n}\tau$, where $j, j+n \in \{1, 2, \ldots, l\}$, then for the smallest j with this property we define the left n-factorization as $z_1 z_2 \ldots z_{j-1} z^1 z^2 \ldots z^{n+1}$, where $z^i = z_{j+i-1}$, $z^{n+1} = z_{j+n} \ldots z_l$, $i=1, 2, \ldots, n$.

(ii) Otherwise, if such j does not exist, we define the left n-factorization as $z_1 z_2 \ldots z_l z^1 z^2 \ldots z^{n+1}$, where $z^i = \lambda$ i. e. $z^i \tau = \emptyset$ for $i=1, 2, \ldots, n+1$.

By the left-right duality we also define the right n-factorization of z in the form $z^{n+1} z^n \ldots z^1 z_g z_{g-1} \ldots z_1$ for $g \geqq 0$.

The following observation follows directly from the definitions:

LEMMA 4: *Let* $z, u \in A^*$ *and* $|z\tau| = k$. *Then* $z \,{}_q\!\sim u$ (*or* $z\tau_q = u\tau_q$) *for* $q \geqq (n+1) \sum_{i=1}^{k} i$ *implies that left n-factorizations of z and u are the same in the sense that* $z = z_1 z_2 \ldots z_h z^1 z^2 \ldots z^{n+1}$, $u = u_1 u_2 \ldots u_h u^1 u^2 \ldots u^{n+1}$ *and* $z_i \tau = u_i \tau$, $(i=1, 2, \ldots, h)$, $z^1 \tau = u^1 \tau$.

The similar observation is true for the right factorizations.

So far, in case (2), we have that $x = x_1 x_2 \ldots x_{r-1} zbwat$ and $y = y_1 y_2 \ldots y_{r-1} ubsav$, and $x \,{}_m\!\sim y$ for $m = m(n, k+1, r)$. Now, let us observe that $(A^{r-1}, a_1, a_2, \ldots, a_q, b) \underline{\in} x (\underline{\in} y)$ iff $(a_1, a_2, \ldots, a_q) \underline{\in} z$ ($\underline{\in} u$, respectively). Hence, $x \,{}_m\!\sim y$ implies that $(zb)\tau_q = (ub)\tau_q$ for $q = m(n, k+1, r)-(r-1)(k+1)-1$. Since $q > (n+1) \sum_{i=1}^{k} i$, by Lemma 4 the left n-factorizations of zb and ub are respectively, $z_1 z_2 \ldots z_h z^1 z^2 \ldots z^{n+1}$ and $u_1 u_2 \ldots u_h u^1 u^2 \ldots u^{n+1}$, where $u_i \tau = z_i \tau$ $(i=1, 2, \ldots, h)$, $h \geqq 0$, and $z^1 \tau = u^1 \tau$.

Similarly, $(A^{r-1}, a, a_1, a_2, \ldots, a_q) \underline{\in} x (\underline{\in} y)$ iff $(a_1, a_2, \ldots, a_q) \underline{\in} t$ ($\underline{\in} v$, respectively) for $q \geqq 1$. Hence, $x \,{}_m\!\sim y$ implies that $at \,{}_q\!\sim av$ for $q = m(n, k+1, r)-(r-1)(k+1)-1$. Again, since $q > (n+1) \sum_{i=1}^{k} i$, by the right-left duality and Lemma 2, the right n-factorizations of at and av are respectively:

$$t^{n+1} t^n \ldots t^1 t_g t_{g-1} \ldots t_1 \quad \text{and} \quad v^{n+1} v^n \ldots v^1 v_g v_{g-1} \ldots v_1,$$

where $t^1 \tau = v^1 \tau$ and $t_j \tau = v_j \tau$ $(j=1, 2, \ldots, g)$, $g \geqq 0$.

Our proving way now will depend on the letter content of w:

2(A) if $w = w_1 w_2$ for some $w_1, w_2 \in A^*$ such that $w_1 \tau \subseteq z^1 \tau$ and $w_2 \tau \subseteq t^1 \tau$;

2(B) if $w=w_1\gamma w_2\beta w_3$ for $w_1, w_2, w_3\in A^*, \gamma, \beta\in A$ such that $w_1\tau\subseteqq z^1\tau$, $w_3\tau\subseteqq t^1\tau$ and $\gamma\notin z^1\tau$, $\beta\notin t^1\tau$.

2(C) if $w=w_1\gamma w_3$ for $\gamma\notin z^1\tau\cup t^1\tau$, $w_1\tau\subseteqq z^1\tau$ and $w_3\tau\subseteqq t^1\tau$.

Now, if w is of type 2(B), then $x\,_m\!\sim y$ implies that $(A^{r-1}, z_1\tau, z_2\tau, \ldots, z_h\tau, \gamma, \beta, t_g\tau, t_{g-1}\tau, \ldots, t_1\tau)\underline{\in} x$ iff $(A^{r-1}, z_1\tau, z_2\tau, \ldots, z_h\tau, \gamma, \beta, t_g\tau, t_{g-1}\tau, \ldots, t_1\tau)\underline{\in} y$, because:

$$m=m(n, k+1, r)>(r-1)(k+1)+2n\sum_{i=1}^{k} i+2$$

$$\geqq(r-1)(k+1)+\sum_{i=1}^{h}|z_j\tau|+\sum_{j=1}^{g}|t_j\tau|+2,$$

and $z_i\tau=u_i\tau$, $t_j\tau=v_j\tau$ $(i=1, 2, \ldots, h, j=1, 2, \ldots, g)$. Hence, the conditions $\gamma\notin z^1\tau=u^1\tau$ and $\beta\notin t^1\tau=v^1\tau$ imply that $s=s_1\gamma s_2\beta s_3$ for some $s_1, s_2, s_3\in\Sigma^*$ such that $\gamma\notin s_1\tau$ and $\beta\notin s_3\tau$, but *not* necessarily $s_1\tau\subseteqq z^1\tau$, $s_3\tau\subseteqq t^1\tau$. If $w=w_1\gamma w_3$, then similarly $s=s_1\gamma s_3$. By this, if w is of type 2(A), then $s=s_1 s_2$ for $s_1\tau\subseteqq z^1\tau$ and $s_2\tau\subseteqq t^1\tau$.

2 (A) We have:

$$x=x_1 x_2 \ldots x_{r-1} z_1 z_2 \ldots z_h z^1 z^2 \ldots z^{n+1} w_1 w_2 t^{n+1} t^n \ldots t^1 t_g t_{g-1} \ldots t_1$$

and:

$$y=y_1 y_2 \ldots y_{r-1} u_1 u_2 \ldots u_h u^1 u^2 \ldots u^{n+1} s_1 s_2 v^{n+1} v^n \ldots v^1 v_g v_{g-1} \ldots v_1.$$

Since $z^1\tau=u^1\tau$, there are factorizations $z^1=z_1^1 z_2^1$ and $u^1=u_1^1 u_2^1$ such that $z_1^1\omega=u_1^1\omega=z_2^1\propto=u_2^1\propto$ for some $z_1^1, z_2^1, u_1^1, u_2^1\in A^*$. Similarly, $t^1\tau=v^1\tau$ implies that $t^1=t_1^1 t_2^1$, $v^1=v_1^1 v_2^1$ such that $v_1^1\omega=v_2^1\propto=t_1^1\omega=t_2^1\propto$. Hence by the definition of $\underset{n}{\simeq}$, since $s_1\tau\cup w_1\tau\subseteq z^1\tau$ and $s_2\tau\cup w_2\tau\subseteq t^1\tau$ we have:

$$z_2^1 z^2 \ldots z^{n+1} w_1 w_2 t^{n+1} t^n \ldots t^2 t_1^1\,{}_n\!\simeq u_2^1 u^2 \ldots u^{n+1} s_1 s_2 v^{n+1} v^n \ldots v^2 v_1^1.$$

Also, by the definition of ${}_n\!\simeq$, $z_2^1 z^2 \ldots z^{n+1}\,{}_n\!\simeq u_2^1 u^2 \ldots u^{n+1}$ and $t^{n+1} t^n \ldots t^2 t_1^1\,{}_n\!\simeq v^{n+1} \ldots v^2 v_1^1$.

On the other hand, from the choosing of letter b *it follows that* $x_1 x_2 \ldots x_{r-1} zb\,{}_{q_1}\!\!\sim y_1 y_2 \ldots y_{r-1} ub$ for $q_1=m(n, k+1, r)-1$ and from the choosing of letter a, it follows that $at\,{}_{q_2}\!\!\sim av$ for $q_2=m(n, k+1, r)-(r-1)(k+1)-1$. Consequently, since $(zb)\tau=(ub)\tau\subseteqq A$, $(at)\tau=(ar)\leftrightarrow\tau\subseteqq A$ and $q_1>m(n, k+1, r-1)$, $q_2>m(n, k)$, then by the induction assumption for $p=r-1$:

$$x_1 x_2 \ldots x_{r-1} z_1 z_2 \ldots z_h z^1 z^2 \ldots z^{n+1}$$

${}_n\simeq y_1 y_2 \dots y_{r-1} u_1 u_2 \dots u_h u^1 u^2 \dots u^{n+1}$ and by the general induction assumption:

$$t^{n+1} t^n \dots t^1 t_g t_{g-1} \dots t_1 {}_n\simeq v^{n+1} v^n \dots v^1 v_g v_{g-1} \dots v_1.$$

Thus, since ${}_n\simeq$ is a graph congruence, $x {}_n\simeq y$.

2(B) In this subcase, we have:

$$x = x_1 x_2 \dots x_{r-1} z_1 z_2 \dots z_h z^1 z^2 \dots z^{n+1} w_1 \gamma w_2 \beta w_3 t^{n+1} t^n \dots t^1 t_g t_{g-1} \dots t_1,$$

$$y = y_1 y_2 \dots y_{r-1} u_1 u_2 \dots u_h u^1 u^2 \dots u^{n+1} s_1 \gamma s_2 \beta s_3 v^{n+1} v^n \dots v^1 v_g v_{g-1} \dots v_1^1,$$

where $\gamma \notin z^1 \tau = u^1 \tau$, $\beta \notin t^1 \tau = v^1 \tau$, $\gamma \notin s_1 \tau$, $\beta \notin s_3 \tau$ and $w_1 \tau \subseteqq z^1 \tau$, $w_3 \tau \subseteqq t^1 \tau$.

Now, $(a_1, a_2, \dots, a_q, \beta, t_g \tau, t_{g-1} \tau, \dots, t_1 \tau) \in x \,(\in y)$ iff $(a_1, a_2, \dots, a_q) \in x_1 x_2 \dots x_{r-1} zbw_1 \gamma w_2$ ($\in y_1 y_2 \dots y_{r-1} ubs_1 \gamma s_2$, respectively), for $q \geqq 0$. Hence, $x {}_m\sim y$ implies that:

$$x_1 x_2 \dots x_{r-1} zbw_1 \gamma w_2 {}_q\sim y_1 y_2 \dots y_{r-1} ubs_1 \gamma s_2,$$

for:

$$q = m(n, k+1, r) - \sum_{j=1}^{g} |t_j \tau| - 1 \geqq m(n, k+1, r) - n \sum_{i=1}^{k} i - 1.$$

Thus, $(zbw_1 \gamma w_2)\tau \subseteqq A$ and $q > m(n, k+1, r-1)$ imply by the induction assumption for $p = r-1$ that:

$$x_1 x_2 \dots x_{r-1} zbw_1 \gamma w_2 {}_n\simeq y_1 y_2 \dots y_{r-1} ubs_1 \gamma s_2.$$

Similarly, $(A^{r-1}, z_1 \tau, z_2 \tau, \dots, z_h \tau, \gamma, a_1, a_2, \dots, a_q) \in x \,(\in y)$ iff $(a_1, a_2, \dots, a_q) \in w_2 \beta w_3 at$ ($\in s_2 \beta s_3 av$, respectively) for $q \geqq 0$. Hence, $x {}_m\sim y$ implies that $w_2 \beta w_3 at {}_q\sim s_2 \beta s_3 av$ for:

$$q = m(n, k+1, r) - (r-1)(k+1) - \sum_{i=1}^{h} |z_i \tau| - 1 \geqq m(n, k+1, r) - (r-1)(k+1) - n \sum_{i=1}^{k} i - 1.$$

Since $q > m(n, k)$ and $(w_2 \beta w_3 at)\tau = (s_2 \beta s_3 av)\tau \nsubseteqq A$, by the general induction assumption we have:

$$w_2 \beta w_3 at {}_n\simeq s_2 \beta s_3 av.$$

Finally, $(A^{r-1}, z_1\tau, z_2\tau, \ldots, z_h\tau, \gamma, a_1, a_2, \ldots, a_q, \beta, t_g\tau, t_{g-1}\tau, \ldots, t_1\tau) \underline{\in} x$ ($\underline{\in} y$) iff $(a_1, a_2, \ldots, a_q) \underline{\in} w_2$ ($\underline{\in} s_2$, respectively); $q \geqq 0$. Hence, $x\ {}_m{\sim}\ y$ implies that $w_2\ {}_q{\sim}\ s_2$ for:

$$q = m(n, k+1, r) - (r-1)(k+1) - \sum_{i=1}^{h} |z_i\tau|$$

$$- \sum_{j=1}^{g} |t_g\tau| - 2 \geqq m(n, k+1, r) - (k+1) - 2n \sum_{i=1}^{k} i - 2.$$

Since $q > m(n, k)$ and $w_2\tau = s_2\tau \notin A$, then by the general induction assumption:

$$w_2\ {}_n{\simeq}\ s_2.$$

Thus, since ${}_n{\simeq}$ is a graph congruence, $x\ {}_n{\simeq}\ y$.

2(C) The proof follows as in 2(B), it is sufficient to regard γ and β as the same letter and $w_2 = s_2 = \lambda$. □

$(d) \Rightarrow (b)$: Congruence ${}_m{\sim}$ satisfies (b) for $n = m$, consequently, also $\sim$ satisfies (b). □

$(b) \Rightarrow (a)$: Let $x = x_1 x_2 \ldots x_h$ and $y = y_1 y_2 \ldots y_h$ be coterminal paths such that $x_i \mathscr{J} y_i$ for $i = 1, 2, \ldots, h$ and $h \geqq 1$. Then, by the definition of relation $\mathscr{J}$, $x_i \sim z_1^i y_i z_2^i$ and $y_i \sim z_3^i x_i z_4^i$ for some paths $z_1^i, z_2^i, z_3^i, z_4^i$. Consequently:

$$x_i \sim (z_1^i z_3^i)^n x_i (z_4^i z_2^i)^n$$

and:

$$x_1 x_2 \ldots x_h \sim (z_1^1 z_3^1)^n x_1 (z_4^1 z_2^1)^n (z_1^2 z_3^2)^n x_2 (z_4^2 z_2^2)^n \ldots (z_1^h z_3^h)^n x_h (z_4^h z_2^h)^n,$$

for $n \geqq 0$. Since x and y are coterminal, then z_1^1 and z_3^1 are loops about the same vertex. Similarly, z_4^h and z_2^h are loops about the same vertex. By (b) for sufficiently large n and since $\sim$ is a graph congruence:

$$x_1 x_2 \ldots x_h \sim z_3^1 (z_1^1 z_3^1)^n x_1 (z_4^1 z_2^1)^n z_4^1 z_3^2 (z_1^2 z_3^2) x_2 (z_4^2 z_2^2)^n z_4^2 \ldots$$

$$\ldots z_3^h (z_1^h z_3^h)^n x_n (z_4^h z_2^h)^n z_4^h.$$

Note that for $i = 1, 2, \ldots, h-1$ z_4^i and z_1^{i+1} are coterminal. Next, since:

$$y_i \sim z_3^i (z_1^i z_3^i)^n x_i (z_4^i z_2^i) z_4^i,$$

we obtain $x \sim y$. Thus $\dot{\sim}$ is dot ($\mathscr{J}$)-trivial. □

REFERENCES

1. J. A. Brzozowski and I. Simon, *Characterization of Locally Testable Events*, Discrete Math., Vol. 4, 1973, pp. 243-271.
2. S. Eilenberg, *Automata, Languages and Machines*, Vol. B, 1976, Academic Press.
3. R. Knast, *A Semigroup Characterization of Dot-Depth One Languages* R.A.I.R.O., Informatique théorique, Vol. 17, No. 4, 1983, pp. 321-330.
4. I. Simon, *Piecewise Testable Events* (*2nd GI-Professional Conference on Automata Theory and Formal Languages*, L.N. in Computer Science, Vol. 33, 1975, Springer Verlag, pp. 214-222).