SOME THEOREMS ON LOCALLY PRODUCT RIEMANNIAN SPACES

SHUN-ICHI TACHIBANA

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In a previous paper [3][1], we have generalized the notion of analytic tensors and obtained the one of Φ-tensors. As a natural developement, we shall deal with locally product Riemannian spaces. Since such a space is formally analogous to a Kählerian space, it seems to be interesting to translate well known theorems in the latter to the former.

We shall devote § 1 to preliminaries. In § 2, we shall obtain an integral formula for a tensor field in a compact orientable space and give an application on harmonic tensors. In § 3 another application will be given and we shall see that in a compact orientable locally product Riemannian space an infinitesimal projective (or conformal) transformation is necessarily an isometry. In § 6 we shall discuss infinitesimal product-projective transformations which correspond to holomorphically projective transformations in a Kählerian space. Its preliminary results are given in § 4 and § 5. In § 7 infinitesimal product-conformal transformations are defined and discussed.

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1. Preliminaries. Let us consider an n-dimensional locally product Riemannian space. Then, by definition, there exists a system of coordinate neighborhoods $\{U_\alpha\}$ such that in each U_α the line element is given by the form

$$(1.\ 1) \qquad ds^2 = \sum_{\lambda,\mu=1}^{p} g_{\lambda\mu}(x^\nu)\, dx^\lambda dx^\mu + \sum_{a,b=p+1}^{n} g_{ab}(x^c)\, dx^a dx^b,^{2)}$$

and in $U_\alpha \cap U_\beta$ the coordinate transformation $(x^\lambda, x^a) \rightarrow (x^{\lambda'}, x^{a'})$ is given by the form

$$(1.\ 2) \qquad x^{\lambda'} = x^{\lambda'}(x^\mu), \qquad x^{a'} = x^{a'}(x^b).$$

Such a coordinate system (x^λ, x^a) will be called a separating coordinate

1) See the Bibliography at the end of the paper.

2) As to the notations and the terminologies, we follow [3]. We agree to use the following ranges of indices throughout the paper $1 \leqq \lambda, \mu, \ldots \leqq p < n$, $p+1 \leqq a, b, \ldots \leqq p+q=n$, $1 \leqq i, j, k, \ldots, r, s, \ldots \leqq n$.

system.

If we define $\varphi_i^{\ h}$ by

$$(\varphi_i^{\ h}) = \begin{pmatrix} \delta_\lambda^\mu & 0 \\ 0 & -\delta_a^{\ b} \end{pmatrix} \tag{1. 3}$$

in each U_α, then they define a tensor field and satisfy

$$\varphi_i^{\ r}\varphi_r^{\ h} = \delta_i^{\ h}, \tag{1. 4}$$

$$g_{ri}\varphi_j^{\ r} = g_{jr}\varphi_i^{\ r}, \tag{1. 5}$$

$$\nabla_j\varphi_i^{\ h} = 0, \tag{1. 6}$$

where ∇_j denotes the operator of the Riemannian covariant derivation. (1. 4) shows that $\varphi_i^{\ h}$ assigns an almost-product structure to the space [1], [2], [5], [6]. (1. 5) means that the Riemannian metric tensor g_{ji} is pure in the sence of [3].

Conversely consider an n-dimensional Riemannian space M which admits a tensor field $\varphi_i^{\ h}$ ($\neq \delta_i^{\ h}$) satisfying (1. 4), (1. 5) and (1. 6). By virtue of (1. 4), the matrix $(\varphi_i^{\ h})$ has ± 1 as its proper values. Let us denote by $T(P)$ the tangent vector space of M at a point P and let $E(P)$ and $F(P)$ be the proper vector spaces corresponding to the proper values $+1$ and -1 respectively. If we put $\dim E(P) = p$ and $\dim F(P) = q$, then they are constant and it holds that $\varphi \equiv \varphi_r^{\ r} = p - q = n - 2q$. By virtue of (1. 6) the field of vector spaces $E(P)$ (resp. $F(P)$), $P \in M$, constitutes a p-(resp. q-) dimensional involutive distribution [6]. Consequently there exists a system of coordinate neighborhoods such that (1. 3) holds good. Since $\varphi_i^{\ h}$ is a tensor, coordinate transformations among the coordinate systems are the type of (1. 2). In such a coordinate system, (1. 5) is equivalent to $g_{\lambda a} = 0$, from which and (1. 6) we have $g_{\lambda\mu} = g_{\lambda\mu}(x^\nu)$ and $g_{ab} = g_{ab}(x^c)$. Thus the space under consideration is nothing but the locally product Riemannian space.

Throughout the paper we shall assume that M is an n-dimensional locally product Riemannian space whose positive definite metric tensor is given by g_{ji} and that p and q are greater than 1.

We shall say that a vector field v^i is decomposable[3], if its covariant derivative is pure, i.e., $\underset{v}{\pounds}\, \varphi_i^{\ h} = 0$ is valid[4]. With respect to a separating coordinate system (x^λ, x^a), it is equivalent to the fact that $\partial_\lambda v^a = 0$ and $\partial_a v^\lambda = 0$ are valid.

A tensor field will be called decomposable if it and its covariant deri-

3) A covariant vector field u_i is called decomposable, if $u^i = g^{ir}u_r$ is decomposable. This is equivalent to the fact that $\nabla_j u_i$ is pure.

4) $\underset{v}{\pounds}$ denotes the operator of Lie derivation with respect to v^i.

vative are both pure. Hence a tensor $\xi_i^{\ h}$ is decomposable if $\xi_\lambda^{\ a} = \xi_a^{\ \lambda} = 0$, $\xi_\lambda^{\ \mu} = \xi_\lambda^{\ \mu}(x^\nu)$ and $\xi_a^{\ b} = \xi_a^{\ b}(x^c)$ are valid in a separating coordinate system. In particular, g_{ji} is decomposable.

Let $R_{kji}^{\ \ \ h}$, $R_{ji} = R_{rji}^{\ \ \ r}$ and $R = R_{ji}g^{ji}$ be the Riemannian curvature tensor, the Ricci tensor and the scalar curvature formed from g_{ji} respectively. Then the following lemma has been known [3].

LEMMA 1. *The Riemannian curvature tensor and its succesive covariant derivatives are decomposable.*

The following identity is well known

$$\nabla_j R = 2\,\nabla_r R_j^{\ r}. \tag{1. 7}$$

Since we have known that $\nabla_k R_{ji}$ is pure by virtue of Lemma 1, the identity

$$\nabla_j R^* = \varphi_j^{\ r}\nabla_r R \tag{1. 8}$$

is obtained, where we have put $R^* = \varphi^{rt}R_{rt}$[5].

2. An integral formula. In this section we shall only consider a compact orientable space M. Let $\xi_{(i)} \equiv \xi_{i_p\ldots i_1}$[6] be a tensor field and define

$$\xi^*_{(i)} \equiv \xi^*_{i_p\ldots i_1} \equiv \xi_{i_p\ldots i_2 r}\varphi_{i_1}^{\ r},$$
$$a_{j(i)}(\xi) \equiv (\varphi_l^{\ r}\nabla_r\xi_{(i)} - \nabla_l\xi^*_{(i)})\varphi_j^{\ l}$$
$$= \nabla_j\xi_{(i)} - \varphi_j^{\ l}\nabla_l\xi^*_{(i)}.$$

If $\xi_{(i)}$ is pure, then $a_{j(i)}(\xi) = 0$ means that $\xi_{(i)}$ is decomposable, i. e., $\nabla_l\xi_{(i)}$ is pure.

Denoting the square of $a_{j(i)}(\xi)$ by $a^2(\xi)$, we obtain easily

$$\nabla^r(a_{r(i)}\xi^{(i)}) = (\nabla^r a_{r(i)})\xi^{(i)} + (1/2)\,a^2(\xi),$$

from which and Green's theorem we have

THEOREM 1. *In a compact orientable space M, the integral formula*

$$\int_M [(\nabla^r\nabla_r\xi_{(i)} - \varphi^{rt}\nabla_r\nabla_t\xi^*_{(i)})\xi^{(i)} + (1/2)\ a^2(\xi)]\,d\sigma = 0$$

is valid for a tensor field $\xi_{(i)}$, where $d\sigma$ means the volume element of M.

COROLLARY. *In a compact orientable space M, a necessary and sufficient condition in order that a pure tensor $\xi_{(i)}$ is decomposable is that*

5) The tensor φ_{ji} is a Riemannian metric tensor whose inverse is given by $\varphi^{ji} = \varphi_r^{\ i}g^{rj}$. The scalar R^* is nothing but the scalar curvature with respect to the Riemannian metric φ_{ji}.

6) In this section, p does not mean the dimension of $E(P)$.

$$\nabla^r\nabla_r\xi_{(i)} = \varphi^{rt}\nabla_r\nabla_t\xi^*_{(i)}.$$

On the other hand we have known [7] that in a compact orientable Riemannian space a skew-symmetric tensor $\xi_{(i)}$ is harmonic when and only when it satisfies that

$$\text{(2. 1)} \qquad \nabla^r\nabla_r\xi_{(i)} - \sum_{k=1}^{p} R_{i_k}{}^r\xi_{i_p\ldots r\ldots i_1} + \sum_{l>k} R_{i_l i_k}{}^{rs}\xi_{i_p\ldots r\ldots s\ldots i_1} = 0.$$

Now let $\xi_{(i)}$ be a pure harmonic tensor, then $\xi^*_{(i)}$ is a skew-symmetric pure tensor [3]. Since we can see that $\xi^*_{(i)}$ also satisfies the equation of the same form as (2. 1), we have

LEMMA 2. *In a compact orientable space M, if a pure tensor $\xi_{(i)}$ is harmonic, then so is $\xi^*_{(i)}$.*

If $\xi_{(i)}$ is pure harmonic, then it holds that

$$\nabla_l\xi_{(i)} = \sum_{k=1}^{p}\nabla_{i_k}\xi_{i_p\ldots l\ldots i_1}, \quad \nabla_l\xi^*_{(i)} = \sum_{k=1}^{p}\nabla_{i_k}\xi^*_{i_p\ldots l\ldots i_1}.$$

Hence we have

$$\varphi^{rt}\nabla_r\nabla_t\xi^*_{(i)} = \varphi^{rt}\nabla_r\Big[\sum_{k=1}^{p}\nabla_{i_k}\xi^*_{i_p\ldots t\ldots i_1}\Big] = \nabla_r\Big[\sum_{k=1}^{p}\nabla_{i_k}(\varphi^{rt}\xi^*_{i_p\ldots t\ldots i_1})\Big]$$

$$= \nabla^r\Big[\sum_{k=1}^{p}\nabla_{i_k}\xi_{i_p\ldots r\ldots i_1}\Big] = \nabla^r\nabla_r\xi_{(i)}.$$

Thus we get

THEOREM 2. *In a compact orientable space M, a pure harmonic tensor is decomposable.*

3. Infinitesimal transformations. As a corollary of Theorem 1, we have

THEOREM 3. *In a compact orientable space M, the integral formula*

$$\int_M [(\nabla^r\nabla_r v_i - \varphi^{rt}\nabla_r\nabla_t v^*_i)\, v^i + (1/2)\, a^2(v)]\, d\sigma = 0$$

is valid for a vector field v^i.

In this section we shall give some applications of this theorem. Let us consider a vector field v^i and put

$$t_{ji}{}^h \equiv \underset{v}{\mathfrak{L}}\{{}^{\,h}_{ji}\} = \nabla_j\nabla_i v^h + R_{rji}{}^h v^r.$$

Taking account of the purity of $R_{rji}{}^h$, if we transvect the last equation with

$\varphi^{ji}\varphi_h{}^l$, then we have

$$\varphi^{ji}\varphi_h{}^l t_{ji}{}^h = \varphi^{ji}\nabla_j\nabla_i v^{*l} + R_r{}^l v^r.$$

Hence we find

(3. 1) $$g^{ji}t_{ji}{}^h - \varphi^{rt}\varphi_s{}^h t_{rt}{}^s = \nabla^r\nabla_r v^h - \varphi^{rt}\nabla_r\nabla_t v^{*h}.$$

Now consider an infinitesimal projective transformation v^i, then it satisfies by definition

(3. 2) $$t_{ji}{}^h = \rho_j\delta_i^h + \rho_i\delta_j^h,$$

where ρ_i is a certain vector.

Substituting this into (3. 1) it follows that

(3. 3) $$\nabla^r\nabla_r v^h = \varphi^{rt}\nabla_r\nabla_t v^{*h},$$

which and Theorem 3 show that v^i is decomposable, i. e., it satisfies $\underset{v}{\pounds}\,\varphi_i{}^h = 0$.

On the other hand, since the identity

$$\underset{v}{\pounds}\nabla_j\varphi_j{}^h - \nabla_j\underset{v}{\pounds}\varphi_i{}^h = t_{jr}{}^h\varphi_i{}^r - t_{ji}{}^r\varphi_r{}^h$$

holds good, we have

$$t_{jr}{}^h\varphi_i{}^r = t_{ji}{}^r\varphi_r{}^h,$$

from which and (3.2) we obtain $\rho_i = 0$. Thus

THEOREM 4[7]. *In a compact orientable space M, an infinitesimal projective transformation is necessarily an isometry.*

COROLLARY. *In a compact orientable space M, a Killing vector is decomposable.*

In the next place we consider an infinitesimal conformal transformation v^i. It satisfies by definition

(3. 4) $$\underset{v}{\pounds}g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji},$$

where ρ is a scalar, from which we have

(3. 5) $$t_{ji}{}^h = \rho_j\delta_i^h + \rho_i\delta_j^h - \rho^h g_{ji}, \qquad \rho_j = \partial_j\rho.$$

Hence taking account of (3. 1) it follows that

(3. 6) $$\nabla^r\nabla_r v_i - \varphi^{rt}\nabla_r\nabla_t v_i^* = -n\rho_i + \varphi\rho_i^*.$$

On the other hand, from (3. 4) we have

$$\nabla_r v^r = n\rho, \qquad \nabla_r v^{*r} = \varphi\rho.$$

7) Cf. Tashiro, Y. [4].

Hence it holds that

$$(3.\ 7)\qquad \begin{aligned} -n\,\rho_i v^i &= -(\nabla_i \nabla_r v^r) v^i \\ &= (n\,\rho)^2 - \nabla_i (v^i \nabla_r v^r), \\ n\,\rho^*_i v^i &= (\varphi_i{}^t \nabla_t \nabla_r v^r) v^i \\ &= \nabla_t (v^{*t} \nabla_r v^r) - n\,\varphi\rho^2. \end{aligned}$$

Substituting (3. 6) and (3. 7) into the integral formula in Theorem 3, we obtain

$$\int_M [(n^2 - \varphi^2)\rho^2 + (1/2)\, a^2(v)]\ d\sigma = 0,$$

which shows that $\rho = 0$. Consequently we have

THEOREM 5. *In a compact orientable M, an infinitesimal conformal transformation is necessarily an isometry.*

4. Separately Einstein spaces. If the Ricci tensor of our space M satisfies the relation

$$(4.\ 1)\qquad R_{ji} = a\, g_{ji} + b\, \varphi_{ji},$$

then we shall call the space a separately Einstein space. If we make use of a separating coordinate system $(x^\lambda,\ x^a)$, then (4. 1) becomes

$$R_{\lambda\mu} = (a+b) g_{\lambda\mu}, \qquad R_{\lambda c} = 0, \qquad R_{ce} = (a-b) g_{ce}.$$

Let us consider such a space, then from (4. 1) we have

$$(4.\ 2)\qquad R = n\,a + \varphi\, b, \qquad R^* = \varphi\, a + n\, b,$$

from which it holds that

$$a = \alpha_0 R + \beta_0 R^*, \qquad b = \alpha_0 R^* + \beta_0 R,$$

where we have put

$$\alpha_0 = n/(n^2 - \varphi^2), \qquad \beta_0 = -\varphi/(n^2 - \varphi^2),$$

If we substitute (4. 2) into (1. 8), then we have

$$n(b_j - \varphi_j{}^r a_r) = \varphi(\varphi_j{}^r b_r - a_j),$$

where $a_j = \partial_j a$ and $b_j = \partial_j b$. Transvecting this with $\varphi_i{}^j$, then it follows

$$\varphi(b_i - \varphi_i{}^r a_r) = n(\varphi_i{}^r b_r - a_i).$$

Thus from the last two equations we get

$$(4.\ 3)\qquad a_j = \varphi_j{}^r b_r.$$

On the other hand, if we substitute (4. 1) and (4. 2) into (1. 7), then we obtain $(n-4)a_j + \varphi b_j = 0$, from which we get, taking account of (4. 3), $a_j = 0$ provided that p and q are different from 2. Therefore we have

THEOREM 6. *In a separately Einstein space, the scalar curvature is constant if p and q are different from* 2.

5. Spaces of serarately constant curvature. Let us consider an arbitrary but fixed point P in our space M. In this section we shall restrict our attention to the tangent space $T(P)$ and write $E = E(P)$ and $F = F(P)$.

We shall assume that the dimensions p and q are greater than 2.

In the first place we have

LEMMA 3. *If $a^i \in E$ and $b^i \in F$, then it holds that*

$$R_{kjih}a^i b^h = 0.$$

This follows from the facts that R_{kjih} is pure and $a^i b^h$ is hybrid [3].

As a trivial consequence, we obtain the following

THEOREM 7. *The sectional curvature determined by $a^i \in E$ and $b^i \in F$ vanishes.*

A vector u^i is uniquely decomposed in the form

$$u^i = a^i + b^i, \quad a^i \in E,\ b^i \in F. \tag{5. 1}$$

Let v^i be another vector and put

$$v^i = r^i + s^i, \quad r^i \in E,\ s^i \in F. \tag{5. 2}$$

If we put $R(u, v) = R_{kjih}\, u^k v^j u^i v^h$, then we have by means of Lemma 3

$$R(u, v) = R(a, r) + R(b, s). \tag{5. 3}$$

Now we assume that the sectional curvature of 2-planes in E and the one of 2-planes in F have values λ and μ respectively which are independent of the direction of 2-planes.

From the assumption and (5. 3), we get

$$R(u, v) = \lambda[a^2 r^2 - (a, r)^2] + \mu[b^2 s^2 - (b, s)^2],$$

where

$$a^2 = a_i a^i, \quad (a, r) = a_i r^i, \text{ etc.}$$

By virtue of (5. 1) and (5. 2), the last equation is written in the following form

$$\begin{aligned} R(u, v) = {} & (1/4)(\lambda + \mu)[u^2 v^2 + (u^*, u)(v^*, v) - (u, v)^2 - (u^*, v)^2] \\ & + (1/4)(\lambda - \mu)[(u^*, u)v^2 + (v^*, v)u^2 - 2(u, v)(u^*, v)] \end{aligned} \tag{5. 4}$$

$$= (A\, r_{kjih} + B\, r^*_{kjih}) u^k v^j u^i v^h,$$

where

(5. 5)
$$\begin{aligned} r_{kjih} &= g_{ki}g_{jh} - g_{ji}g_{kh} + \varphi_{ki}\varphi_{jh} - \varphi_{ji}\varphi_{kh}, \\ r^*_{kjih} &= r_{kjit}\varphi_h{}^t, \\ A &= (1/4)(\lambda + \mu), \qquad B = (1/4)(\lambda - \mu). \end{aligned}$$

It is evident that r_{kjih} is a pure tensor and satisfies

$$r_{(kj)ih} = 0, \qquad r_{kjih} = r_{ihkj},$$
$$r_{[kji]h} = 0.$$

Since (5. 4) holds good for any u^i and v^i, we have

(5. 6)
$$R_{kjih} = A\, r_{kjih} + B\, r^*_{kjih}.$$

Conversely, if the Riemannian curvature tensor takes the form (5. 6), A and B being scalars, then we can prove that the sectional curvature of 2-planes in E (resp. F) has a value which is independent of the direction.

We call the space satisfying (5, 6) at any point of M a space of separately constant curvature.

THEOREM 8. *If the sectional curvature of 2-planes in E and the one of 2-planes in F have values which are independent of the direction respectively at any point, p and q being greater than 2, then the space is of separately constant curvature. The converse is also true.*

From (5. 6) we have $R_{ji} = a\, g_{ji} + b\, \varphi_{ji}$, where

$$a = -(n-2)A - \varphi B, \qquad b = -(n-2)B - \varphi A.$$

Hence a space of separately constant curvature is separately Einsteinian. From Theorem 7 we get

THEOREM 9. *In a space of separately constant curvature* $(p, q > 2)$, *the scalar curvature R is constant.*

6. Infinitesimal product-projective transformations. We can easily obtain the following

LEMMA 4. *A vector field* $v_i = \partial_i f$ *is decomposable if there exists a scalar function g such that* $\partial_i f = \varphi_i{}^r \partial_r g$.

We shall call a vector field v^i an infinitesimal product-projective transformation or, for brevity, a *PP*-transformation, if it satisfies[8)]

8) More generally, we may consider a transformation such that
$$t_{ji}{}^h = \sigma_j \delta_i^h + \sigma_i \delta_j^h + \rho_j \varphi_i{}^h + \rho_i \varphi_j{}^h.$$
If v^i under consideration is decomposable, then we have $\sigma_i = \rho^*_i$.

(6. 1) $$t_{ji}{}^h = \rho^*_j\delta_i{}^h + \rho^*_i\delta_j{}^h + \rho_j\varphi_i{}^h + \rho_i\varphi_j{}^h,$$

where ρ_i is a vector and $\rho^*_i = \varphi_i{}^r\rho_r$.

In this case we shall call ρ_i the associated vector of v^i.

Let v^i be a *PP*-transformation, then we know easily that (3. 3) holds good. Hence

THEOREM 10. *In a compact orientable space M, an infinitesimal product-projective transformation is decomposable.*

If we take account of the purity of $R_{kji}{}^h$, then we can obtain

THEOREM 11. *If v^i is an infinitesimal product-projective transformation whose associated vector is ρ_i, then so is v^{*i} and its associated vector is ρ^*_i.*

Let us consider a *PP*-transformation v^i, then from (6. 1) we have

$$\nabla_j\nabla_r v^r = (n+2)\rho^*_j + \varphi\rho_j, \qquad \nabla_j\nabla_r v^{*r} = (n+2)\rho_j + \varphi\rho^*_j.$$

The left hand members are gradient, so we have that ρ_i and ρ^*_i are gradient. If we put $\rho_i = \partial_i\rho$ and $\rho^*_i = \partial_i\rho^*$, where ρ and ρ^* are scalars, then we have $\partial_i\rho^* = \varphi_i{}^r\partial_r\rho$, which and Lemma 4 show that ρ_i and ρ^*_i are decomposable.

Substituting (6. 1) into the identity

$$\underset{v}{\pounds}\, R_{kji}{}^h = \nabla_k t_{ji}{}^h - \nabla_j t_{ki}{}^h,$$

we get

(6. 2) $$\underset{v}{\pounds}\, R_{kji}{}^h = \delta_j{}^h\nabla_k\rho^*_i - \delta_k{}^h\nabla_j\rho^*_i + \varphi_j{}^h\nabla_k\rho_i - \varphi_k{}^h\nabla_j\rho_i,$$

from which it follows that

(6. 3) $$\underset{v}{\pounds}\, R_{ji} = -\varphi\nabla_j\rho_i - (n-2)\nabla_j\rho^*_i.$$

Now we assume that v^i under consideration is decomposable, then it holds that

(6. 4) $$\underset{v}{\pounds}\, R^*_{ji} = -(n-2)\nabla_j\rho_i - \varphi\nabla_j\rho^*_i.$$

From (6. 3) and (6. 4) we have

(6. 5) $$\underset{v}{\pounds}\, U_{ji} = -\nabla_j\rho_i,$$

where

(6. 6) $$U_{ji} = \beta_1 R_{ji} + \alpha_1 R^*_{ji},$$

(6. 7) $$\alpha_1 = (n-2)/\{(n-2)^2 - \varphi^2\}, \qquad \beta_1 = -\varphi/\{(n-2)^2 - \varphi^2\}.$$

From (6. 2) and (6. 5) we obtain

$$\underset{v}{£} P_{kji}{}^{h} = 0,$$

where we have put

(6. 8) $$P_{kji}{}^{h} = R_{kji}{}^{h} + U^{*}_{ki}\delta_{j}{}^{h} - U^{*}_{ji}\delta_{k}{}^{h} + U_{ki}\varphi_{j}{}^{h} - U_{ji}\varphi_{k}{}^{h},$$

which will be called the product-projective curvature tensor.

If we substitute (6. 6) into (6. 8), then $P_{kji}{}^{h}$ is also written as follows

$$\begin{aligned} P_{kji}{}^{h} = R_{kji}{}^{h} &+ \alpha_1\{R_{ki}\delta_{j}{}^{h} - R_{ji}\delta_{k}{}^{h} + R^{*}_{ki}\varphi_{j}{}^{h} - R^{*}_{ji}\varphi_{k}{}^{h}\} \\ &+ \beta_1\{R^{*}_{ki}\delta_{j}{}^{h} - R^{*}_{ji}\delta_{k}{}^{h} + R_{ki}\varphi_{j}{}^{h} - R_{ji}\varphi_{k}{}^{h}\}. \end{aligned}$$

The following theorem is a consequence by some calculations.

THEOREM 12. *In order that the product-projective curvature tensor $P_{kji}{}^{h}$ vanishes at any point, it is necessary and sufficient that the space under consideration is of separately constant curvature.*

7. Infinitesimal product-conformal transformations. We call a vector field v^i an infinitemisal product-conformal transformation or, for brevity, a *PC*-transformation, if it satisfies

(7. 1) $$\underset{v}{£} g_{ji} = 2(\rho g_{ji} + \sigma\varphi_{ji}),$$

where ρ and σ and scalar functions such that

(7. 2) $$\partial_i\rho = \varphi_{i}{}^{r}\partial_r\sigma.$$

With respect to a separating coordinate system, (7. 1) and (7. 2) are written as follows

$$\underset{v}{£} g_{\lambda\mu} = 2(\rho + \sigma)g_{\lambda\mu}, \qquad \underset{v}{£} g_{\lambda a} = 0, \qquad \underset{v}{£} g_{ab} = 2(\rho - \sigma)g_{ab},$$

$$\partial_a(\rho + \sigma) = 0, \qquad \partial_\lambda(\rho - \sigma) = 0.$$

By virtue of Lemma 4 and (7. 2), the vectors $\rho_i = \partial_i\rho$ and $\sigma_i = \partial_i\sigma$ are decomposable.

If we take account of the identity

$$\nabla_k \underset{v}{£} g_{ji} - \underset{v}{£} \nabla_k g_{ji} = t_{kj}{}^{r} g_{ri} + t_{ki}{}^{r} g_{jr},$$

then we have

$$t_{ji}{}^{h} = \rho_j\delta_{i}{}^{h} + \rho_i\delta_{j}{}^{h} - \rho^{h}g_{ji} + \sigma_j\varphi_{i}{}^{h} + \sigma_i\varphi_{j}{}^{h} - \sigma^{h}\varphi_{ji}.$$

From the last equation we can see that v^i satisfies (3. 3). Thus

THEOREM 13. *In a compact orientable space M, an infinitesimal pro-*

duct-conformal tansformation is decomposable.

For a *PC*-transformation, the following equations are valid.

$$\underset{v}{\pounds} g^{ji} = -2(\rho g^{ji} + \sigma\varphi^{ji}),$$

$$\underset{v}{\pounds} R_{kji}{}^{h} = \rho_{kji}{}^{h},$$

$$\underset{v}{\pounds} R_{ji} = -(n-4)\nabla_j\rho_i - \varphi\nabla_j\sigma_i - xg_{ji} - y\varphi_{ji},$$

$$\underset{v}{\pounds} R_j{}^h = -(n-4)\nabla_j\rho^h - \varphi\nabla_j\sigma^h - x\delta_j^h - y\varphi_j{}^h - 2(\rho R_j{}^h + \sigma R_j^{*h}),$$

$$\underset{v}{\pounds} R = -2[(n-2)x + \varphi y + \rho R + \sigma R^*].$$

where we have put $x = \nabla_r\rho^r$, $y = \nabla_r\sigma^r$ and

$$\rho_{kji}{}^h = \delta_j^h\nabla_k\rho_i + \varphi_j{}^h\nabla_k\sigma_i + g_{ki}\nabla_j\rho^h + \varphi_{ki}\nabla_j\sigma^h - \delta_k^h\nabla_j\rho_i - \varphi_k{}^h\nabla_j\sigma_i - g_{ji}\nabla_k\rho^h - \varphi_{ji}\nabla_k\sigma^h, \tag{7. 3}$$

which is pure.

In the following we shall assume that v^i is decomposable *PC*-transformation and p and q are greater than 2.

We shall obtain a tensor which is invariant under such transformations.

Since we have $\underset{v}{\pounds}\varphi_i{}^h = 0$ by the assumption, it follows that

$$\underset{v}{\pounds}\varphi_{ji} = 2(\rho\varphi_{ji} + \sigma g_{ji}), \qquad \underset{v}{\pounds}\varphi^{ji} = -2(\rho\varphi^{ji} + \sigma g^{ji}),$$

$$\underset{v}{\pounds} R^*_{ji} = -(n-4)\nabla_j\sigma_i - \varphi\nabla_j\rho_i - x\varphi_{ji} - yg_{ji},$$

$$\underset{v}{\pounds} R_j^{*h} = -(n-4)\nabla_j\sigma^h - \varphi\nabla_j\rho^h - x\varphi_j{}^h - y\delta_j^h - 2(\rho R_j^{*h} + \sigma R_j{}^h),$$

$$\underset{v}{\pounds} R^* = -2[(n-2)y + \varphi x + \rho R^* + \sigma R].$$

If $r_{kji}{}^h$ is the tensor defined by (5. 5), then we have

$$\underset{v}{\pounds} r_{kji}{}^h = 2(\rho r_{kji}{}^h + \sigma r^*_{kji}{}^h).$$

Now we define a tensor $s_{kji}{}^h$ by

$$s_{kji}{}^h = \delta_j^h R_{ki} + \varphi_j{}^h R^*_{ki} + g_{ki}R_j{}^h + \varphi_{ki}R_j^{*h} - \delta_k^h R_{ji} - \varphi_k{}^h R^*_{ji} - g_{ji}R_k{}^h - \varphi_{ji}R_k^{*h}, \tag{7. 4}$$

then it is pure, as known by the similarity between (7. 3) and (7. 4). Thus we get

$$\underset{v}{\pounds} s_{kji}{}^h = -\varphi\rho^*_{kji}{}^h - (n-4)\rho_{kji}{}^h - 2x\, r_{kji}{}^h - 2y\, r^*_{kji}{}^h,$$

$$\underset{v}{\pounds} s^*_{kji}{}^h = -(n-4)\rho^*_{kji}{}^h - \varphi\rho_{kji}{}^h - 2y\, r_{kji}{}^h - 2x\, r^*_{kji}{}^h.$$

From these equations we get

$$\underset{v}{\pounds}\, C_{kji}{}^{h} = 0,$$

where $C_{kji}{}^{h}$ is defined by

$$C_{kji}{}^{h} = R_{kji}{}^{h} + \alpha_2 s_{kji}{}^{h} + \beta_2 s^{*}_{kji}{}^{h}$$
$$- (\alpha_2 U^{*} + \beta_2 U) r_{kji}{}^{h} - (\alpha_2 U + \beta_2 U^{*}) r^{*}_{kji}{}^{h}$$

and

$$U = U_{ji} g^{ji} = \beta_1 R + \alpha_1 R^{*}, \qquad U^{*} = U^{*}_{ji} g^{ji} = \alpha_1 R + \beta_1 R^{*},$$
$$\alpha_2 = (n-4)/\{(n-4)^2 - \varphi^2\}, \qquad \beta_2 = -\varphi/\{(n-4)^2 - \varphi^2\}.$$

We shall call $C_{kji}{}^{h}$ the product-conformal curvature tensor.

It is written also in the following form

$$C_{kji}{}^{h} = R_{kji}{}^{h} + \alpha_2 \{ s_{kji}{}^{h} - (\alpha_1 R + \beta_1 R^{*}) r_{kji}{}^{h} - (\alpha_1 R^{*} + \beta_1 R) r^{*}_{kji}{}^{h} \}$$
$$+ \beta_2 \{ s^{*}_{kji}{}^{h} - (\alpha_1 R + \beta_1 R^{*}) r^{*}_{kji}{}^{h} - (\alpha_1 R^{*} + \beta_1 R) r_{kji}{}^{h} \}.$$

After some complicated calculations we get the following

THEOREM 14. *In a space of separately constant curvature, the product-conformal curvature tensor vanishes at any point.*

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OCHANOMIZU UNIVERSITY, TOKYO.