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Some Topics in the Dynamics of Group Actions on Rooted Trees

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To the cherished memory of Evgenii Frolovich Mishchenko

Abstract—This article combines the features of a survey and a research paper. It presents a review of some results obtained during the last decade in problems related to the dynamics of branch and self-similar groups on the boundary of a spherically homogeneous rooted tree and to the combinatorics and asymptotic properties of Schreier graphs associated with a group or with its action. Special emphasis is placed on the study of essentially free actions of self-similar groups, which are antipodes to branch actions. At the same time, the theme "free versus nonfree" runs through the paper. Sufficient conditions are obtained for the essential freeness of an action of a self-similar group on the boundary of a tree. Specific examples of such actions are given. Constructions of the associated dynamical system and the Schreier dynamical system generated by a Schreier graph are presented. For groups acting on trees, a trace on the associated C^*-algebra generated by a Koopman representation is introduced, and its role in the study of von Neumann factors, the spectral properties of groups, Schreier graphs, and elements of the associated C^*-algebra is demonstrated. The concepts of asymptotic expander and asymptotic Ramanujan graph are introduced, and examples of such graphs are given. Questions related to the notion of the cost of action and the notion of rank gradient are discussed.

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1. INTRODUCTION

The modern theory of dynamical systems studies systems defined by group actions, i.e., systems of the form (G, X, μ), where the measure μ is invariant or at least quasi-invariant (semigroup actions are also considered, but this subject is much less developed compared with group actions). The theory also deals with topological dynamical systems of the form (G, X), where X is a topological space and the group G acts by homeomorphisms (topological dynamics). An important class of actions that are considered in modern dynamics is formed by the actions of countable groups, among which a special role is played by the actions of finitely generated groups. The study of rough properties (such as the structure of the partition into orbits) of the actions of countable groups is closely related to the study of countable Borel partitions, while the latter direction is

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closely linked with modern studies in descriptive set theory (which traces its roots to the Russian scientific school due to the pioneering works of N.N. Luzin and M.Ya. Suslin). The group aspect of dynamical systems theory is also largely due to the Russian mathematical school and is associated with the fundamental studies by N.N. Bogolyubov, I.M. Gel'fand, Yu.V. Linnik, M.L. Gromov, G.A. Margulis, and other outstanding mathematicians. From among the Western school, we should mention, first of all, the studies by J. von Neumann, H. Furstenberg, A. Connes, and R. Zimmer.

In the studies carried out until recently, the (essentially) free actions played a major role, while nonfree actions appeared episodically. Among the first works that dealt with nonfree actions were the studies by the present author [70, 72] and by Vershik and Kerov [185]; the results obtained by the author in the early 1980s were mainly of an algebraic character (related to the geometric and asymptotic directions in group theory) and gave rise to the theory of branch groups and self-similar groups, whereas the studies by Vershik and Kerov were mainly related to representation theory and concentrated around the analysis of the infinite symmetric group $\mathcal{S}(\infty)$ and some other locally finite groups. In the last decade, especially after the publications [87, 16], it has become clear that it is important to study group actions on individual orbits for nonfree actions on measure spaces and on topological spaces. This led to the study of Schreier graphs and orbital graphs (associated with actions on orbits). At the same time, two years ago, Vershik put forward a new idea related to the study of the so-called totally nonfree actions. It turned out that the approach of the present author and Vershik's approach have common points; in particular, branch-type actions are totally nonfree (we will touch upon this question below). At the same time, dealing with nonfree actions for many years, I realized the importance of free actions in the case of actions on the boundaries of rooted trees. Therefore, in this paper I pay approximately equal attention to both types of actions and to their relation to various topics.

Originally this paper was planned as a review of a certain range of problems concerning group dynamics on rooted trees, the problems that were first considered about ten years ago in [79, 80, 16, 87, 95]. However, while writing this paper, I came up with new ideas, revealed new relations, and the contours of new directions of investigations started to emerge. Therefore, the paper turned out to be not a pure survey; it contains a lot of new observations and sketches. I devote the following part of the Introduction to the brief description of my philosophy concerning group-action dynamics and then list the contents of the sections of the paper.

There is a close relationship between noncommutative dynamical systems and operator algebras (first of all, von Neumann algebras and C^*-algebras). Any action with a quasi-invariant measure generates a unitary representation of a group; thus, the problems and methods of noncommutative dynamics are often intertwined with the problems and methods of representation theory (which, in turn, are intertwined with the problems and methods of the theory of operator algebras). It is well known that the spectrum of a representation (i.e., its decomposition into irreducible ones) may be either a pure point spectrum (i.e., it may contain only finite-dimensional subrepresentations), a continuous spectrum (i.e., it may contain only infinite-dimensional subrepresentations), or a mixed spectrum (i.e., it may contain both finite-dimensional and infinite-dimensional subrepresentations). For the dynamics of a single automorphism (meaning an action of the cyclic group $\mathbb{Z}$), it is well known that an action with a pure point spectrum is isomorphic to a shift action on a compact abelian group. This classical result by von Neumann and Halmos was generalized by Mackey [131], who proved that faithful ergodic actions with invariant measure and pure point spectrum of a locally compact topological group are isomorphic to the actions of this group on spaces of the form K/H equipped with the image λ of the normalized Haar measure on K, where K is a compact topological group that contains a subgroup isomorphic to the given group or its homomorphic image and H is a closed subgroup of K.

In the classical situation of a single automorphism of a measure space, the discrete case (i.e., the case of a pure point spectrum) is considered trivial (or at least very simple). For the actions of noncommutative groups, the case of a pure point spectrum is not any simpler (and maybe even more complicated) than the case of a continuous spectrum. Of special interest are the actions of discrete groups G on homogeneous spaces K/P of profinite groups (i.e., when K is a compact totally disconnected group). If such an action is faithful, then the group G is embeddable in a profinite group and hence is residually finite (i.e., it has a large family of finite-index subgroups; namely, the intersection of these subgroups is a trivial subgroup). Actually, residually finite groups give precisely the class of groups that have a faithful action with an invariant measure and a pure point spectrum. The Mackey realization of such an action on a homogeneous space K/H may lead to the case when the group K is either connected (and then it is a Lie group), is totally disconnected (the profinite case), or is of mixed type (i.e., it has a nontrivial connected subgroup such that the quotient by this subgroup is totally disconnected). The case of connected K seems to be the simplest case and is the most studied one. All of what is written in the few paragraphs above is well known. Less known are the following facts.

It turns out that dynamical systems of the form $(G, K/P, \lambda)$, $G \leq K$, where K is a profinite group, arise on a seemingly very different basis; namely, they are isomorphic to systems of the form $(G, \partial T, \nu)$, where T is a spherically homogeneous rooted tree, ∂T is its boundary, G acts by tree automorphisms, and ν is a uniform measure on the tree boundary (Theorem 2.9). The first nontrivial actions of this type were considered in [70, 72, 99]; the results obtained there are more related to algebra. The dynamical aspect was given greater attention in [80, 87, 16, 18, 142, 86] and other papers, which initiated a number of new directions of research at the interface between algebra, dynamical systems theory, holomorphic dynamics, theory of operator algebras, discrete mathematics, and other fields of mathematics. Although the theory of actions on trees and their various generalizations (such as $\mathbb{R}$-trees, hyperbolic spaces, buildings, CAT(0)-spaces, etc.) has long become a well-developed theory (an excellent example is given by the material of Serre's book *Trees* [168]), the study of actions on rooted trees has required new concepts and methods and allowed one to reformulate many results related to the widely used group-theory operation of taking the wreath product in geometric and dynamic terms. This fact has made it possible to significantly extend the application domain of this operation (especially under its iteration) and to better understand it. The concept of branch group introduced by the present author [80, 79] is one of the key concepts related to the actions on rooted trees. In terms of dynamical systems, the definition of a branch group looks as follows.

Definition 1.1. (a) A group G acts on a space (X, μ) with invariant measure in a *weakly branch way* if there exists an increasing sequence $\{\xi_n\}_{n=1}^{\infty}$ of finite G-invariant partitions of X that tends to the partition into points and is such that the action of G is transitive on the set of atoms of each partition ξ_n and, for any n and any atom $A \in \xi_n$, there exists an element $g \in G$ acting nontrivially on A and, at the same time, acting trivially on the complement A^{c} of A.

(b) An action (G, X, μ) belongs to a *branch type* if it is weakly branch and, in addition, for any atom A of any of the partitions ξ_n, the subgroup $\mathrm{rist}_G(A) < G$ consisting of elements that act trivially on the complement A^{c} has finite index in the restriction $\mathrm{st}_G(A)|_A$ of the stabilizer $\mathrm{st}_G(A)$ of the set A to this set, provided that this subgroup is identified with the restriction $\mathrm{rist}_G(A)|_A$. A group is called a branch group if it has a faithful action of branch type.

Note that the definition of branch groups has never been presented in this form; instead, either a purely algebraic definition or a geometric definition in the language of actions on rooted trees was used [80, 19]. In Section 2, we present a geometric definition and prove that it is equivalent to the one given above.

Branch (just-infinite) groups constitute one of the three subclasses into which the class of just-infinite groups (i.e., infinite groups each proper quotient group of which is finite) is naturally split; it is this fact that primarily determines the importance of branch groups in group theory.

Another important class of groups that act on rooted trees is formed by self-similar groups, in other words, groups generated by Mealy-type automata. Mealy-type automata are automata that operate as transducers, or sequential machines; i.e., these are automata operating as synchronous transducers of information that transmit, letter by letter, an input sequence of letters from a certain alphabet into an output sequence. Invertible initial synchronous automata (more precisely, their equivalence classes) constitute a group with a well-known (in informatics) operation of composition of automata [56, 119]. This group depends on the cardinality of the alphabet; i.e., in fact, there exists a sequence of groups that is indexed by positive integers (by the cardinality of the alphabet).

If we consider a more general class of automata, namely, the asynchronous automata, then, as shown in [87], there is only one universal group, independent of the cardinality of the alphabet, in which all groups of synchronous automata are embedded. In [87] this group was called the group of rational homeomorphisms of the Cantor set. In addition to self-similar groups, it contains other quite interesting subgroups, for example, the famous Thompson groups. A group is said to be self-similar if it is isomorphic to a group generated by the states of a noninitial invertible synchronous automaton.

The groups generated by finite automata (we call them strongly self-similar groups in this paper) are specially distinguished. A simple example of a self-similar group is the infinite cyclic group, which can be realized by the action of an odometer (also called an *adding machine* in the English-language literature and often translated into Russian as a *d-adic counter*, where d is the cardinality of the alphabet). The odometer acts in the space of right-infinite sequences of letters in an alphabet of cardinality $d \geq 2$ equipped with a uniform Bernoulli measure, or, equivalently, on the boundary of a d-regular rooted tree (there is a generalization of the concept of odometer to the case when the phase space is the Cartesian product of a sequence of various alphabets). This dynamical system with discrete spectrum is well known in ergodic theory. A considerably more complex example of a (strongly) self-similar group is given by a group $\mathcal{G}$ that was constructed by the author in [70] and then studied in [72] and many other papers. The main properties of this group are the periodicity, intermediate growth (between polynomial and exponential), and nonelementary amenability.

Self-similar groups, especially those possessing the branch property, form quite an interesting class of groups related to many aspects of dynamical systems theory and other fields of mathematics. The theory of iterated monodromy groups developed by Nekrashevych [142] has breathed new life into holomorphic dynamics and found wide applications in the study of Julia sets and other fractal objects [142].

Actions on rooted trees turned out to be useful for the theory of profinite groups, since any profinite group with a countable base of open sets is embedded in the automorphism group (equipped with the natural topology) of an appropriate rooted tree T. Moreover, if a group G acts transitively on the levels of a tree (or, equivalently, its action on the boundary is ergodic), then the closure $\overline{G}$ in $\mathrm{Aut}(T)$, which is a profinite group, acts transitively on the boundary ∂T, and the uniform measure ν becomes the image of the Haar measure on $\overline{G}$. In this case, the dynamical system $(G, \partial T, \nu)$ is isomorphic to the system $(G, \overline{G}/P, \nu)$, where $P = \mathrm{st}_{\overline{G}}(\xi)$, $\xi \in \partial T$. As already mentioned, the converse is also true; namely, any action with pure point spectrum of type $(G, K/P, \mu)$, where K is a profinite group, is isomorphic to the action on the boundary of an appropriate rooted tree (which can easily be verified by applying the construction of the action of a residually finite group on a coset tree as described in the next section; see Theorem 2.9 there). Thus, in the Mackey theorem, the profinite case corresponds to actions on rooted trees. Another argument in favor of rooted trees is that, as noticed in [87], any compact homogeneous ultrametric space is isometric to the boundary of a rooted tree with an appropriate metric on it (a weaker version of this statement is contained in [59]).

When studying group actions, one usually assumes that the actions are essentially free, i.e., for any nonidentity element of a group, the measure of the fixed point set is zero. One of the first attempts to draw attention to the case of actions that are not essentially free was made in [70] and the following studies [72, 73, 99], which led to the concept of branch action and, accordingly, branch group. Obviously, a weakly branch action is not essentially free. Paper [185] by Vershik and Kerov was also one of the pioneering works on the use of actions that are not essentially free. The importance of studying nonfree actions has recently been pointed out by Vershik in [184]. The theme of the "free versus nonfree action" alternative runs through the larger part of our paper.

An important object that arises when studying actions that are not essentially free is an orbital graph Γ_ξ, $\xi \in X$, of an action (on the space X). The vertices of this graph are points of the orbit and the edges correspond to transitions from one vertex to another under the action of a generator (in this case, the edges are labeled by the respective generators). If the action is essentially free, then such graphs are almost surely isomorphic to the Cayley graph constructed for the group by means of the same system of generators. For actions that are not essentially free, the graphs Γ_ξ are almost surely nonisomorphic to the Cayley graphs but are isomorphic to Schreier graphs, i.e., to graphs of the form $\Gamma = \Gamma(G, H, A)$, where $H \leq G$ is a subgroup (corresponding to the stabilizer of some boundary point) and A is a system of generators. The vertices of such a graph are left (one may also consider right) cosets gH, and two vertices fH and gH are connected by an oriented edge labeled by a generator $a \in A$ if $gH = afH$. The Cayley graphs are obtained in this construction when H is the trivial subgroup. Depending on the situation, one can consider various modifications of the concept of a Schreier graph: one can make edges nonoriented, remove labels from them, distinguish a vertex in a graph and consider it as a root, etc. According to the category chosen, it is expedient to consider appropriate spaces of graphs with natural compact topology and speak of the convergence of graphs in this topology. For example, a topology in the space of Cayley graphs was first defined in [72] and used for studying group properties such as intermediate growth, impossibility of presentations by a finite set of relations, Kolmogorov complexity of the word problem [74], etc. Later, this topology and its variations were examined more carefully (first of all in [43]), and now it plays a significant role in many investigations. Note that, in the much earlier work [42], Chabauty introduced a topology on the set of closed subgroups of a locally compact group and applied it to the study of lattices in such groups. The Chabauty topology is widely used in the studies of lattices in Lie groups (see [158]). In terms of this topology one can also interpret topologies in the spaces of Cayley graphs and Schreier graphs.

The first publications in which the authors realized the importance of studying Schreier graphs that arise as orbital graphs of actions are [16, 87]. For example, the following simple but important fact is borrowed from [87, Proposition 6.22].

Proposition 1.1 [87]. *Let G be a finitely generated group that acts ergodically on a space (X, μ) by transformations that preserve the measure μ (i.e., the measure μ is quasi-invariant). Then the Schreier graphs of the action on orbits are μ-almost surely locally isomorphic to each other.*

In this proposition the local isomorphism of two graphs means that, for any radius r and an arbitrary vertex of any of the graphs, there exists a vertex of the other graph such that the neighborhoods (subgraphs) of radius r around the chosen vertices in the two graphs are isomorphic. A similar proposition is valid in the topological situation as well, but it requires the concept of a G-typical point and a slight correction in the formulation of the proposition above; moreover, there are examples of graphs that are generic in the topological sense but are not generic in the metric sense [2].

Schreier graphs associated with the actions of self-similar groups and branch-type groups on the levels and the boundary of a tree are important both for solving various problems of graph theory and for studying asymptotic problems involving graphs and groups. These graphs model various phenomena and reflect the complexity of many related problems. For example, the classical Tower of Hanoi problem with four or more pegs is equivalent to calculating the distance between specific vertices in these graphs and finding the shortest path between them in an algorithmic manner.

See [91–93] for more details on this subject. There are a lot of questions that arise when studying Schreier graphs; first of all, these questions are related to group theory and dynamical systems. These are questions on the number of ends of the graphs, on the growth, amenability, and the possibility to define the graph by a finite system of substitution rules, on the possibility of reconstructing a system from a generic Schreier graph, on the asymptotic behavior of the first nonzero eigenvalue of the discrete Laplace operator, on the spectrum of the discrete Laplace operator, on the construction of expanders, on the asymptotic behavior of random walks, on the calculation of the cost of actions and cost of groups, etc. Many of these questions are touched upon in the present paper or in the references cited. One of the new results given below is the construction of asymptotic expanders on the basis of finite automata (and on the basis of related self-similar groups). It is an interesting open problem to find out whether these graphs are true expanders. Another circle of questions related to the study of actions on rooted trees is the study of infinite decreasing chains of finite-index subgroups in residually finite groups, in particular, the study of the rank gradient of these subgroups [121, 5].

As already mentioned, actions on rooted trees and the problems of self-similar groups are mysteriously related to many questions of dynamical systems theory. These questions arise when restricting the actions to Lyapunov stable attractors [87, Theorem 6.16]. They are related to substitution dynamical systems (which arise when finding presentations of groups by generators and relations) [129, 97, 14]. The description of invariant subsets of multidimensional rational mappings served as a basis for a new unexpected method of solution of the spectral problem for the discrete Laplace operator [16]. Owing first of all to the studies of Nekrashevych, the theory of iterated monodromy groups has led to significant changes in the strategy of studies on holomorphic dynamics [142].

In Section 8, we propose a construction of a Schreier dynamical system: given a combinatorial structure (a Schreier graph) or algebraic data (a pair consisting of a group and its subgroup), one can use this construction to obtain a dynamical system. The examples presented in Section 8 show that in many cases the original dynamical system can be recovered from this construction if one takes the orbital graph of the action on a specific orbit as the Schreier graph, or if one considers the stabilizer of a point of the phase space as a subgroup of the acting group.

In essence, all new results concerning the structure of the class of amenable groups, which was introduced by von Neumann and independently by Bogolyubov, as well as the class of intermediate growth groups (about which Milnor asked whether it is empty) have been obtained on the basis of studying group actions on rooted trees. An original method for proving the amenability, called a "Münchhausen trick," was developed in [23, 109]. Various operator algebras associated with actions on rooted trees (as well as with Cuntz algebras in some cases) were defined and studied in [16, 141, 86]. It turned out that among these algebras there are both simple C^*-algebras and algebras that can be approximated by finite-dimensional algebras, similar to residually finite groups. The classical method known as the "Schur complement" found an unexpected application to these algebras in [86]. This list of research directions related to actions on rooted trees is far from complete but we stop here.

Now we briefly outline the contents of the paper. Section 2 is of preliminary character and contains many definitions used in the paper. First of all, we define the main concepts related to spherically homogeneous rooted trees and actions on them. We give a different (compared with Definition 1.1) definition of branch groups and explain how to construct a rooted tree by a decreasing chain of subgroups of finite index. We define just-infinite groups and hereditary just-infinite groups and formulate a theorem describing the trichotomy of the structure of the class of just-infinite groups. We give examples of groups and of their actions.

Section 3 is devoted to groups of automata and self-similar groups. We explain what wreath recursions are. We give definitions of a contracting group and a self-replicating group.

Section 4 is devoted to studying essentially free actions on the boundary of a tree. Abért and Virág [6] proved that a randomly chosen action of a group with a finite number of generators on a binary tree is free (moreover, the group itself is free). However, the explicit construction of an essentially free action is, as a rule, a complicated problem. We present a number of conditions of algebraic character that guarantee the freeness of an action and discuss the relationship between the freeness of actions in the topological and metric senses. Although in the general case there is no direct relation between topological freeness and freeness in the metric sense (i.e., with respect to a measure), a remarkable fact is that for the actions of strongly self-similar groups the two concepts are equivalent; this result was obtained by Kambites, Silva, and Steinberg [112].

In Section 5, we give specific examples of essentially free actions. We consider both the actions of well-known groups, such as the lamplighter realized by a two-state automaton [95], and new actions, and discuss an approach to finding out under what conditions a self-similar group acts essentially freely. A certain role in this discussion is played by the Mikhailova subgroups of the direct product of two copies of a free group.

In Section 6, we consider various topologies on the spaces of Schreier graphs and prove the Gross theorem stating that any connected regular graph with even-degree vertices can be realized as a Schreier graph of a free group.

In Section 7, we give examples of Schreier graphs related to self-similar groups. We define various types of substitution rules and recursions for infinite sequences of finite graphs. The main objects here are the Schreier graphs of the group $\mathcal{G}$ of intermediate growth and of the group called the Basilica. The material of this section is mainly based on the publications [16, 142, 93, 31, 81, 48].

In Section 8, we describe a construction that starts with a dynamical system and yields an associated dynamical system in the space of Schreier graphs or in the space of subgroups of a group. This material correlates with some questions touched upon in [184]. In addition, we describe a technique that allows one to construct an action of a group on a certain compact set by an infinite Schreier graph of this group. This technique leads to interesting actions when the automorphism group of the graph is small (for example, trivial). We show how this technique works in the case of the group $\mathcal{G}$ and the Thompson group (for the latter we use the results of Vorobets [188] and Savchuk [166]). An interesting fact exhibited in these examples is that on the metric level a dynamical system is reconstructed by the Schreier graph, whereas on the topological level the arising space and action are simple perturbations of the original phase space and an action on it. In general, the approach proposed in this part of the paper to the study of dynamical systems should have been called an *orbit method* in the dynamics of finitely generated groups, by analogy with Kirillov's orbit method in representation theory. We stress that while Kirillov's orbit method is mainly used in the representation theory of Lie groups, our approach applies to actions of countable groups equipped with a discrete topology. In this section, we also emphasize the importance of weakly maximal subgroups for the orbital approach to dynamical systems and present some results from [16, 17] (most of which are known) that are related to this class of subgroups. We also present E. Pervova's nontrivial example of a weakly maximal subgroup of the intermediate growth group $\mathcal{G}$.

In Section 9, we discuss unitary representations of groups acting on trees and consider C^*-algebras associated with these representations (we also touch upon von Neumann algebras). On one of these C^*-algebras, we define a trace, which is called a recurrent (or self-similar) trace, and describe some of its properties. Here we mainly use the results obtained in [16, 95, 141, 86, 184]. The recurrent trace has additional useful properties in the case when a group is strongly self-similar. For the intermediate growth group $\mathcal{G}$, we give an explicit description of the values of the trace on the elements of the group. We discuss some properties of the C^*-algebras under consideration. We show that the weakly branch groups belong to the class of ICC (infinite conjugacy class) groups, which possess infinite (nontrivial) conjugacy classes of elements.

In Section 10, we consider questions related to random walks on groups and graphs, the spectral properties of the discrete Laplace operator (or, equivalently, of the Markov operator related to a random walk), as well as the Kesten spectral measure and the so-called KNS (Kesten–von Neumann–Serre) spectral measure, which was introduced and examined in [16, 98]. Examples of a self-similar essentially free action of a free rank 3 group and of the free product of three copies of an order 2 group and results on the recurrent trace are used for constructing asymptotic expanders. We discuss various questions concerning the asymptotic behavior of infinite graphs and infinite covering sequences of finite graphs.

Finally, in Section 11 we discuss questions related to the concept of the cost of actions of countable groups and of countable Borel equivalence relations, as well as the concept of rank gradient of infinite decreasing sequences of finite-index subgroups. This material is based on the studies by G. Levitt, D. Gaboriau, M. Lackenby, M. Abért, and N. Nikolov. We discuss the problems of amenability and hyperfiniteness of groups and equivalence relations and present classical results associated with the names of H. Dye, J. Feldman, C. Moore, A. Connes, and B. Weiss. We introduce the concepts of self-similar and self-replicating equivalence relations and show that the latter are "cheap" in the sense of cost.

This paper is mainly a survey that summarizes the results of research carried out during the last decade in a certain direction. However, it also presents some new observations. Moreover, the paper formulates many open questions. I hope that this paper will stimulate further investigations in the field of dynamics with a pure point spectrum, dynamics of actions on trees, and other related fields of mathematics.

Since the paper is addressed to readers involved in different fields of mathematics, starting from algebraists and ending with specialists (or beginners) in dynamical systems theory, theory of operator algebras, and discrete mathematics, in many places I go into greater detail than I should have to if the paper was addressed only to the reader involved in one specific field. Sometimes, I do not consider it beneath me to remind an already introduced notion or an already formulated result. I hope that the reader will not judge me harshly for this.

2. ACTIONS ON ROOTED TREES

Let $\overline{m} = \{m_n\}_{n=1}^{\infty}$, $m_n \geq 2$, be a sequence of positive integers (called a branch index in what follows) and $T_{\overline{m}}$ be a spherically homogeneous rooted tree defined by the sequence $\overline{m}$. This tree has a root vertex denoted by $\varnothing$, m_1 vertices of the first level, $m_1 m_2$ vertices of the second level, and generally $m_1 m_2 \dots m_n$ vertices of the nth level, $n = 1, 2, \dots$. Each vertex of level n has m_{n+1} "successors" situated at the next level and connected by an edge with this vertex. A clear idea of a rooted tree is given by Fig. 2.1. Note that according to the tradition established in Russian mathematics, a tree is depicted top down.

The norm of a vertex u (denoted by $|u|$) is the level to which this vertex belongs. When the sequence $\overline{m}$ is constant, i.e., $m_n = d$ for some $d \geq 2$ and any n, the tree $T_{\overline{m}}$ is called a *regular*

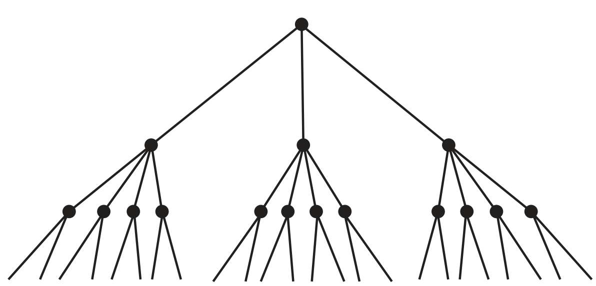

Fig. 2.1. A spherically homogeneous rooted tree.

rooted tree of degree d or a d-regular tree and is denoted by T_d. In what follows, we will usually omit the word *rooted* and the branch index. An automorphism of a tree T is any bijection on the vertex set that preserves the incidence relation of vertices and maps the root vertex to itself. The automorphisms of a tree form a group $\mathrm{Aut}(T)$ with respect to the operation of composition. An arbitrary subgroup $G \le \mathrm{Aut}(T)$ can be considered as a group acting faithfully (i.e., each nonidentity element acts nontrivially) on T. One can also consider finite trees defined by finite sequences $\overline{m}$, and their automorphism groups; in particular, some of the arguments below may involve subtrees of an infinite tree with vertices up to the nth level inclusive. In what follows, unless otherwise stated, we will deal with infinite trees. For such trees, the concept of boundary ∂T (or the space of ends) is defined; geometrically, this boundary can be viewed as consisting of geodesic paths in T that connect the root vertex with infinity. On ∂T there is a natural topology, in which two paths are close if their common beginning is large, and the greater the common beginning, the closer the paths. This topology is naturally metrizable with the use of an arbitrary decreasing sequence $\bar\lambda = \{\lambda_n\}_{n=1}^{\infty}$ of positive numbers that tends to zero. The distance $\mathrm{dist}_{\bar\lambda}(\alpha,\beta)$, $\alpha,\beta \in \partial T$, is equal to λ_n if the sequences α and β diverge at the level n. The space $(\partial T, \mathrm{dist}_{\bar\lambda})$ is an ultrametric space, and $\mathrm{Aut}(T)$ serves as its isometry group; moreover, as shown in [87, Proposition 6.2], the above-described metric structure associated with a spherically homogeneous tree is a general model of a homogeneous ultrametric space. If $G < \mathrm{Aut}(T)$ is a subgroup, then the pair $(G, \partial T)$ is a compact topological dynamical system (i.e., the group acts by homeomorphisms of the compact set ∂T), which will play an important role in what follows. This system can be made into a metric dynamical system by adding an invariant probability measure, which will be discussed below.

Alternatively, the boundary can be described as follows. Let X_n, $n = 1, 2, \dots$, be alphabets of cardinality $|X_n| = m_n$ with fixed orders on them, where $\overline{m} = \{m_n\}_{n=1}^{\infty}$ is a branch index. The set of nth-level vertices of the tree $T = T_{\overline{m}}$ is naturally identified with the set $X_1 \times X_2 \times \ldots \times X_n$, on which a lexicographic order is introduced. This order is used when drawing the tree on a plane; namely, the vertices on a level are arranged according to their order. The boundary ∂T is naturally identified with the Tikhonov product $\prod_{n=1}^{\infty} X_n$ of the sets X_n, each of which is equipped with a discrete topology. Thus, the boundary can be thought of as the space of infinite sequences of symbols of the alphabets X_n (the nth term of a sequence belongs to the set X_n) equipped with the topology of pointwise convergence. This description is simplified when the tree is regular. Then all alphabets X_n can be assumed to coincide, and the boundary is the space of right-infinite sequences of letters in this alphabet. As a topological space, the boundary ∂T is homeomorphic to the Cantor perfect set. Naturally, ∂T is a metrizable compact set and, as mentioned above, possesses a family of metrics $d_{\bar\lambda}$ parameterized by decreasing sequences $\bar\lambda = \{\lambda_n\}_{n=1}^{\infty}$ of positive numbers that tend to zero. Any such metric is an ultrametric, is invariant under the action of the whole group $\mathrm{Aut}(T)$, and induces the Tikhonov topology on the boundary.

To study the dynamics of actions on the boundary, it is relevant to introduce a uniform probability measure ν on the boundary. This measure is defined on the σ-algebra of Borel sets by its values on cylindrical sets of the form C_u, where u is a vertex and C_u consists of geodesic paths that connect the root vertex with infinity and pass through u. If $|u| = n$, then $\nu(C_u) = 1/(m_1 \dots m_n)$. Alternatively, the measure ν can be defined as $\nu = \bigotimes_{n=1}^{\infty} \nu_n$, where ν_n is the uniform measure on the alphabet X_n, provided that the boundary is represented as $\partial T = \prod_{n=1}^{\infty} X_n$. The measure ν is invariant with respect to the whole automorphism group of the tree; therefore, it is also invariant with respect to the action of any subgroup of $\mathrm{Aut}(T)$. When the action (G, T) is spherically transitive, this is a unique invariant probability measure; this follows from Proposition 4.1 (see below). Throughout the paper, we will keep the notation ν exclusively for this measure and call it a *uniform* measure on the boundary of the tree.

Denote by τ the shift in the space of right-infinite sequences that serve as a branch index (i.e., applying τ to a sequence means removing the first term of this sequence). The tree $T_{\overline{m}}$ consists of m_1 copies of the tree $T_{\tau(\overline{m})}$ that are connected by edges with the root vertex. Accordingly, the group $\mathrm{Aut}(T)$ is isomorphic to the semidirect product

$$\big(\mathrm{Aut}(T_{\tau(\overline{m})}) \times \ldots \times \mathrm{Aut}(T_{\tau(\overline{m})})\big) \rtimes \mathrm{Sym}(m_1) \tag{2.1}$$

of the product of m_1 copies of the automorphism group of the subtree $T_{\tau(\overline{m})}$ and the symmetric group $\mathrm{Sym}(m_1)$ acting on this product by permutations of the factors. This semidirect product is often called a permutational wreath product and is denoted here as $\mathrm{Aut}(T_{\tau(\overline{m})}) \wr_{\mathrm{perm}} \mathrm{Sym}(m_1)$. An arbitrary automorphism $g \in \mathrm{Aut}(T_{\overline{m}})$ can be represented as

$$g = (g_1, \ldots, g_{m_1})\sigma, \tag{2.2}$$

where $\sigma \in \mathrm{Sym}(m_1)$ and $g_i \in \mathrm{Aut}(T_{\tau(\overline{m})})$, $i = 1, \ldots, m_1$. The elements g_i are called sections (or projections) of the element g at vertices of the first level. By induction on the level, one defines sections g_u of the element g at an arbitrary vertex u. Denote by T_u a (complete) subtree of the tree T with origin at vertex u, which serves as the root of the subtree. Throughout this paper, we will use the following notations: $x^y = y^{-1}xy$ and $[x, y] = x^{-1}y^{-1}xy$.

Definition 2.1. Let G be a group acting by automorphisms on a rooted tree T, u be a vertex of this tree, and T_u be the subtree with root vertex u.

(a) The *stabilizer of* u is the subgroup $\mathrm{st}_G(u) = \{g \in G\colon g(u) = u\}$.

(b) The *nth-level stabilizer* $\mathrm{st}_G(n)$ is defined as the intersection of the stabilizers of all vertices of this level (i.e., it is a subgroup consisting of elements that fix all vertices of the nth level).

(c) The *rigid stabilizer of* u is a subgroup $\mathrm{rist}_G(u)$ in G that consists of automorphisms $g \in G$ acting trivially on the complement of the subtree T_u.

(d) The *nth-level rigid stabilizer* $\mathrm{rist}_G(n)$ is a subgroup in G generated by the rigid stabilizers of all vertices of the nth level.

Since the rigid stabilizers of vertices of the same level commute with each other, the nth-level rigid stabilizer $\mathrm{rist}_G(n)$ is the (inner) direct product of the rigid stabilizers of the nth-level vertices:

$$\mathrm{rist}_G(n) = \langle \mathrm{rist}_G(v)\colon |v| = n\rangle = \prod_{v\colon |v|=n} \mathrm{rist}_G(v). \tag{2.3}$$

Note that the stabilizer of a vertex is a subgroup of finite index, the stabilizer of a level is a normal subgroup of finite index, the rigid stabilizer of a level is a normal subgroup, and the rigid stabilizer of a vertex is a subnormal subgroup of degree 1 (namely, a normal subgroup in $\mathrm{st}_G(n)$, where n is the level of the vertex). The rigid stabilizers of a vertex or level may be trivial, in contrast to the stabilizers of vertices and levels, which always have finite index and are therefore nontrivial if the group is infinite and acts faithfully. In what follows, unless otherwise stated, we will consider only actions of groups by automorphisms on rooted trees and the induced actions on the boundaries of the trees.

Definition 2.2. (a) An action of a group G on a spherically homogeneous tree T (henceforth denoted as (G, T)) is said to be *spherically transitive* if it is transitive on each level of the tree.

(b) An action (G, T) is of *branch* type if it is spherically transitive and the rigid stabilizer $\mathrm{rist}_G(n)$ has finite index in G for any $n \geq 1$.

(c) An action (G, T) is of *weakly branch* type if it is spherically transitive and the rigid stabilizer $\mathrm{rist}_G(n)$ is infinite for any n (equivalently, for any vertex v of the tree, the rigid stabilizer $\mathrm{rist}_G(v)$ is nontrivial).

(d) An action (G,T) is of *weakly nonbranch* type if, starting from a certain level, the rigid stabilizers are trivial.

(e) An action (G,T) is of *nonbranch* type if the rigid stabilizers of all levels are trivial.

(f) An action (G,T) is of *diagonal* type if, for any n and arbitrary $g \in \mathrm{st}_G(n)$, either all sections g_v at the vertices v of the nth level are nonidentity elements or all of them are equal to the identity element (i.e., g is the identity element).

(g) An action of a group is said to be *locally trivial* if there exists a vertex v such that $\mathrm{st}_G(v)$ acts trivially on the subtree T_v with root at the vertex v. If there is no such a vertex, then the action is said to be *locally nontrivial*.

Remark 2.1. In the case of a self-similar group (self-similar groups will be defined in the next section) acting on a binary tree, the action is locally nontrivial if and only if the rigid stabilizer $\mathrm{rist}_G(1)$ of the first level is trivial.

In some publications, the condition of spherical transitivity in the definitions of branch and weakly branch actions is omitted. However, we will stick to our (original) definition. Below we will prove the equivalence between the definitions of branch and weakly branch actions and Definition 1.1.

It is obvious that the levels of a tree are invariant under an arbitrary automorphism preserving the root vertex; therefore, the spherical transitivity is the strongest possible transitivity condition for actions on rooted trees. In this case, a group, acting on itself by conjugations, transitively permutes the stabilizers (respectively, the rigid stabilizers) of vertices on each level. Therefore, if an action is spherically transitive, then (2.3) contains a product of isomorphic groups.

In Section 4, we discuss topologically free and essentially free actions on the boundary of a tree. Obviously, the topological or essential freeness implies that the action is not of weakly branch type. The converse is not true, which follows, for example, from the results of [40].

The following proposition is useful for verifying the transitivity of an action on the levels.

Proposition 2.1. *Let $G \le \mathrm{Aut}(T)$. Then G acts transitively on the levels if and only if*

(a) *G acts transitively on the first level of the tree, and, for some vertex u of the first level, the stabilizer $\mathrm{st}_G(u)$ acts transitively on the levels of the subtree T_u;*

(b) *for any $n \ge 1$, the group G acts transitively on the n-th level of the tree, and, for some vertex v, $|v| = n$, the stabilizer $\mathrm{st}_G(v)$ acts transitively on the subtree T_v.*

The proof is almost obvious. For details we refer the reader to [80, Lemma A].

Definition 2.3. (a) A group G is said to be a *branch group* if it has a faithful *spherically transitive* action of branch type.

(b) G is said to be a *weakly branch group* if it has a faithful *spherically transitive* action of weakly branch type.

The automorphism group $\mathrm{Aut}(T)$ is an example of a branch group because the rigid stabilizer of any vertex u coincides with $\mathrm{Aut}(T_u)$. However, this group is not finitely generated even as a topological group, because the abelianization of this group is an elementary 2-group of infinite rank (below we will discuss the structure of a profinite group on $\mathrm{Aut}(T)$). Soon we will give examples of finitely generated groups of branch type; the main such example is the group $\mathcal{G} = \langle a,b,c,d\rangle$ generated by four transformations of the interval $[0,1]$ that preserve the Lebesgue measure (Example 2.3). We will also point out another extremal property of nonfree actions that correlates with Definition 1.10 of a finite-type action from Zalesskii's paper [196].

The class of branch groups is important due to the fact that branch groups make up one of three subclasses into which the class of just-infinite groups (i.e., infinite groups any of whose proper quotient groups is finite) is naturally partitioned. A more detailed account of this will be given below. The concept of weakly branch group has emerged as a generalization of the concept of branch group, because it turned out that many statements on the general properties of branch groups can be proved under a more general assumption of weak branching. Examples of such statements are the Abért theorem on the absence of nontrivial identities in weakly branch groups [1] and the proposition that these groups have no faithful finite-dimensional representations, which was established by Delzant and the present author (as pointed out by Abért in a paper[1] of 2006).

Vershik [184] introduced the concept of *extremely* nonfree action; in one of the variants of its definition, an action is extremely nonfree if the stabilizers of points of a full-measure subset of the space on which the group acts are pairwise different. The following proposition immediately implies that weakly branch actions are extremely nonfree.

Proposition 2.2. *Let G be a group acting on a tree T in a weakly branch way. Then, for any two different points ζ and η of the boundary ∂T, the stabilizers $\mathrm{st}_G(\zeta)$ and $\mathrm{st}_G(\eta)$ are different.*

Proof. Let us apply the following lemma, which was first proved in [17].

Lemma 2.3. *Let G be a group acting on a rooted tree T in a weakly branch way, u be a vertex of the tree, and ξ be a point of the boundary that passes through u. Then the orbit of the point ξ under the action of the group $\mathrm{rist}_G(u)$ is infinite.*

Proof. Since $\mathrm{rist}_G(u)$ is a nontrivial subgroup, it acts nontrivially on some vertex v situated under u. Let $g(v) = w$, $w \neq v$, $g \in \mathrm{rist}_G(u)$, and let u' be a vertex that belongs to the path ξ and to the same level as v. According to Proposition 2.1, the stabilizer $\mathrm{st}_G(u)$ acts transitively on the subtree T_u. Let $h \in \mathrm{st}_G(u)$ and $h(u') = v$. Then $g^h(u') \neq u'$ and $g^h \in \mathrm{rist}_G(u)$. Thus, the orbit $\mathrm{rist}_G(u)(\xi)$ consists of at least two points. Applying similar arguments to the vertex u' and its images under the action of $\mathrm{rist}_G(u)$, below each of these vertices we find at least a pair of vertices belonging to the $\mathrm{rist}_G(u)$-orbit of the point ξ; thus, $\mathrm{rist}_G(u)(\xi)$ consists of at least four points. Continuing this argument by induction, we arrive at the desired conclusion. $\square$

Let v be a vertex that belongs to η but does not belong to ζ. Then $\mathrm{st}_G(\eta)$ contains $\mathrm{rist}_G(v)$. At the same time, according to Lemma 2.3 and the fact that G is a weakly branch group, the $\mathrm{rist}_G(v)$-orbit of the point η is infinite; therefore, $\mathrm{st}_G(\zeta) \neq \mathrm{st}_G(\eta)$. $\square$

In [184] Vershik also introduced the concept of *totally nonfree* action. Let (G, X, B, μ) be an action of a group G on a Lebesgue space (X, B, μ) with continuous measure; if the sigma algebra generated by the sets of fixed points of the elements of the group coincides with the whole sigma algebra B, then the action is said to be totally nonfree. In the general case, totally nonfree actions constitute a narrower class of actions compared with extremely nonfree actions (to see this, it suffices to notice that according to another definition an action is extremely nonfree if the mapping Ψ_G defined in [184] is monomorphic). However, for countable groups these two concepts coincide.

It turns out that weakly branch actions satisfy this more restrictive condition as well; this is proved by applying the same Lemma 2.3. It would be interesting to find out what actions of nonbranch type belong to the class of totally nonfree actions.

Theorem 2.4. *Suppose that an action (G, X, μ) of a group G belongs to a weakly branch type. Then this action is totally nonfree.*

Proof. In view of Corollary 2.7 proved below, we can assume that the action is defined on the boundary of a tree T; i.e., we deal with a system of the form $(G, \partial T, \nu)$. Let us prove that for any vertex v the cylindrical set C_v belongs to the sigma algebra generated by the sets of fixed points of the elements of the group. For an arbitrary element $g \in \mathrm{rist}_G(v)$, the set $\mathrm{Fix}(g)$ of fixed points contains the complement C_v^{c} of this set. We argue that

$$C_v = \bigcup_{g \in \mathrm{rist}_G(v)} \mathrm{Fix}^{\mathrm{c}}(g). \tag{2.4}$$

[1] M. Abért, "Representing Graphs by the Non-commuting Relation," Publ. Math. **69** (3), 261–269 (2006).

Indeed, the inclusion $C_v \supseteq \bigcup_{g\in G} \mathrm{Fix}^{\mathrm{c}}(g)$ is obvious. If the complement $U_v = C_v \setminus \bigcup_{g\in \mathrm{rist}_G(v)} \mathrm{Fix}^{\mathrm{c}}(g)$ is not empty, take $\xi \in U_v$. The path ξ obviously passes through the vertex v, and the inclusion $\xi \in U_v$ implies that $\xi \in \mathrm{Fix}(g)$ for any $g \in \mathrm{rist}_G(v)$. However, this contradicts Lemma 2.3. □

Theorem 2.5. *Suppose that an action $(G, \partial T)$ of a countable group G on the boundary of a rooted tree is of weakly branch type. Then it is totally nonfree (in the topological sense).*

The proof is similar to the proof of Theorem 2.4, and we omit it.

In essence, weakly branch actions appear in Theorem 3 in Rubin's paper [162], which gives sufficient conditions under which a topological space can be reconstructed from its group of homeomorphisms.

Now we prove the equivalence of the two definitions of a branch group.

Theorem 2.6. (a) *Definition* 1.1 *of a branch group is equivalent to Definition* 2.3(a).

(b) *The same is true for the definition of a weakly branch group.*

Proof. Let us prove the more difficult part (a) of the theorem. Suppose that a branch group G acts faithfully and in a branch way on a spherically homogeneous tree T. Then the system $(G, \partial T, \nu)$ satisfies the conditions of Definition 1.1 if we take the partition into cylindrical sets C_v corresponding to the vertices of the nth level as ξ_n. Indeed, since G is a branch group, the rigid stabilizer $\mathrm{rist}_G(n)$ has finite index in G and, hence, in $\mathrm{st}_G(n)$ for any n. We have a chain of embedded subgroups of finite index,

$$\mathrm{rist}_G(n) < \mathrm{st}_G(n) < \mathrm{st}_G(v), \tag{2.5}$$

where v is an arbitrary nth-level vertex. The restriction $\mathrm{rist}_G(n)|_{T_v}$ to the subtree T_v coincides with the restriction $\mathrm{rist}_G(v)|_{T_v}$ (and the corresponding restriction homomorphism $\rho_v\colon \mathrm{rist}_G(v) \to \mathrm{rist}_G(v)|_{T_v}$, $g \to g_v$ is an isomorphism). This implies that the action $(G, \partial T, \nu)$ belongs to a branch type according to Definition 1.1.

Conversely, if the group G is defined by an action (G, X, μ) satisfying the conditions of Definition 1.1, we construct a spherically homogeneous tree T whose vertices are in one-to-one correspondence with the atoms of the partition ξ_n (the root vertex corresponds to the trivial partition consisting of a single atom, the entire space X), and two vertices $A \in \xi_n$ and $B \in \xi_{n+1}$ are connected by an edge if $B \subset A$. The action of the group on the set of atoms of the partition ξ_n induces an action on the vertex set of the nth level, and these actions are consistent so that one obtains a faithful spherically transitive action of G on the constructed tree (because the sequence of partitions ξ_n tends to the partition into points). Here we have an isomorphism $(G, X, \mu) \simeq (G, \partial T, \nu)$. The fact that the rigid stabilizer $\mathrm{rist}_G(A)$ has finite index in $\mathrm{st}_G(A)|_{T_v}$ for any atom A implies that the first two subgroups in (2.5) have finite index in $\mathrm{st}_G(v)$ and, hence, also in G. Thus, G is a branch group in the sense of Definition 2.3(a). □

While proving this theorem, we have also proved

Corollary 2.7. *An arbitrary action (G, X, μ) of weakly branch type is isomorphic to an action of the form $(G, \partial T, \nu)$ for an appropriate spherically transitive tree T.*

Below we give examples of groups and their actions on a regular (namely, on a binary) tree; in these examples the groups are defined by recurrent relations. This method for defining groups will be used throughout the paper. It corresponds to the representation of an element in the form (2.2), but also employs the self-similarity of a regular tree, which allows one to identify sections (acting on subtrees) with automorphisms of the entire tree. In the next section, we interpret this method for defining groups in the language of automata and define the class of self-similar groups. All the examples given below present self-similar groups. When discussing these groups, we sometimes refer to the properties that will be defined only in the subsequent sections. The reader may omit appropriate places and return to them later, or, conversely, look into Sections 3 and 4, where these concepts are defined.

Example 2.1 (infinite cyclic group $\mathbb{Z}$ generated by an odometer, or by a dyadic shift)**.** Define an automorphism α of a binary tree by recurrent relations on the set of vertices represented by dyadic binary sequences:

$$\alpha(0w) = 1w, \qquad \alpha(1w) = 0\alpha(w), \tag{2.6}$$

$w \in \{0,1\}^*$. Here $\{0,1\}^*$ denotes the set of finite sequences (we will also use the term *words*) over the alphabet $\{0,1\}$. Note that in the recurrent relations that define this automorphism (as well as other automorphisms defined below by recurrent relations), the sequence w can be assumed to run through the set of right-infinite sequences over the alphabet $\{0,1\}$ (i.e., the set of points of the boundary ∂T).

In this example, the group $\mathbb{Z}$ is represented as a group generated by the automorphism α, which is called an *adding machine* in the English-language literature (in the Russian literature, it is translated as a dyadic odometer or a dyadic shift) because, under the natural identification of infinite dyadic sequences with the corresponding integer 2-adic numbers, the transformation α turns into the operation of adding unity. The odometer transformation is also defined for an arbitrary finite alphabet of cardinality ≥ 2. It is obvious that the above action of $\mathbb{Z}$ is topologically and essentially free.

The following is one of the basic examples illustrating essentially free actions.

Example 2.2 (the lamplighter group, or the group of dynamical configurations in the terminology of V. Kaimanovich and A.M. Vershik, $\mathcal{L}$)**.** This is a solvable group of derived length 2 (i.e., a metabelian group) well known in group theory, which is defined as the semidirect product

$$\Big(\bigoplus_{\mathbb{Z}} \mathbb{Z}_2\Big) \rtimes \mathbb{Z} \tag{2.7}$$

($\mathbb{Z}_2$ is a group of second order) in which the generator of the active factor $\mathbb{Z}$ acts on the direct sum by a right shift by one step; in other words, $\mathcal{L}$ is isomorphic to the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$. This is an infinitely presented group of exponential growth that plays an important role in geometric group theory (see, for example, [100, 137]). The importance of this group and its multidimensional analogs for the random walk theory was demonstrated by Vershik and Kaimanovich in [111]. In this paper we will use the shorter name *lamplighter*, which is conventional in the Western literature. In [95], the lamplighter was realized as a self-similar group generated by a two-state automaton over the alphabet $\{0,1\}$. Namely, $\mathcal{L}$ is isomorphic to the group generated by two automorphisms a and b of a binary tree that are defined via the recurrent relations

$$a(0w) = 1a(w), \qquad a(1w) = 0a(w), \qquad b(0w) = 0a(w), \qquad b(1w) = 1b(w), \tag{2.8}$$

$w \in \{0,1\}^*$, or by the automaton defined by the Moore diagram shown in Fig. 3.1, which is described in Section 3, where we discuss such diagrams.

The realization of the lamplighter as a self-similar group was used in [95] to calculate the spectral measure of the Markov operator on $\mathcal{L}$ (spectral measures are discussed in Section 10), which, in turn, made it possible to answer in [84] the question of Atiyah [10] as to whether there exist compact manifolds with noninteger l^2 Betti number. For our study, this example is important because the action is essentially free (which played a significant role in calculating the spectral measure in [95]). This is in fact the first nontrivial example of an essentially free action on a rooted tree.

Example 2.3 (torsion branch group of intermediate growth)**.** This group was first considered in [70] as a simple example of an infinite finitely generated 2-group and was later analyzed in many papers, including [78, 71, 80, 72, 90].

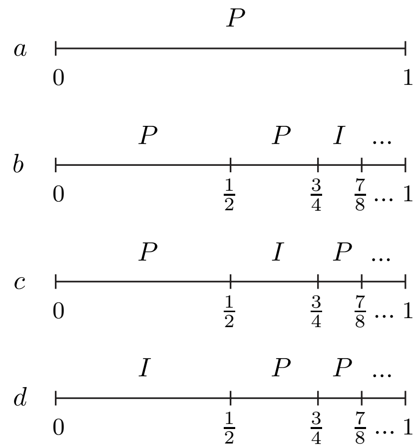

Fig. 2.2. Transformations of an interval that generate the group $\mathcal{G}$.

Let us define four automorphisms a, b, c, and d of a binary tree by the recurrent relations

$$\begin{aligned} &a(0w)=1w, && b(0w)=0a(w), && c(0w)=0a(w), && d(0w)=0w,\\ &a(1w)=0w, && b(1w)=1c(w), && c(1w)=0d(w), && d(1w)=1b(w), \end{aligned} \tag{2.9}$$

$w\in\{0,1\}^*$, and denote by $\mathcal{G}$ the group generated by these automorphisms. Originally this group was defined as a group generated by four transformations a, b, c, and d of the interval $[0,1]$ from which dyadic rational points are removed (we do not change the notation of generators deliberately because the new generators correspond to the old ones under a natural isomorphism that arises when dyadic irrational points of the interval are identified with the corresponding binary sequences). These transformations are defined in Fig. 2.2, where the letter P over an interval denotes the permutation of the two halves of the interval and the letter I denotes the identity transformation.

The group $\mathcal{G}$ is a torsion branch group (i.e., each element $g\in\mathcal{G}$ has a finite order; in the present case, the order has the form 2^n, $n=n(g)$); moreover, $\mathcal{G}$ has intermediate growth between polynomial and exponential. In other words, if we denote by $\gamma(n)$ the growth function of $\mathcal{G}$, i.e., the number of elements that can be represented as products of at most n generators, then this function grows faster than any polynomial but slower than any exponential $(\lambda)^n$, $\lambda>1$ [71, 72]. The group $\mathcal{G}$ acts on the tree in a branch way and is therefore a branch group [80].

Notice another extremal property of nonfree actions; namely, it is clear from the definition of the group $\mathcal{G}$ that the whole space of sequences $\Omega=\{0,1\}^{\mathbb{N}}$ is a union of the sets of fixed points of the generators b, c, and d. In [196, Definition 1.10], an action of a group on a set Y is called an action of finite type if Y is a union of the sets of fixed points of a finite set of nonidentity elements of the group. In our example, the set on which the group acts (namely, Ω) has the cardinality of the continuum; however, one can easily construct an action of $\mathcal{G}$ on the set of integers $\mathbb{Z}$ such that this action simulates the action $(\mathcal{G},\Omega)$ and $\mathbb{Z}$, as before, is a union of the sets of fixed points of the generators b, c, and d. To this end, we should imitate the structure of a binary tree by partitioning $\mathbb{Z}$ into cosets with respect to the subgroups $2^n\mathbb{Z}$, $n=1,2,\dots$, and construct an action of $\mathcal{G}$ on $\mathbb{Z}$ that mimics the action of $\mathcal{G}$ on a binary tree. (For example, a acts by the permutation of neighboring even and odd numbers $2i$ and $2i+1$, $i\in\mathbb{Z}$, b acts on even numbers as a acts on $\mathbb{Z}$ (if one identifies $2i$ with i), while on odd numbers it acts as c, etc.)

Example 2.4. The group $\mathcal{B}=\langle a,b\rangle$, called the Basilica, is defined by the relations

$$a(0w)=0w,\qquad a(1w)=1b(w),\qquad b(0w)=1w,\qquad b(1w)=0a(w), \tag{2.10}$$

where $w\in\{0,1\}^*$.

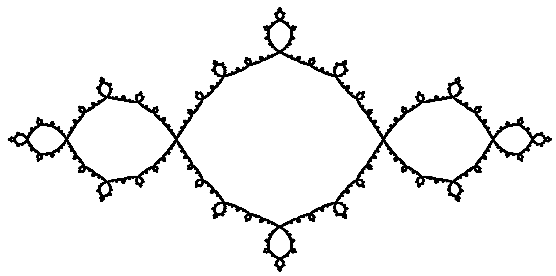

Fig. 2.3. The Julia set of the polynomial z^2-1.

The action of $\mathcal{B}$ on a binary tree belongs to the weakly branch type. The group $\mathcal{B}$ is weakly branch but not branch, because some of its quotients are not virtually abelian (while any proper quotient group of a branch group is virtually abelian [80]). The group got its name due to the similarity of the limit space (shown in Fig. 2.3) of Schreier graphs associated with $\mathcal{B}$ to the reflection in water of Saint Mark's Basilica in Venice. This is one of the most well-known self-similar groups in view of the fact that it is isomorphic to the iterated monodromy group of the quadratic polynomial x^2-1. The theory of iterated monodromy groups and their relation to self-similar groups is the subject of monograph [142] by Nekrashevych, who is mainly responsible for the development of this theory.

Example 2.5 (the Baumslag–Solitar group)**.** The Baumslag–Solitar group $\mathrm{BS}(1,3)=\langle x,y\colon x^{-1}yx=y^3\rangle$ is a representative of one of the most popular (in combinatorial group theory) series of groups $\mathrm{BS}(1,n)=\langle x,y\colon x^{-1}yx=y^n\rangle$. The map $x\to b$, $y\to b^{-1}a$ establishes an isomorphism between $\mathrm{BS}(1,3)$ and the group generated by three automorphisms of a binary tree that are defined by the recurrent relations

$$\begin{aligned} &a(0w)=1c(w), \qquad b(0w)=0a(w), \qquad c(0w)=0b(w),\\ &a(1w)=0b(w), \qquad b(1w)=1c(w), \qquad c(1w)=1a(w). \end{aligned} \tag{2.11}$$

As is shown in [22], the solvable Baumslag–Solitar groups $\mathrm{BS}(1,n)$ can be represented as self-similar groups for any integer n; note that $\mathrm{BS}(1,3)$ can be defined as a self-similar group in a considerably different way compared with that described here (see Section 4 for more details on this subject).

All the groups presented above are self-replicating groups (see the definition in the next section). Consider an example of a non-self-replicating action.

Example 2.6 (free group F_3 of rank 3)**.** The relations

$$\begin{aligned} &a(0w)=0b(w), \qquad b(0w)=1a(w), \qquad c(0w)=1c(w),\\ &a(1w)=1b(w), \qquad b(1w)=0c(w), \qquad c(1w)=0a(w) \end{aligned} \tag{2.12}$$

define a group isomorphic to F_3; this statement was conjectured by S. Sidki and proved by Yaroslav and Mariya Vorobets [186]. The action is transitive on the levels and essentially free, which follows from the results formulated below (for example, from Proposition 4.11, or Proposition 4.12 together with Corollary 9.8). However, this is not a self-replicating action, which can easily be verified by calculating the generators of the first-level stabilizer and projecting them onto the left subtree T_0 of the binary tree T.

Example 2.7 (the group $C_2*C_2*C_2$)**.** The group $C_2*C_2*C_2$ (C_2 is a group of order 2) can be realized by the Bellaterra automaton, which has number 846 in the Atlas of self-similar groups [35]. This automaton got its name from the locality (near Barcelona, Spain) where it was identified by my students E. Muntyan and D. Savchuk as an automaton generating the above-mentioned free product (the relevant proof can be found, for example, in [142, Theorem 1.10.3]). The generators satisfy the following recurrent relations:

$$\begin{aligned} a(0w) &= 0c(w), \qquad & b(0w) &= 0b(w), \qquad & c(0w) &= 1a(w),\\ a(1w) &= 1b(w), \qquad & b(1w) &= 1c(w), \qquad & c(1w) &= 0a(w). \end{aligned} \tag{2.13}$$

The class of groups acting faithfully on rooted trees coincides with the class of residually finite groups, i.e., groups such that, for an arbitrary nonidentity element, there exists a homomorphism into a finite group such that the image of this element is different from the identity. Indeed, if a group G acts faithfully on a tree T, then it is approximated by a sequence of finite groups $G/\mathrm{st}_G(n)$, $n \ge 1$. The reverse implication results from the following proposition.

Proposition 2.8. *Every finitely generated residually finite group has a faithful spherically transitive action of nonbranch type on a spherically homogeneous rooted tree. More precisely, the following assertions are valid.*

(i) *Let $\{H_n\}_{n=1}^{\infty}$, $H_1 = G$, be a decreasing sequence of finite-index subgroups in a group G with trivial core (i.e., the only normal subgroup contained in the intersection $\bigcap_{n=1}^{\infty} H_n$ is trivial). Let $m_n = [H_n : H_{n+1}]$ be the sequence of values of the indices of the subgroups. Then there exist a rooted tree $T_{\overline{m}}$ with branch index $\overline{m} = \{m_n\}_{n=1}^{\infty}$ and a canonically defined faithful action of the group G on it.*

(ii) *If $\{H_n\}_{n=1}^{\infty}$ is a sequence of normal subgroups, then the construction of the action in* (i) *leads to a nonbranch action. Moreover, the stabilizer of any vertex in this case coincides with the stabilizer of the level to which the vertex belongs.*

Proof. (i) Let us construct a rooted tree T in which the nth-level vertices are left cosets with respect to the nth subgroup and two vertices gH_n and fH_{n+1} are connected by an edge if $gH_n \supset fH_{n+1}$. The action of the group G on this set by left multiplication induces an action on the vertex set of the tree T; this action preserves the incidence of vertices and is faithful since the core of the sequence is trivial. The subgroup H_n is the stabilizer of the vertex H_n, and $\bigcap_{g\in G} H_n^g$ is the nth-level stabilizer. It is also obvious that the action is transitive on the levels.

Now, let us prove (ii). First, notice that, replacing each term of the sequence by the intersection of its conjugations by all elements of the group G (there are only a finite number of these conjugations), one can make it so that the sequence from (i) consists of normal subgroups (recall that a residually finite group always has a decreasing sequence of normal subgroups of finite index with trivial intersection). If H_n are normal subgroups, then $H_n = \mathrm{st}_G(n)$ in the previous construction.

The action constructed in (i) is of nonbranch type. Indeed, since the action is transitive on the levels, it suffices to show that for any $n \ge 1$ the rigid stabilizer of some nth-level vertex is trivial. Take $v_n = 1\cdot H_n$ as a vertex of the nth level, and let $g \in G$, $g \notin H_n$. Suppose that $1 \ne f \in \mathrm{st}_G(v_n)$, i.e., $f \in H_n$. Then $fgH_n = gH_n$ for any $g \in G$, and thus $f \in \mathrm{st}_G(n)$; i.e., the stabilizer of the vertex coincides with the stabilizer of its level. Suppose that the rigid stabilizer $\mathrm{rist}_G(u)$ of some vertex $u = gH_n$ of the nth level is nontrivial, $1 \ne f \in \mathrm{rist}_G(u)$, and an element $h \in G$ satisfies $hH_n \ne gH_n$. Then, for any k, $k \ge n$, the vertex hH_k (situated under hH_n) is fixed for f; hence, $f \in \mathrm{st}_G(k)$, and since this is valid for any $k \ge n$ and the intersection of the stabilizers is trivial, we arrive at a contradiction. $\square$

Definition 2.4. The tree constructed when proving this theorem is called a *coset tree*.

Thus, any residually finite group has a nonbranch action. At the same time, by no means every residually finite group has a branch (or even a weakly branch) action, since this requires the presence of nontrivial commutation relations.

In this paper, we mainly consider the actions of abstract groups. However, the set of the topics presented also makes sense and is important for the theory of profinite groups (i.e., compact totally disconnected topological groups or, which is the same, projective limits of finite groups), and we will sometimes briefly discuss some questions related to profinite groups (one can learn about the theory of these groups, for example, in books [54, 192]).

If a tree T is infinite, then $\mathrm{Aut}(T)$ is naturally an infinite profinite group; namely, it is the projective limit of the sequence $\{\mathrm{Aut}(T_{[n]})\}_{n=1}^{\infty}$, where $T_{[n]}$ is the subtree defined by vertices up to the nth level inclusive. In this case, the topology is defined by the system of neighborhoods $\mathrm{st}(n)$, $n \ge 1$, consisting of the stabilizers of the levels. Two automorphisms are close in this topology if their actions coincide up to the nth level inclusive, where n is large. This topology is metrizable, and $\mathrm{Aut}(T)$ has a countable base of open sets. The sequence $\{\mathrm{st}(n)\}_{n\ge 1}$ is called the *principal congruence sequence*.

In the theory of actions on rooted trees and various problems of its applications, it is of interest to study closed subgroups of the profinite group $\mathrm{Aut}(T)$. This interest is motivated by the facts that every profinite group with a countable base of open sets is embedded in $\mathrm{Aut}(T)$ under an appropriate choice of the tree T, and that quite often the properties of an abstract self-similar group are connected with the properties of its closure in $\mathrm{Aut}(T)$. For profinite groups, one can introduce, in a similar way, the notions of branch group and self-similar group and consider the types of actions that were defined above for abstract groups (Definition 2.2). These issues are discussed, in particular, in [3, 79, 80, 82, 153, 6, 1, 13] and other publications.

An efficient method for analyzing an abstract group G is to consider its profinite completion $\widehat{G}$ and to study the properties of this completion and the dynamics of the action of G on $\widehat{G}$ by left translations. This action is viewed as a dynamical system with an invariant measure (the Haar measure). When the pair (G,T) possesses the congruence property with respect to the principal sequence of subgroups $\mathrm{st}_G(n)$, $n=1,2,\dots$, the profinite completion $\widehat{G}$ is isomorphic to the closure $\overline{G}$ in $\mathrm{Aut}(T)$. When the action is transitive on the levels, the action of G on the boundary of the tree is isomorphic to the action of G on the homogeneous space $\overline{G}/P$, where the subgroup P is the stabilizer of a boundary point (the choice of the point does not matter). It would be interesting to find out what faithful actions of a residually finite group with pure point spectrum are isomorphic to the actions by translations on the homogeneous space of the profinite completion. At least, the following proposition holds.

Theorem 2.9. *Suppose that (G,X,μ) is a dynamical system with a pure point spectrum, where G is a countable group acting faithfully on X by transformations preserving the probability measure μ, and the system (G,X,μ) is isomorphic to a system $(G',K/P,\lambda)$, where K is a profinite group and G' is a group isomorphic to G under an isomorphism $\varphi\colon G\to G'$* (*one of the cases of the Mackey theorem* [131] *mentioned in the Introduction*)*. Then there exist a spherically homogeneous rooted tree T and an embedding $\phi\colon G\to \mathrm{Aut}(T)$ such that the following isomorphism of dynamical systems holds*:

$$(G,X,\mu)\simeq(\varphi(G),K/P,\lambda)\simeq(\phi(G),\partial T,\nu),$$

where ν is a uniform measure on the boundary of the tree.

Proof. Since the group is countable, the profinite group K has a countable base of open sets and any closed subgroup is the intersection of a sequence of open subgroups (see, for example, [192, Proposition 0.3.3]). In particular, the subgroup P can be represented as $P=\bigcap_{n=1}^{\infty}P_n$, where $\{P_n\}_{n=1}^{\infty}$ is a decreasing sequence of open subgroups. Using this sequence, we construct a coset tree T in the same way as in the proof of Proposition 2.8 (obviously, this tree is of type $T_{\overline{m}}$ for an appropriate branch index $\overline{m}$). Associating a point gP of the space K/P with a boundary point $\xi\in\partial T$ specified by the condition $\xi\in gP_n$, $n=1,2,\dots$, defines a homeomorphism from K/P

onto ∂T that maps the action of G on K/P to the corresponding action on the boundary and the measure μ to ν. $\square$

If G is a regularly branch group (see Definition 3.5 below), then the closure $\overline{G}$ is described by a finite number of local prohibitions on its action, as demonstrated in [81, 175]. This fact is a noncommutative analog of an ergodic-theory phenomenon associated with the concept of a finite-type translation, when the phase space of a symbolic system is described by a finite set of prohibitions. Therefore, group actions on a regular tree that are defined by a finite set of prohibitions should be called *actions of finite type*. Let us explain this idea in more detail. Any automorphism of a rooted tree T can be identified with a coloring of vertices of the tree with the elements of appropriate symmetric groups (namely, a vertex u of level n can be labeled by an arbitrary element of the symmetric group $\mathrm{Sym}(m_{n+1})$; such a coloring is called a portrait of the automorphism). If the tree T is regular of degree d, then the set of such colorings is a configuration space on the vertex set $V = V(T)$ with values in $\mathrm{Sym}(d)$, i.e., the space $\mathrm{Sym}(d)^V$ equipped with the Tikhonov topology. In this approach, the elements of the group $\mathrm{Aut}(T)$ are identified with elements of the set $\mathrm{Sym}(d)^V$. If we introduce a set of local prohibitions, i.e., for every vertex u, we declare that some colorings of the tree T_u of depth k, where k is a preassigned number, are inadmissible, then, under the condition that the set of admissible colorings (considered as tree automorphisms) is closed with respect to the operation of composition and makes up a group, we obtain a closed subgroup of the group $\mathrm{Aut}(T)$.

The simplest example of this kind of prohibitions is a prohibition on the set of elements of a symmetric group that can be used for labeling vertices. In other words, the choice of a subgroup $H < \mathrm{Sym}(d)$ defines a subgroup $G_H < \mathrm{Aut}(T)$ whose elements are in bijection with elements of the set H^V. For example, as H, one can take a cyclic subgroup of order d (which acts by cyclically permuting the edges emanating from an arbitrary vertex). If $d = p$ is a prime number, then the group thus obtained is a Sylow p-subgroup in $\mathrm{Aut}(T)$. If $H = \mathrm{Alt}(d)$ is the alternating group, then we obtain another interesting example of a group, which we denote by $\mathrm{Alt}(T)$. The profinite group $\mathrm{Alt}(T)$, just as $\mathrm{Aut}(T)$, is a branch group and is isomorphic to the infinite iterated wreath product of copies of the group $\mathrm{Alt}(d)$. For $d \geq 5$, this group is finitely generated as a topological group (moreover, it is 2-generated, and the set of pairs of elements that generate this group has a positive measure; i.e., two randomly chosen elements of this group generate it with positive probability [29]). Bondarenko [32] found necessary and sufficient conditions for the infinite iterated permutational wreath product of copies of a finite transitive group of permutations, considered as a profinite group, to be topologically finitely generated.

However, even in the case of prohibitions of depth 2, a group defined by prohibitions may not be finitely generated as a topological group [175, 176]. Note that the closure of the intermediate growth group $\mathcal{G}$ is defined by prohibitions of depth 4, and it is quite possible that in the case of a binary tree (i.e., when $d = 2$) there do not exist any topologically finitely generated groups defined by prohibitions of depth $k < 4$. It is the group $\mathcal{G}$ that in fact prompted the present author to introduce the concept of a group of finite type.

Let us introduce two more concepts related to abstract group theory (one of these concepts has already been mentioned above).

Definition 2.5. (1) A group G is said to be *just-infinite* if it is infinite while any of its proper quotient groups is finite.

(2) A group is said to be *hereditary just-infinite* if it is residually finite and any finite-index subgroup is just-infinite (obviously, “finite-index subgroup” in this definition can be replaced by “normal subgroup of finite index”).

Any infinite finitely generated group can be mapped by a homomorphism onto a just-infinite group. In other words, the following proposition is valid.

Proposition 2.10 [79]. *Let G be finitely generated and infinite. Then there exists an $H \triangleleft G$ such that G/H is a just-infinite group.*

Proof. Consider the partially ordered (by inclusion) set $\mathcal{N}$ of normal subgroups of infinite index of the group G. This set is nonempty (the trivial subgroup belongs to $\mathcal{N}$). Let us show that any chain in $\mathcal{N}$ has a maximal element that belongs to $\mathcal{N}$. Suppose the contrary: let H_α be a chain consisting of normal subgroups of infinite index in G. Let $H = \bigcup_\alpha H_\alpha$. It is obvious that H is a normal subgroup and a maximal element for the chain H_α. Suppose that H does not belong to $\mathcal{N}$, i.e., H has finite index in G. Then H is a finitely generated group and, hence, coincides with H_α for some α. A contradiction. □

Thus, if one faces the problem of constructing an infinite finitely generated group possessing a certain property $\mathcal{P}$ that is preserved under homomorphisms, then it is natural to try to construct such an example in the class of just-infinite groups, because if such an example exists, then it exists in this class as well.

Just-infinite groups generalize, in a sense, the class of simple groups and occupy a kind of intermediate position between finite and infinite groups. The following theorem, which was derived in [80] from Wilson's results [191], elucidates the algebraic meaning of the concept of a branch group. The role of this concept in problems of dynamical systems theory is discussed in [80, 87, 81].

Theorem 2.11. *Let G be a just-infinite group. Then either G is a branch group or there exists a finite-index subgroup $H \leq G$ isomorphic to a finite power K^m of some group K that is either a hereditary just-infinite group or a simple group.*

We should warn the reader that the concept of hereditary just-infinite group adopted in the present paper differs from that used by Wilson in [193] (namely, he did not assume the residual finiteness of groups).

3. GROUPS OF AUTOMATA AND SELF-SIMILAR ACTIONS

Now we consider a narrower class of actions on regular rooted trees, namely, the actions defined by invertible initial Mealy-type automata. The groups defined by such actions are called groups of automata. An especially important subclass of the class of groups of automata is given by self-similar groups.

Let X be a finite alphabet consisting of d letters. The set X^* of all finite words over X is identified with the vertex set of a d-regular rooted tree, which we denote by T_d, $T(X)$, or T if it is clear what alphabet is meant (the empty word corresponds to the root vertex). For any word $v \in X^*$ and an arbitrary letter $x \in X$, the vertices v and vx are neighbors in T_d, and this defines the edge set. Usually, we will consider alphabets of the form $\{1, 2, \dots, d\}$ or $\{0, 1, \dots, d-1\}$. Important is the case of a prime number d, and a special role is played by the binary alphabet $\{0, 1\}$, which is associated with a binary rooted tree. A d-regular tree T_d is an example of a spherically homogeneous tree $T_{\overline{m}}$ and corresponds to the case when the branch index $\overline{m}$ is a constant sequence: $m_n = d$, $n = 1, 2, \dots$.

The group $\mathrm{Aut}(X^*)$ of all automorphisms of the tree X^* has the structure of an infinite iterated permutational wreath product $\wr_{\mathrm{perm}}^{i\geq 1} \mathrm{Sym}(d)$ (since $\mathrm{Aut}(X^*) \cong \mathrm{Aut}(X^*) \wr_{\mathrm{perm}} \mathrm{Sym}(d)$, where the symmetric group $\mathrm{Sym}(d)$ acts naturally by permutations on the first level of a tree, which consists of vertices numbered by elements of X). This gives a convenient means of representing the automorphisms in $\mathrm{Aut}(X^*)$ as

$$g = (g_0, g_1, \dots, g_{d-1})\sigma_g \tag{3.1}$$

(this relation is analogous to (2.2)). Here $g_0, g_1, \dots, g_{d-1}$ are the automorphisms induced by the automorphism g on the subtrees growing from the first-level vertices, and σ_g is a permutation of the first-level vertices that is induced by this element (in other words, $\sigma_g(x) = g(x)$, where x is an arbitrary vertex of the first level). The automorphisms $g_i = g|_i$ are called sections (or projections) of the automorphism g at the first-level vertices.

The concept of section is generalized to an arbitrary vertex as follows. For any vertex $u \in X^*$, the section $g|_u$ at u is the automorphism induced by the element g on the subtree T_u starting at the vertex u. This automorphism $g|_u$ is uniquely defined by the relation $g(uw) = g(u)g|_u(w)$, which holds for any word $w \in X^*$. The multiplication of automorphisms expressed in the form (3.1) is performed as follows. If $h = (h_0, h_1, \dots, h_{d-1})\sigma_h$, then

$$gh = \big(g_{\sigma_h(0)}h_0, \dots, g_{\sigma_h(d-1)}h_{d-1}\big)\sigma_g\sigma_h.$$

Definition 3.1. An action of a group $G \le \mathrm{Aut}(T)$ is said to be *self-similar* if every section $g|_u$ of any element $g \in G$ belongs to the group G under the canonical identification of T_u with T.

Self-similar actions can also be defined in the following equivalent manner.

Definition 3.2. An action of a group G of automorphisms of a tree T is said to be *self-similar* if, for any element $g \in G$ and any $x \in X$, there exist $h \in G$ and $y \in X$ such that the following equality holds for an arbitrary $w \in X^*$:

$$g(xw) = yh(w). \tag{3.2}$$

Definition 3.3. A group is said to be *self-similar* if it has a faithful self-similar action. A group is said to be *strongly self-similar* if it has a faithful self-similar action and a finite system of generators that is closed with respect to the operation of transition from an element to its section.

A convenient means of defining a self-similar group G generated by automorphisms g_i, $i \in I$, is given by *wreath recursions* (or recurrent relations). They have the form

$$g_i = \big(w_1(g_1, \dots, g_n, \dots), \dots, w_d(g_1, \dots, g_n, \dots)\big)\sigma_{g_i}, \tag{3.3}$$

where w_i, $i \in I$, are some words in the alphabet $\{g_n^{\pm 1}\}$ (in other words, elements of the corresponding free group of rank $|I|$). If the group is strongly self-similar, then the system of relations (3.3) is finite. By an appropriate choice of generators, the words w_i in (3.3) can be taken of length ≤ 1 and in the positive alphabet of generators (i.e., each w_i is either an empty word representing the identity or a word of length 1 representing a generator g_j, $j = j(i)$). This form of recurrent relations corresponds to representing a group by a Mealy-type automaton as described below.

These relations describe the action of every generator g_i on the first level, as well as the action of its sections in terms of the same set of generators. Iterating these relations and "unwinding" them, one can derive ordinary (from the viewpoint of the combinatorial group theory) relations between the generators (namely, one can find words, in the alphabet of generators and their inverse elements, that define the identity element).

There exists a canonical embedding

$$\Psi\colon\ \mathrm{st}_G(1) \hookrightarrow G \times G \times \ldots \times G \tag{3.4}$$

(d factors) defined as

$$g \overset{\Psi}{\mapsto} (g|_0, g|_1, \dots, g|_{d-1}).$$

Iterating Ψ, we obtain a sequence of embeddings

$$\Psi_n\colon\ \mathrm{st}_G(n) \hookrightarrow G^{d^n} \tag{3.5}$$

(in the last expression, the right-hand side denotes the direct product of d^n copies of the group G).

Definition 3.4. Let $K, K_0, \dots, K_{d-1}$ be subgroups of a self-similar group G that acts on a tree T. We will say that K *geometrically* contains the product $K_0 \times \ldots \times K_{d-1}$ and use the notation

$$K_0 \times \ldots \times K_{d-1} \preceq K$$

if $K_0 \times \ldots \times K_{d-1} \le \Psi(\mathrm{st}_G(1) \cap K)$.

Definition 3.5. Let G be a self-similar group of automorphisms of a tree T that acts transitively on the levels of the tree and K be a nontrivial subgroup of G. The group G is called a *regularly weakly branch* group over K if

$$K \times \ldots \times K \preceq K.$$

If, in addition, the index of K in G is finite, then we say that the group G is *regularly branch* over K.

The meaning of this definition is that the group K is *co-self-similar*; i.e., placing arbitrary elements of the group K at the vertices of an arbitrary finite set Y of mutually orthogonal vertices (i.e., $T_u \cap T_v = \varnothing$, $u, v \in Y$, $u \neq v$) and making these elements act on the subtrees with roots at these vertices, we obtain an automorphism of the entire tree (acting trivially outside the union $\bigsqcup_{u \in Y} T_u$) that belongs to the group K. Note that a co-self-similar subgroup of a self-similar group may not be self-similar. The property of co-self-similarity of a finite-index subgroup in a self-similar group G is useful and allows one to solve various problems related to the group G (for the group $\mathcal{G}$, examples of application of co-self-similar groups are given in [81]).

Note that if G is a regularly branch group, then G is also a branch group because

$$\Psi_n(\mathrm{st}_G(n)) \ge \Psi_n(\mathrm{rist}_G(n)) \ge K^{d^n}$$

is a subgroup of finite index in G^{d^n} and so $\mathrm{rist}_G(n)$ is a subgroup of finite index in G.

Definition 3.6. (a) A self-similar group G is said to be *self-replicating* (the terms *recurrent group*, *self-reproducing group*, and *fractal group* are also used in the literature) if, for any vertex $u \in X^*$, the map $\varphi_u \colon \mathrm{st}_G(u) \to G$ defined by the relation $\varphi_u(g) = g_u$ (recall that g_u is the section of g at u) is an epimorphism.

(b) A self-similar group G is said to be *strongly self-replicating* if the projection of the stabilizer $\mathrm{st}_G(1)$ to an arbitrary vertex coincides with the entire group G.

Obviously, in the first part (a) of this definition, it suffices to restrict the consideration to vertices of the first level because the property of self-reproducibility extends to the other vertices automatically. If an action is transitive on the levels, then it suffices to check the property of self-reproducibility for any specific vertex of the first level. It is also clear that condition (b) implies that the projection of the stabilizer $\mathrm{st}_G(n)$ of any level n to an arbitrary vertex of this level is the entire group G. The group $\mathcal{G}$ is an example of a strongly self-replicating group.

In the study of self-similar groups, an important role is played by contracting groups defined below. These groups possess many useful properties. For example, for these groups, there exists a branch algorithm for solving the word problem (which solves this problem in polynomial time [72, 163]), and the so-called *limit space* and *limit solenoid* [142] are defined; these objects give additional means for studying groups, as demonstrated in many papers by Nekrashevych and in his monograph [142]. In the definition, we use the standard (in group theory) concept of *length* of an element of a group with respect to a system of its generators; the length of an element is the minimal length of the word that represents this element as a product of generating elements of the group and their inverses. Recurrent relations for the generators of a self-similar group in the case when $|w_i| \le 1$ (i.e., when a group is given by an automaton presentation as will be discussed below) obviously imply that the length of an element does not increase when passing to projections. Contracting groups possess a stronger property.

Definition 3.7. A strongly self-similar group G is said to be *contracting* if there exist constants λ, $0 < \lambda < 1$, and C such that, for any element $g \in G$, the length (with respect to the system of generators defined by wreath recursions of type (3.3)) of the section g_u at an arbitrary first-level vertex u of the tree satisfies the following upper estimate:

$$|g_u| < \lambda |g| + C.$$

If

$$|g| > C/(1-\lambda),$$

then the lengths of all projections of the first level (and hence of other levels as well) are strictly less than the length of the original element. There exists only a finite set of elements (contained in the ball of radius $C/(1-\lambda)$) whose lengths do not decrease when passing to projections. This set serves as a kind of core of the group, because, when one describes the action of an arbitrary element on a tree in terms of its projections at vertices of level n for sufficiently large n, everything is ultimately reduced to the elements of the core. The concept of contracting group can also be introduced in a wider context of self-similar groups. In this case, a self-similar group G acting on a regular rooted tree is said to be contracting if there exists a finite set (core) such that, for any element $g \in G$, there exists a level for which all projections of the element g to the vertices of this level belong to the core. A minimal set possessing this property is called a *nucleus*. It is easily seen that for strongly self-similar groups this definition of contractibility is equivalent to the one given above.

Historically, the first example of a contracting self-similar group was the group $\mathcal{G}$. For this group, the constants λ and C are equal to $1/2$ and 1, respectively, and the nucleus is $\mathcal{N} = \{1, a, b, c, d\}$.

An alternative language in which it is convenient to deal with self-similar groups is the language of automata. Here we mean Mealy-type automata; a convenient means of describing these automata is the Moore diagrams. The theory of such automata is presented, for example, in [119] (we also recommend the introductory part of paper [35] for preliminary acquaintance with the subject, as well as the survey article [87], in which the basic information is presented for a wider class of automata (transducers), namely, for asynchronous automata). Paper [87] also contains detailed information on the groups of automata that was available at the time of writing.

Definition 3.8. A *Mealy-type automaton* is defined as a quadruple (Q, X, π, λ), where Q is a set called the set of states, X is a finite alphabet, $\pi\colon Q \times X \to Q$ is a map called the transition function, and $\lambda\colon Q \times X \to X$ is a map called the output function. If the set of states Q is finite, then the automaton is said to be *finite*.

An automaton thus defined is said to be noninitial. Denote it by $\mathcal{A}$. An initial automaton is obtained from a noninitial one by choosing a certain state $q \in Q$ as the initial state. An initial automaton with initial state q is denoted by $\mathcal{A}_q = (Q, X, \pi, \lambda, q)$. It can be interpreted as a sequential machine (or transducer), i.e., an information processing machine. Namely, given a word $w \in X^*$, an automaton acts on it as follows. It reads the first letter x in the word w and, depending on this letter and on the state q, outputs a letter $\lambda(q, x) \in X$ and changes its internal state to $\pi(q, x)$. This new state defines the action of the automaton on the remaining part of the word w, which proceeds in the same manner. In other words, having read the next letter of the word w at the input, the automaton immediately outputs a letter defined by the input letter and the current state (according to the function λ), and then goes over to the next state according to the command of the transition function π. The automaton thus defined belongs to the class of synchronous automata (i.e., for any input symbol, an output symbol immediately appears at the output).

The output function λ can be extended to a function $\lambda\colon Q \times X^* \to X^*$; for fixed q, this map preserves the length of words and is consistent with taking the beginnings (prefixes) of words. Thus, each state q of the automaton defines a map (which is also denoted by q) from X^* to X^* according to the relation $q(w) = \lambda(q, w)$. We pay special attention to the case when the map $\lambda(q, \cdot)$ is a permutation on X for any $q \in Q$; then the map $q \colon X^* \to X^*$ is invertible and hence (in view of the consistency of the action on prefixes) defines an automorphism of the tree X^*. In this case, the automaton $\mathcal{A}_q$ is said to be *invertible*. A noninitial automaton $\mathcal{A}$ is invertible if all initial automata $\mathcal{A}_q$, $q \in Q$, are invertible. In what follows, we will consider only invertible automata, although the study of semigroups generated by noninvertible automata is also of interest [21].

Two initial Mealy automata $\mathcal{A}_q$ and $\mathcal{B}_s$ with the same alphabet X are said to be equivalent if they define the same map on the set of words X^* (or, which is the same in the case of invertible automata, define the same automorphism of the tree $T(X)$). Two noninitial automata $\mathcal{A}$ and $\mathcal{B}$ are equivalent if, for any state q of the automaton $\mathcal{A}$, there exists a state s of the automaton $\mathcal{B}$ such that the automata $\mathcal{A}_q$ and $\mathcal{B}_s$ are equivalent and, conversely, for any state of the automaton $\mathcal{B}$, there exists a state of the automaton $\mathcal{A}$ such that the corresponding initial automata are equivalent. For any finite automaton, there exists a minimal automaton (i.e., an automaton with the minimum number of states) that is equivalent to it; the minimal automaton is unique up to equivalence. There exists a classical minimization algorithm for finite automata [119], which was generalized to asynchronous automata in [87].

On the set of initial automata, the operation of composition of automata is defined. This operation corresponds to the operation of composition of the respective maps of the set X^* (or the respective tree automorphisms); i.e., the transformation defined by a composition of two automata is the composition of the transformations defined by these automata. The formal definition is as follows. The composition of automata $\mathcal{A}_q = (Q, X, \pi, \lambda, q)$ and $\mathcal{B}_s = (S, X, \mu, \rho, s)$ is an automaton $\mathcal{C}_{(q,s)}$ with the set of states $Q \times S$, the initial state (q, s), and the transition function γ and output function δ defined as follows:

$$\gamma((r,t),x) = (\pi(r,x), \mu(t, \lambda(r,x))), \qquad \delta((r,t),x) = \rho(t, \lambda(r,x)).$$

The composition of automata is consistent with the equivalence of automata (i.e., replacing the automata in a composition by equivalent ones, we obtain an automaton equivalent to the composition of the original automata). Thus, the equivalence classes of invertible initial automata form a group with respect to the operation of composition (in what follows, we will omit the word "class"). This group is isomorphic to the automorphism group of the tree $T(X)$ and is denoted by $\mathcal{GA}(X)$. The composition of finite automata is a finite automaton with the number of states equal to the product of the numbers of states of the factors (such an automaton may be equivalent to an automaton with a smaller number of states; therefore, after applying the formal rule of composition, one usually applies the procedure of minimization).

For each invertible initial automaton A_q, there exists an initial automaton A_q^{-1} inverse to it, which defines a transformation of sequences that is the inverse of the transformation defined by the automaton A_q (thus, after minimization, the compositions $A_q \circ A_q^{-1}$ and $A_q^{-1} \circ A_q$ are trivial automata, which define the identity transformation). In the language of Moore diagrams (which will be discussed soon), when edges are labeled by pairs of alphabet symbols (input–output), the automaton A_q^{-1} is obtained from the automaton A_q by reversing the orientations of all edges (without changing the labels) and preserving the state q as an initial one.

If A is a noninitial invertible automaton, then A^{-1} is a noninitial automaton obtained from A by reversing the orientations of the edges in the same way as above. For each state of the automaton A, there is a state of the inverse automaton A^{-1} such that the corresponding initial automata are inverses of each other in the sense of the composition operation.

The group of all invertible initial automata contains the subgroup $\mathcal{FGA}(X) \subset \mathcal{GA}(X)$ of finite invertible automata. This subgroup is isomorphic to a certain countable subgroup of the group $\mathrm{Aut}(T(X))$, namely, to the group of tree automorphisms defined by finite automata, and depends on the cardinality of the alphabet X. An embedding of the alphabet in a larger alphabet defines an embedding of the corresponding groups of automata. Therefore, there exists an increasing sequence of groups of automata indexed by positive integers.

If we also consider asynchronous automata (as suggested, for example, in [87]), then there exists only one (universal) group of finite invertible asynchronous automata, denoted by $\mathcal{Q}$ in [87], that does not depend on the cardinality of the alphabet (if we consider only alphabets of cardinality >1) and contains all the groups $\mathcal{FGA}(X)$, $|X| = 2, 3, \dots$. This group (called the group of *rational homeomorphisms of the Cantor set*) is of greatest interest; however, efficient methods for studying this group have not yet been found, and very little is known about it. In particular, it is not known whether or not this group is finitely presented; moreover, it is even not known whether or not it is finitely generated. The only known thing is that the automorphism groups of cellular automata are embedded in this group and that it contains the famous Thompson group F (as well as simple Thompson groups and the Thompson–Higman groups), which will be touched upon in Section 8. Note that the groups $\mathcal{FGA}(X)$ and $\mathcal{GA}(X)$ are not finitely generated because they can be mapped by a homomorphism onto an elementary 2-group of infinite rank.

An important property of the group $\mathcal{Q}$ is that the word problem is solvable in this group. Namely, to find out if a product of finite invertible asynchronous automata is the identity element in the group $\mathcal{Q}$, one should multiply the automata using the method described in [87] for calculating the composition of asynchronous automata (it generalizes the classical method of composition of automata in the synchronous case) and then minimize the product using the minimization algorithm for asynchronous automata described in [87] (which again generalizes the classical minimization algorithm for synchronous automata). In particular, the word problem is solvable in each of the groups $\mathcal{FGA}(X)$ and, hence, in each finitely generated subgroup of the group of finite automata. It is still unknown whether the isomorphism problem for groups generated by finite (synchronous or asynchronous) automata is solvable; however, the recent result of Z. Šunić and E. Ventura shows that the conjugacy problem may be unsolvable [177]. At the same time, for many groups generated by automata, for example for the group $\mathcal{G}$, the conjugacy problem is sometimes solvable even by a polynomial algorithm in nonobvious situations [123, 161, 81, 130].

Definition 3.9. Let $\mathcal{A}$ be a noninitial invertible automaton. Each state of $\mathcal{A}$ defines an automorphism of a tree T. The group generated by all such automorphisms (i.e., by the automorphisms defined by the initial automata $\mathcal{A}_q$, $q \in Q$) is called the *automaton group* defined by $\mathcal{A}$.

The class of automaton groups coincides with the class of self-similar groups. Indeed, an action on X^* of an arbitrary element g of a self-similar group G can be represented as an automaton map $\mathcal{A}_q$, where the states of the automaton $\mathcal{A}$ are in bijection with the sections g_w, $w \in X^*$, and the transition and output functions are defined by relations (3.1); namely, $\pi(g|_u, x) = g|_{ux}$ and $\lambda(g|_u, x) = g|_u(x)$ for any $u \in X^*$.

Of special interest are self-similar groups defined by a finite automaton (i.e., groups generated by all states of one finite noninitial automaton). We call them strongly self-similar groups (it is these groups that are called self-similar in some publications).

A convenient illustrative means for defining automata are *Moore diagrams*. Such a diagram is an oriented graph whose vertex set is identified with the set of automaton states Q and, for each vertex $q \in Q$ and each letter $x \in X$, there exists an oriented edge connecting q with $\pi(q, x)$; this edge is labeled by the pair $(x|\lambda(q, x))$ (often the parentheses are omitted and the separating vertical line is replaced by a comma or another separating mark; often multiple edges between two vertices are replaced by a single edge with a multiple lable). Examples of such diagrams are given by the diagrams shown in Figs. 3.1 and 3.2.

The automata defined by these diagrams generate the lamplighter group mentioned in the preceding section and the free product of three copies of an order 2 group, respectively (i.e., the groups from Examples 2.2 and 2.7).

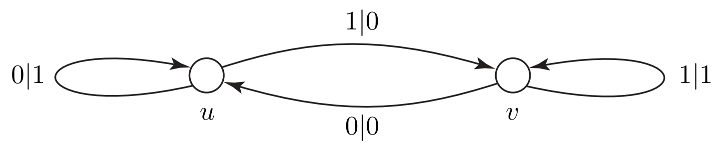

Fig. 3.1. Automaton generating the group $\mathcal{L}$.

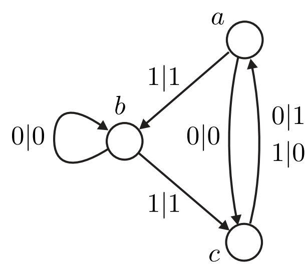

Fig. 3.2. Bellaterra automaton.

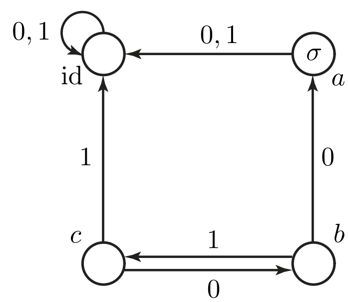

Fig. 3.3. Automaton generating $\mathrm{IMG}(z^2+i)$; $\sigma = (0,1) \in \mathrm{Sym}(2)$.

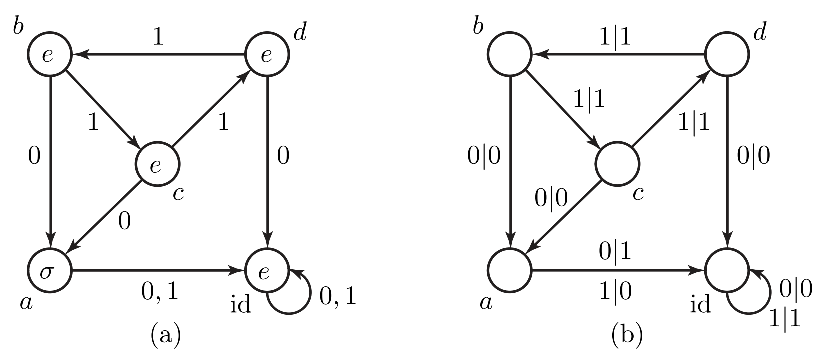

Fig. 3.4. Automaton generating the group $\mathcal{G}$; $\sigma = (0,1) \in \mathrm{Sym}(2)$.

Recall that for an invertible automaton and any of its states $q \in Q$, the map $\lambda(q,\cdot)\colon X \to X$ is a permutation of the alphabet. Therefore, it is often convenient to describe the output function by labeling the states of the automaton with the respective elements of the symmetric group $\mathrm{Sym}(X)$ and keeping the first part of the label (which describes the transition function) on the edges, as shown in Figs. 3.3 and 3.4a (Fig. 3.4 represents the same automaton in two ways). For the states labeled by the identity element, the label is sometimes omitted (as is the case for the automaton shown in Fig. 3.3), or the symbol e is used to denote the identity element of the symmetric group $\mathrm{Sym}(d)$ (as in Fig. 3.4a).

The automata shown in Figs. 3.1–3.4, generate the lamplighter group, the Bellaterra group, the iterated monodromy group $\mathrm{IMG}(z^2+i)$ of the map $z \to z^2+i$ (for the iterated monodromy groups, see [18, 142]), and the intermediate growth group $\mathcal{G}$, respectively.

The groups defined by finite automata are objects of greatest interest; however, these groups are hard to analyze. One of the natural problems arising when studying these groups is the problem of their classification in the class (m, n) consisting of groups generated by m-state invertible automata over an n-letter alphabet, $m, n \geq 2$ (at least for small values of the parameters m and n). This problem is easily solved for the class $(2, 2)$ (see [87], where it is shown that there exist six such groups, namely, $\{1\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}$, the infinite dihedral group D_∞, and the lamplighter group). The classes $(3, 2)$ and $(2, 3)$ have much higher degree of complexity and contain at most 115 and 139 groups, respectively [35, 139]. To date, the class $(3, 2)$ of groups has been studied in much more detail compared with the class $(2, 3)$.

4. FREE ACTIONS ON THE BOUNDARY

In this section, we focus our attention on the actions on a rooted tree that induce free (in the topological or metric sense) actions on the boundary ∂T of the tree.

However, before passing to the consideration of free actions, we recall some concepts related to dynamical systems and formulate an important proposition. An action (G, X) (G is a group and X is a topological space) is minimal if it has no nonempty proper closed subsets (in other words, the closure of the orbit of any point is equal to the entire space X). A dynamical system $(G, \partial T)$ is topologically transitive if, for any two open sets $U, V \subset X$, there exists an element $g \in G$ such that $gU \cap V \neq \varnothing$. For separable metric spaces, this is equivalent to the existence of a point with a dense orbit. Finally, the unique ergodicity means the uniqueness of an invariant probability measure. The unique ergodicity implies ergodicity. Many of the definitions given below (as well as some of the propositions) make sense for actions on general topological spaces; however, we will mainly restrict ourselves to actions on the boundary of a rooted tree.

Proposition 4.1 [87, Proposition 6.5]**.** *If a group acts on a spherically homogeneous tree, then the following conditions are equivalent*:

(i) *The action is spherically transitive.*

(ii) *The dynamical system $(G, \partial T)$ is minimal.*

(iii) *The dynamical system $(G, \partial T)$ is topologically transitive.*

(iv) *The dynamical system $(G, \partial T, \nu)$ is ergodic.*

(v) *The measure ν is a unique invariant σ-additive probability measure.*

Recall that the uniform measure ν on the boundary of a tree was defined in Section 2.

We restrict ourselves to the proof of the equivalence of conditions (i) and (iv) and refer the reader to [87] for all the details.

Proof. Indeed, if for some level there exists a proper nonempty invariant subset U of its vertices, then the union $\bigcup_{u \in U} C_u$ of the corresponding cylinders yields a nontrivial invariant subset of the boundary.

There are at least two natural methods for proving the converse statement. One can apply the Lebesgue density theorem and use a G-invariant metric of the form $d_{\bar\lambda}$ on the boundary and the fact that under the action of the group elements the images of a ball $B_x(\delta)$ of radius δ centered at a point x are also balls of the same radius and either are disjoint or coincide with each other (see [87, Proposition 6.5] for details). One can also consider, as suggested in [81], a unitary representation π of the group G in the Hilbert space $H = L^2(\partial T, \nu)$; the representation is defined by the action of G on the boundary by the transformations $(\pi(g)f)(x) = f(g^{-1}x)$, which preserve the measure ν. The ergodicity of the dynamical system $(G, \partial T, \nu)$ is equivalent to the fact that the G-invariant functions in H are constant. Denote by H_n the subspace of dimension $m_1 \dots m_n$ in H generated by the characteristic functions of cylindrical sets C_w: $|w| = n$. Then the spaces H_n, $n = 1, 2, \dots,$

form an increasing system of subspaces whose union is dense in H. The spaces H_n are invariant with respect to π, and the restrictions $\pi_n = \pi|_{H_n}$ are isomorphic to permutation representations for the actions of G on levels. Since the action on levels is transitive, only constant functions are invariant for the representations π_n. Assuming that $f \in H$ is a nonconstant invariant function and expanding it (as an element of the Hilbert space) in a sum $\sum_{i=0}^{\infty} f_i$ (corresponding to the decomposition $H = \mathbb{C} \oplus \bigoplus_{i=1}^{\infty} H_n^{\perp}$, where $H_n^{\perp} = H_n \ominus H_{n-1}$ and the first term $H_0 = \mathbb{C}$ corresponds to constant functions), we arrive at a contradiction. □

Let G be a countable group acting on a complete metric space X. Denote by X_- the set of points with nontrivial stabilizer and by X_+ the set of points with trivial stabilizer.

Definition 4.1. 1. The action (G, X) is said to be *absolutely* free if all points have trivial stabilizer.

2. The action (G, X) is said to be *relatively* free if there exists at least one point with trivial stabilizer.

3. The action (G, X) is *topologically* free if X_- is a meager set (i.e., it can be represented as a countable union of nowhere dense sets).

4. Suppose that the action (G, X) has a G-invariant (not necessarily finite) Borel measure μ. The system (G, X, μ) is said to be *essentially* free if $\mu(X_-) = 0$.

For countable groups, the concepts of topological freeness and essential freeness can be formulated in terms of the sets of fixed points of individual elements (namely, for any $1 \neq g \in G$, the set $\mathrm{Fix}(g)$ is meager or has measure 0, respectively).

In what follows, we will use the terminology introduced mainly for the case of actions on rooted trees; however, notice that below we sometimes deal with general topological spaces. So, suppose given a faithful action of an infinite countable group G on a rooted tree T, $\xi \in \partial T$. Denote by $\mathrm{st}_G(\xi)$ the stabilizer of a point ξ on the boundary of the tree, by $\mathrm{Fix}(g)$, as above, the set of g-fixed points of the boundary, by $(\partial T)_+$ the set of points with trivial stabilizer (we will call these points free points), and by $(\partial T)_-$ its complement.

The following concept proves useful for constructing examples of free actions.

Definition 4.2. For a vertex u of a tree, the subgroup $\mathrm{triv}_G(u)$ consisting of elements that fix u and act trivially on the subtree T_u is called the *trivializer* of the vertex u.

Thus, an action is locally nontrivial (see Definition 2.2) if and only if the trivializers of all vertices are trivial.

Proposition 4.2. *The action of a countable group on the boundary of a tree is topologically free if and only if it is locally nontrivial.*

Proof. Indeed, if the action is topologically free and we assume that $\mathrm{triv}_G(u) \neq 1$, then all points of the boundary that belong to the cylindrical set C_u have a nontrivial stabilizer; hence, $C_u \subset (\partial T)_-$; however, since C_u is an open set, we arrive at a contradiction.

Now, suppose that the action is not topologically free. The set of points with nontrivial stabilizers can be represented as a union $\bigcup_{1 \neq g \in G} \mathrm{Fix}(g)$. Hence, for some $g \in G$, $g \neq 1$, the set $\mathrm{Fix}(g)$ contains an open subset and, hence, also contains a certain cylindrical set C_u. In this case, $g \in \mathrm{triv}_G(u)$, which proves that the trivializer $\mathrm{triv}_G(u)$ is nontrivial. □

The following statement is a corollary of the proposition proved above and of the fact that the uniform measure on the boundary is uniformly "distributed" over this boundary. This statement also follows from a more general proposition (see Corollary 4.10).

Corollary 4.3. *A spherically transitive essentially free action on the boundary of a tree is topologically free.*

It is obvious that the absolute freeness of an action implies other types of freeness formulated in Definition 4.1, which in turn imply the faithfulness of the action; however, there exist faithful actions that have no free orbit. For example, the group $\mathcal{G}$ from Example 2.3 acts nonfreely on each of its orbits. This follows from the fact that $\mathcal{G}$ is a contracting self-similar group; for such groups, it is shown in [16, 17] that each orbit has polynomial growth. Since the growth of the group $\mathcal{G}$ is intermediate between polynomial and exponential, the action on orbits is nonfree. In [87], the authors raised the problem of constructing an ergodic action on a rooted tree for which the Schreier graphs (which will be considered in the subsequent sections) generic in the topological sense would differ from the Schreier graphs generic in the metric sense. Such an example can be derived from [28]; in an explicit form it was constructed in [2]. Moreover, in [2] there is a construction of a free group acting on a rooted tree such that the action is free in the topological sense (i.e., a generic Schreier graph is a Cayley graph) but not free in the metric sense. However, there are no such examples for actions of strongly self-similar groups because of the following theorem established by Kambites, Silva, and Steinberg on the basis of the results and methods of [95].

Theorem 4.4 [112, Theorem 4.2]**.** *For strongly self-similar groups (i.e., groups defined by finite automata), any topologically free action is essentially free.*

Proof. For the proof of this theorem, see Corollary 9.8 below. $\square$

The following simple proposition will be useful in Section 5 for proving the essential freeness of some actions.

Proposition 4.5. *For strongly self-similar groups acting on a binary tree, the action is essentially free if and only if the rigid stabilizer of the first level is trivial.*

Proof. Indeed, if $\mathrm{rist}_G(1) \neq 1$, then, obviously, the action is not essentially free. By Proposition 4.2, the essential freeness is equivalent to local nontriviality. Let u be a vertex with the minimum level that has a nontrivial trivializer $\mathrm{triv}_G(u)$ in G and $1 \neq g \in \mathrm{triv}_G(u)$. Let v be the vertex preceding the vertex u in the tree. Then the section g_v is nontrivial (since $|u|$ is minimal) and belongs to $\mathrm{rist}_G(1)$. $\square$

The following proposition is proved in a similar way.

Proposition 4.6. *Let G be a strongly self-similar group acting on a regular rooted tree T. Then the action $(G, \partial T, \nu)$ is essentially free if and only if the trivializers of all first-level vertices are trivial.*

Proof. If the trivializer of some first-level vertex is nontrivial, then, obviously, the action is not essentially free. Suppose that, conversely, the action is not essentially free. By Proposition 4.2 and Theorem 4.4, the action is locally trivial. Let u be a minimum-level vertex that has a nontrivial trivializer $\mathrm{triv}_G(u)$ in G, and $1 \neq g \in \mathrm{triv}_G(u)$. Let v be the vertex preceding the vertex u in the tree. Then the section g_v is nontrivial (since $|u|$ is minimal), belongs to G, and belongs to the trivializer of some first-level vertex. $\square$

Remark 4.1. A similar proposition holds without the assumption of strong self-similarity of a group provided that the essential freeness is replaced by the topological freeness.

For an automorphism defined by a finite initial automaton $\mathcal{A}_q$ on a homogeneous tree, the set of fixed boundary points can be described algorithmically and has a transparent structure. Let us describe this set.

We will assume that an automaton $\mathcal{A}_q$ defined over a d-letter alphabet X is minimal and defines a nonidentity tree automorphism. Denote by $\mathrm{Fix}(q)$ the set of fixed boundary points of the transformation defined by the automaton $\mathcal{A}_q$. In order for $\mathrm{Fix}(q)$ to be nonempty, it is necessary that the state q in the Moore diagram of the automaton be labeled by the identity element e. Suppose that $\mathcal{A}_q$ has a state t that defines the identity automorphism of the tree (i.e., a state t such that it is labeled by the identity element e of the symmetric group and one cannot leave it by means of the transition map in the automaton because all edges emanating from t in the Moore diagram also end in t; obviously, if there is such a state in a minimal automaton, then it is unique). Denote by $\mathcal{R}$ the set consisting of words that define paths in the Moore diagram which lead from q to t and pass only through the states labeled by the identity element. Denote by F the set of states of the automaton $\mathcal{A}_q$ that are labeled by e but are different from the state t (recall that $q \in F$). Let $\mathcal{L}$ be the set of words in the alphabet X of the automaton that define paths starting in the state q and passing only through the states in F. Introduce the following subtree T' of the d-regular tree defined by the alphabet X. The subtree T' consists of all vertices defined by the words in $\mathcal{L}$ and of all intermediate vertices and edges connecting them with the root vertex. Let $\partial T'$ be the boundary of the tree T'. It is a closed subset of the boundary ∂T and is nonempty if and only if the Moore diagram of the automaton $\mathcal{A}_q$ contains at least one cycle all of whose states lie in the set F. The languages $\mathcal{R}$ and $\mathcal{L}$ are regular (according to the classical classification of Chomsky of formal languages, see [104]) languages defined by the acceptor automaton obtained from the Moore diagram of the automaton $\mathcal{A}_q$ by a simple reconstruction.

Proposition 4.7. *The set of fixed points can be decomposed into a disjoint union*

$$\mathrm{Fix}(q) = \partial T' \sqcup \bigsqcup_{w\in\mathcal{R}} C_w$$

(*where, as before, C_w stands for the cylindrical set defined by a word w*).

Recall that for actions on the boundary of a tree, the minimality is equivalent to the topological transitivity by Proposition 4.1.

Proposition 4.8. *Suppose that an action of a countable group G on a complete metric space X is minimal and the stabilizer $\mathrm{st}_G(x)$ is trivial for some point $x \in X$. Then the action is topologically free.*

Proof. To prove this proposition, we need the following concept.

Definition 4.3. Let X be a topological G-space.

(α) For an element $g \in G$, a point $x \in X$ is said to be *g-typical* if either $gx \neq x$ or g acts trivially in some neighborhood of the point x.

(β) A point $x \in X$ is said to be *G-typical* (or simply typical) if it is g-typical for any element $g \in G$.

As before, denote by X_+ the set of free points (points with trivial stabilizer), by O_g the set of g-typical points, and by O the set of G-typical points, and let $N_g = X \setminus O_g$. Then it is easily seen that O_g is an open dense set, and hence $O = \bigcap_{g\in G} O_g$ is the intersection of a countable number of open dense sets. We argue that the inclusion $O \subseteq X_+$ holds. Indeed, suppose that some point $y \in O$ has a nontrivial stabilizer. Let $f \in \mathrm{st}_G(y)$, $f \neq 1$. Let us show that an arbitrary neighborhood U_y of the point y contains a point that can be moved by the element f (i.e., it is not f-fixed); this leads to a contradiction. Thus, $y \notin O_f$ in this case.

Let x be a point from the hypothesis of the proposition. In view of the minimality, the orbit of the point x is dense and G acts freely on this orbit. Let z be a point of the orbit Gx that belongs to U_y. Then $fz \neq z$, and since such a point exists in any neighborhood of the point y, we arrive at a contradiction. Thus, the complement of X_+ is a meager set. The proposition is proved. □

While proving the above proposition, we have also proved

Corollary 4.9 [87, Proposition 6.20]**.** *The set of G-typical points of a countable group is comeager* (*i.e., its complement is meager*).

Corollary 4.10. *For minimal actions of a countable group G on complete metric spaces, the essential freeness with respect to at least one G-invariant measure μ implies the topological freeness.*

Proof. Indeed, in the case of an essentially free action, the set of points with trivial stabilizer has full measure; hence this set is nonempty, and it remains to apply Proposition 4.8. □

It follows from the results of Abért and Virág obtained in [6] that m, $m \geq 2$, randomly taken automorphisms of a rooted tree generate a free group that acts essentially freely on the boundary with respect to the uniform measure. Moreover, in [3], Abért and Elek developed a method that allows one to construct a continuum of pairwise weakly inequivalent free actions for a wide class of groups, including free groups and groups with Kazhdan's T-property. Some conditions of algebraic character that guarantee the topological freeness of specific actions on the boundary of a tree have been announced by Abért in a number of his talks. Below we present a few conditions of algebraic character that we could find; in some cases, these conditions may coincide with Abért's conditions.

Definition 4.4. 1. We say that a group G has the property of *hereditary nontriviality of intersections* if, for any subgroup $H \leq G$ of finite index, the intersection $K \cap L$ of any two nontrivial normal subgroups $K, L \trianglelefteq H$ is nontrivial (this is equivalent to the fact that any subgroup $H \leq G$ of finite index is not a subdirect product of two nontrivial groups).

2. We say that a group G is *hereditary free of finite normal subgroups* if any subgroup of finite index in G does not contain nontrivial finite normal subgroups.

Proposition 4.11. *Suppose that a countable group G acts faithfully and spherically transitively on a rooted tree T and that at least one of the following conditions is satisfied*:

(1) *G is torsion-free and possesses the property of hereditary nontriviality of intersections*;

(2) *G is hereditary free of finite normal subgroups*;

(3) *the tree T is binary, and G has the property of hereditary nontriviality of intersections*;

(4) *G is a hereditary just-infinite group* (*see Definition* 2.5).

Then the action $(G, \partial T)$ is topologically free.

Proof. 1. Let us prove that each of the sets $\mathrm{Fix}(g)$, $g \neq 1$, is nowhere dense and thus $(\partial T)_- = \bigcup_{g \in G,\, g \neq 1} \mathrm{Fix}(g)$ is meager.

Suppose that this is not so, and let, for some element $g \in G$, $g \neq 1$, the set of fixed points $\mathrm{Fix}(g)$ (which is closed) contain a neighborhood U_x of some point $x \in \partial T$. Then there exists a vertex u such that the cylindrical set C_u is contained in U_x. In this situation, the element g fixes the vertex u and acts trivially on the subtree T_u. Consider the trivializer $\mathrm{triv}_G(u)$. The intersection $K_u = \mathrm{triv}_G(u) \cap \mathrm{st}_G(n)$, $n = |u|$, is nontrivial since the group $\mathrm{triv}_G(u)$ is infinite ($g \in \mathrm{triv}_G(u)$ and g is an element of infinite order), whereas $\mathrm{st}_G(n)$ is a subgroup of finite index in G. It is easily seen that K_u is a normal subgroup in $\mathrm{st}_G(n)$.

Since the group G acts transitively on the levels, for every nth-level vertex v one can similarly define a normal subgroup $K_v < \mathrm{st}_G(n)$, which consists of elements acting trivially on the subtree T_v. The subgroup K_v is conjugate to K_u in G. Since G has the property of hereditary nontriviality of intersections, for any vertices v and w, $|v| = |w| = n$, the intersection $K_{v,w} = K_v \cap K_w$ is nontrivial and is a normal subgroup in $\mathrm{st}_G(n)$. For a subset E of the set of nth-level vertices, denote by K_E the subgroup in $\mathrm{st}_G(n)$ that consists of elements acting trivially on the union of subtrees T_e, $e \in E$. We have shown that for sets E of cardinality 1 or 2 the subgroup K_E is a nontrivial normal subgroup in $\mathrm{st}_G(n)$. However, the intersection of any such groups is again a nontrivial normal subgroup in $\mathrm{st}_G(n)$, and, arguing by induction on the cardinality of E, we conclude that K_{L_n} is a nontrivial group, where L_n is the set of nth-level vertices, which contradicts the faithfulness of the action of G.

2. The proof of this case is similar to that of the previous case. Since the trivializer $\mathrm{triv}_G(u)$ is a normal subgroup in $\mathrm{st}_G(u)$, it is infinite. Further, the proof repeats the arguments of the preceding part.

3. If the tree is binary, then the stabilizer of the first level coincides with the stabilizer of each of the two vertices of the first level. Take a minimum-level vertex u such that the trivializer $\mathrm{triv}_G(u)$ is nontrivial. Since G acts faithfully, u is not a root vertex. Let v be the vertex preceding the vertex u in the tree, $|u| = n$, and w be a neighboring vertex for v that belongs to the same level as u (i.e., $|w| = n$). Let us replace in our arguments the pair (G, T) by (G_v, T_v), where G_v is the restriction of the action of $\mathrm{st}_G(v)$ to the subtree T_v (note that the group G_v may not act faithfully on T_v, but G_v acts transitively on the levels of the tree T_v by virtue of Proposition 2.1, which gives necessary and sufficient conditions for the action of a group G to be transitive). The trivializers $\mathrm{triv}_G(u)$ and $\mathrm{triv}_G(w)$ are conjugate in G_v and are normal subgroups in the subgroup $M_v \le G_v$ of index 2 that stabilizes the vertices u and w. Since G_v has finite index in G, M_v also has finite index in G. Therefore, the intersection $R_{u,w} = \mathrm{triv}_G(u) \cap \mathrm{triv}_G(w)$ is nontrivial and is contained in the trivializer $\mathrm{triv}_G(v)$. We have obtained a contradiction to the assumption that the level of the vertex u is minimal.

4. Let G be hereditary just-infinite. If the trivializer $\mathrm{triv}_G(u)$ is different from 1 for some vertex u, then $\mathrm{triv}_G(u)$ is a nontrivial normal subgroup in $\mathrm{st}_G(u)$ and hence has finite index in $\mathrm{st}_G(u)$. Then the action of the group $\mathrm{st}_G(u)$ is nontransitive on the levels of the subtree T_u, which contradicts the transitivity of the action of G. □

Here is another condition of algebraic character that guarantees the topological freeness of an action.

Proposition 4.12. *Let p be a prime number and a sequence $\overline{m} = \{m_n\}_{n=1}^{\infty}$ defining a branch index consist of powers of the number p. Suppose that a countable group G acts faithfully and spherically transitive on $T_{\overline{m}}$ and, for each n and each nth-level vertex w, any element of the group acts on the set of edges emanating from w by powers of the cyclic permutation of order m_{n+1}. Suppose that all abelian subgroups in G are cyclic. Then the action $(G, \partial T)$ is topologically free.*

Before proving this proposition, notice that the condition on the abelian subgroups in the proposition is automatically satisfied if G is a torsion-free Gromov hyperbolic group.

Proof. Suppose the contrary, and let $1 \ne g \in \mathrm{triv}_G(u)$ for some vertex u. It is obvious that u is a nonroot vertex. Take u with the minimum norm $|u|$, and let v be a predecessor of u, $G_v = \mathrm{st}_G(v)$, and $\overline{G}_v = G_v|_{T_v}$ be the restriction of G_v to the subtree T_v (the kernel of the homomorphism $G_v \to \overline{G}_v$ is the trivializer $\mathrm{triv}_G(v)$). The group G_v, just as its homomorphic image $\overline{G}_v$, acts spherically transitively on the subtree T_v. The element g fixes v, and its projection $\bar{g}$ to the subtree T_v stabilizes all first-level vertices of this tree. Let $L = \{w_1, \dots, w_c, \dots, w_q\}$ be the set of first-level vertices of the tree T_v and $\bar{g} = (g_{w_1}, \dots, g_{w_c}, \dots, g_{w_q})$ be a decomposition of the element $\bar{g}$ into projections at the first-level vertices; here the index c corresponds to the vertex u, and $q = m_{k+1}$ denotes the number of first-level vertices of the tree T_v, where k is the level of the vertex v in the tree T; let $q = p^l$. In view of the minimality of $|u|$, the element $\bar{g}$ can be assumed to be nontrivial. The projection g_{w_c} is the identity element.

Denote by E the nonempty subset consisting of vertices x for which the corresponding projection g_x is trivial. In addition to the assumption that $|u|$ is minimal and $\bar{g}$ is nonidentity, we require that the element g should be chosen so that the cardinality $|E|$ is maximal (in this case, obviously, $E \ne L$). Suppose that $a \in G_v$ acts on the first level of the subtree T_v by a cyclic permutation σ of order $q = p^l$ (such an element exists in view of the spherical transitivity of the action of G_v on T_v), and $\bar{a}$ is the projection of a onto this subtree. Then the conjugate element $\bar{g}^{\bar{a}}$ has $\sigma^{-1}(E)$ as the set of vertices in L with trivial projections. Consider the commutator $f = [g, g^a]$ and its projection $\bar{f} = [\bar{g}, \bar{g}^{\bar{a}}] \in \overline{G}_v$. Since f has at least $|E \cup \sigma^{-1}(E)| > |E|$ identity components in the decomposition, we have $f = 1$; i.e., g and g^a commute. Hence, by virtue of the hypothesis of the proposition, there exists an element $h \in G_v$ such that $g, g^a \in \langle h \rangle$ with $h \in \mathrm{st}_{G_v}(1)$ because $h \in \langle g, g^a \rangle$. Let $g = h^i$ and $g^a = h^j$.

Let $\bar{h}$ be the projection of h onto T_v. Then $\bar{h} \in \mathrm{st}_{\overline{G}_v}(1)$ and $\bar{h} = (h_1, \dots, h_q)$. Let F be the set of identity coordinates of this vector and F^{c} be its complement. Suppose that $F^{\mathrm{c}} \cap E \neq \varnothing$. Then $i = p^t$ for some positive t. Similarly, if $F^{\mathrm{c}} \cap \sigma^{-1}(E) \neq \varnothing$, then $j = p^r$ for some r. If both above intersections are nontrivial, then, assuming that $t \geq r$, we find that the element $\bar{g}^{\bar{a}} = \bar{h}^j$, represented as a vector, has identity components at the vertices from the set $E \cup \sigma^{-1}(E)$, which contains E as a proper subset. This contradicts the assumption of the maximality of $|E|$.

Suppose that $F^{\mathrm{c}} \cap E = \varnothing$. Then $F \supseteq E$, and the equality $F = E$ holds in view of the maximality of $|E|$. In this case, the set $\sigma^{-1}(E)$ of identity components of the element $\bar{g}^{\bar{a}} = h^j$ contains the set E and does not coincide with it, a contradiction. The case of $F^{\mathrm{c}} \cap \sigma^{-1}(E) = \varnothing$ is considered similarly. $\square$

5. EXAMPLES OF ESSENTIALLY FREE ACTIONS

We begin this section with a few specific examples of essentially free actions of strongly self-similar groups and end it with the discussion of the strategy of classification of all essentially free actions of strongly self-similar groups acting on a binary tree. The first two examples, modulo the algebraic result behind them that identifies the groups generated by the respective automata, are a simple application of the general results obtained in the previous section.

Example 5.1. We begin with an example of the free rank 3 group F_3 realized as the group generated by the automaton shown in Fig. 5.1.

This automaton, with number 2240 in the classification of groups generated by three-state automata (see [35]), was constructed by Aleshin in [7] along with another five-state automaton $\mathcal{B}$ with the aim of generating the free group F_2 by the initial automata $\mathcal{A}_q$ and $\mathcal{B}_s$ for appropriate states q and s. Unfortunately, in [7] Aleshin presented only a scheme of proof.

A complete proof, involving new ideas and techniques compared with the lemmas formulated in [7], was published by M. Vorobets and Ya. Vorobets [186], who proved not only that $\mathcal{A}_q$ and $\mathcal{B}_s$ indeed generate a group isomorphic to F_2, but also that $G(\mathcal{A}) \simeq F_3$, i.e., that the states of the automaton $\mathcal{A}$ are independent with respect to the operation of composition of automata. Later on, they constructed infinite series of automata that generate free groups of different ranks [187]. Note that prior to the publication of [186] free groups of some (sufficiently high) ranks were realized as self-similar automaton groups by Glasner and Mozes [68]. In addition to the difference in the scheme of the proofs of propositions in these papers, an important difference in the realization of free groups by automata in them lies in the fact that the states of automata in [68] define not only basis elements but also their inverses, whereas in [186] different states of an automaton correspond to different basis elements.

Since a noncommutative free group obviously satisfies both the condition of hereditary nontriviality of intersections and the condition of hereditary freeness of finite subgroups, it follows from Proposition 4.11 that any self-similar spherically transitive action of this group is topologically free. However, in this example the action is not absolutely free, because the stabilizers of some points are nontrivial (for example, Ya. Vorobets pointed out (in a private communication) that the stabilizer

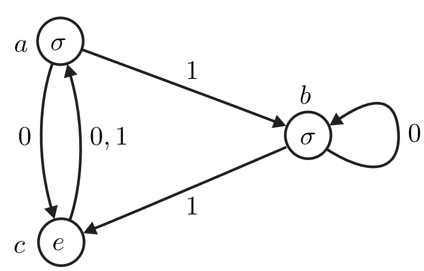

Fig. 5.1. The Aleshin automaton.

of the point 1^∞ is an infinitely generated group). It would be interesting to describe, for this and the next example, all boundary points with nontrivial stabilizer as well as the stabilizers of these points themselves. It would also be interesting to find a realization of the free group of rank ≥ 2 as a strongly self-similar group acting absolutely freely on the boundary of an appropriate tree (if such a realization exists).

Example 5.2. The second interesting example of an essentially free action is given by the Bellaterra automaton, which is shown in Fig. 3.2 and has number 846 in the classification in [35]. This automaton generates the free product $C_2 * C_2 * C_2$ of three copies of an order 2 group. The relevant result was obtained by my students E. Muntyan and D. Savchuk, and its proof can be found in [142, p. 25].

Since this group obviously satisfies both conditions of Definition 4.4, it follows from Proposition 4.11 that the group $C_2 * C_2 * C_2$ realized by the automaton in Fig. 3.2 acts essentially freely on the boundary of a binary tree.

The realizations of F_3 and $C_2 * C_2 * C_2$ by finite automata will allow us to effectively construct (in Section 10) sequences of asymptotic expanders, which will be defined below.

Example 5.3. Now, consider the group $\mathcal{L}$ from Example 2.2, which is realized by the automaton in Fig. 3.1 and is called the lamplighter group. The lamplighter group acts on a tree essentially freely, as was first shown in [95] with the help of relatively complex (though useful for some problems) arguments. In particular, it was found in [95] that if the stabilizer of some boundary point of the binary tree is nontrivial for the action of the group $\mathcal{L}$ defined by the automaton in Fig. 3.1, then this stabilizer is an infinite cyclic group (see Proposition 4 in [95]).

In [87], the authors proposed a more algebraic approach to the analysis of the action of $\mathcal{L}$ on a tree. Recall that the generators corresponding to the states a and b of the automaton satisfy the recurrent relations $a = (a,b)\sigma$ and $b = (a,b)$, which imply that the element $c = b^{-1}a = \sigma$ is an involution and a simplest finitary (i.e., acting nontrivially only in a neighborhood of the root vertex) automorphism that permutes the two vertices of the first level and acts trivially on the subtrees that grow from these vertices.

Denote by α and γ the standard generators of the lamplighter defined as the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$. Namely, $\alpha \in \bigoplus_{\mathbb{Z}} \mathbb{Z}_2$ is defined by the relation

$$\alpha = (0, \dots, 0, 0, 1, 0, 0, \dots),$$

where the group of order 2 is represented in the additive form and consists of elements 0 and 1; the only nonzero component of the vector that defines α occupies the zeroth coordinate, while γ is the generator of the infinite cyclic group that plays the role of the active group in the wreath product. Then the map $\alpha \to c$, $\gamma \to b$ induces an isomorphism between the group represented by the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$ and the group generated by the automaton. It is easily seen that the elements b and c act on the boundary points (i.e., on infinite binary sequences) as follows:

$$b(x_1x_2x_3\dots) = (x_1 + x_2)(x_2 + x_3)(x_3 + x_4)\dots,$$
$$c(x_1x_2x_3\dots) = (x_1 + 1)x_2x_3\dots$$

(addition is modulo 2). Consider the ring $R = \mathbb{Z}_2[[t]]$ of formal power series over a two-element field. The elements of the ring are in natural bijection with the elements of the boundary of the tree (namely, a formal series is associated with a binary sequence of its coefficients).

Under such an identification, the generators of the lamplighter act by transformations of the ring R as $\phi_\alpha\colon F(t) \to F(t) + 1$ and $\phi_\gamma\colon F(t) \to (1+t)F(t)$; hence, the lamplighter acts on the ring R by transformations of the form

$$F(t) \to (1+t)^m F(t) + \sum_{s\in\mathbb{Z}} k_s(1+t)^s, \qquad k_s \in \mathbb{Z}_2,$$

and this action is conjugate to the lamplighter's action defined by the automaton in Fig. 3.1 on the boundary of the tree.

Hence we can easily infer that for the relation $g(F(t)) = F(t)$ to hold for some nonidentity element g of the group $\mathcal{L}$ and some series $F(t)$, the series $F(t)$ should represent a rational function $\frac{U(t)}{V(t)}$, where $U(t)$ and $V(t)$ are polynomials of the form $U(t) = (1+t)^{n_1} + (1+t)^{n_2} + \ldots + (1+t)^{n_k}$ and $V(t) = 1 + (1+t)^l$ with integer $n_1, n_2, \ldots, n_k$ and l. Thus, for almost every point (more precisely, for all points except for a countable number of them), the stabilizer is trivial. It is shown in [145] that the stabilizer is nontrivial if the boundary point is strictly periodic (i.e., if it has the form $www\ldots$ for some binary word w); hence, the stabilizer is an infinite cyclic group.

Example 5.4. Our next example is the Baumslag–Solitar group $\mathrm{BS}(1,3) = \langle x, y \mid x^y = x^3\rangle$. Bartholdi and Šunić developed a general approach to the realization of groups in the whole family $\mathrm{BS}(1,n)$, $n \in \mathbb{Z}$, $n \neq \pm 1$, of Baumslag–Solitar groups (and even for a more general class of groups that are ascending HNN extensions of abelian groups) as self-similar groups [22]. For simplicity, here we consider only the case of $n = 3$. Consider the ring $\mathbf{Z}_2$ of integer 2-adic numbers and three affine transformations on it: $a(x) = 3x$, $b(x) = 3x+1$, and $c(x) = 3x+2$. Then the transformations $x = a$ and $y = ab^{-1}$ satisfy the relation $x^y = x^3$, and, as shown in [22], the group isomorphism $\langle a,b,c\rangle \simeq \langle x,y\rangle \simeq \mathrm{BS}(1,3)$ holds. Identifying the elements of the ring $\mathbf{Z}_2$ with the binary sequences of coefficients of the expansion in powers of two, we find that the transformations a, b, and c act on binary sequences in a self-similar way; namely, the following recurrent relations hold:

$$a = (a,b), \qquad b = (a,c)\sigma, \qquad c = (b,c).$$

Thus, we obtain a realization of the group $\mathrm{BS}(1,3)$ as a group generated by a three-state automaton over a binary alphabet. This automaton has number 2083 in [35]. The inverse automaton (which, obviously, also generates the same group) has number 924 in the same paper.

These automata define an essentially free action of the group $\mathrm{BS}(1,3)$ on the boundary of the tree, which follows immediately from the affine representation of the group $\mathrm{BS}(1,3)$ by transformations of the ring of 2-adic numbers (each such transformation has at most one fixed point).

It is interesting that in [35] the Baumslag–Solitar group is also represented by automaton 870; however, the action defined by this representation on the boundary is not apparently conjugate to the action defined by automata 924 and 2083. Therefore, the essential freeness of this action is not backed by the arguments presented above. However, now we will show that in fact this action is also essentially free.

Example 5.5. The recurrent relations defined by the Moore diagram of automaton 870 in [35] are as follows:

$$a = (c,b)\sigma, \qquad b = (a,c), \qquad c = (b,a)$$

(note that, in contrast to [35], we write an element of the symmetric group $\mathrm{Sym}(2)$ on the right rather than on the left). The first-level stabilizer of the group $G = G(\mathcal{A}_{870}) \simeq \mathrm{BS}(1,3)$ is generated by the following elements:

$$s_1 = b = (a,c), \qquad s_2 = c = (b,a), \qquad s_3 = a^2 = (cb, bc),$$
$$s_4 = aba^{-1} = (c, bab^{-1}), \qquad s_5 = aca^{-1} = (cac^{-1}, b);$$

this, in particular, implies that the group $\mathrm{BS}(1,3)$ in this realization is a self-replicating group (since the projections of the elements s_1, s_2, and s_3 yield the set $\{a,b,cb\}$ of generators of the group). Applying the Nielsen transformations to this set, we transform it into

$$\begin{aligned}
t_1 &= (a,c) = b = s_1,\\
t_2 &= (b,a) = c = s_2,\\
t_3 &= (c, ab^{-1}c) = ac^{-1}ba^{-1}b = s_5^{-1}s_4s_1,\\
t_4 &= (1, ac^{-1}ab^{-1}) = ca^{-1}ba^{-1} = s_2s_3^{-1}s_4,\\
t_5 &= (1, ab^{-1}cba^{-1}b^{-1}) = ac^{-1}ba^{-1}bab^{-1}a^{-1} = s_5^{-1}s_4s_1s_4^{-1}.
\end{aligned}$$

If at least one of the elements $\alpha = ac^{-1}ab^{-1}$ or $\beta = ab^{-1}cba^{-1}b^{-1}$ was nonidentity, then this would contradict the essential freeness of the action of generators on the boundary. However, in fact these elements are equal to the identity element. Indeed, the following relations are valid:

$$\begin{aligned}
\alpha &= (ca^{-1}ba^{-1}, 1) = (\alpha^{ba^{-1}}, 1),\\
\beta &= (1, ba^{-1}bab^{-1}c^{-1}) = (1,\gamma),\\
\gamma &= (ab^{-1}cba^{-1}b^{-1}, 1) = (\beta, 1),
\end{aligned}$$

which imply that $\alpha,\beta,\gamma = 1$.

Let us apply Proposition 4.5 in order to prove the triviality of $\mathrm{rist}_G(1)$. Let $\phi\colon a \to c$, $b \to a$, $c \to ab^{-1}c$ (so that ϕ maps the left projections of the elements s_1, s_2, and s_3 to the right projections). The map ϕ extends to an automorphism of the whole group G. Indeed, as shown in [35], the relations $c = ab^{-1}a$ and $\mu^b = \mu^3$ with $\mu = b^{-1}a = a^{-1}c$ are valid; these relations imply that the group G is in fact 2-generated and isomorphic to the group $\mathrm{BS}(1,3)$ (one can easily verify that this group has no additional relations). Using the Tietze transformations (see [128]), one can easily show that ϕ extends to an epimorphism of G to itself; this, in view of the Hopf property of the group $\mathrm{BS}(1,3)$ (which follows from the fact that this group is residually finite), implies that in fact ϕ extends to an automorphism. Hence, any element g of the stabilizer $\mathrm{st}_G(1)$ has the form $g = (h, \phi(h))$, $h \in G$, which shows that the first-level rigid stabilizer is trivial and, hence, the action of the group $\mathrm{BS}(1,3)$ is essentially free.

The method used in the above example for proving the essential freeness of the action can also be applied to other examples of actions on a binary tree for which the isomorphic type of the group is known (hence, one can determine whether a map defined on the set of generators extends to an automorphism of the entire group). Namely, if G is a strongly self-similar group acting on a binary tree, with a set of generators $A = \{a_1,\dots,a_m\}$, then the method consists in the following. First, we calculate generators s_j, $j \in J$ (where J is a finite set of indices), of the first-level stabilizer by using, say, the Reidemeister–Schreier method (see [128]). We represent these generators as pairs $s_j = (s_j^{(0)}, s_j^{(1)})$ using the recursions induced by the Moore diagrams of the automaton that defines the group. Consider s_j and these pairs as elements of a free group F_A and of the direct product $F_A \times F_A$, respectively, where A is an alphabet that is in bijection with the states of the automaton (i.e., with the generators of the group G). Applying the Nielsen transformations to these generators, we turn them into a set of generators $\{t_j,\ j \in J\}$ whose projection onto the first coordinate is Nielsen reduced (see [128]); i.e., it starts with a Nielsen set of generators b_k, $k \in K$, of some subgroup $H \le F_A$ and is extended by a sequence of identity elements. If G is a self-replicating group, then $H = F_A$ and (possibly, after the application of additional Nielsen transformations) the set $\{t_j,\, j \in J\}$ takes the form

$$t_1 = (a_1, w_1), \quad \dots, \quad t_m = (a_m, w_m), \quad t_{m+1} = (1, r_1), \quad \dots, \quad t_{m+l} = (1, r_l),$$

where $m + l = |J|$.

Suppose that the elements $w_1, \dots, w_m$ generate the entire free group F_A (this condition holds in many examples from the Atlas of self-similar groups, whose development was initiated in [35]). Then the map $\phi\colon a_i \to w_i$, $i = 1, \dots, m$, defines an automorphism of the free group $F_m = F_A$. If ϕ is the identity automorphism, then the subgroup generated by elements of the above type in the direct product $F_m \times F_m$ is a subgroup of the form that was used by Mikhailova [138] to prove that the inclusion problem for direct products of free groups is algorithmically unsolvable. Even if $\phi \neq 1$, we will call the subgroups generated by elements of the above type Mikhailova subgroups (obviously, this is in fact the same class of subgroups as in the case of $\phi = 1$). If at least one of the elements r_i is different from the identity element, then $\mathrm{rist}_G(1) \neq 1$ and the action is not essentially free. Suppose that all r_i are identity elements. In this case, we say that the definition of the group G by a finite automaton belongs to the diagonal type (this is consistent with Definition 2.2(f)). This condition does not depend on how the pairs of elements are reduced to the Mikhailova form by the Nielsen transformations.

Proposition 5.1. *Suppose that G is a strongly self-similar group acting on a binary tree and having a first-level stabilizer that can be reduced by the Nielsen transformations to the diagonal type. Let ϕ be the above-constructed automorphism of the free group F_A. Then the action is essentially free if and only if ϕ induces an automorphism of the group G.*

Proof. If ϕ does not induce an automorphism, then this means that $\mathrm{st}_G(1)$ contains an element of the form $(g, \phi(g))$, $g = 1$, $\phi(g) \neq 1$, and thus the action is not essentially free. If ϕ induces an automorphism of G, then any element of $\mathrm{st}_G(1)$ has the form $(g, \phi(g))$, and the equality $g = 1$ implies the equality $\phi(g) = 1$. Thus, $\mathrm{rist}_G(1) = 1$, and, applying Proposition 4.5, we obtain the assertion. $\square$

The verification of the fact that ϕ induces an automorphism of G is obviously equivalent to the verification of whether ϕ translates the defining relations of the group G into defining relations. Indeed, in this case ϕ induces a homomorphism of the group G into itself; by the Hopf property (i.e., any proper quotient group is not isomorphic to the group itself) of finitely generated residually finite groups, ϕ is in fact an isomorphism. If the presentation of the group by generators and relations is known, then usually such a verification does not face any difficulties. If the presentation is unknown and, moreover, the isomorphic type of the group is unknown, then serious complications may arise in verifying the essential freeness of the action.

It is desirable to have a classification of all groups in the classes $(2,3)$, $(3,2)$, and $(2,4)$ of groups acting essentially freely on the boundary of a binary tree. For the classes $(2,3)$ and $(3,2)$, this must be a relatively simple problem; however, for the class $(2,4)$, the problem may turn out to be much more difficult.

6. SCHREIER GRAPHS

Some of the first applications of Schreier graphs (or orbital Schreier graphs) to the analysis of the properties of dynamical systems were proposed in [106, 87, 16, 40]. The recent paper [83] gives another example of usefulness of Schreier graphs in ergodic theory and related problems. There are some nuances in the approach to defining a graph and in the methods for studying its properties; therefore, we first describe the main terminology and conventions to which we will adhere.

We will consider only locally finite graphs, denoting them by uppercase Greek letters, mainly by Γ. Such a graph consists of a pair (V, E), where V is the vertex set and E is the edge set, together with a map $\delta\colon E \to (V \times V)/\asymp$, where $\asymp$ is the equivalence relation that identifies the pairs (u,v) and (v,u) (i.e., an edge is associated with an unordered pair of vertices, the endpoints of the edge). In this case, u and v are said to be adjacent vertices (which is denoted as $u \sim v$), and an edge e is said to connect these vertices. We will also say that the edge e is incident to the vertices u and v, while the vertices u and v are incident to the edge e. We admit loops (i.e., edges with coinciding endpoints) and multiple edges (i.e., the number of edges connecting two vertices may be greater than 1). In the literature on graph theory, such graphs are usually called multigraphs, but we will omit the prefix "multi." The graphs are visualized as diagrams that are geometric representations (realizations) of graphs. In this case the edges are visually represented as arcs (curves homeomorphic to a closed interval) on the plane or in the space. A path in a graph is a sequence of edges in which the starting point of the next edge coincides with the endpoint of the preceding edge. The combinatorial length of a path is the number of edges in it; i.e., the length of an edge is assumed to be 1. One can also introduce more complex metrics on a graph, but we will not do this here.

A graph is said the be connected if any two vertices in it are connected by a path. For each vertex $v \in V$, the concept of degree $\deg(v)$ is defined. By this is meant the number of edges incident to v. We will assume that a loop attached to a vertex v contributes number 2 to the multiplicity. In some cases (for example, when a graph is a Schreier graph of a group generated by elements of order 2), it is convenient to assume that loops make a contribution of 1 to the degree (as, for example, in Example 7.1, considered below, of Schreier graphs associated with the group $\mathcal{G}$, which is generated by four involutions). Below, when defining Schreier graphs, we will specify what contribution to the degree is made by loops.

Up to this point, we have dealt with nonoriented graphs. However, along with nonoriented graphs, sometimes we will consider oriented graphs. An oriented graph is a graph in which an arbitrary edge $e \in E$ is defined by an ordered pair of vertices $(\alpha(e), \beta(e))$, the first of which plays the role of the starting point, and the second, the endpoint of the edge. Thus, every edge has a beginning and an end (which coincide when the edge is a loop). Visually, the orientation is described by the choice of direction on each edge represented by an arc in a geometric realization of a graph.

In the oriented case, each loop makes a unit contribution to the degree of a vertex. Of greatest interest for us (both in the oriented and nonoriented cases) is the class of regular graphs, i.e., graphs in which all vertices have the same degree equal to a number $d \geq 3$. Obviously, the cases of regular graphs of degrees 1 and 2 are rather simple to analyze, while the study of regular graphs of degree 3 has actually the same order of complexity as the study of higher degree graphs.

The next class of graphs that we are going to discuss is colored graphs. By a *colored* graph we mean an oriented or nonoriented graph whose edges are labeled by letters of a finite alphabet. Graphically, this is implemented by labeling edges with appropriate letters. If the letters correspond to colors, then we can assume that the edges are colored in the corresponding way and use color diagrams in the geometric realization of a graph.

An important example of a colored oriented graph is a Cayley graph $\Gamma(G,A)$ of a finitely generated group G generated by a set $A = \{a_1, \dots, a_m\}$. The vertex set of the graph is identified with the set of elements of the group, and the edges are given by ordered pairs (g, ag), $g \in G$, $a \in A$ (the order of a pair of vertices (g, ag) determines which vertex of the edge is the starting point and which is the endpoint). Thus, the generators serve as labels of edges, and the corresponding alphabet that describes formally the set of labels is the alphabet A. The above-described Cayley graph is a left Cayley graph (because the left multiplication by the generators is used when constructing this graph). A right Cayley graph is defined in a similar way. The group G acts by right multiplications on the sets of vertices and edges of the left Cayley graph; in this case the right multiplication by an arbitrary element $g \in G$ defines an automorphism of the graph $\Gamma(G,A)$. Thus, the Cayley graph is vertex-transitive (i.e., its automorphism group acts transitively on the vertex set). The Cayley graph is homogeneous of degree $2m$, where m is the number of generators, if we consider the graph as a nonoriented graph, and homogeneous of degree m if we consider it as an oriented graph. This graph depends on the group and on the system of generators; however, its *rough* properties such as the number of ends, amenability, growth, etc., are independent of the choice of generators. Formally speaking, the notation $\Gamma(G, A)$ should be supplemented with a symbol indicating which (left or right) Cayley graph is meant; however, one usually does not do so (but settles this from the very beginning). Often, when studying Cayley graphs, one removes the orientation and coloring by generators. In this case, the asymptotic properties of the group are still reflected in the asymptotic properties of the graph, but the information about the algebraic properties is largely lost. It may turn out that nonisomorphic groups have isomorphic (nonoriented and noncolored) Cayley graphs [36, 100]. The study of groups with the use of the language of Cayley graphs and other geometric tools (van Kampen diagrams, boundaries, etc.) is the subject of geometric group theory; the foundations of this theory can be found in books [128, 100, 137].

A generalization of the concept of Cayley graph is a Schreier graph, which we will now define. Suppose that, in addition to a group G with a system of generators A, a subgroup $H \leq G$ is defined. The vertex set of a Schreier graph (G, H, A) consists of the left cosets gH, while the edges are given by ordered pairs (gH, agH), $a \in A$, supplemented with the label a. The graph thus defined is a left Schreier graph. A right Schreier graph is defined in a similar way. Note that when a generator $a \in A$ is an involution (i.e., $a^2 = 1$), each edge labeled by this generator should be assumed nonoriented and the corresponding arrow indicating the orientation of the edge in the graphical representation of the graph should be omitted. So, we have thus defined oriented Schreier graphs (which are in fact partially oriented when there are elements of order 2 among the generators). If we remove the orientation of edges in a Schreier graph, we obtain a nonoriented Schreier graph. However, one should bear in mind that in the nonoriented variant of a Schreier graph the loops that are labeled by a noninvolution make a contribution of 2 to the degree of the relevant vertex, while those labeled by an involution make a contribution of 1.

In contrast to Cayley graphs, there does not exist a natural action of the base group G by automorphisms on Schreier graphs; moreover, there are examples of Schreier graphs with trivial automorphism group. However, there is a natural action of the base group on the vertex set (by left multiplication in the case of a left Schreier graph), which will be used in Section 8 for constructing a dynamical system by a graph. Obviously, the Schreier graphs (just as the Cayley graphs) are connected and homogeneous of degree $2m$, where m is the number of generators, if there are no involutions among them and one ignores the orientation of edges. If there are involutions and one follows the above agreement on the contribution of loops to the degree of vertices, then the degree of vertices is $i + 2j$, where i is the number of involutions and j is the number of noninvolutions among generators. Note that a different (but essentially equivalent) definition of Schreier graphs that is based on the Serre approach to the definition of graphs can be found in [87]. In many questions related to the study of the asymptotic behavior of graphs and groups, it is expedient to deal with *rooted* graphs, i.e., with graphs (Γ, o) one of whose vertices (denoted here by o) is distinguished as the initial vertex. Choosing an initial point o, one can measure the distance from o to other points and thus determine a sequence of balls in a graph with center at o, the growth function of the graph $\gamma_\Gamma(n)$, which counts the number of vertices situated at a distance $\leq n$ from o, and so on. One may initiate a random walk on a graph with initial position at o and with transitions from a current position to neighboring positions along edges emanating from the current position with equal probabilities (the random walk thus defined is called a simple random walk) and analyze its asymptotic properties.

Schreier graphs are closely related to *orbital graphs* (or *action graphs*). Namely, suppose that a finitely generated group G with a system of generators A acts on a set X. An orbital graph

$\Gamma = \Gamma(G, X, A)$ of the action is a graph whose vertices are given by the elements of X and two vertices $x, y \in X$ are connected by an oriented edge labeled by letter $a \in A$ if $y = a(x)$. It is obvious that the action graph is connected if and only if the action is transitive. An orbital graph Γ_x, $x \in X$, is a subgraph of the action graph whose vertices are points of the orbit $G(x)$. The orbital graph is connected.

Often, when studying the asymptotic properties of graphs, one needs to use the natural topology in the space of connected regular rooted graphs. This topology was first used in [72] when studying the algebraic properties of intermediate growth groups and later in [94] in connection with random walks. The base of open sets in this topology is given by the sets $B_{(\Gamma,o)}(n)$ consisting of rooted graphs such that the subgraph with the set of vertices situated at distance $\leq n$ from the distinguished vertex and with the induced set of edges is isomorphic to the similarly defined subgraph in (Γ, o). We will denote the space of d-regular rooted graphs with the above-mentioned topology by $\mathcal{X}_d$. Similar notation will be used for the space of $2m$-regular rooted nonoriented Cayley graphs $\mathcal{X}_{2m}^{\mathrm{Cay}}$ and $2m$-regular rooted nonoriented Schreier graphs $\mathcal{X}_{2m}^{\mathrm{Sch}}(G)$ of an m-generated group G (whose generators do not contain involutions), respectively. One can also consider similar spaces of Cayley and Schreier graphs in the situation when there are involutions among the generators (one just should introduce special notations). If the argument G in $\mathcal{X}_{2m}^{\mathrm{Sch}}(G)$ is missing, then the free group F_m of rank m is implied (which is a universal object in the category of m-generated groups).

All the spaces introduced are metrizable totally disconnected topological spaces. The distance can be defined, for example, by the relation $d((\Gamma_1, o_1), (\Gamma_2, o_2)) = 2^{-n}$, where n is the greatest positive integer such that the neighborhoods of the points o_1 and o_2 (subgraphs) of radius n in the graphs (Γ_1, o_1) and (Γ_2, o_2) are isomorphic. Instead of the sequence $\{2^{-n}\}$, one can take any other decreasing sequence of positive numbers that tends to zero. One can also consider the spaces $\mathcal{X}_{\leq d}$ of rooted graphs of degree $\leq d$ and similar spaces for the case of oriented graphs or graphs with coloring by the symbols of a finite alphabet. All the spaces introduced are compact totally disconnected separable spaces. Recall that the concept of the Cantor–Bendixson rank is defined for complete separable metric spaces (which are also called Polish spaces).

Problem 6.1. What is the Cantor–Bendixson rank of each of the above spaces?

This question is of special interest for the space of Cayley graphs.

Each of the spaces of graphs considered can be equipped with a probability measure, and the typical properties of graphs with respect to this measure can be studied. The choice of a measure may be suggested by a range of questions to be considered; however, it seems that the most natural choice is given by measures obtained as limit points in the weak topology of a sequence of measures μ_n, where μ_n is any discrete measure for which all cylindrical sets defined by a subgraph-neighborhood of radius n around the distinguished point have the same measure $1/l_n$, where l_n is the number of such neighborhoods for a given type of graphs.

Benjamini and Schramm [27] proposed an interesting approach to constructing measures in spaces of graphs. Namely, given a sequence $\{\Gamma_n\}_{n=1}^{\infty}$ of finite graphs, they suggested considering an associated sequence of measures μ_n, where μ_n is a uniform probability measure concentrated on rooted graphs (Γ_n, v) with v running through the vertex set of the graph Γ_n. Although this is not mentioned in [27], it is expedient to identify (Γ_n, v) and (Γ_n, w) if they are isomorphic as rooted graphs. Any limit point (in the weak topology) of the sequence $\{\mu_n\}_{n=1}^{\infty}$ will be a measure in the space of graphs, and this measure will be invariant under the replacement of the root (i.e., the distinguished vertex) of the graph with any other vertex. A similar operation is applied in Section 8 in order to construct a Schreier dynamical system.

When a measure is defined, one can speak of random graphs. Although random graphs have been studied extensively enough, it seems that the proposed model of random regular graphs has not yet been considered in detail. However, a uniform measure on the set of finite regular graphs has long been studied [194, 136]. It seems likely that a probabilistic model for infinite regular graphs can be obtained by a passage to the limit from the model for finite graphs (which, however, may face problems in the case of Cayley graphs). A brief discussion of questions related to the probabilistic model in the space $\mathcal{X}_{2m}^{\mathrm{Cay}}$ can be found in [81].

Kaimanovich observed [107, 108, 110] that the space of rooted graphs is equipped with a natural equivalence relation that arises when the root is carried to another vertex; therefore, one can consider measures invariant with respect to this equivalence relation. Kaimanovich referred to the resulting random graphs as stochastically homogeneous. For Schreier graphs, stochastic homogeneity is equivalent to the fact that the corresponding measure is invariant under the group action. The condition of stochastic homogeneity can also be interpreted in terms of stationary measures for the corresponding simple random motion on the equivalence classes (cf. Sections 8 and 11).

For the spaces $\mathcal{X}_{2m}^{\mathrm{Cay}}$ and $\mathcal{X}_{2m}^{\mathrm{Sch}}$, there exists another natural method (equivalent to the previous one) for introducing topology. Namely, to define a Cayley graph $\Gamma \in \mathcal{X}_{2m}^{\mathrm{Cay}}$ is the same as to define a pair (G, A), where A is an ordered system of generators of the group G, which is in turn equivalent to defining a triple (F_m, N, A'), where F_m is a free group with ordered basis A', which is in bijection with the set A, and N is a normal subgroup in F_m such that G is isomorphic to the quotient group F_m/N under an isomorphism that maps the image of the basis A' to A and preserves the order of generators.

Consider a two-point set $\{0,1\}$ with the discrete topology and raise it to the power F_m (i.e., consider the Cartesian product $\mathcal{Y} = \prod_{F_m}\{0,1\}$ with the Tikhonov topology). To an arbitrary subset E in F_m there corresponds a point in the space $\mathcal{Y}$ that is the characteristic function of E. Let $\mathcal{Z} \subset \mathcal{Y}$ be the subset consisting of points corresponding to normal subgroups. Then the subset $\mathcal{Z}$ with the topology induced from $\mathcal{Y}$ is homeomorphic to the space $\mathcal{X}_{2m}^{\mathrm{Cay}}$. Analogously, the subset $\mathcal{S} \subset \mathcal{Y}$ consisting of points corresponding to subgroups in F_m with the induced topology is homeomorphic to the space $\mathcal{X}_{2m}^{\mathrm{Sch}}$. This follows from the fact that the Schreier graph $\Gamma(G, H, A)$ of a group G with m generators is isomorphic to a Schreier graph of the free group of rank m. Indeed, representing G as a quotient group of the free group F_m and taking the preimage $\overline{H}$ of the subgroup H under the canonical homomorphism, we find that the graphs $\Gamma(G, H, A)$ and $\Gamma(F_m, \overline{H}, A')$ are isomorphic (A' is a system of generators of the free group that are projected to A under the canonical homomorphism).

For an arbitrary countable group G, one can similarly define a space $\mathcal{Y}(G)$ of subgroups of the group G. In Section 8, we consider the actions of groups by conjugations on spaces of this type and the duals of these actions (in the case of finitely generated groups) on the spaces of rooted Schreier graphs.

The space $\mathcal{X}_{2m}^{\mathrm{Cay}}$, which was first defined in [72] and then studied in [43] and [44], has been examined in considerable detail, although there are still many unsolved problems. It is known that this space has a closed subset without isolated points (i.e., a subset homeomorphic to the Cantor perfect set) that consists (except for a countable subset) of intermediate growth groups and has a dense G_δ subset consisting of periodic groups. The groups with Kazhdan's T-property form an open subset in this space [170]. The so-called limit groups (also called totally freely approximable groups in the Russian literature), which were first defined by G. Baumslag and then studied by V.N. Remeslennikov, O. Kharlampovich, A. Myasnikov, Z. Sela, and others, are the limits of free groups in the space $\mathcal{X}_{2m}^{\mathrm{Cay}}$, as proved in [44]. For the present, almost nothing is known about the space $\mathcal{X}_{2m}^{\mathrm{Sch}}$. The properties of the space $\mathcal{X}_d$, at least for even values of d, are largely related to the properties of the space $\mathcal{X}_{2m}^{\mathrm{Sch}}$ in view of the fact that every nonoriented $2m$-regular graph can be transformed (by adding an orientation and labels of edges) into a Schreier graph of a free group. Below we will prove a relevant statement (Theorem 6.1).

Note that in the same way as this was done in [94], we can also consider the spaces $\mathcal{Y}_n$, $n = 2, 3, \dots$, of inhomogeneous graphs the degree of whose vertices is bounded from above by a number

$n \geq 2$, as well as the inductive limit $\mathcal{Y}$ of these spaces. In particular, the convergence in the spaces $\mathcal{Y}_n$ was used in [94] to prove some statements on the spectra of limit graphs and spectral measures.

The theorem stating the realizability of a regular graph as a Schreier graph of a free group, which we will now present, has been proved by Gross in the case of finite graphs (its proof is given in Lubotzky's paper [126]). The fact that its proof can be adapted to the infinite case was pointed out in de la Harpe's book [100, p. 83].

Theorem 6.1. *Let Γ be a connected nonoriented $2m$-regular graph. Then there exists a subgroup $H \leq F_m$ of the free group F_m with basis A such that the Schreier graph $\Gamma(F_m, H, A)$ with removed labels and orientation is isomorphic to the graph Γ.*

To prove this theorem, we need the following lemma.

Lemma 6.2. *Every finite nonoriented $2m$-regular graph Δ possesses a 2-factor (i.e., a 2-regular subgraph with the same vertex set as Δ).*

Proof. Let us prove that Δ has a subgraph that is a union of pairwise disjoint cycles. We will prove this by joint induction on the parameters m, $m \geq 1$, and n, the latter being the number of vertices in the graph. The cases of $m = 1$ or $n = 1$ are trivial. If the graph is disconnected, then we can apply the induction hypothesis. Therefore, we will assume that Δ is connected. Since the degree of each vertex is an even number, there exists an Eulerian cycle, i.e., a closed path γ in the graph that passes through each edge precisely once. Note that the standard theorem on an Eulerian cycle is proved for graphs without multiple edges and loops. However, this theorem is also valid for multigraphs if we assume that the loops contribute multiplicity 2 to the degree of a vertex. Indeed, first, loops can be removed (if there is an Eulerian cycle for a graph with removed loops, then it is obvious how to construct such a cycle in the graph with loops). Next, if there are multiple edges, then each of them can be "doubled" by adding a vertex that divides this edge into halves. As a result of this procedure, we obtain a graph without multiple edges (and without loops) and such that the degree of each vertex is even. This graph has an Eulerian cycle, which induces in an obvious way an Eulerian cycle on the original graph.

Let us transform γ into an oriented path by arbitrarily choosing the direction of motion along it. This defines an orientation of edges of the graph Δ. Then the number of edges entering each vertex is m, and the same number of edges emanate from the vertex. Let us construct a new graph $\widetilde{\Delta}$ by performing a "surgery" at each vertex $v \in V(\Delta)$; the surgery consists in splitting the vertex v into two components v^- and v^+. The edges entering v turn into edges that enter v^-, while the edges leaving v turn into edges that leave v^+. The vertex set of the graph $\widetilde{\Delta}$ is partitioned into two disjoint subsets V^- and V^+ that consist of "sinks" and "sources," respectively; only pairs of vertices that belong to different parts of this partition may be connected by edges (the arrows point from a source to a sink). The graph $\widetilde{\Delta}$ is bipartite (i.e., its vertex set is divided into two disjoint subsets, and no pair of vertices in the same subset is connected by an edge) and, by Hall's theorem, has a perfect matching, i.e., a set of edges such that each vertex of the graph is incident to one and only one edge. Choosing one of such matchings, we construct a 2-factor on Δ. To this end, we begin the construction from an arbitrary vertex $u \in \Delta$ and choose an edge that emanates from this vertex and corresponds to an edge from the perfect matching of the graph $\widetilde{\Delta}$. Let u_1 be the endpoint of this edge. Take an edge that emanates from u_1 and belongs to the already constructed 1-factor, and so on. Since the graph is finite, in a finite number of steps we return to one of the points that has already been passed. If it turned out that this was not the initial point, then we would arrive at a contradiction with the properties of perfect matching. Thus, we have constructed in Δ a closed cycle δ_1 that does not pass more than once through any vertex. If there is a vertex that does not belong to δ_1, then we apply the same procedure to it as that for u. We obtain a cycle δ_2 that has no common vertices with δ_1 and passes through every vertex at most once. Proceeding in this way, we partition the vertex set of the graph into pairwise disjoint subsets that are the vertex sets of disjoint cycles passing through each of their vertices precisely once. It is this partition that defines a 2-factor. Note that we have obtained, in addition, an oriented 2-factor. □

Proof of Theorem 6.1. If the Schreier graph is finite, then we apply the lemma proved and construct for it a 2-factor δ_1 that is oriented by our construction (a nonoriented factor can always be transformed into an oriented by an arbitrary choice of orientation on each cycle of the factor). Let $A = \{a_1, \dots, a_m\}$. Let us supplement the edges of the factor δ_1 with the letter a_1 and remove them from our graph Γ (which we denote by $\Delta = \Delta_1$ to apply induction). We obtain a $(2m-2)$-regular graph Δ_2 for which there exists an oriented 2-factor δ_2, whose edges we label by the letter a_2. Proceeding in the same way, we obtain a chain of subgraphs Δ_i, $i = 1, 2, \dots, m$, and 2-factors δ_i. The union of edges of these 2-factors coincides with the set of all edges of the graph Δ, and the edges of δ_i are labeled by the symbol a_i. The labeling obtained determines on Δ the structure of a Schreier graph of a free group (because a graph labeled by the elements of the basis A of a free group is a Schreier graph of this group if and only if exactly one edge labeled by symbol $a \in A$ enters and leaves each vertex and this is true for every generator $a \in A$).

The case of an infinite graph requires additional arguments. The idea is to approximate such a graph by finite graphs, introducing appropriate labelings on them, and then apply a diagonal process. This is formalized as follows.

An oriented graph Γ labeled by symbols of a set of generators A is called a pre-Schreier graph of degree $m = |A|$ if, for every symbol $a \in A$ and each vertex v of the graph, there exist at most one edge that enters v and is labeled by a and at most one edge that emanates from v and is labeled by a. Such labelings are said to be admissible. Every finite graph the degrees of whose vertices are not greater than $2m$ can be extended to a finite $2m$-regular graph. Indeed, if there are two different vertices u and v whose degrees are less than $2m$, then, connecting them by an edge, we increase the degrees of these vertices by 1. Let us repeat this operation until no such a pair of vertices remains. If there is only one such vertex, then its degree must be an even number. Adding the necessary number of loops (each of which increases the degree of the vertex by 2) to this vertex, we obtain a $2m$-regular graph.

Let $\Gamma = \Delta$ be an infinite $2m$-regular graph, and let $\{\Delta_n\}_{n=1}^{\infty}$ be a sequence of finite subgraphs, where Δ_n consists of vertices situated at a distance of at most n from the initial vertex (which can be chosen arbitrarily from the very beginning) and those edges in Δ both of whose endpoints belong to Δ_n. On each of the graphs Δ_n, we introduce an orientation and label the edges by elements of the set A, making the graph into a pre-Schreier graph of degree m. We construct an infinite rooted tree $\mathcal{M}$ whose elements are pairs (Δ_n, R), where R is an admissible labeling of the graph Δ_n, the root vertex corresponds to the graph Δ_0 consisting of a single vertex, and two pairs (Δ_n, R) and (Δ_{n+1}, Q) are connected by an edge if the restriction of Q to Δ_n yields the labeling R. Since every finite graph has only a finite number of admissible labelings, the tree $\mathcal{M}$ is locally finite, and since each of the graphs Δ_n has at least one appropriate labeling, the tree is infinite. By Koenig's theorem, there exists an infinite path in $\mathcal{M}$ that connects the root vertex with infinity. It is this path that determines a labeling on the entire graph Δ that is consistent with the labelings on all the graphs Δ_n simultaneously, thus making Δ into a Schreier graph. The theorem is proved. □

In spite of the fact that a Schreier graph is not usually vertex-transitive and, moreover, often has a rather small automorphism group, the realization of a regular graph as a Schreier graph of some group often brings a kind of (in a sense, hidden) symmetry to the structure of the graph, which allows one to apply algebraic methods or even the methods of the theory of operator algebras to its analysis, as it was done in [16, 86].

7. SUBSTITUTION RULES FOR GRAPHS AND EXAMPLES OF SCHREIER GRAPHS

In this section, we consider a number of examples of Schreier graphs that are constructed by means of automaton groups (i.e., self-similar groups). As repeatedly discussed above, these groups act by automorphisms on a d-regular rooted tree. Therefore, for every $n = 1, 2, \dots$, one can construct a graph Γ_n defined by the action on the corresponding level n and thus obtain a sequence of graphs $\{\Gamma_n\}$. This is a covering sequence; namely, Γ_{n+1} covers Γ_n in the sense of graph theory (which corresponds to the covering in the topological sense if one deals with nonoriented noncolored graphs). Indeed, it is easily seen that the projection map that takes an $(n+1)$th-level vertex to its nth-level predecessor induces a covering of the graph (edges are projected to edges and labels to labels). This construction can be applied to an arbitrary finitely generated group acting on any rooted spherically homogeneous tree. When the action is transitive on the levels, the graph Γ_n is isomorphic to the Schreier graph $\Gamma(G, \mathrm{st}_G(u), A)$, where u is a point of the nth level. In addition to the sequence of finite graphs $\{\Gamma_n\}_{n=1}^{\infty}$, with the action of a group on an infinite rooted tree T one naturally associates the graph of the action on the boundary ∂T of the tree. If the group is finite or countable, then this graph is decomposed into an uncountable union of connected components Γ_ξ, $\xi \in \partial T$, where Γ_ξ is the orbital graph for the action on the orbit of a point ξ; the graph Γ_ξ is isomorphic to the Schreier graph $\Gamma(G, \mathrm{st}_G(\xi), A)$. We will call the graph $\Gamma(G, \mathrm{st}_G(\xi), A)$ a boundary Schreier graph.

When the action is transitive on the levels, there exists a close relationship between the sequence $\{\Gamma_n\}$ and the boundary Schreier graphs. Let $\xi \in \partial T$, $\{u_n\}_{n=1}^{\infty}$ be the sequence of vertices of the path ξ, $P = \mathrm{st}_G(\xi)$, and $P_n = \mathrm{st}_G(u_n)$. Then $\{P_n\}$ is a decreasing sequence, and the following relation obviously holds:

$$P = \bigcap_{n=1}^{\infty} P_n. \tag{7.1}$$

Proposition 7.1. *The relation*

$$(\Gamma(G, P, A), P) = \lim_{n\to\infty} (\Gamma(G, P_n, A), P_n) \tag{7.2}$$

holds in the sense of the topology of the space $\mathcal{X}_{2m}^{\mathrm{Sch}}$.

Proof. Note that the neighborhood of the Schreier graph $\Gamma(G, P, A)$ of radius n with center at an arbitrary vertex is determined by this vertex and the set of words of length $\le 2n$ over the alphabet A that define the elements of the subgroup P. By virtue of (7.1), for any k, there exists an N such that, for $n \ge N$, the sets of words of length $\le 2k$ that define the elements in the subgroups P and P_n coincide. Hence, the graphs $\Gamma(G, P, A)$ and $\Gamma(G, P_n, A)$, $n \ge N$, have isomorphic neighborhoods of radius k with centers at the distinguished vertices represented by the cosets P and P_n of the identity element, which was to be proved. □

We will write relation (7.2) in the abridged form $\Gamma_\xi = \lim_{n\to\infty} \Gamma_n$, omitting the indication of distinguished points. The questions about the structure of the graphs Γ_n and Γ_ξ, the asymptotic properties of the covering sequence $\{\Gamma_n\}_{n=1}^{\infty}$ and the infinite graphs Γ_ξ, and, in particular, their spectral properties are important for solving many problems.

The sequence $\{\Gamma_n\}_{n=1}^{\infty}$ associated with a finite automaton (i.e., defined by the action of a strongly self-similar group with the set of generators that corresponds to the set of states of the finite automaton) is a recurrently defined sequence; i.e., it is defined by the first graph Γ_1 and a substitution rule that describes how to obtain the graph Γ_{n+1} from the graph Γ_n. Let us give a formal definition.

Definition 7.1. Let $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of finite Schreier graphs associated with a group G generated by a set A and acting on a regular rooted tree defined by an alphabet $X = \{x_1, \dots, x_m\}$.

The sequence $\{\Gamma_n\}_{n=1}^{\infty}$ is *recurrent* if there exists a rule according to which every edge (u,v,a), $u,v\in X^n$, $v=a(u)$, $a\in A$, of the graph Γ_n and every symbol $x\in X$ define an edge (ux,wy,a), with some $y\in X$ and some $w\in X^n$, of the graph Γ_{n+1} so that the graph obtained from Γ_n by this substitution applied to all edges is isomorphic to the graph Γ_{n+1}, and this is valid for any n.

In this definition, we have fibbed a little because we have not defined what the *rule* means. Nevertheless, taking this definition on trust, we prove the following proposition.

Proposition 7.2. *For any finite automaton, the sequence $\{\Gamma_n\}$ is recurrent.*

Proof. The proof of this proposition is obvious. The automaton defines self-similarity relations $a(xu)=ya'(u)$, $x,y\in X$, $a,a'\in A$, $u\in X^*$, of the form (3.2), which show that if $w=a'(u)$, then $yw=a(xu)$. $\square$

The above definition of when the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ is recurrent is in fact equivalent to the fact that the sequence of graphs is generated by a finite automaton, and then by the rule one means the rule of action on the sequences that is involved in the definition of a Mealy automaton. The definition of a recurrent sequence of graphs can be extended by including partially defined Mealy automata, i.e., automata such that some of their states may not recognize some of the symbols at the input (or chains of symbols; i.e., here we mean asynchronous automata, the algebraic and algorithmic theory of which is discussed in [87]). The definition of the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ of oriented graphs in this case is corrected as follows. The set of vertices that serve as the starting points of edges is the set of words W of length n over the alphabet of the automaton that are recognized by at least one state q, while the set of vertices U that serve as the endpoints of edges consists of words that are the images of the words W in the first set under the action of states that recognize W. In this case, the corresponding edges (W,U) are colored with the state that translates W into U.

In our view, the definition of sequences $\{\Gamma_n\}_{n=1}^{\infty}$ of regular graphs by a finite invertible Mealy automaton is the most efficient method for describing complicated graphs. Even rather simple automata with a small number of states can generate very complicated sequences of finite graphs and infinite graphs that are the limits of finite graphs. At the same time, it is worth mentioning that the software implementation of such graphs does not require any tricks and that computers, depending on their memory capacity and speed, can construct the graph Γ_n for a given finite automaton and number n and store this graph in their memory. In particular, in this way one can construct asymptotic expanders considered in Section 10 and, possibly, even real expanders if the answer to Problem 10.1 will turn out to be positive.

Now we are going to discuss another, more conventional, method for obtaining sequences of finite graphs by means of iterative procedures, namely, by iterating a *graph substitution*. This approach has long been used, since the time of construction of graphs associated with fractals known under the name of *Sierpinski triangle* or *diamond fractal*, and including relatively recent "inventions" such as *pentagon* graphs [156]. The idea is as follows.

Given a graph Γ, one can construct a new graph $\mathcal{R}(\Gamma)$ by replacing some subgraphs in Γ with more complex graphs using a certain substitution rule $\mathcal{R}$. Here some variations are possible. The simplest type of substitutions, which was considered, for example, by Previte in [155, 156], belongs to the class of *vertex* substitution rules. For example, a substitution $\mathcal{R}$ can be defined by a finite set $(H_1,\partial H_1),\dots,(H_m,\partial H_m)$ of finite graphs H_i with distinguished subsets ∂H_i of vertices, called boundary vertices. In this case, it is assumed that the following two conditions are satisfied:

(1) H_i is symmetric with respect to ∂H_i in the sense that every permutation of the set ∂H_i can be realized by an automorphism of the graph H_i;

(2) $|\partial H_i|\ne|\partial H_j|$ for $i\ne j$, where $|\cdot|$ denotes the cardinality of a set.

A vertex v of a graph Γ is said to be replaceable if its degree is equal to $|\partial H_i|$ for some i. The rule $\mathcal{R}$ acts on Γ as follows. The graph $\mathcal{R}(\Gamma)$ is obtained from Γ by replacing each replaceable vertex v

with a graph H_i for which $|\partial H_i| = \deg(v)$. The vertex v itself is removed, while the edges incident to it are connected (no matter how, because of the symmetry) with the boundary vertices of a copy of the graph H_i, which is thus "attached" to Γ, and this procedure is applied to each replaceable vertex. If a substitution rule $\mathcal{R}$ is defined, then, starting from any graph $H = H_0$ (axiom), one can construct a sequence $\{H_n\}_{n=1}^{\infty}$, where $H_{n+1} = \mathcal{R}(H_n)$, $n = 0, 1, \dots$, which may turn out to be finite in case of inappropriate rule and initial graph; however, in the typical case, this is an infinite sequence of graphs that has interesting asymptotic and combinatorial properties [155, 156]. One may also consider more complex local substitution rules in which not only vertices but also edges, and even more complicated subgraphs, are involved in the substitution.

Let us describe a more general scheme of local recurrent rules. Suppose given two finite disjoint alphabets A and B. Consider graphs whose edges are colored with letters in the alphabet A and vertices are colored with words in the set B^* of words over the alphabet B. Different edges may have the same labels, whereas vertices ought to have different labels. Denote the category of such graphs by $\mathcal{W}_{\mathcal{R}}$.

A local rule $\mathcal{R}$ consists of a finite set of pairs (Δ, Ξ) of graphs whose edges are colored with letters of the alphabet A (however, the vertices have no labels), a pair of maps φ, ψ that will be described below, and a local surgery rule. The latter rule means an arbitrary rule that allows one to uniquely replace, in a given graph $\Gamma \in \mathcal{W}_{\mathcal{R}}$, any inclusion of Δ into Γ by Ξ, by performing a local surgery according to this rule. The graphs Δ are called templates, and Ξ, duplicates. To every vertex $v \in V(\Delta)$, the map φ assigns a subset of vertices $\varphi(v) \subset V(\Xi)$, and each vertex u in $\varphi(v)$ is additionally equipped with a letter x_u in the alphabet B. In addition, $\varphi(v) \cap \varphi(u) = \varnothing$ if $v \neq u$, and $\bigcup_{v \in V(\Delta)} \varphi(v) = V(\Xi)$. If the label of a vertex v in the graph Γ was $w \in B^*$, then, after the surgery, each of the vertices u of the image $\varphi(v)$ is labeled by xw, where $x = x_u \in B$ is the symbol corresponding to this vertex in the surgery rule. The map ψ assigns to each edge $e \in E(\Delta)$ a subset of the set of edges $E(\Xi)$. It is assumed that $\psi(e) \cap \psi(e') = \varnothing$ if $e \neq e'$, $\bigcup_{e \in E(\Delta)} \psi(e) = E(\Xi)$, and each edge in $\psi(e)$ is equipped with a label from A, which becomes a label of this edge after the surgery. The surgery is performed by simultaneously replacing all inclusions of templates in Γ with the respective duplicates. The graph obtained from Γ by the local rule is denoted by $\mathcal{R}(\Gamma)$. The substitution rule should satisfy the condition that if the inclusions of two templates in a graph have a nonempty intersection, then the surgery rule is consistent on the common part. One can complicate the procedure by introducing one more alphabet C (disjoint from A and B) for additional labels of some vertices of the graphs. The vertices labeled by symbols from C are called key (or boundary) vertices. If a graph Γ has key labels, then key labels may also appear in the templates and their duplicates. After the surgery, the graph will also have key vertices, to which one should apply a local surgery according to an additional instruction (for example, one may need to add loops at these vertices).

Starting from a graph Γ_0 (axiom) and applying a local rule $\mathcal{R}$, one can construct a sequence $\{\Gamma_n\}_{n=1}^{\infty}$, $n = 1, 2, \dots$, where $\Gamma_{n+1} = \mathcal{R}(\Gamma_n)$. In interesting cases, this should be an infinite sequence with (as a rule, exponentially) growing size of the graphs. Having constructed the sequence $\{\Gamma_n\}_{n=1}^{\infty}$, $n = 1, 2, \dots$, depending on the situation and the aims, one may forget about labels (or a part of them) and delete them, thus obtaining an ordinary sequence of graphs. It is obvious that Previte's scheme of constructing graphs (see above) is included in this more general scheme. Here we avoid a rather burdensome condition that the templates should be symmetric. The proposed scheme works well for the Schreier graphs generated by bounded automata (see Definition 7.2 below), which will be demonstrated in examples below. It can be additionally complicated by making the description of the substitution periodic with some period k. Thus, in fact, one deals with k substitution rules $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_k$ and applies them consecutively in cyclic order. The simplest type of substitution rules is when templates are given only by the vertices and edges of graphs. The examples given below belong precisely to this type of substitutions.

The second approach to constructing a sequence of graphs $\{\Gamma_n\}_{n=0}^{\infty}$ consists in applying *global* substitution rules, which will be denoted by $\mathcal{S}$. In this case, just as above, three disjoint alphabets A, B, and C are used; as in the case of local rules, A is used for the labels of edges, the words in B^* are used for the labels of vertices, and the letters in C are used to label key (boundary) vertices. Let d be the number of letters in the alphabet B. The graph Γ_0 (axiom) is assumed to be given, and the construction of the whole sequence starts from this graph. Moreover, for any pair (x,y), $x,y\in C$, a graph $\Theta_{x,y}$ is defined that may either be empty or consist of one or two vertices and some set (of cardinality $\le |B|$) of edges colored with letters from the alphabet A. We will call the graphs $\Theta_{x,y}$ bridges in view of the role they play in the construction. For $n\ge 0$, the graph Γ_{n+1} is obtained from the graph Γ_n in the following way. Take a disjoint union of d copies $\Gamma_{n,x}$, $x\in B$, of the graph Γ_n, which we temporarily denote by $\overline{\Gamma}_{n+1}$; the vertices of these copies acquire labels of the form vx, $x\in B$, if the corresponding vertex in Γ_n has label $v\in B^*$. After that, following the instruction given by the rule $\mathcal{S}$, a local surgery is performed in the neighborhoods of key vertices depending on their C-colors (for example, a loop is removed). Then, again according to the instruction defined by $\mathcal{S}$, for every pair u,v of key vertices that have labels x and y from C and belong to different copies of Γ_n, the graph $\Theta_{x,y}$ is attached to $\overline{\Gamma}_{n+1}$. The key vertices at which the graphs $\Theta_{x,y}$ with more than one vertex are attached cease to be key vertices. The other key vertices remain key as before; however, the C-colors of these vertices are changed according to the instruction. This completes the construction of the graph Γ_{n+1}. Thus, formally $\Gamma_{n+1}=\mathcal{S}(\Gamma_n)=\mathcal{S}^n(\Gamma_0)$. Having constructed the sequence $\{\Gamma_n\}_{n=0}^{\infty}$, depending on the situation and the aims, one may delete all the labels (or only a part of them). If one does not need a one-to-one correspondence between the vertices of the graph Γ_n and the words of length n over the alphabet B, then one may not care about the coloring of vertices by words from B^* and choose more complicated bridge graphs $\Theta_{x,y}$ (i.e., with more than two vertices); however, in the latter case one should label a pair of key (boundary) vertices in each of these graphs, equipping them with labels from the alphabet C, and, when attaching $\Theta_{x,y}$ to $\overline{\Gamma}_{n+1}$, should follow the convention that the key vertices to be identified have identical labels. Again, just as in the local rule, the procedure can be complicated by admitting the periodicity of the construction rule with some period k.

Note that one of the most significant differences between the description of local and global rules for constructing graphs is that in the local case we write a symbol $x\in B$ on the left of the label of a vertex, whereas in the global rule we write the same symbol on the right.

This approach to the construction of graphs on the basis of global substitution rules, just as the local approach, goes back to the studies on fractal theory. With the classical fractals such as the Sierpinski triangle, the Koch fractal, or the diamond fractal [65], one usually associates a sequence of graphs that approximates these fractals. To generalize these and other examples, the concepts of a finitely branched self-similar fractal and a sequence of finite graphs associated with this fractal have been introduced (see the studies by Kigami, Barlow, Strichartz, Nekrashevych, Teplyeaev, and others [117, 12, 174, 182, 146]). The fact that many fractal sets can be represented as boundaries of naturally arising Gromov hyperbolic graphs, which provides new means for analyzing them, was first pointed out by Kaimanovich [108].

A new turn in the theory of fractals and the associated self-similar graphs was due to the application of the ideas and methods of the theory of self-similar groups, which was initiated by the present author, Gupta and Sidki, Sushchansky, Nekrashevych, Bartholdi, and other mathematicians, as well as the related theory of iterated monodromy groups, developed mainly by Nekrashevych [142, 18], and the theory of limit spaces associated with contracting self-similar groups [142]. This gave a new insight into classical objects such as Julia sets and provided a new approach to the approximation of these objects by discrete objects such as graphs and even cell complexes [144].

As repeatedly discussed, an important role in the study of dynamical systems is played by the results that describe the typical properties of a graph. One of the results of this kind is the recent

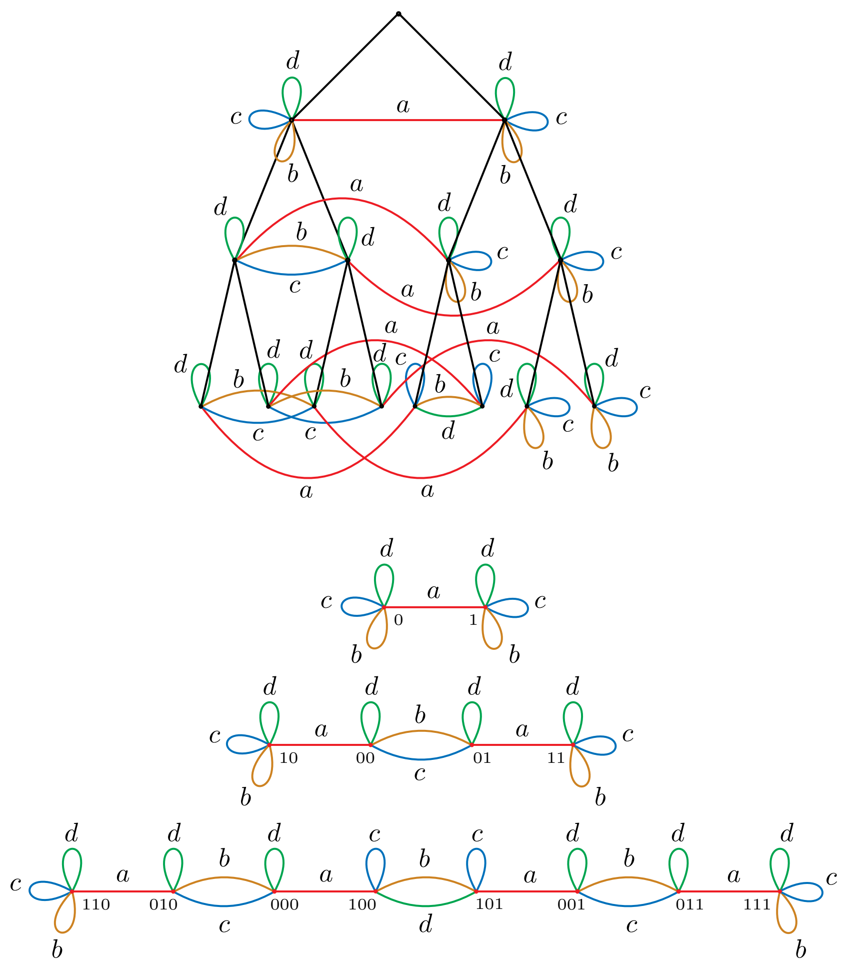

Fig. 7.1. Orbital graphs of action on levels 1, 2, and 3 of a tree and their "ruled" representation.

result of I. Bondarenko, D. D'Angeli, and T. Nagnibeda stating that the number of ends in a typical Schreier graph for the action of a self-similar group on the boundary of a tree is fixed (and thus is an invariant of the dynamical system) [34].

After all these discussions, now it is time to present specific examples.

Example 7.1. We begin with the group $\mathcal{G}$ of intermediate growth, whose generators a, b, c, and d have order 2 (and thus make a contribution of 1 to the degree of a vertex of a Schreier graph if this vertex is mapped into itself under the action of a generator). The Schreier graphs Γ_1, Γ_2, and Γ_3 associated with the first three levels of a tree are shown in Fig. 7.1, and the local substitution rules are demonstrated by the diagrams in Figs. 7.2 and 7.3. Infinite Schreier graphs are shown in Figs. 7.4 and 7.5; they are shown in two variants: without and with the labels of edges. Below we examine the properties of these graphs in greater detail; meanwhile we describe a global substitution rule for this case.

To construct Γ_{n+1} from Γ_n, we first construct two copies $\Gamma_{n,0}$ and $\Gamma_{n,1}$ of the graph Γ_n by adding a symbol 0 or 1 to the right of the label of each vertex. If $n = 3m + 1$, we remove the loops labeled by symbols b and c at the vertices $1^{n-1}00$ and $1^{n-1}01$ in $\Gamma_{n,0}$ and $\Gamma_{n,1}$, respectively, and replace them by edges with labels b and c that connect the vertices $1^{n-1}00$ and $1^{n-1}01$. Similarly, if $n = 3m + 2$, we replace loops with labels b and d at the vertices $1^{n-1}00$ and $1^{n-1}01$ by two edges that have the same labels and connect the same vertices. Finally, if $n = 3m$, we apply the same procedure for the pair of symbols c, d. The procedure described is periodic with period 3. The role of the alphabet A in this example is played by the set of generators $\{a, b, c, d\}$, and $B = \{0, 1\}$. We

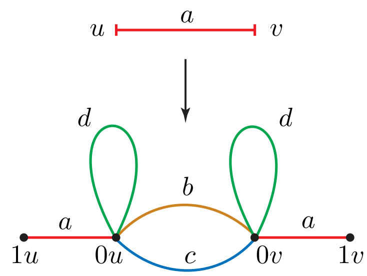

Fig. 7.2. The first substitution rule.

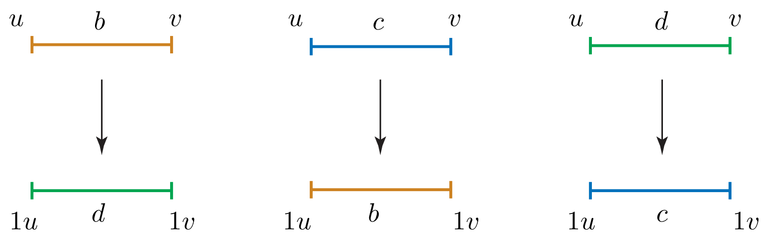

Fig. 7.3. The second substitution rule.

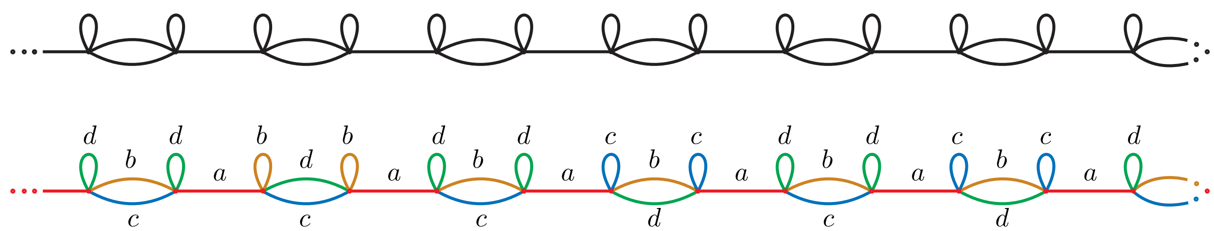

Fig. 7.4. Infinite Schreier graph with two ends.

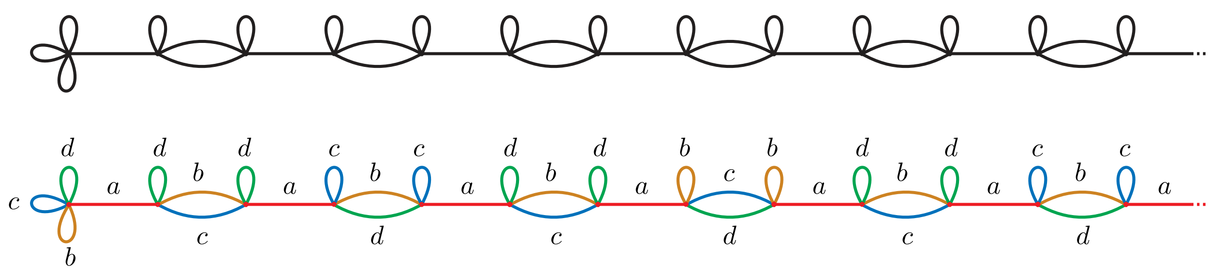

Fig. 7.5. Infinite Schreier graph with one end.

did not use the third alphabet C; however, we could also consider it, say, by setting $C = \{R, W\}$ and assigning a label R to the left vertex 0 of the graph Γ_1 (we assume that the axiom corresponds to the value $n = 1$) and a label W to the right vertex. Then the rule prescribes that one should remove loops at the key vertices with label R in the copies $\Gamma_{n,0}$ and $\Gamma_{n,1}$ and connect these vertices by a pair of edges, equipping the latter by labels according to the above description (and adhering to the periodicity). At the same time, the key vertices labeled by symbol W remain key vertices, but the first of them (namely, $1^{n-1}0$) changes its label to R.

This example, as well as other examples related to a group $\widetilde{\mathcal{G}}$ that envelopes $\mathcal{G}$ and to the Gupta–Sidki 3-group [99], was first considered in [16].

Theorem 7.3. (a) *Two boundary points* $\xi = \{u_n\}_{n=1}^{\infty}$ *and* $\zeta = \{v_n\}_{n=1}^{\infty}$, $u_n, v_n \in \{0, 1\}$, *of the binary tree lie in the same orbit of the group* $\mathcal{G}$ *if and only if there exists an* n_0 *such that* $u_n = v_n$ *for all* $n \geq n_0$.

(b) *The sequence* $\{\Gamma_n\}_{n=1}^{\infty}$ *of finite Schreier graphs of the group* $\mathcal{G}$ *with the system of generators* a, b, c, d *is substitutional and is described by the above local and global substitution rules. In the graph* Γ_n *drawn in the plane as a linear graph, as shown in Fig.* 7.1, *the extreme vertices* 0^n *and* 1^n *occupy positions with numbers* p_n *and* 2^n, *respectively, where* $p_n = \frac{1}{3}(2^n - 1)$ *if* n *is even and* $p_n = \frac{1}{3}(2^n + 2)$ *if* n *is odd.*

(c) *There exist only two types of infinite nonoriented noncolored Schreier graphs for the group* $\mathcal{G}$ *with the system of generators* a, b, c, d. *They are shown in Figs.* 7.4 *and* 7.5 *and have one and two ends, respectively. The graph* Γ_ξ *has one end if and only if the point* ξ *belongs to the orbit of the point* 1^∞.

(d) *The graphs* Γ_ξ, $\xi \in \partial T$, *are pairwise nonisomorphic* (*as oriented colored graphs*).

Proof. (a) According to the recurrent relations (2.9), the action of any generator on an arbitrary sequence of symbols changes at most one symbol. Therefore, it is obvious that if two infinite sequences lie in the same orbit, then, starting from some place, they should coincide. Suppose, conversely, that n is the place starting from which the sequences ζ and η coincide, and u and v, $u \neq v$, are their initial segments of length n (which are different). By induction on n we prove that the sequences lie in the same orbit. Let $u = u_1, \dots, u_n$ and $v = v_1, \dots, v_n$, $u_n \neq v_n$. If $u_1 \neq v_1$, then, acting on ζ by the generator a, we can make it so that the first symbols coincide. Now, let $u_1 = v_1$. By the induction hypothesis, there exists an element g that maps $\{u_n\}_{n=2}^{\infty}$ to $\{v_n\}_{n=2}^{\infty}$. Since $\mathcal{G}$ is a self-replicating group (Definition 3.6), the section $\mathcal{G}_{u_1}$ of the stabilizer of the vertex u_1 is equal to the copy of the group $\mathcal{G}$ acting on the subtree T_{u_1}. In other words, for any element $g \in \mathcal{G}$, there exists an element $h \in \mathcal{G}$ such that the relation $h(u_1 w) = u_1 g(w)$ holds for any sequence $w \in \{0,1\}^{\mathbb{N}} = \partial T$. Hence, there also exists an element h that maps ζ to η.

(b) Let u and v be vertices of the nth level of the tree and $a(u) = v$. Recall that a changes the first symbol of any nonempty sequence to the opposite; the elements b and c act on a sequence of the form $0w$ according to the relations $b(0w) = 0a(w)$ and $c(0w) = 0a(w)$, and on a sequence of the form $1w$ according to the relations $b(1w) = 1c(w)$ and $c(1w) = 1d(w)$; finally, the action of the element d is defined by the relations $d(0w) = 0w$ and $d(1w) = 1b(w)$. This implies that the graph on the four vertices $0u$, $1u$, $0v$, and $1v$ that appears in the substitution rule illustrated in Fig. 7.2 is a subgraph of the graph Γ_{n+1}, and this part of the graph corresponds to an edge in Γ_n that connects the vertices u and v and is labeled by the generator a. In exactly the same way, the recurrent relations between generators define the correspondence shown in Fig. 7.3 between the edges labeled by the generators b, c, and d in the graphs Γ_n and Γ_{n+1}. Thus, we have established the local substitution rule.

The character of the local substitution relations shows that the graphs Γ_n have the structure of a linear chain of length 2^n (more precisely, they are isomorphic to such a structure) and that, at the nth iteration step, the vertex 1^n of the tree occupies the rightmost position, i.e., it has number 2^n. To find the position of the vertex 0^n in the linear structure, we will call the arrangement of vertices of the graph Γ_n drawn as a linear chain a nonstandard order. If the $(n-1)$th-level vertices are arranged in the nonstandard order $v_1, \dots, v_{2^{n-1}}$, then, in view of the substitution rules, the nonstandard order on the set V_n of nth-level vertices is defined by the sequence $1v_1, 0v_1, 0v_2, 1v_2, \dots, 1v_{2^{n-1}-1}, 0v_{2^{n-1}-1}, 0v_{2^{n-1}}, 1v_{2^{n-1}}$. Thus, $p_n = 2p_{n-1}$ if n is even and $p_n = 2p_{n-1} - 1$ if n is odd; this leads to the formulas given in the formulation of the theorem. The global substitution rule also becomes obvious.

(c) With a boundary point ξ represented by a path consisting of vertices u_n, $n \geq 1$, one can associate a sequence $\{(\Gamma_n, u_n)\}_{n=1}^{\infty}$ of finite rooted Schreier graphs. If the position of the current point u_n (the root) in the representation of the graph Γ_n as a linear chain is at a distance tending to infinity as $n \to \infty$ from the left and right boundaries, then we obtain in the limit a linear graph with two ends; otherwise, we obtain a linear graph with one end. If the sequence representing a boundary point contains infinitely many symbols 0, then, according to the substitution rule, each next symbol 0 shifts the vertices at least by one away from the boundaries (but the distance cannot increase more than twice) upon every application of the substitution. Thus, we conclude that in the limit the distinguished vertex turns out to be situated at an infinite distance from both ends of the graph. If the symbol 0 is encountered in the sequence ξ a finite number of times, then, starting from some number k, all of its symbols are 1. Then, for any n, the vertex u_n in the graph Γ_n is located at a distance of at most 2^k from the right end; i.e., in the limit we obtain a graph with one end, and the reference point ξ is situated at distance $\le 2^k$ from the rightmost vertex, which corresponds to the point 1^∞. In fact, the location of the distinguished point in the graph Γ_ξ with one end can be exactly calculated by the prefix U_k consisting of the first k symbols of the sequence ξ.

(d) This assertion of the theorem follows from Proposition 2.2. $\square$

Remark 7.1. Assertion (a) of the theorem proved shows that the decomposition into orbits of the action of the group $\mathcal{G}$ is a *cofinal* equivalence relation, which plays an important role in the theory of countable Borel equivalence relations [113]. We touch upon the issue of countable decompositions in Section 11.

After removing the labels of edges in the graph Γ shown in Fig. 7.4, we obtain a graph associated with a one-dimensional lattice (at least $\mathbb{Z}$ acts on this graph by automorphisms and this action is cocompact). However, being equipped with labels, this is a rather complicated graph that allows one to completely reconstruct the group $\mathcal{G}$, which is itself a group with rather complicated structure and properties. Indeed, to find out whether a word $W = W(a, b, c, d)$ defines the identity element in the group $\mathcal{G}$, one should check whether any path defined by the word W with beginning at an arbitrary vertex of the graph is closed. This follows from the fact that, for any point $\xi \in \partial T$ and its stabilizer $P = P_\xi$, the intersection $\bigcap_{g\in G} P^g$ is trivial. Indeed, every nontrivial normal subgroup in $\mathcal{G}$ has finite index (the property of maximal minimality), and the action on the levels of a binary tree is transitive.

Consider a few more examples of substitution rules for Schreier graphs.

Example 7.2. The following example is of interest for a number of reasons. On the one hand, it is related to the Sierpinski triangle, and the corresponding sequence of graphs $\{\Gamma_n\}_{n=1}^\infty$ may serve as a discrete approximation of this classical fractal. However, it differs from the standard approximation sequence used for studying the Sierpinski fractal by approximation techniques. The graphs Γ_n (a clear idea of which is given by Fig. 7.6) are called Pascal graphs, because they can be described combinatorially by a recursive procedure similar to the construction of Pascal's triangle.

On the other hand, this sequence of graphs is related to the combinatorial problem known as the *Towers of Hanoi* game. The essence of this game is as follows. Three pegs of the same length are attached to a support (a small plate) in the perpendicular direction. On one of these pegs, a pile of $n \ge 1$ disks of different diameters is placed in the order of decreasing size. The problem is to move the disks to another peg while observing the rules of the game; moreover, this should be done in the minimum number of moves (one may move only one disk at a time). The main rule of the game is that one cannot place a larger disk on top of a smaller one. For the values of n from 5 to 8, this game can be purchased in stores as a game for children of age above three years. If, in addition, the solution must be algorithmic, then such a problem is suggested to junior students of universities as an exercise on the subject of recursion. One can easily verify that the problem is solved in $2^n - 1$ steps.

There exist various modifications and complications of the game. The main of them is increasing the number of pegs. It is clear that the larger the number of pegs, the wider the possibilities to move the disks, and thus the faster the solution of the problem. In spite of numerous attempts, the problem of finding a minimal and algorithmic solution in the case of four or greater number of pegs has not yet been completely solved. For this famous problem, its history, generalizations and variations, failed

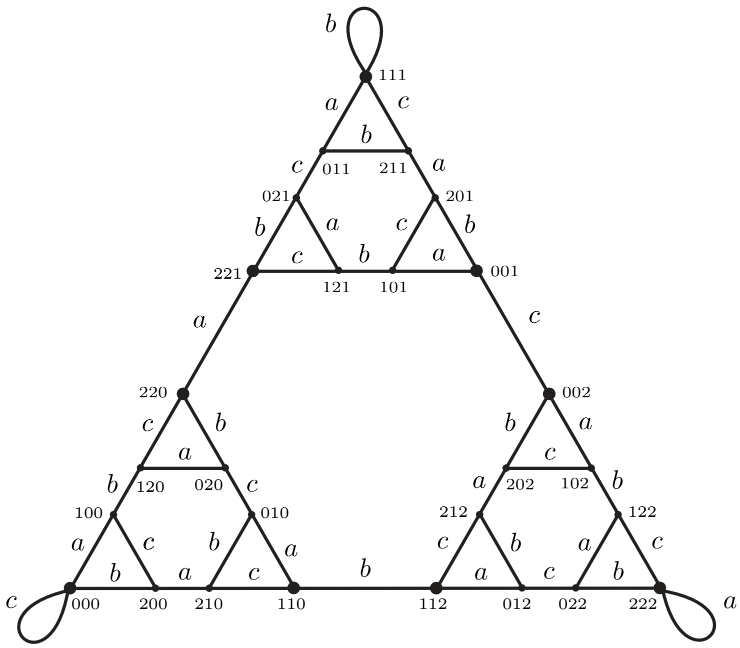

Fig. 7.6. Pascal graph as a Schreier graph of level 3.

attempts at solving it, and some advances, see [102, 91] and references therein. We just note that an asymptotic (i.e., approximate) solution of the problem was obtained by Szegedy in [178], where the author showed that, for a number $k \geq 3$ of pegs, the problem is solved in $\sim 2^{n^{1/(k-2)}}$ steps. Thus, for $k > 3$, the number of moves necessary for transferring n disks from one peg to another grows (as a function of n) in an intermediate manner between polynomial and exponential growth. Note that Szegedy did not discuss how to find an asymptotically minimal path algorithmically, although it seems that a recursive method for the asymptotic solution of the problem can be derived from the proof given in [178].

Surprisingly, the Towers of Hanoi problem is directly related to the theory of self-similar groups; this fact was first noticed by Z. Šunić and described in [89, 88, 93, 91], where the groups $\mathcal{H}^k$, $k \geq 3$, called Hanoi Towers groups were introduced and some results related to the algebraic properties of these groups and the asymptotic theory of Schreier graphs were presented. A part of these results is obtained on the basis of known information on the asymptotic properties of the game.

The Hanoi Towers group $\mathcal{H}^3$ is defined by the automaton shown in Fig. 7.7. This group acts on a ternary tree and is generated by generators a, b, and c that correspond to the automaton states a_{01}, a_{02}, and a_{12}. Recurrent relations between the generators are as follows:

$$a = (1, 1, a)(01), \qquad b = (1, b, 1)(02), \qquad c = (c, 1, 1)(12).$$

Acting on a ternary tree, $\mathcal{H}^3$ defines a sequence of finite Schreier graphs $\{\Gamma_n\}_{n=1}^{\infty}$ (called Pascal graphs, as already mentioned; one can get a clear idea of these graphs from Fig. 7.6), as well as a continuum family of infinite Schreier graphs associated with the action on the boundary of the tree (the connected components of the orbital graph). A global rule that describes the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ is as follows. To construct Γ_{n+1} from Γ_n, one takes three copies of the graph Γ_n, denoted by $\Gamma_{n,0}$, $\Gamma_{n,1}$, and $\Gamma_{n,2}$, that differ from Γ_n only by the labels of the vertices; namely, each label u, $u \in X^* = \{0, 1, 2\}^*$, of a vertex in Γ_n is replaced by the label ux in $\Gamma_{n,x}$, $x \in \{0, 1, 2\}$. To obtain Γ_{n+1}, one removes loops at the vertices $z^n x$ and $z^n y$ for every pair $x, y \in X$, $x \neq y$ (z is a letter different from x and y), in the graphs $\Gamma_{n,x}$ and $\Gamma_{n,y}$, respectively, and adds an edge that connects these vertices and is colored with the symbol a_{xy}. One can describe this procedure more formally by introducing the third alphabet and key vertices, but we will not do this.

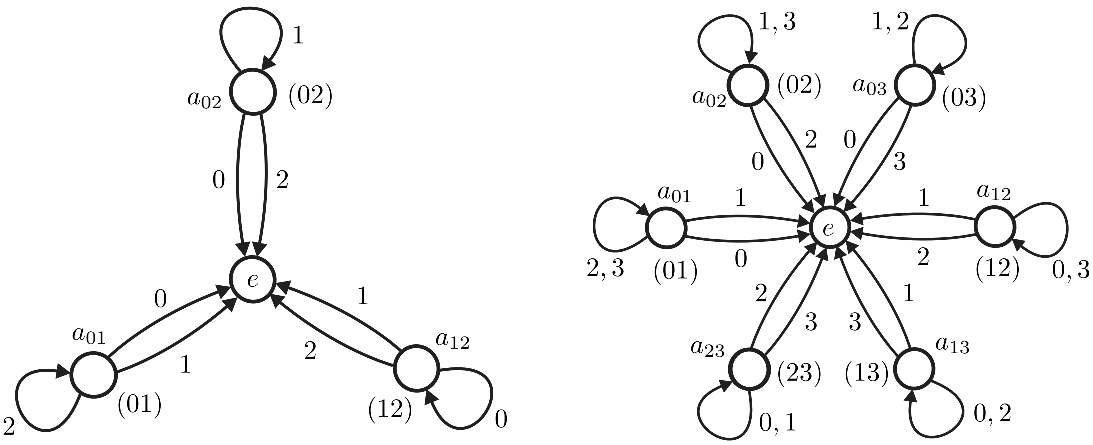

Fig. 7.7. Automata that generate the Hanoi Towers groups $\mathcal{H}^3$ and $\mathcal{H}^4$.

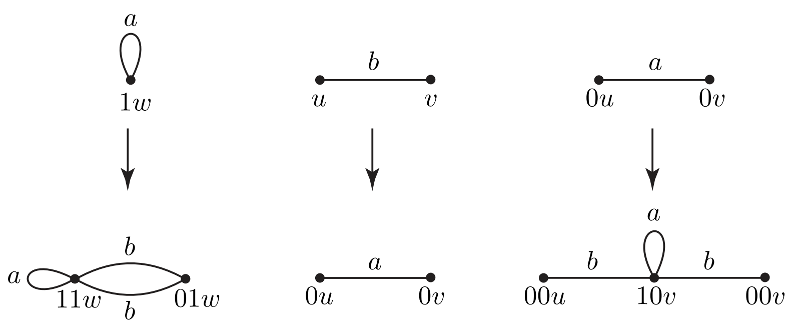

Fig. 7.8. Substitution rules for the Schreier graphs of the Basilica.

After deleting the labels of edges and the loops at the boundary vertices, the graphs Γ_n become identical to the graphs considered in relation to different aspects of the Towers of Hanoi game with three pegs. The spectral properties of these graphs were studied in [157, 93].

As already pointed out, to the Towers of Hanoi game with $k \geq 4$ pegs, there corresponds a group $\mathcal{H}^k$ that is self-similar and is defined by an automaton with $\frac{k(k-1)}{2} + 1$ states over a k-letter alphabet. The Schreier graphs Γ_n associated with this game are much more complicated, which is indicated by the fact that the problem of calculating the distance between vertices 0^n and 1^n in these graphs is equivalent to finding the minimum number of moves needed to transfer a tower of n disks from one peg to another (which is still unsolved). The problem of calculating the diameters of these graphs (which in this case are greater than the distance between the vertices 0^n and 1^n when the number of disks is large) is also unsolved; and the spectral problem (discussed later) has not been solved either.

Example 7.3. The following sequence of graphs is related to the self-similar group called the *Basilica*, which was considered in Example 2.4. This group has played an important role in solving a problem connected with amenable groups (namely, it served as the first example of an amenable, but not subexponentially amenable, group) [96, 23], and it also served as the first nontrivial example that has led to the development of the theory of iterated monodromy groups [142, 18].

Local substitution rules for this case are shown in Fig. 7.8, and a clear idea of the Schreier graphs is given by Figs. 7.9–7.12. It is easily noticed that the graphs Γ_n converge to a set similar to the Julia set of the map $z \to z^2 - 1$. That this is indeed the case under a reasonable definition of convergence is proved in [142]. Obviously, the structure of graphs associated with the Basilica is much more complicated than in the previous two examples. In [48], the authors proved that the infinite Schreier graphs of the action on the boundary have one, two, or four ends, and gave a full

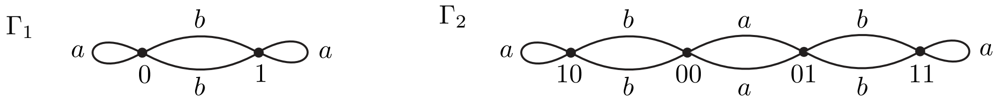

Fig. 7.9. Schreier graphs of the Basilica of levels 1 and 2.

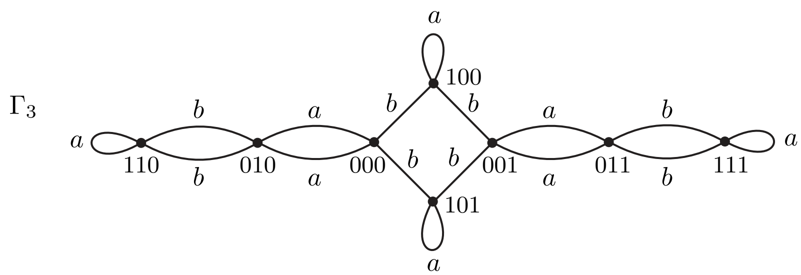

Fig. 7.10. Schreier graph of the Basilica of level 3.

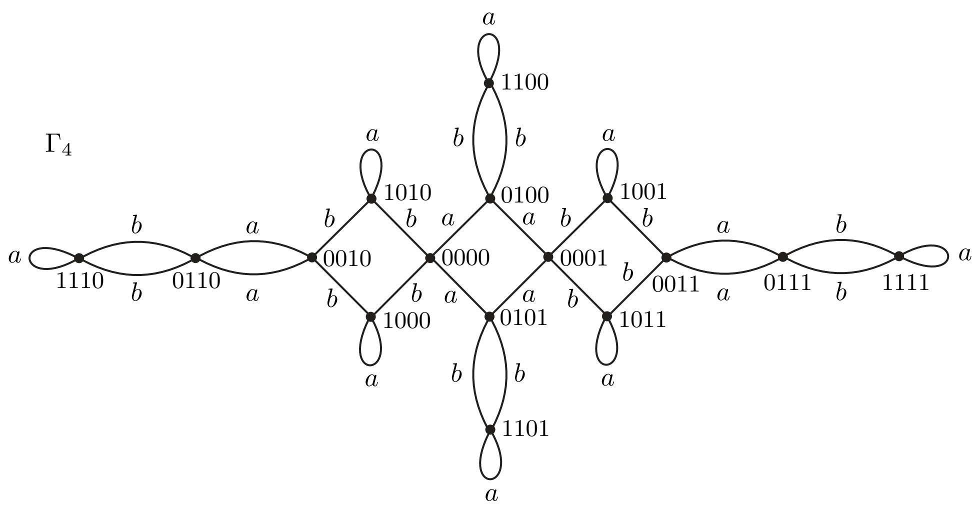

Fig. 7.11. Schreier graph of the Basilica of level 4.

description of these graphs up to isomorphism (the graphs were considered without labels). There are uncountably many such graphs, and some invariants that allow one to distinguish these graphs were also found in [48].

Interesting classes of automata and the corresponding classes of groups were introduced by Sidki [172]; these are the classes of polynomially growing automata and groups generated by these automata. An important characteristic of an automorphism g of a degree d tree is its activity, which is the growth of the number $a(n)$ of nonidentity elements of the symmetric group $\mathrm{Sym}(d)$ in the portrait of the element g on the nth level as $n \to \infty$ (recall that the portrait of a tree automorphism is a coloring of the vertices of the tree with the elements of a symmetric group that describe locally the action of the automorphism). This concept was already used in the discussion of the concept of finite-type action (after Theorem 2.9).

Definition 7.2. (a) If the growth of the sequence $a(n)$ associated with a tree automorphism defined by a finite initial automaton is polynomial of degree k, then the corresponding initial automaton is said to be polynomially (in the sense of Sidki) growing of degree k.

(b) A noninitial automaton $\mathcal{A}$ is said to be polynomially growing of degree k if, for any of its states q, the corresponding initial automaton is polynomially growing of degree at most k and there exists a state for which the growth is polynomial of degree k.

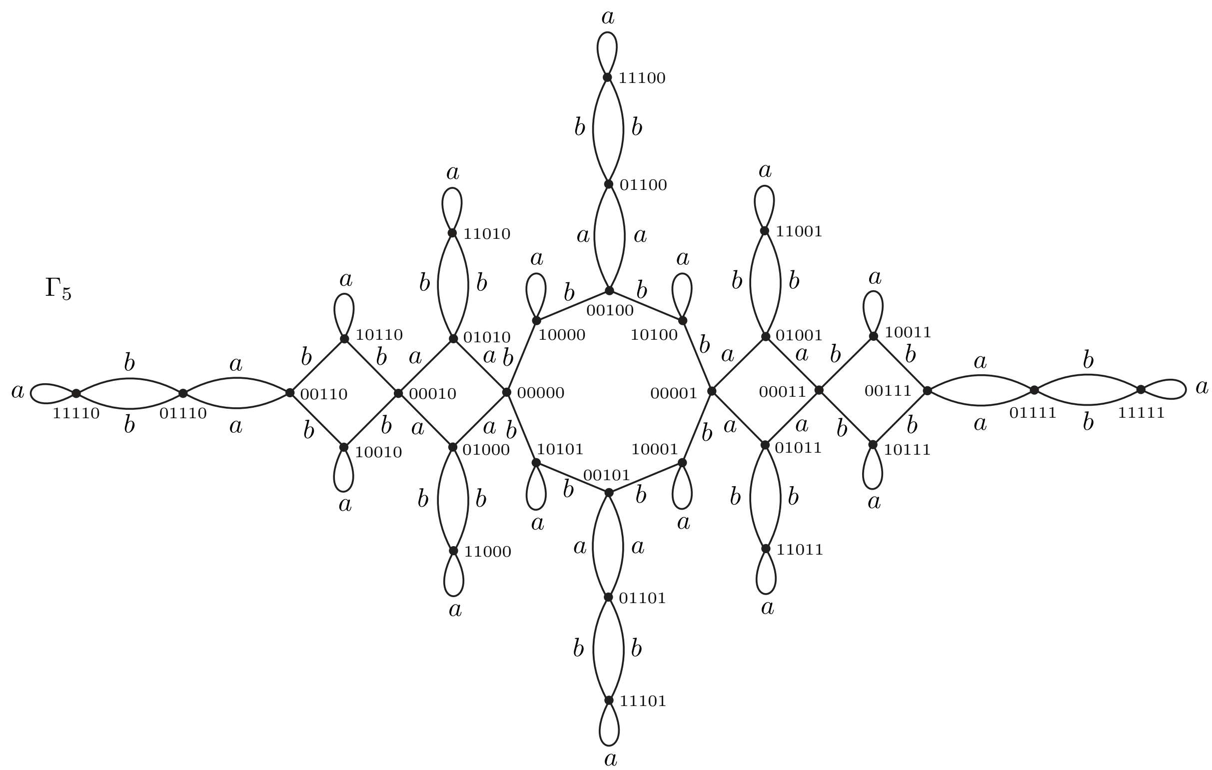

Fig. 7.12. Schreier graph of the Basilica of level 5.

Thus, one distinguishes the classes of bounded, linear, quadratic, etc. automata and the corresponding classes of groups.

Note that this terminology is not consistent with the concept of growth introduced in [75] (see also [87]); however, now we will follow precisely this terminology. There exists a simple algorithm for finding out whether or not a given automaton has a polynomial growth in the sense of Sidki and for calculating the degree of the polynomial growth [172, 142]. We will assume that an automaton is minimal, which, recall, means that different states induce different tree automorphisms. First, in order for an automaton to be polynomially growing, it should have an identity state id, i.e., a state that induces the trivial tree automorphism. Then the automaton is polynomial if and only if any two cycles in its Moore diagram that do not pass through the identity state are disjoint. Moreover, the automaton is bounded if and only if any two cycles in its Moore diagram that do not pass through the identity state are disjoint and there does not exist an oriented path leading from one of these cycles to the other. An alternative method for verifying the polynomiality of growth is to find out if the spectral radius (the Perron–Frobenius number in the present case) of the adjacency matrix of the Moore diagram of the automaton (considered as an oriented graph) with the identity state removed is equal to 1; the degree of polynomial (in the sense of Sidki) growth of the automaton in this case is equal to $n-1$, where n is the multiplicity of the eigenvalue 1.

The Schreier graphs associated with bounded automata have been studied rather intensively [16, 142, 31]. They have polynomial growth (possibly, with noninteger and even irrational exponent), which follows from the fact that the group generated by such an automaton is contracting [16, 142]. Among the recent results in this direction, we mention the results obtained by D'Angeli, Donno, and Nagnibeda concerning the structure of Schreier graphs associated with the Basilica [48].

For polynomially growing but unbounded automata, the Schreier graphs have a much more complicated structure. For example, the recent result of Bondarenko [33] shows that the growth of infinite Schreier graphs associated with polynomially growing automata is subexponential with

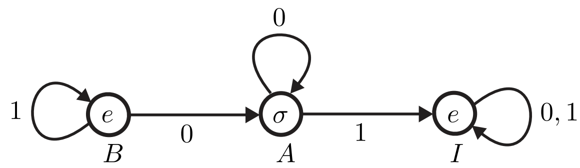

Fig. 7.13. Automaton of linear growth.

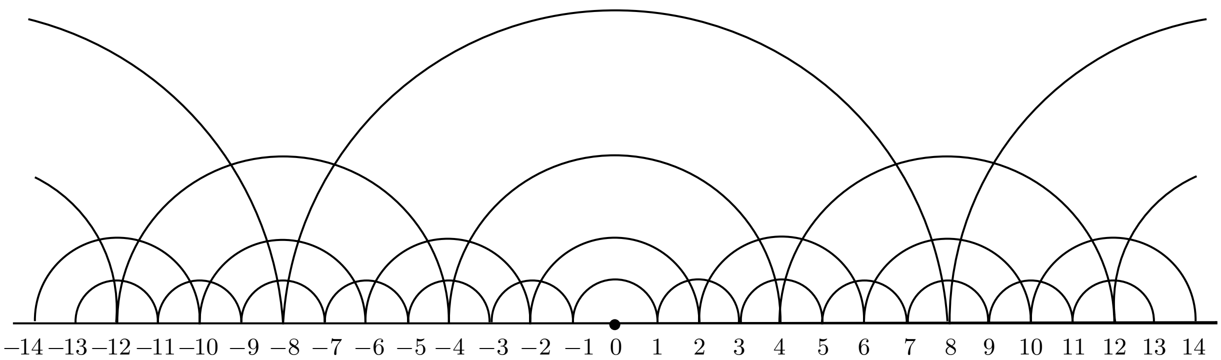

Fig. 7.14. Schreier graph of intermediate growth.

an upper bound of the form $n^{(\log n)^m}$, where m is a positive constant; for many of these automata, the growth is indeed intermediate between polynomial and exponential. For exponentially growing automata, the complexity of finite Schreier graphs can be still higher, even if the group generated by such an automaton is isomorphic to a well-known group such as a free group. However, one should distinguish between the complexity of graphs in the chains $\{\Gamma_n\}$ of finite graphs and the complexity of infinite Schreier graphs Γ_ξ, $\xi \in \partial T$. For example, in the case of a free group, infinite Schreier graphs are typically Cayley graphs (i.e., regular trees), whereas their finite "relatives" Γ_n have a rather complicated structure that cannot be described recursively in the spirit of substitution rules. These questions will be considered in more detail in Section 10.

Example 7.4. We conclude the series of examples of Schreier graphs with a graph generated by one of the simplest automata of linear growth. The automaton is shown in Fig. 7.13.

This graph was studied by Benjamini and Hoffman [26], as well as by Bondarenko, Nekrashevych, and Ceccherini-Silberstein. An idea of the infinite Schreier graph Γ corresponding to the sequence 0^∞ is given by Fig. 7.14.

The formal description of the representation of the graph Γ on the plane is as follows. As the vertex set, we use integer numbers. Define the set of edges by the relation $E = \bigcup_{k\ge 0} E_k$, where

$$E_0 = \{(i, i+1) \colon i \in \mathbb{Z}\} \qquad \text{and} \qquad E_k = \bigl\{\bigl(2^k(n-1/2), 2^k(n+1/2)\bigr)\bigr\}$$

for all $n \in \mathbb{Z}$ and $k > 0$. The main feature of the graph is that it has intermediate growth between polynomial and exponential (here we mean the growth of the number of vertices of the graph that are situated at distance n from a certain distinguished vertex, say, the vertex represented by number 0 in our case). Moreover, as shown in [26], the growth of the graph is asymptotically equal to $n^{\log_4 n}$. Another interesting feature of this graph is that, according to the terminology of [26], this graph is ω-periodic; i.e., it is the inductive limit of a sequence of periodic graphs with vertex sets in $\mathbb{Z}$.

There also exist other exotic types of behavior of the growth of infinite Schreier graphs associated with self-similar groups. For example, as pointed out in [91], infinite Schreier graphs associated with the Hanoi Towers group $\mathcal{H}^k$, $k \ge 4$, have intermediate growth of degree $2^{(\log n)^{k-2}}$.

In this connection, recall that, as shown in [72], there exist Cayley graphs of intermediate growth of finitely generated groups. In spite of extensive results on the class of intermediate growth groups, there did not exist until recently any example of calculating the exact growth asymptotics for a Cayley graph of intermediate growth. For example, even for $\mathcal{G}$, it is still unknown whether the growth is asymptotically equal to 2^{n^α} for some $0 < \alpha < 1$. However, using the group $\mathcal{G}$, Bartholdi and Erschler constructed an infinite sequence of groups with growth of the form 2^{n^α}, and the exponent α for this sequence is accumulated around 1 (remaining less than 1) [15].

The problem of determining for what automata the sequence of finite graphs $\{\Gamma_n\}$ is substitutional in one or another sense is far from being solved. All the available examples are related to contracting self-similar groups. For graphs and limit spaces defined by bounded automata, substitution-type recursions were studied by Nekrashevych [142, 144] and Bondarenko [31]. Perhaps, the existence of a substitution rule depends on the existence of a finite L-presentation of a group by generators and relations (i.e., on whether the set of defining words of a group can be obtained from a finite set of words by iteration with the use of a substitution) [19, 81].

We conclude this section with another definition of a variant of a recurrent sequence of graphs. To this end, we need the concept of acceptor automaton (language recognizer), which is widely used in informatics. Omitting a formal definition (see [104] for details), we only note that such an automaton (denote it by $\mathcal{A}_{\mathrm{acc}}$) can be described by a finite oriented graph whose vertices are called the states of the automaton and edges are colored with symbols of a finite alphabet X. From each state, there emanate $|X|$ edges colored with the symbols of the alphabet, and each symbol is encountered once as a label among the edges emanating from a given vertex. There exist an initial state q_0 and a nonempty set F of final states. The automaton $\mathcal{A}_{\mathrm{acc}}$ defines (recognizes) the language $\mathcal{L}$ (i.e., a subset of the set of words over the finite alphabet X) consisting of words that are read along the paths in the graph of the automaton that start at the initial state q_0 and end at a state belonging to the set F. A language is said to be *regular* if it is recognizable by a finite automaton of the type indicated above. There are various generalizations of the concept of regular languages. For example, Chomsky's classical hierarchy of formal languages consists of classes of regular, context-free, context-sensitive languages, and languages defined by grammars without restrictions. Alternatively, these languages are defined by finite, stack, linear bounded automata, and by the Turing machines, respectively [104].

Definition 7.3. Let $\{\Gamma_n\}$ be a sequence of finite graphs. This sequence is said to be recurrent in the broad sense of the word if there exist finite alphabets X and Y and automata $\mathcal{A}_{\mathrm{acc}}$ and $\mathcal{B}_{\mathrm{acc}}$ over X and Y, respectively, such that there exist bijections between the sets of vertices and edges of the graph Γ_n and the sets of words of length n recognizable by the automata $\mathcal{A}_{\mathrm{acc}}$ and $\mathcal{B}_{\mathrm{acc}}$, respectively.

The type of the automaton under consideration, i.e., finite automaton, stack automaton, etc., determines the relevant class of the graphs. It seems that such an approach to the definition of a recursively defined sequence of graphs has not yet been considered. The definition above is in a sense analogous to the definition of an *automaton group*, or a group with automaton structure, which is widely used in geometric group theory.

8. ACTIONS ON THE SPACE OF SUBGROUPS AND SCHREIER DYNAMICAL SYSTEMS

With every topological dynamical system (G, X) or metric dynamical system (G, X, μ), where G is a finitely generated group with a system of generators $A = \{a_1, \dots, a_m\}$, one can associate a subset $\mathcal{S} \subset \mathcal{X}_{2m}^{\mathrm{Sch}}(G)$ (we call it a pencil of Schreier graphs) consisting of pairs (Γ_x, x), $x \in X$, where Γ_x is the orbital graph constructed on the orbit of the point x with the use of the system of generators A. Then one can try to determine the properties of the dynamical system based on the information about $\mathcal{S}$ or, conversely, study the properties of the pencil $\mathcal{S}$ of graphs based on the information about the properties of the dynamical system. As will be shown in an example below, it may turn out that the dynamical system is reconstructed from a single representative of the pencil. Also, assigning to every point $x \in X$ its stabilizer, we obtain a map from X into the space $\mathcal{Y}(G)$ of subgroups of the group G; in this case, the action of G on X induces the action of G by conjugations on $\mathcal{Y}(G)$ (the adjoint action). For actions that are far from free (in one or another sense), information on the original action may be encoded in the new actions (more precisely, in the dynamical systems with phase spaces lying in $\mathcal{X}_{2m}^{\mathrm{Sch}}(G)$ or $\mathcal{Y}(G)$). Let us explain this idea in greater detail.

The topology on the space $\mathcal{X}_{2m}^{\mathrm{Sch}}(G)$ has already been defined. The topology on $\mathcal{Y}(G)$ is induced by the Tikhonov topology of the space $\{0,1\}^G$ (as before, we assume that G is countable) if each subgroup $H \le G$ is identified with its characteristic function $\omega_H\colon\ \omega_H(g) = 1 \Leftrightarrow g \in H$. For a given finite subset $F \subset G$, a neighborhood U_F^H of a subgroup H in this topology is the set of subgroups $L \le G$ such that $H \cap F = L \cap F$, and neighborhoods of this kind (when F runs through the set of finite subsets of G and H runs through the set of subgroups of the group G) generate the topology on $\mathcal{Y}(G)$. Note that the sets U_F^H are at the same time closed. This topology is a particular case of the Chabauty topology defined on the space of closed subgroups of a locally compact topological group [42]. For a countable group, this topology is metrizable and the space $\mathcal{Y}(G)$ is compact and totally disconnected. As is well known and was already pointed out, such spaces are characterized by the Cantor–Bendixson rank. For the free group F_n, $n \ge 2$, the rank is obviously equal to zero and the space $\mathcal{Y}(G)$ is homeomorphic to the Cantor perfect set. At the same time, for the Tarski monsters $\mathcal{Y}(G)$ constructed by Ol'shanskii [148], which are simple p-groups (p is a large number), the space $\mathcal{Y}(G)$ consists of a countable number of isolated points that accumulate to a point corresponding to the trivial subgroup. This result follows from the fact that any proper subgroup in $\mathcal{Y}(G)$ is cyclic of order p.

Remark 8.1. If we consider the lattice of subgroups of a discrete group with operations $\wedge$ and $\vee$ defined as the intersection of two subgroups and their group union (i.e., $H \vee K = \langle H, K\rangle$ is the subgroup generated by H and K), respectively, then the first operation is obviously continuous, whereas the second is not generally continuous, as shown by the following simple example proposed by Ya. Vorobets.

In the group $\mathbb{Z}$, both sequences $n\mathbb{Z}$ and $(n+1)\mathbb{Z}$, $n = 1, 2, \dots$, converge to the trivial subgroup, but $n\mathbb{Z} \vee (n+1)\mathbb{Z} = \mathbb{Z}$. Thus, the topology introduced is a topology on the set of subgroups of the group G rather than on the lattice of subgroups of G (when one speaks of a topology on a lattice, it is implied that both operations are continuous).

Problem 8.1. (a) What is the range of values of the Cantor–Bendixson rank of the space $\mathcal{Y}(G)$ (G runs through the set of countable groups)?

(b) The same question as above but for finitely generated groups, or even for 2-generated groups.

Let (G, X) be a topological dynamical system and $\alpha\colon X \to \mathcal{Y}(G)$ be the map that assigns the stabilizer $\mathrm{st}_G(x)$ to every point x. This map is measurable (with respect to the Borel structures of the spaces X and $\mathcal{Y}(G)$). Indeed,

$$\alpha^{-1}(U_F^H) = \bigl\{x \in X\colon f(x) = x\ \forall f \in F_0,\ g(x) \ne x\ \forall g \in F \setminus F_0,\ F_0 = F \cap H\bigr\},$$

and this set is obviously measurable. However, the map α may not be continuous, as shown in the example of the group $\mathcal{G}$ considered in Theorem 8.1 below.

Suppose, in addition, that the group G is finitely generated and $A = \{a_1, a_2, \dots, a_m\}$ is an ordered set of its generators. Define a map $\beta\colon X \to \mathcal{X}_{2m}^{\mathrm{Sch}}(G)$ by assigning a rooted graph (Γ, v_x) to a point $x \in X$, where $\Gamma = \Gamma(G, \mathrm{st}_G(x), A)$ is the corresponding Schreier graph and the vertex $v_x = \mathrm{st}_G(x)$ plays the role of the root.

Let us also construct a map $\gamma\colon \mathcal{Y}(G) \to \mathcal{X}^{\mathrm{Sch}}_{2m}(G)$ by assigning the rooted Schreier graph $(\Gamma(G,H,A),H)$ to each subgroup $H \le G$. It is easily seen that this map is continuous and has an inverse $\delta\colon \mathcal{X}^{\mathrm{Sch}}_{2m}(G) \to \mathcal{Y}(G)$, which maps a rooted graph $(\Gamma(G,H,A),H)$ to a subgroup in G consisting of elements defined by the words in the alphabet of generators A and their inverses A^{-1} that are read along closed paths in the graph $\Gamma(G,H,A)$ with the beginning and end at the root (represented by the coset H of the identity element), and δ is also continuous. We stress that γ and δ depend on the system of generators.

The action of G on X naturally extends to the spaces $\mathcal{Y}(G)$ and $\mathcal{X}^{\mathrm{Sch}}_{2m}(G)$. Indeed, for the transition from a point x to a point gx, the corresponding transition from the stabilizer $\mathrm{st}_G(x)$ to the stabilizer $\mathrm{st}_G(gx)$ obviously yields the conjugate subgroup $g^{-1}\mathrm{st}_G(x)g$. Therefore, we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \mathcal{Y}(G) \arrow[d, "\gamma"] \\ & \mathcal{X}^{\mathrm{Sch}}_{2m}(G) \end{tikzcd}$$

in which all maps are equivariant (with respect to the actions of the group), all the actions of the group G are topological (i.e., are implemented by homeomorphisms), and the vertical arrow defines an isomorphism of topological dynamical systems.

Now suppose that a metric dynamical system (G,X,μ) is given with either a finite or an infinite, invariant or quasi-invariant measure μ. Then the previous commutative diagram can be converted into the commutative diagram

$$\begin{tikzcd} (G,X,\mu) \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & (G,\mathcal{Y}(G),\zeta) \arrow[d, "\gamma"] \\ & (G,\mathcal{X}^{\mathrm{Sch}}_{2m}(G),\eta) \end{tikzcd}$$

in which the measures ζ and η are defined by the relations $\zeta = \alpha_*\mu$ and $\eta = \beta_*\mu$. If the map α (respectively, β) is injective modulo a measure-zero set, then β (respectively, α) is also injective modulo a measure-zero set; therefore, there arises an isomorphism between the dynamical system (G,X,μ) and its image under the map α or β by the Lusin–Suslin theorem on Borel monomorphisms. Thus, totally nonfree dynamical systems of a countable group G can be modeled by dynamical systems whose phase spaces are subspaces of the space $\mathcal{Y}(G)$, or subspaces of the space $\mathcal{X}^{\mathrm{Sch}}_{2m}(G)$ if the group is generated by a set of m elements. In this case, the action is adjoint (i.e., by conjugations). The measures ζ and η are invariant or quasi-invariant, depending on whether the measure μ is invariant or quasi-invariant.

If the system (G,X) is topologically transitive, then the image system is also topologically transitive. Corollary 8.9 and Proposition 8.11 proved below show that for minimal actions the trajectory of a typical Schreier graph in the image $\beta(X) \subset \mathcal{X}^{\mathrm{Sch}}_{2m}(G)$ is dense; therefore, the β-image of the system (G,X) is at least topologically transitive.

If the system (G,X,μ) is ergodic, then its image is also ergodic. In this connection Vershik in [184] posed the problem of describing the set of ergodic continuous invariant measures for the adjoint action. By the Bogolyubov–Day theorem [30, 52] (a generalization of the Bogolyubov–Krylov theorem), this set is nonempty at least for the actions of *amenable* groups. Amenable groups and amenable actions are discussed a little at the end of this section and also appear in a number of problems discussed in the subsequent sections.

We also believe that it would be interesting to describe invariant measures on the closure of the orbit of a subgroup H of a group G for important examples of pairs (G,H). Namely, the problem

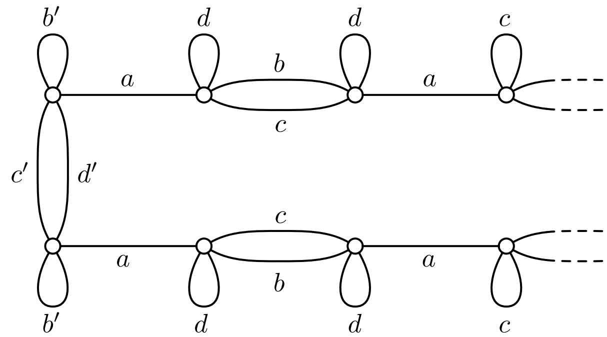

Fig. 8.1. The graph used for defining the graphs Δ_1, Δ_2, and Δ_3.

is formulated as follows. Let $H \le G$, $C(H) = \{g^{-1}Hg\colon g \in G\}$ be the conjugacy class of the subgroup H, and $\overline{C(H)}$ be its closure in $\mathcal{Y}(G)$.

Problem 8.2. What conditions on the system (G,X,μ) guarantee that for a typical point $x \in X$ the system (G,X,μ) is isomorphic to the system $(G,\overline{\mathrm{st}_G(x)},\overline{\zeta})$ for some measure $\overline{\zeta}$ concentrated on the closure $\overline{\mathrm{st}_G(x)}$?

In fact, the question is in what cases the set $\overline{\mathrm{st}_G(x)}$ serves as a support of the measure ζ, and the problem is aimed at finding out in what cases the action is reconstructed from the action on one typical orbit.

According to Corollary 4.9, the set of G-typical points for a topological system (G,X) is comeager; therefore, it seems likely that for extremely nonfree actions the action $(G,\overline{C(\mathrm{st}_G(x))})$ on the closure of the orbit of the stabilizer of a G-typical point should "almost" completely reconstruct the system (G,X). For the present, there is no general result on this subject; however, here we present an example—considered in detail by Ya. Vorobets at our request—that confirms the thesis that in many interesting cases the action is completely (in the metric situation) or almost completely (in the topological situation) reconstructed from the action on one orbit.

Example 8.1. Let $\mathcal{G} = \langle a,b,c,d\rangle$ be the group of intermediate growth from Example 2.3, which acts on the boundary ∂T of a binary tree, and $\beta\colon \partial T \to \mathcal{X}_4^{\mathrm{Sch}}(\mathcal{G})$ be the map defined above. Recall that since the generators of the group $\mathcal{G}$ are involutions, the corresponding Schreier graphs, which were described in the previous section, are 4-regular. Let Δ_i, $i=1,2,3$, be the three graphs defined by replacing the labels b', c', and d' in Fig. 8.1 with the labels c, b, d, the labels d, b, c, and the labels b, c, d, respectively.

Theorem 8.1 [188]**.** *The following assertions hold.*

(i) *The map β is injective with respect to the uniform measure on ∂T.*

(ii) *The map β is continuous everywhere except for a countable set of points that belong to the orbit of the point $\xi = 1^\infty$.*

(iii) *The isolated points of the image $\beta(\partial T)$ consist of rooted graphs $(\Gamma_\xi, g\xi)\colon g \in \mathcal{G}$ (i.e., of labelings of the Schreier graph corresponding to the stabilizer of the point ξ that are obtained by choosing an arbitrary vertex as the root).*

(iv) *The closure of the set $\beta(\partial T)$ in the space of rooted Schreier graphs consists of this set and a countable set of points obtained from the graphs Δ_i, $i=1,2,3$, by making them into rooted graphs by choosing an arbitrary vertex as the root.*

One can easily deduce from this theorem that the image ζ of the uniform measure ν on the boundary ∂T is concentrated on the closure of the orbit of an arbitrary graph (Γ_x, x), where $x \in \partial T$ is an arbitrary point that does not belong to the orbit of the point ξ.

Problem 8.3. (a) Under what conditions on the pair (G, H) does there exist an invariant measure for the action of G on $\overline{C(H)}$?

(b) Under what conditions is the system $(G, \overline{C(H)})$ uniquely ergodic (i.e., an invariant measure exists and is unique)?

Note that in the Russian literature the term "strictly ergodic" is used instead of "uniquely ergodic." At the end of Section 10, we will briefly discuss the concept of strong ergodicity. Note also that the answer to the first part of the above-formulated problem is affirmative if there exists an invariant mean for the action of the group G by left multiplication on the space of left cosets G/H (i.e., if the action is amenable in the sense of von Neumann; see the end of this section).

In addition to the questions raised, one may also study various questions related to the topology of sets of the type $\overline{C(H)}$ and their subsets $C(H)$ and to the relative topology of the embedding $C(H) \hookrightarrow \overline{C(H)}$; however, these questions are more peculiar to lattice theory in locally compact groups (in particular, in Lie groups) (see, for example, [158]).

An action (G, X, μ) is obviously nonfree if the stabilizers of points are almost surely pairwise different. As pointed out in Section 2, Vershik suggested calling such actions extremely nonfree actions. This terminology can also be translated to the topological situation by saying that a system (G, X) is extremely nonfree if the stabilizers of points are pairwise different for a comeager set. A narrower (in the case of topological groups) class of actions consists of totally nonfree actions, which are defined, as has already been said, as those actions with invariant measure for which the sigma algebra of sets generated by the sets of fixed points of the elements of the group coincides with the sigma algebra of all measurable sets [184] (however, for countable groups the classes of extremely nonfree and totally nonfree actions coincide). The importance of this concept is demonstrated by Theorem 10 in [184]. In the topological situation, a totally nonfree action can be defined as an action whose algebra of Borel sets coincides with the algebra generated by the (closed) sets of fixed points of the elements.

Returning to the reconstruction of a dynamical system from the action on one orbit, we reproduce in more detail the scheme that we have actually already described. For an arbitrary Schreier graph, this scheme allows one to construct a dynamical system associated with this graph. Let $\Gamma = \Gamma(G, H, A)$ be a Schreier graph of a group G with a system of generators $A = \{a_1, \dots, a_m\}$. In the space $\mathcal{X}_{2m}^{\mathrm{Sch}}$, consider the subset $\mathcal{Z}$ consisting of points of the form (Γ, v), where v runs through the vertex set of the graph Γ. In other words, we do not change the graph, but we change the initial point (the root). The group G acts on this set on the second coordinate of the pair (Γ, v) by changing the initial point but not changing the graph. This action looks like a translation in the graph Γ from the vertex v to a neighboring vertex under the action of an appropriate generator. Obviously, if two pairs (Γ, v) and (Δ, w) are close in the space $\mathcal{X}_{2m}^{\mathrm{Sch}}$, then the pairs (Γ, v') and (Δ, w') are also close, where v' and w' are neighbors of the vertices v and w, respectively, such that the edges connecting these pairs of vertices are colored with a symbol a, $a \in A$. Thus, the action described is continuous. Therefore, the action of G on $\mathcal{Z}$ extends by continuity to the closure $\overline{\mathcal{Z}}$ (which also consists of Schreier graphs of the group G). The topological dynamical system $(G, \overline{\mathcal{Z}})$ is referred to as a Schreier dynamical system associated with the Schreier graph $\Gamma(G, H, A)$ (or simply an associated dynamical system).

If we apply this scheme to a Cayley graph, we obtain nothing interesting, because the space $\overline{\mathcal{Z}}$ consists of a single point (since a change of the root in a Cayley graph leads to a graph isomorphic to the original one). It is also clear that the case of a finite-index subgroup $H \leq G$ is of little interest too, because in this case one obtains a dynamical system with a finite phase space. Therefore, our scheme is of interest for infinite Schreier graphs that have small (or better trivial) automorphism groups. The following proposition, which is similar to a relevant statement in the theory of coverings [134], provides a clue to calculating the automorphism group of a Schreier graph.

Proposition 8.2. *The automorphism group of a Schreier graph* $\Gamma(G,H,A)$ *is isomorphic to the quotient group* $N_G(H)/H$, *where* $N_G(H)$ *is the normalizer of* H *in* G.

Proof. The proof of this proposition is analogous to the proof of Corollary 7.3 in [134], and we omit it. □

The scheme considered is of interest, first of all, when one takes a maximal (or weakly maximal) subgroup of the group G. While the concept of maximal subgroup (i.e., a proper subgroup such that there are no intermediate subgroups between it and the entire group) is well known in group theory, the concept of weakly maximal subgroup is known much less. It seems that for the first time it appeared in Shalev's paper [169].

Definition 8.1. Let G be an infinite group. A subgroup $H \le G$ is said to be *weakly maximal* if it has infinite index and is maximal with respect to this property (i.e., any intermediate subgroup $H \le K \le G$ either coincides with H or has finite index in G).

The following theorem (the first part of which is well known) shows that maximal and weakly maximal subgroups always exist in an infinite finitely generated group. However, while there are a large number of weakly maximal groups (at least every subgroup of infinite index is contained in a weakly maximal subgroup), the number of maximal subgroups in an infinite group may be small, even finite. For example, any maximal subgroup of the group $\mathcal{G}$ has index 2 [152], and there are only seven such subgroups.

Theorem 8.3. *In an infinite finitely generated group, any proper subgroup is contained in a maximal subgroup, and any subgroup of infinite index is contained in a weakly maximal subgroup.*

Proof. Let G be an infinite finitely generated group and $H < G$. Consider a set $\mathcal{S}$, partially ordered by inclusion, that consists of proper subgroups of G containing H. Any chain $\{H_n\}$ in $\mathcal{S}$ has a maximal element M that belongs to $\mathcal{S}$. Indeed, define a maximal element M by the relation $M = \bigcup_n H_n$. If $M = G$, then all generators of the group G belong to H_n for some n because G is finitely generated; thus, $M = H_n$, which is a contradiction. By Zorn's lemma, the set $\mathcal{S}$ has a maximal element, which is a maximal subgroup in G.

Similarly, if H has infinite index, then the partially ordered set $\mathcal{U}$ consisting of infinite-index subgroups of G that contain H possesses the property that every chain $\{H_n\}$ has a maximal element $M = \bigcup_n H_n$. Indeed, if we assume that M has finite index in G, then M is a finitely generated subgroup (since G is finitely generated) and, hence, coincides with H_n for some n. Again, we arrive at a contradiction. □

Of special interest are maximal and weakly maximal subgroups $H < G$ that have a trivial core $\mathcal{K} = \bigcap_{g\in G} H^g$. Maximal subgroups with trivial core are directly related to primitive actions of groups, i.e., actions that do not have invariant equivalence relations different from the trivial ones (in other words, when the entire set is a single equivalence class, or when each point of the set is an equivalence class). The question of whether a group is primitive (i.e., whether it has a faithful primitive action) is one of the cornerstones of group theory, and extensive literature has been devoted to this question. Primitive groups include finitely generated noncommutative free groups, nonelementary hyperbolic (in Gromov's sense) groups and their generalizations, the so-called convergence groups, nonsolvable linear groups, and most of the mapping class groups [66]. At the same time, Pervova [154, 152] proved that the group $\mathcal{G}$, its generalizations $\mathcal{G}_\omega$, and the Gupta–Sidki p-groups have only maximal subgroups of finite index. Therefore, they and many other self-similar groups of branch type are not primitive.

It seems that weakly maximal subgroups should also play an important role in the theory of permutation groups. If a group G acts transitively on a set X and the stabilizer H of some point is a weakly maximal subgroup, then any proper quotient system (G,Y) of the system (G,X) (i.e., Y is a quotient set of the set X and the projection commutes with the action of the group) is finite (i.e., $|Y| < \infty$). Thus, transitive actions with weakly maximal stabilizers play a role similar to the role of just-infinite groups in group theory; therefore, by analogy with the group case, they should be called just-infinite actions. Theorem 8.4 proved below shows that branch groups act in a just-infinite way on the orbits of the boundary points of the tree. We call an action (G, X) residually finite if, for any pair of elements $x, y \in X$, there exists a finite quotient system in which the images of the elements x and y are different. Then, for transitive actions of countable groups, the condition of residual finiteness is equivalent to the fact that the stabilizer H can be represented as the intersection $\bigcap_{n=1}^{\infty} H_n$ of finite-index subgroups; this, in turn, is equivalent to the fact that H is closed in the profinite topology.

For some time, it was an open question whether any maximal subgroup in any finitely generated branch group has finite index. For example, this question was formulated in [19]. However, Bondarenko constructed a counterexample [32]. Nevertheless, taking into account the specific features of this example (namely, the fact that the group in this example acts locally by elements of a finite group that coincides with its derived subgroup), we formulate the following question.

Problem 8.4. Is it true that all maximal subgroups in any finitely generated branch p-group (p is a prime number) have finite index?

In contrast to maximal subgroups, the group $\mathcal{G}$ has an extensive set of weakly maximal subgroups, among which the stabilizers of the boundary points are primarily distinguished. Moreover, the following general fact is valid.

Theorem 8.4 [16, 17]**.** *Let G be a branch group acting on a rooted tree T. For any boundary point $\xi \in \partial T$, the stabilizer $P = \mathrm{st}_G(\xi)$ is a weakly maximal subgroup.*

Proof. Suppose that P is a proper subgroup of a subgroup $H \le G$ and a vertex v belongs to the path $\xi \in \partial T$ and is not a fixed vertex for H, $h(v) = w \ne v$ for some $h \in H$. It is obvious that the rigid stabilizers of all vertices of level n, where $n = |v|$, except for the vertex v, are subgroups of P. Since $hPh^{-1}(w) = w$, it follows that hPh^{-1} contains the rigid stabilizers of all nth-level vertices except for the rigid stabilizer of the vertex w. Thus, H contains the rigid stabilizers of all nth-level vertices and, hence, also contains $\mathrm{rist}_G(n)$, which has finite index in G. $\square$

The next example (Proposition 8.7), which belongs to E. Pervova, shows that the list of weakly maximal subgroups of the group $\mathcal{G}$ is not exhausted by groups of the form $\mathrm{st}_G(\xi)$, $\xi \in \partial T$. However, we first prove a statement that we will need when considering this example and that is useful in itself.

Lemma 8.5. *Let G be a branch just-infinite group acting faithfully on a rooted tree T. Suppose that $H < G$ is a subgroup of finite index that acts spherically transitively on T. Then H is just-infinite.*

Proof. Let $x \in H$ be an element different from the identity and $N = \langle x \rangle^H$ be its normal closure. To prove the lemma, it suffices to prove that N has finite index in G. It is proved in [80, Theorem 4] that every nontrivial normal subgroup of a branch group contains $(\mathrm{rist}_G(n))'$ for some n. There exists a vertex v such that x fixes v but acts nontrivially on the set of vertices situated under v. We may assume that the level k to which v belongs is greater than n. Indeed, if this is not so, then we apply the arguments used in the proof of Theorem 4 in [80]. Namely, we replace the element x by an appropriate (see [80]) element of the commutator $[N, \mathrm{rist}_G(w)]$, where the vertex w is situated one level below v and is such that x acts on it nontrivially; thus we obtain a new nonidentity element in N that belongs to a lower level of the tree than v. Repeating this procedure several times, we obtain a nonidentity element in N that belongs to a level $> n$.

Then, applying again the procedure of taking the commutator of N with $\mathrm{rist}_G(w)$, similar to the way it was done in the concluding part of the proof of Theorem 4 in [80], we find that N contains the derived subgroup $(\mathrm{rist}_G(u))'$ (which is a nontrivial group), where the vertex u situated immediately under w (i.e., one level below) is such that at least one element of N acts on it nontrivially. Taking into account that H acts transitively on the levels and conjugating $(\mathrm{rist}_G(u))'$ by the elements of H, we find that N contains $L = (\mathrm{rist}_G(|w|+1))'$. The group L is normal in G and, hence, also in H. It follows from [80, Theorem 4] that the quotient group of a branch group by a nontrivial normal subgroup is virtually abelian and, hence, is finite in our case in view of the just-infinite property. Hence, N has finite index in H. $\square$

Example 8.2. Let L be the normal closure of the element b in $\mathcal{G}$, K be the normal closure of the element $(ab)^2$, and $\widehat{L} = \langle (ad)^2, L\rangle$. Note that $L \le \mathrm{st}_{\mathcal{G}}(1)$ and that $\mathcal{G}$ is a regularly branch group over K (in the sense of Definition 3.5) [80]. For any subgroup X in $\mathcal{G}$ and for an arbitrary vertex u, define X_u as $\mathrm{st}_X(u)|_u$. Since the group $\langle ad, L\rangle$ acts spherically transitively on the tree and since $\mathcal{G}$ is a branch just-infinite group, it follows from the above lemma that this group is just-infinite.

Lemma 8.6. *Let $x \in \widehat{L}$. Then the normal closure $x^{\widehat{L}}$ has infinite index in $\widehat{L}$ if and only if $x \in \mathrm{rist}_{\widehat{L}}(u)$ for some first-level vertex u.*

Proof. Since $\widehat{L} \le \mathrm{st}_{\mathcal{G}}(1)$, the implication in one direction is obvious. Suppose that x does not belong to $\mathrm{rist}_{\widehat{L}}(u)$ for any first-level vertex, i.e., both projections x_0 and x_1 in $x = (x_0, x_1)$ are nonidentity elements. Then there exist elements $y_0, y_1 \in K$ such that $[x_i, y_i] \ne 1$ for $i = 0, 1$. Let Y_i be the normal closure of the element $[x_i, y_i]$ in $\mathcal{G}$. Then $x^{\widehat{L}} \ge (Y_0 \times Y_1)_1$,[2] and the latter group has finite index in $\widehat{L}$, which was to be proved. $\square$

Proposition 8.7. *Let $W = \langle a, \mathrm{diag}(\widehat{L} \times \widehat{L})_1, (1 \times K \times 1 \times K)_2\rangle$. Then W is a weakly branch group in $\mathcal{G}$.*

Proof. It is obvious that W has finite index in $\mathcal{G}$ (note that all elements in $(K \times 1 \times 1 \times 1)_2$ belong to different cosets in $\mathcal{G}/W$). Let $x \in \mathcal{G} \setminus W$; we have to prove that $\widehat{W} = \langle x, W\rangle$ has finite index in $\mathcal{G}$. Obviously, we can assume that $x \in \mathrm{st}_{\mathcal{G}}(1)$, so that $x = (x_0, x_1)$. Moreover, we can assume that $x_0 \in \{1, a, ad\}$. Consider three cases.

Case 1. Suppose that $x_0 = 1$. Then $x_1 \in L \setminus \{1\}$ and

$$\mathrm{rist}_{\widehat{W}}(1) \ge (x_1^{\widehat{L}} \times x_1^{\widehat{L}})_1.$$

Let $u_0 = (0)$ and $u_1 = (1)$ be the two vertices of the first level. Note that x_1 does not belong to $\mathrm{rist}_{\mathcal{G}}(u_1)$ in view of the conditions imposed on the group W and element x. If the element x_1 does not belong to $\mathrm{rist}_{\mathcal{G}}(u_0)$, then $\widehat{W}$ has finite index by Lemma 8.6. Suppose the contrary. Then the decomposition $x = (1, 1, y, 1)$ is valid for some nontrivial $y \in K$. Note that $\widehat{L}_u = \mathcal{G}$ for an arbitrary nonroot vertex u. Let Y be the normal closure of the element y in $\mathcal{G}$. Then $\widehat{W} \ge (Y \times K \times Y \times K)_2$ and, hence, $\widehat{W}$ has finite index in $\mathcal{G}$.

Case 2. Suppose that $x_0 = a$. Then $x_1 \in dL$, and the equality $\widehat{W_u} = \mathcal{G}$ holds for an arbitrary nonroot vertex u. Let $u = (0)$ be a first-level vertex, and let $X = \mathrm{rist}_{\widehat{W}}(1)|_u$. Then X is normal in $\mathcal{G}$; therefore, it is either trivial or of finite index in $\mathcal{G}$. In the latter case, $\widehat{W}$ also has finite index in $\mathcal{G}$. However, since $\widehat{W} \ge (1 \times K \times 1 \times 1)_2$, the first case is impossible, which implies the conclusion for this case.

Case 3. Let $x_0 = ad$. Then $x_1 \in daL$, and we have $\widehat{W_u} = \langle ad, L\rangle$ for an arbitrary first-level vertex u. As before, let $u = (0)$ denote a first-level vertex and $X = \mathrm{rist}_{\widehat{W}}(1)|_u$. Then X is normal in $\langle ad, L\rangle$. Since $\langle ad, L\rangle$ is just-infinite, it follows that X is either trivial or of finite index in $\mathcal{G}$. In the latter case, the conclusion follows immediately. However, since $\widehat{W} \ge (1 \times K \times 1 \times 1)_2$, the first case is impossible, which completes the proof of the proposition. $\square$

[2] Here and below, the index 1 (or 2) denotes the level of the tree on the vertices of which the subgroup under consideration is represented as the product of its projections (this corresponds to relations (2.1) and (2.2)). Note that in the arguments below, when considering group inclusions denoted by the sign $\le$, we will omit the signs of embeddings Ψ and Ψ_2 defined in (3.4) and (3.5).

Although $\mathcal{G}$ has an extensive set of weakly maximal subgroups, the following problem does not seem hopeless.

Problem 8.5. Describe all weakly maximal subgroups in $\mathcal{G}$.

If this is done, then all just-infinite actions of the group $\mathcal{G}$ will be described, and this will be the first nontrivial example of this kind. The problem of describing weakly maximal subgroups in other self-similar branch groups is on the agenda as well.

Definition 8.2. Two colored graphs Γ_1 and Γ_2 are said to be *locally isomorphic* if, for an arbitrary $r \in \mathbb{N}$ and an arbitrary vertex u of one of the graphs, there exists a vertex v of the other graph such that the subgraphs in Γ_1 and Γ_2 representing neighborhoods of radius r with centers at the points u and v, respectively (we call them graph neighborhoods), are isomorphic as colored rooted graphs.

Recall that Γ_ξ denotes the Schreier graph associated with a point ξ; the vertex set of Γ_ξ is the orbit of ξ. The following two statements are obvious generalizations of statements formulated in [87, Propositions 6.21 and 6.22].

Proposition 8.8. *Let a group G act minimally on a topological space X. Then, for any positive integer r, any G-typical point $\omega \in X$, and an arbitrary point $\eta \in X$, there exists a vertex v of Γ_η such that the graph neighborhoods of radius r with centers at the vertices ω and v in the graphs Γ_ω and Γ_η are isomorphic.*

Proof. Let $B_\omega(r)$ be the subgraph representing the neighborhood of radius r in the graph Γ_ω with a distinguished vertex ω (the subgraph $B_\omega(r)$ includes all vertices of the graph Γ_ω that lie at a combinatorial distance $\le r$ from the vertex ω and all edges incident to these and only to these vertices in Γ_ω, including loops). Suppose that u is a vertex of $B_\omega(r)$ such that $a(u) \ne u$, $a \in A$, and $a(u)$ also belongs to $B_\omega(r)$. Thus, there is an edge in $B_\omega(r)$ that is labeled by the symbol a and connects the vertices u and $a(u)$. Since the action is continuous, the point u has a neighborhood U_u in X such that $a(w) \ne w$ for all points $w \in U_u$. Let us construct neighborhoods of this kind for all vertices of the graph $B_\omega(r)$ that are shifted by the generator a; we choose these neighborhoods so small that they are pairwise disjoint. Now, let η be an arbitrary boundary point. Since the action is minimal, the orbit of this point is dense in X. In the orbit of the point η, we find a point v that is so close to ω that the relation $\theta = W(\omega)$ implies $W(v) \in U_\theta$ for an arbitrary word W of length $\le r$ over the alphabet A. Thus, the W-images of the point v are close to the corresponding W-images of the point ω (we will call such pairs partners). Connecting a-neighboring (but different) W-images (for the words W of length $\le r$) of the point v by edges, labeling them by a, and applying this procedure to every symbol $a \in A$, we obtain a graph $\Delta_v(r)$ with the same number of vertices as $B_\omega(r)$. The graph $\Delta_v(r)$ may differ from the graph $B_\omega(r)$ only by loops, if they are present in $B_\omega(r)$. Note that up to now we have not used the fact that ω is a G-typical point.

Now, suppose that the vertex u of the graph $B_\omega(r)$ is fixed under the action of the generator a. Then the point u has a neighborhood V_u in X such that the element a acts trivially in this neighborhood. Taking the point v sufficiently close to ω and using the fact that ω is a G-typical point, we can make it so that every vertex of the graph $\Delta_v(r)$ for which its partner vertex in the graph $B_\omega(r)$ is a-fixed for some $a \in A$ is also a-fixed. Supplementing the graph $\Delta_v(r)$ with the corresponding loops, we obtain a graph isomorphic to $B_\omega(r)$. $\square$

Corollary 8.9. *If the action is minimal, then the Schreier graphs of G-typical points are locally isomorphic. In particular, this is true for spherically transitive actions on rooted trees. If X is a metric compact set, then the Schreier graphs Γ_ω are pairwise locally isomorphic for values of the parameter ω that fill a comeager set.*

We stress that in this corollary, just as in the above definition of local isomorphism of graphs, we deal with colored graphs. The concept of local isomorphism is defined in an arbitrary category of graphs (either colored or not, oriented or not, etc.). It is clear that a local isomorphism of colored graphs implies a local isomorphism of the graphs obtained by sweeping the colors away, but not vice versa.

Corollary 8.10. *Suppose that $(G,\overline{\mathcal{Z}})$ is a Schreier dynamical system*[3] *defined by a minimal action of a group G on a topological space X. Then this system is topologically transitive, and the orbit of each point (Γ_ω,ω), where ω is a G-typical point for the system (G,X), is dense in $(G,\overline{\mathcal{Z}})$.*

Now we prove a metric variant of Proposition 8.8. The proof will follow the same strategy as the proof of Proposition 8.8.

Proposition 8.11. *Let (G,X,μ) be an ergodic dynamical system with an invariant (not necessarily finite) measure μ and a finitely generated group G. Then, for almost every $x\in X$, the graphs Γ_x are pairwise locally isomorphic.*

Proof. We have to prove that there exists a measurable subset $X_*\subset X$ such that $\mu(X\setminus X_*)=0$ and the graphs Γ_α and Γ_β are locally isomorphic for any $\alpha,\beta\in X_*$. We will say that a subset $X'\subset X$ has full measure if its complement has measure zero. Denote by $B_\omega(r)$ a part of the graph Γ_ω, $\omega\in X$, that represents the neighborhood of radius r of the point ω in the graph Γ_ω (i.e., $B_\omega(r)$ includes vertices lying in the graph Γ_ω at a distance of at most r from ω, and a pair of such points is connected by an edge if it is connected by an edge in Γ_ω). A graph Δ with a distinguished vertex is said to be admissible of radius r if it is isomorphic to the graph $B_\omega(r)$ for some point ω. For an admissible graph Δ, denote by $X_\Delta(r)$ the set of points $\omega\in X$ such that $B_\omega(r)$ is isomorphic to Δ (as a rooted graph). The set $X_\Delta(r)$ is measurable because it consists of the points $x\in X$ that satisfy the inequalities $gx\ne hx$ and the equalities $sx=tx$, where the pairs (g,h) and (s,t) run through certain finite sets R_Δ and S_Δ defined by the structure of the graph Δ and composed of elements satisfying the inequalities $|g|,|h|,|s|,|t|\le r$. The space X is covered by the sets $X_\Delta(r)$, and since there are only a finite number of these sets, there exists at least one set of positive measure among them. We say that a graph Δ is positively admissible if the measure of the set $X_\Delta(r)$ is positive. Let D_r be the set of positively admissible graphs of radius r. For $\Delta\in D_r$, define

$$\overline{X}_\Delta=\bigcup_{g\in G} g(X_\Delta(r)).$$

Then $\overline{X}_\Delta$ is an invariant subset of positive measure. Since the action is ergodic, this set has full measure. Finally, introduce the set

$$X_*=\bigcap_{r\ge 1}\bigcap_{\Delta\in D_r}\overline{X}_\Delta,$$

which is also invariant and has full measure. We argued that for any pair $\omega,\eta\in X_*$ the graphs Γ_ω and Γ_η are locally isomorphic. Indeed, let $\xi\in G(\omega)$ and $B_\xi(r)\simeq\Delta\in D_r$. Since $\eta\in\overline{X}_\Delta(r)$, it follows that there exists a point $\zeta\in G(\eta)$ such that $B_\zeta(r)\simeq\Delta$. $\square$

Consider another example of a Schreier dynamical system that was analyzed in detail by my student D. Savchuk. This example, related to the famous Thompson group, shows that a Schreier dynamical system may also reconstruct the original action in the case of an action with a quasi-invariant measure.

Definition 8.3. The Richard Thompson group F is the group consisting of all increasing piecewise linear homeomorphisms of the closed interval $[0,1]$ that are differentiable at all points of the interval except for a finite set of dyadic rationals and whose derivatives on the intervals of

[3]We keep the notation used in the definition of a Schreier dynamical system on p. 124.

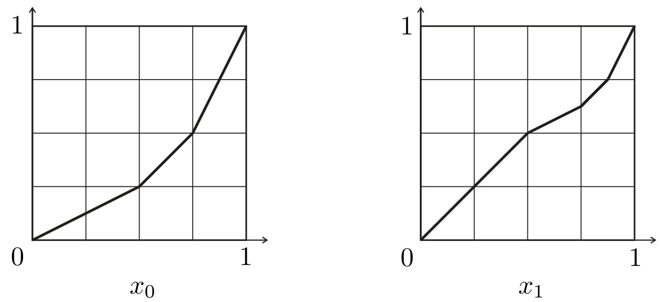

Fig. 8.2. Generators x_0 and x_1 of the Thompson group F.

differentiability are equal to integer powers of two (i.e., have the form 2^n, $n \in \mathbb{Z}$). The group operation is the composition.

In addition to the group F, in the mid-1960s Thompson defined and examined groups V and T that are infinite finitely presented simple groups (these were the first examples of this kind). The main facts regarding these groups and their generalizations constructed by Higman [101] can be found in survey [38]. Although F is not a simple group (note that it is nevertheless close to a simple group; namely, any homomorphism with nontrivial kernel is factored through the abelianization of F, which is equal to $\mathbb{Z}^2$), the group F has another important property: it has no subgroups isomorphic to the free group of rank 2. The group F is a finitely presented group with two generators and two relations, and no nontrivial identities are satisfied in it [37, 1]. The homeomorphisms x_0 and x_1 that generate the group F are defined as follows:

$$x_0(t) = \begin{cases} \dfrac{t}{2}, & 0 \le t \le \dfrac{1}{2}, \\ t - \dfrac{1}{4}, & \dfrac{1}{2} \le t \le \dfrac{3}{4}, \\ 2t - 1, & \dfrac{3}{4} \le t \le 1, \end{cases} \qquad x_1(t) = \begin{cases} t, & 0 \le t \le \dfrac{1}{2}, \\ \dfrac{t}{2} + \dfrac{1}{4}, & \dfrac{1}{2} \le t \le \dfrac{3}{4}, \\ t - \dfrac{1}{8}, & \dfrac{3}{4} \le t \le \dfrac{7}{8}, \\ 2t - 1, & \dfrac{7}{8} \le t \le 1. \end{cases}$$

The graphs of these generators are shown in Fig. 8.2.

Every dyadic irrational point in the interval $[0, 1]$ can be defined by its binary representation. This allows one to extend naturally the action of the group F to the sets $\{0, 1\}^*$ and $\{0, 1\}^\omega$ of finite and infinite words (or, more precisely, sequences) over the alphabet $\{0, 1\}$. The latter set is homeomorphic to the Cantor set, and F acts on it by homeomorphisms. We call this action the *standard action of F on the Cantor set*. In this case the generators x_0 and x_1 act on the sequences as follows:

$$x_0\colon \begin{cases} 0w \mapsto 00w, \\ 10w \mapsto 01w, \\ 11w \mapsto 1w, \end{cases} \qquad x_1\colon \begin{cases} 0w \mapsto 0w, \\ 10w \mapsto 100w, \\ 110w \mapsto 101w, \\ 111w \mapsto 11w, \end{cases}$$

where w is an arbitrary sequence in $\{0, 1\}^\omega$.

In [165], Savchuk completely described the Schreier graph of the action of the group F on the orbit of the point $\frac{1}{2} \in [0, 1]$, which consists of all dyadic rationals of the interval $(0, 1)$. In terms of the action on the set $\{0, 1\}^\omega$, this corresponds to the orbit of the point $1000\dots$, which consists of all sequences with a finite nonzero number of ones. This Schreier graph is shown in Fig. 8.3, and one can see that it has a treelike structure; i.e., it is rather simple in the sense of its geometry

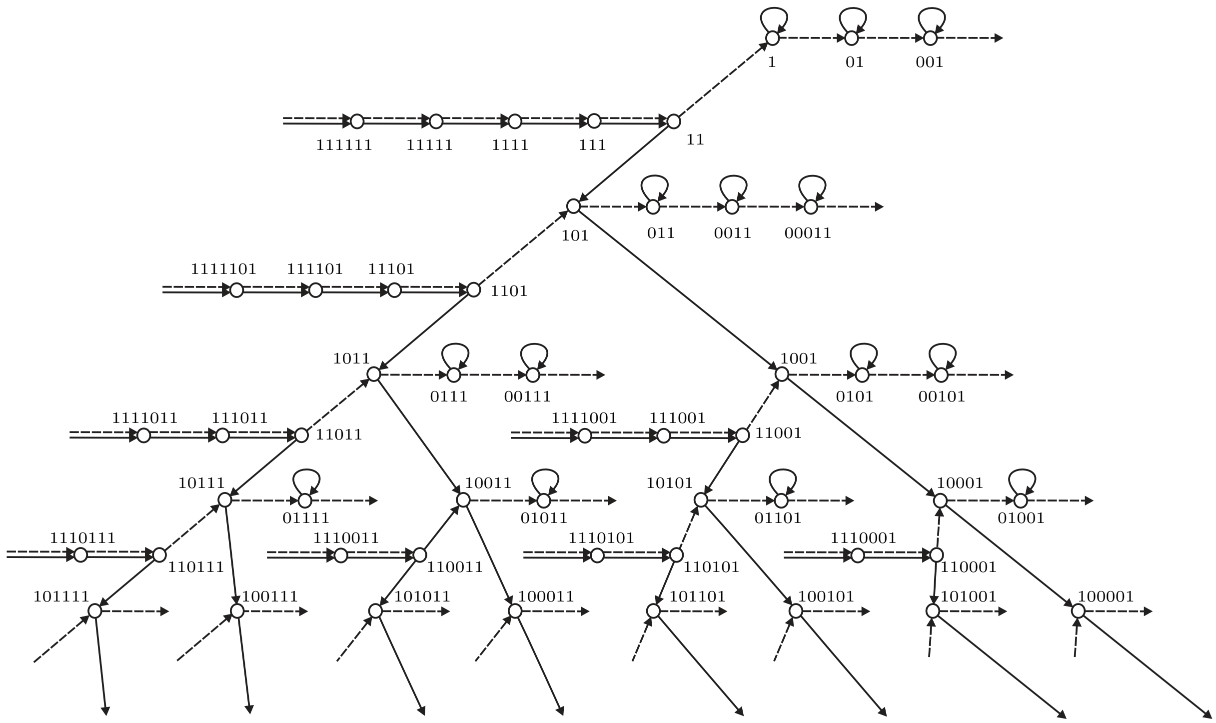

Fig. 8.3. The Schreier graph of the action of the group F on the orbit of the point $1000\ldots$. The dashed edges denote the action of the generator x_0, and the solid edges, the action of x_1.

and combinatorial description. Note that the stabilizer $\mathrm{st}_F\left(\frac{1}{2}\right)$ is a maximal subgroup of F; this fact confirms our thesis that it is the Schreier graphs associated with maximal and weakly maximal subgroups that are of special interest. Since the core of $\mathrm{st}_F\left(\frac{1}{2}\right)$ is trivial (i.e., it does not contain a nontrivial normal subgroup), one can completely reconstruct F from this graph (we have already encountered an analogous situation in the case of the group $\mathcal{G}$).

Let us apply the construction of the dynamical system associated with a Schreier graph. Then we obtain the following statement, which was proved in [166].

Theorem 8.12. *Let $(F, \overline{\mathcal{Z}})$ be a dynamical system associated with the Schreier graph of the Thompson group F.*

(a) *The Cantor–Bendixson rank of the set $\overline{\mathcal{Z}}$ is 1. The perfect kernel $\mathcal{D}$ of the set $\overline{\mathcal{Z}}$ is homeomorphic to the Cantor set.*

(b) *The action of the group F on the set $\mathcal{D}$ is conjugate to the standard action of F on the Cantor set $\{0,1\}^\omega$.*

In other words, after removing a countable set of isolated points from $\overline{\mathcal{Z}}$ and restricting the action to the remaining part, we reproduce the original action.

We complete this section with the definition of some important concepts that will be used in the following sections and have already been used in the present section (when discussing the existence of an invariant measure on the closure of the conjugacy class of a subgroup).

Among the most important concepts of asymptotic group theory that are related to many applied aspects of group theory are the concepts of amenable group and amenable action, which were introduced by von Neumann in 1929. [147]. Independently (and in greater generality for topological, not necessarily locally compact, groups) this concept was discovered by N.N. Bogolyubov [30].

Definition 8.4. A group G is said to be *amenable* if there exists a finitely additive left-invariant measure ν that takes values in the interval $[0,1]$, is defined on the sigma algebra of all subsets of G, and is such that $\nu(G) = 1$.

Definition 8.5. An action (G, X) of a group G on a set X is said to be amenable if there exists a finitely additive G-invariant measure ν that takes values in $[0,1]$, is defined on the sigma algebra of all subsets of X, and is normalized by the condition $\nu(X) = 1$.

Books [69, 189, 39] are good sources for an initial acquaintance with the theory of amenable groups.

Obviously, the amenability of a group is equivalent to the amenability of its action on itself by left (or right) translations. There exist an enormous number of equivalent reformulations of the concept of amenability of a group (perhaps, there is no other concept in mathematics that has as many formulations). Here we restrict ourselves to the formulation of Følner's criterion (also called Følner's condition); in Section 10, we will also mention Kesten's probability criterion.

Theorem 8.13 [60, 41]**.** *An action (G, X) is amenable if and only if, for an arbitrary $\epsilon > 0$ and an arbitrary finite subset $S \subset G$, there exists a finite subset $F \subset X$ such that*

$$|F \triangle gF| \le \epsilon |F| \tag{8.1}$$

for all $g \in S$, where $\triangle$ denotes the symmetric difference of sets.

The set that appears in the formulation of this theorem is called a Følner set. It is easy to understand that if G is a countable group, then Følner's condition that characterizes amenability can be reformulated as the existence of an increasing sequence of Følner sets $\{F_n\}_{n=1}^{\infty}$ that exhausts G and is such that

$$\frac{|F_n \triangle gF_n|}{|F_n|} \to 0$$

for any element $g \in G$.

In the language of Cayley graphs associated with a group or of Schreier graphs associated with the action of a group, Følner's condition means the existence of large subsets of the vertex set with a small boundary compared with the cardinality of the subset itself (here it does not matter how the boundary of a subset of the graph is defined; any reasonable method will do [41]). The class AG of amenable groups is closed with respect to the operations of taking a subgroup, a quotient group, a group extension, and an inductive limit [69]. Finite groups are amenable (which is obvious), as well as all commutative groups are amenable (this is proved by applying the axiom of choice) [69]. Thus, solvable (and nilpotent in particular) groups are amenable, and, moreover, the groups that belong to the class EG of elementary amenable groups—the minimal class of groups that contains finite and commutative groups and is closed with respect to the above-listed operations—are also amenable.

The simplest example of a nonamenable group is the free group F_2 of rank 2. Thus, any group containing a subgroup isomorphic to F_2 is nonamenable. Let NF be the class of groups that do not contain a free subgroup with two generators. The following inclusions hold: $\mathrm{EG} \subset \mathrm{AG} \subset \mathrm{NF}$. In [52], Day raised the question of whether these classes coincide. The fact that $\mathrm{AG} \neq \mathrm{NF}$ was proved in [149]. The fact that $\mathrm{EG} \neq \mathrm{AG}$ was proved in [72]. The latter result follows from the fact that intermediate growth groups are amenable but not elementary amenable [45]. In particular, $\mathcal{G} \in \mathrm{AG} \setminus \mathrm{EG}$. Since there exist uncountably many intermediate growth groups (even with different growth degrees, which, incidentally, implies the existence of an uncountable number of pairwise non-quasi-isometric finitely generated groups) [72], there exist uncountably many essentially different groups that belong to the complement $\mathrm{AG} \setminus \mathrm{EG}$. In [77], we introduced the class SG of subexponentially amenable groups, i.e., the minimal class of groups that contains the groups of subexponential growth (all of which are amenable) and is closed with respect to the four above-listed group operations that preserve amenability. In [77], the question was raised as to whether the classes SG and AG coincide. In [23], applying the results of [94], Bartholdi and Virág proved that the Basilica group defined in Example 2.4 belongs to the complement AG \ SG and thus not only fails to be elementary amenable, but also is a group whose amenability is in no way related to a subexponential growth.

In [20] and [9], it was proved that the groups generated by bounded and linearly growing automata, respectively, are amenable. At the same time, Sidki in [172] and Nekrashevych in [143] proved that the groups generated by polynomially growing automata do not contain a free subgroup with two generators (i.e., they belong to the class NF). The question of their amenability remains open.

At present, all known examples and constructions of amenable but not elementarily amenable groups are in one or another way related to self-similar groups of branch type or to their modifications and schemes based on these groups. It would be interesting to find other constructive types of amenable groups. However, when the work on this paper was almost finished, the author, together with K. Medynets, managed to construct infinite finitely generated simple amenable groups by using minimal homeomorphisms of the Cantor set [85]; moreover, it is proved that there are an uncountable number of pairwise nonisomorphic groups of this form. All these groups are nonelementary amenable, and they are constructed on a completely new basis, which has not been previously applied in group theory. After that paper the central question concerning the construction of new examples of amenable groups is as follows.

Problem 8.6. Do there exist finitely generated hereditary just-infinite nonelementary amenable groups?

Note that finitely generated elementary hereditary just-infinite groups are exhausted by the infinite cyclic and infinite dihedral groups.

To conclude, we formulate a well-known problem.

Problem 8.7. Is the Thompson group F amenable?

This problem was posed by R. Geoghegan as early as the 1970s. In spite of a large number of attempts of many mathematicians to solve this problem, the problem still remains open. Note that the recently published paper [171], which states that F is amenable, is incorrect.

9. REPRESENTATIONS, C^*-ALGEBRAS, AND SELF-SIMILAR TRACE

In this section, we define a certain C^*-algebra associated with a group acting on a rooted tree and construct a self-similar (or recurrent) trace on this algebra in the case when the group is strongly self-similar. We will need this trace and its properties in Section 10, in particular, in order to construct asymptotic expanders. This and other C^*-algebras and recurrent traces on these algebras have been studied in [16, 95, 141, 86]. Below we give some additional information on them.

As already mentioned in the Introduction, with an action of a countable group G on a measure space (X,μ) by measure-preserving transformations, one can associate a number of unitary representations. First, one can study the representation π defined in the Hilbert space $L^2(X,\mu)$ by the relation $\pi_g(f)=f(g^{-1}x)$, $f\in L^2(X,\mu)$ (the Koopman representation). One can also study a pencil of representations ρ_x acting in the spaces $l^2(Gx)$ and indexed by the points $x\in X$, where Gx is the orbit of a point x. The representation ρ_x is defined by left translations $\rho_x(g)f(y)=f(g^{-1}y)$ of functions on the orbit and is isomorphic to the quasiregular representation $\rho_{G/H}$ defined in the space $l^2(G/H)$ by the action $\rho_g(F)(fH)=F(g^{-1}fH)$, $F\in l^2(G/H)$, where $H=\mathrm{st}_G(x)$ is the stabilizer of the point x. In fact, the representation π can also be defined in the case when the measure μ is only quasi-invariant; to define the representations ρ_x, it suffices to have an action of a group on a space X. A unitary representation of a group defines a C^*-algebra generated by the operators of the representation of the group.

One can associate several C^*-algebras with an action of a group. First of all, this is the algebra C_π generated by the unitary representation π. Namely, one extends the representation π by linearity to a representation of the group algebra $\mathbb{C}[G]$ and then takes the closure of this algebra in the operator norm. The second approach consists in extending the quasiregular representation ρ_x to the group algebra and then taking its closure in the norm, which yields a C^*-algebra C^*_x. One can also consider the integral $\int_{x\in X} l^2(Gx)$ of Hilbert spaces, where Gx is the G-orbit of a point x and the group acts by translations in every fiber $l^2(Gx)$, and take the closure of this representation in the norm (again extending it to the group algebra). If X is a topological space, then one can additionally consider the crossed product of the representation π on the commutative C^*-algebra $C(X)$ of continuous functions on X (we refer the reader to [181] for details of this and other constructions). The study of groups on the basis of the analysis of their actions on measure spaces and the analysis of associated C^*-algebras (and von Neumann algebras) falls within the field of measured group theory. Some idea of this theory can be obtained from [181, 62, 113] and other monographs and articles.

From the viewpoint of measured group theory, representation theory, and the theory of operator algebras, the following questions are natural: Are the above-mentioned representations irreducible? What are their properties and the properties of the corresponding algebras and their relation to the asymptotic and geometric properties of a group (for example, amenability, Kazhdan's T-property, Yu's property A [195], etc.)? In particular, it is important to know what properties of the associated C^*-algebras are invariants of a dynamical system. Below (namely, in Section 10), we will show that in some situations the algebras C^*_x, $x\in X$, do not depend on the point x and, moreover, are isomorphic to the algebra C_π. Under some additional assumptions, these algebras are also isomorphic to the reduced algebra $C^*_{\mathrm{r}}(G)$ of the group G (which is the closure in the norm of the left regular representation λ_G, which is a particular case of the quasiregular representation $\lambda_{G/\{1\}}$).

In parallel, we can consider von Neumann algebras associated with an action; however, in this case one should take the weak closure of the corresponding operator algebras instead of the closure in the norm. The construction related to considering the integral of Hilbert spaces in the context of von Neumann algebras is called Krieger's construction (see [181]).

In the case of a group acting on a rooted tree, there are a few more constructions of operator algebras that deserve attention; these constructions were considered in [141, 86] and are based on the use of self-similarities of a Hilbert space (i.e., isomorphisms between the infinite-dimensional Hilbert space and the direct sum of several of its copies). These algebras are defined with the use of the Cuntz algebra (see [51]), which is important for operator algebras, and its generalizations.

Surprisingly, it seems that until recently the algebra C_π was not especially popular in the studies on dynamics. Among the studies in which this algebra plays an important role, we mention the work by Bartholdi and the present author [16], which initiated the research on many problems touched upon in the present paper, as well as the work by Nekrashevych and the present author [86]. Naturally, the Koopman representation appears in many papers and monographs, but, as a rule, without involvement of C^*-algebras associated with it. For example, the Koopman representation is dealt with in Glasner's book [67] and in the paper by Kechris and Tsankov [114]. Note that the question (which, according to Glasner, arose during his discussions with P. de la Harpe) of irreducibility of the Koopman representation (restricted to the orthogonal complement of the space of constant functions; we will say that the Koopman representation is *almost* irreducible in this case) for ergodic actions on spaces with finite measure is discussed in the above-cited book by Glasner (Ch. 5, Section 4), where it is proved that the Koopman representation is almost irreducible for the whole automorphism group $\mathrm{Aut}([0,1],m)$ of the Lebesgue space (this group has a natural topology that turns it into a topological group). In the same book, it is also pointed out that the restriction of the representation of this group to any countable dense subgroup of the group $\mathrm{Aut}([0,1],m)$ is almost irreducible as well. Thus, we can obtain examples of locally finite groups (for example, $S(\infty)$) with irreducible Koopman representation.

In our case, the algebra C^*_π will play an important role. The first fact that we are going to prove is that for groups acting on rooted trees this algebra can be approximated by finite-dimensional algebras. We call such algebras residually finite-dimensional (RFD). The RFD property is equivalent to the fact that the algebra is embedded in a direct product of finite-dimensional matrix algebras (considered as C^*-algebras). It is an analog of the property of residual finiteness for groups, which is inherent in all groups acting faithfully on rooted trees, as already discussed above and is discussed in [25], where the term "direct sum" is erroneously used instead of "direct product." In addition, for our further analysis it is important that there is a special trace on C^*_π. This trace provides a convenient tool for the study of self-similar actions, and we will call it a self-similar trace.

Recall that a normalized trace on a unital C^*-algebra A is a linear functional $\tau\colon A \to \mathbb{C}$ that is positive (i.e., $\tau(x^*x) \geq 0$ for all $x \in A$) and satisfies the relations $\tau(xy) = \tau(yx)$ for arbitrary $x, y \in A$ and $\tau(1) = 1$. A trace is said to be faithful if $\tau(x^*x) \neq 0$ for any $x \in A$, $x \neq 0$. For example, the reduced C^*-algebra $C^*_{\mathrm{r}}(G)$ of a group G has a canonical trace $\tau(a) = \langle a\delta_e, \delta_e\rangle$, where $a \in C^*_{\mathrm{r}}(G)$ and δ_e is a delta function with nonzero value on the identity element e. In this case, $\tau(g) = 0$ if $g \in G$ is a nonidentity element, and $\tau(e) = 1$.

A residually finite-dimensional C^*-algebra always has a faithful trace. Namely, if

$$i\colon\ A \to \prod_{n\geq 1} M_n(\mathbb{C})$$

is an embedding, then the trace

$$\tau\colon\ \prod_{n\geq 1} M_n(\mathbb{C}) \to \mathbb{C}$$

defined by the relation

$$\tau(a_1, a_2, \dots) = \sum_{n\geq 1} 2^{-n}\tau_n(a_n), \tag{9.1}$$

where τ_n stands for the standard normalized trace on matrices of size n, is a faithful trace on the direct sum of matrix algebras, while its restriction to A yields a faithful trace on this subalgebra. Taking another decreasing sequence of positive coefficients with summable series in place of the coefficients 2^{-n} in (9.1) and dividing (9.1) by the sum of the series, we obtain another faithful trace. Thus, we obtain an infinite family of faithful traces (whose restriction to $i(A)$ may nevertheless be the same).

The trace that we are going to define is probably never faithful and is defined only on algebras associated with groups acting on rooted trees; however, it possesses the property of self-similarity, which means that the trace is consistent with matrix recursions used below (see [86] for more details on these recursions).

Let us prove that the RFD property holds.

Proposition 9.1. *Suppose that a countable group G acts faithfully on a rooted tree $T = T_{\overline{m}}$, and let ν be the uniform measure on the boundary ∂T of the tree. Then the algebra C^*_π associated with the dynamical system $(G, \partial T, \nu)$ belongs to the class* RFD.

Proof. Let $\mathcal{H} = L^2(\partial T, \nu)$ and $\mathcal{H}_n$, $n \geq 1$, be the space generated by the characteristic functions χ_{C_u} of the atoms of the partition ξ_n of the boundary ∂T into cylindrical sets C_u, $|u| = n$ (recall that C_u consists of geodesic paths that connect the root vertex with infinity and pass through the vertex u). It is obvious that $\dim_{\mathbb{C}}(\mathcal{H}_n) = |V_n| = m_1 m_2 \dots m_n$, $\mathcal{H}_0 \simeq \mathbb{C}$. Since each rank n atom of the partition (i.e., an atom of the form C_u, $|u| = n$) is a union of m_{n+1} atoms of rank $n+1$ and, hence, each characteristic function of rank n is a sum of m_{n+1} characteristic functions of rank $n+1$, there exists a natural embedding $j\colon \mathcal{H}_n \to \mathcal{H}_{n+1}$. Let $\mathcal{H}_n^\perp$ be the orthogonal complement of $\mathcal{H}_{n-1}$ in $\mathcal{H}_n$. Then $\dim_{\mathbb{C}} \mathcal{H}_n^\perp = m_1 \dots m_{n-1}(m_n - 1) := q_n$ and

$$\mathcal{H} = \mathcal{H}_0 \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_n^\perp.$$

Since each of the partitions ξ_n is G-invariant, the spaces $\mathcal{H}_n$ and $\mathcal{H}_n^\perp$ are also invariant. This shows that the representation π is a direct sum of finite-dimensional representations and that the algebra C_π^* is embedded in

$$\mathbb{C} \times \prod_{n=1}^{\infty} M_{q_n}(\mathbb{C}),$$

which proves the proposition. $\square$

In the case of a group acting on a d-regular tree, the numbers q_n that appeared during the proof of the previous proposition are equal to $d^{n-1}(d-1)$. Assuming that C_π^* is embedded in the direct product of matrix algebras, we will express an element $x \in C_\pi^*$ as $x = (x_0, x_1, \ldots)$, $x_0 \in \mathbb{C}$, $x_n \in M_{q_n}(\mathbb{C})$, $n \geq 1$.

Now, for groups acting on rooted trees, we are going to define a special trace on the algebra C_π^*; in what follows, we will call this trace a recurrent trace, or a self-similar trace if the group is self-similar.

Theorem 9.2. *The limit on the right-hand side of the relation*

$$\tau(x) = \lim_{n\to\infty} \frac{1}{N_n} \operatorname{Tr}[x]_n, \tag{9.2}$$

where $N_n = m_1 \ldots m_n$ is the number of vertices on the n-th level, $[x]_n = (x_1, \ldots, x_n)$, and Tr *is the ordinary matrix trace, exists and defines a normalized trace τ on C_π^*.*

Proof. Denote by π_n the restriction of the representation π to $\mathcal{H}_n$ and by $\pi_n^\perp$ the restriction of π_n to $\mathcal{H}_n^\perp$. Then π is the sum of the representations $\pi_n^\perp$: $\pi = \bigoplus_{n=0}^{\infty} \pi_n^\perp$. The representation π_n is isomorphic to the permutation representation ρ_n in $l^2(V_n)$ induced by the action of the group on the nth level of the tree, where $l^2(V_n)$ is the space of functions on the set V_n of nth-level vertices.

First, we are going to prove the existence of the limit in (9.2) for elements of the group algebra $\mathbb{C}[G]$ and then prove that the functional τ defined on this algebra satisfies the properties of the trace and that, in addition, it is continuous with respect to the norm; after that, it remains to extend this functional to the whole algebra C_π^* by continuity.

Since every element of the group algebra is a finite linear combination of elements of the group, we will prove the existence of the limit in (9.2) under the assumption that $x = g \in G$. Let $[g]_n$ denote a matrix of order N_n corresponding to the operator $\pi_n(g)$ and to the basis in $l^2(V_n)$ consisting of delta functions at the nth-level vertices of the tree. In other words, $[g]_n$ is a permutation matrix (with matrix elements belonging to the set $\{0,1\}$) that describes the permutation of the nth-level vertices under the action of g. In the arguments below, we represent the matrix $[g]_n$ as a block matrix of order N_{n-1} with diagonal blocks g_{ii}, $1 \leq i \leq N_{n-1}$, of order m_n. Denote the diagonal elements of the matrix $[g]_{n-1}$ by $\bar{g}_{ii}$, $1 \leq i \leq N_{n-1}$. Then

$$\bar{g}_{ii} = \begin{cases} 0 & \text{if } g_{ii} = 0, \\ 1 & \text{if } g_{ii} \neq 0, \end{cases}$$

and therefore

$$\frac{1}{N_n} \operatorname{Tr}[g]_n = \frac{1}{N_{n-1}} \sum_{i=1}^{N_{n-1}} \operatorname{Tr}\left(\frac{1}{m_n} g_{ii}\right) \leq \frac{1}{N_{n-1}} \operatorname{Tr}[g]_{n-1},$$

since the matrix $[g]_n$ is obtained from the matrix $[g]_{n-1}$ by replacing the zero matrix elements with zero matrices of order m_n and by replacing the matrix elements equal to 1 with the corresponding permutation matrices of order m_n, whose normalized trace is obviously not greater than 1. This proves the existence of the limit in (9.2) for the elements $x \in G$ and, hence, for the elements $x \in \mathbb{C}[G]$ as well.

Denote the value of the limit in (9.2) on the elements of the group ring by $\tau(x)$. Starting from this place up to Theorem 9.11, we will mean by τ precisely this functional.

Lemma 9.3. *The functional $\tau(x)$ on $\mathbb{C}[G]$ defined by the limit in* (9.2) *is positive and satisfies the commutation relation $\tau(xy)=\tau(yx)$, $x,y\in\mathbb{C}[G]$.*

This lemma follows obviously from analogous properties of the ordinary trace of finite-dimensional matrices.

Lemma 9.4. *Let G be a group acting on a rooted tree and x be a self-adjoint element of the group algebra $\mathbb{C}[G]$. Then the following inequality holds*:

$$|\tau(x)|\le 2\|x\|_\pi, \tag{9.3}$$

where $\|x\|_\pi$ denotes the norm of the operator $\pi(x)$.

Proof. Let $x=(x_0,x_1,\dots)$ be the decomposition of the element x. Then $\|x\|_\pi=\sup_n\|x_n\|$, where $\|\cdot\|$ denotes the standard norm of a matrix (i.e., the norm of the operator defined by the matrix). Thus, using the notation $[x]_n=(x_0,\dots,x_n)$ and the fact that

$$q_n=m_1\dots m_{n-1}(m_n-1),$$

we obtain the estimates

$$\begin{aligned}
\tau(x)&=\lim_{n\to\infty}\frac{1}{N_n}\operatorname{Tr}[x]_n\\
&=\lim_{n\to\infty}\left[\frac{1}{m_1\dots m_n}\operatorname{Tr}(x_0)+\frac{m_1-1}{m_1\dots m_n}\,\frac{\operatorname{Tr}(x_1)}{q_1}+\dots+\frac{m_i-1}{m_i\dots m_n}\,\frac{\operatorname{Tr}(x_i)}{q_i}+\dots+\frac{m_n-1}{m_n}\,\frac{\operatorname{Tr}(x_n)}{q_n}\right]\\
&\le\lim_{n\to\infty}\left[\left(1+\frac{1}{m_n}+\frac{1}{m_{n-1}m_n}+\dots\right)\sup_{1\le i\le n}\frac{\operatorname{Tr}(x_i)}{q_i}\right]\\
&\le\lim_{n\to\infty}\left[\left(1+\frac{1}{m_n}+\frac{1}{m_{n-1}m_n}+\dots\right)\sup_{1\le i\le n}\|x_i\|\right],
\end{aligned}$$

because $\operatorname{Tr}B\le r\|B\|$ for an arbitrary self-adjoint matrix of order r. $\square$

This lemma shows that the functional τ can be extended by continuity to the cone of all self-adjoint elements of the algebra C^*_π. Since any element of the C^*-algebra is a linear combination (with coefficients 1 and $i=\sqrt{-1}$) of two self-adjoint elements, it follows that τ can be defined on the whole algebra C^*_π. This completes the proof of Theorem 9.2. $\square$

As already pointed out, this trace is called a *recurrent* or *self-similar* trace. This name is chosen because in the case of groups defined by finite automata the trace is completely determined by the set of recurrent relations between the generating elements that is defined by the Moore diagram of an automaton.

Now we are going to analyze the range of values of the trace τ on the elements of the group (i.e., on the corresponding operators of the representation of the group). First, we prove that the values of the trace in the case of essentially free actions are the same as in the case of the reduced C^*-algebra C^*_{r} (i.e., 0 or 1); this is one of the advantages of the trace, and we will use it below. Then we produce an example showing that the situation is drastically changed when passing to branch-type actions. However, we will benefit from the trace even in this case.

The following proposition is inspired by Proposition 9 from [95].

Proposition 9.5. *Let $G=G(\mathcal{A})$ be a self-similar group defined by a finite automaton $\mathcal{A}$ over a d-letter alphabet, and suppose that the action of G on the corresponding d-regular tree is essentially* *free. Let C^*_π be the C^*-algebra defined by this action and τ be the self-similar trace on it. Then*

$$\tau(g) = \begin{cases} 1 & \text{if } g = 1, \\ 0 & \text{if } g \neq 1. \end{cases} \tag{9.4}$$

Proof. The equality $\tau(1) = 1$ is obvious. Let us prove that $\tau(g) = 0$ for any nonidentity element g. Let A be the set of states of the automaton $\mathcal{A}$. As usual, we will also consider A as a set of generators of the group G. Suppose that the length of g with respect to the system of generators A is n, $n \geq 1$. This means that g is defined by one of the states s of an automaton $\mathcal{B}$ obtained by minimizing the nth power (with respect to the operation of composition) of the automaton $\mathcal{A} \cup \mathcal{A}^{-1}$ ($\mathcal{A}^{-1}$ is the inverse of the automaton $\mathcal{A}$, and the union of two automata is an automaton obtained by a disjoint union of the Moore diagrams of these automata). Let $\mathcal{B}_*$ be an automaton defined by a connected component containing s in the Moore diagram of the automaton $\mathcal{B}$. The section of the transformation defined by the state s at any vertex of the tree T_d is a transformation corresponding to a certain state of the automaton $\mathcal{B}_*$. For every state q of $\mathcal{B}_*$, there exists a $k(q) \in \mathbb{N}$ such that q acts nontrivially on the set of words of length $k(q)$ (or, which is the same, acts nontrivially on the level $k(q)$ of the tree T_q). Hence we conclude that each state of the automaton $\mathcal{B}_*$ acts nontrivially on the set of words of length k for $k = \prod_{q \in Q} k(q)$.

Consider the matrices $[g]_{kn}$ that describe the action of the element g on the set of words of length kn, $n = 1, 2, \dots$ (similar notation was used in the proof of Theorem 9.2). Let us decompose this matrix into blocks of order $d^{k(n-1)}$, thus obtaining a block matrix of order d^k whose rows and columns are indexed by words of length k that are prefixes of the words of length kn indexing the rows and columns of the matrix $[g]_{kn}$. Denote the diagonal blocks by a_{ii}, $1 \leq i \leq d^k$. These are either zero matrices or permutation matrices representing some elements g_{ii}, $1 \leq i \leq d^k$, of the group G; these elements correspond to some states of the automaton $\mathcal{B}_*$, and the following relations hold: $a_{jj} = [g_{jj}]_{k(n-1)}$. Denote by J the set of values of the index j, $1 \leq j \leq d^k$, for which the corresponding diagonal elements are nonzero (the set of such elements corresponds to the set of states that can be reached from the state s corresponding to the element g by moving in the Moore diagram of the automaton $\mathcal{B}_*$ along paths defined by words of length k such that all the states passed are labeled by the trivial element of the symmetric group $\mathrm{Sym}(d)$). Since the action on the set of words of length k is nontrivial, at least one of the matrices a_{ii}, $1 \leq i \leq d^k$, is nonzero. Thus, we arrive at the inequalities

$$\frac{1}{d^{kn}} \operatorname{Tr}[\pi_{kn}(g)] \leq \frac{d^k - 1}{d^k} \frac{1}{d^{k(n-1)}} \max_{i \in J} \operatorname{Tr}[\pi_{k(n-1)}(g_{ii})] \leq \frac{d^k - 1}{d^k}. \tag{9.5}$$

On the left-hand side of inequalities (9.5), the element g can be replaced by an arbitrary element g_{jj}, $j \in J$. Iterating the first inequality in (9.5), we arrive at the inequality

$$\frac{1}{d^{kn}} \operatorname{Tr}([g]_{kn}) \leq \left(1 - \frac{1}{d^k}\right)^n.$$

Then, passing to the limit as $n \to \infty$, we obtain $\tau(g) = 0$, which was to be proved. $\square$

Let us return to the general situation and assume that G is a group acting on a rooted tree $T = T_{\overline{m}}$ of general form; $N_n = m_1 m_2 \dots m_n$; ν is the uniform measure on the boundary of the tree; C^*_π is the corresponding C^*-algebra; τ is a recurrent trace on it; $g \in G$; $\mathrm{Fix}(g)$, as before, denotes the set of g-fixed points on the boundary; and $\mathrm{Fix}_n(g)$ denotes the set of g-fixed vertices of the nth level. Then, the following relations are valid:

$$\nu(\mathrm{Fix}(g)) = \lim_{n \to \infty} \frac{1}{N_n} |\mathrm{Fix}_n(g)| = \tau(g). \tag{9.6}$$

Indeed, the first relation is a simple consequence of the general properties of the measure, and the second follows from the fact that the number of ones on the diagonal in the matrix representation $\pi_n(g)$ of the element g is $|\mathrm{Fix}_n(g)|$. Thus, we arrive at the following proposition.

Proposition 9.6. *Under the assumptions made above,*

(a) *the value of the trace $\tau(g)$ is zero if and only if g acts essentially freely on the boundary of the tree*;

(b) *the recurrent trace vanishes on all nonidentity elements of a countable group $G \leq \mathrm{Aut}(T_{\overline{m}})$ if and only if the action is essentially free.*

As pointed out in [112, Theorem 4.3], the following important fact holds, whose proof presented here actually reproduces the proof of Theorem 4.3 in [112]; however, it is at the same time close to the proof of Proposition 9 from [95], on which the proof of Proposition 9.5 was also based.

Theorem 9.7. *Suppose that a group G is strongly self-similar* (*i.e., it is defined by a finite automaton*) *and $g \in G$ is an arbitrary element. Then*

(a) *the value of the trace $\tau(g)$ is zero if and only if the set* $\mathrm{Fix}(g)$ *is nowhere dense on the boundary of the tree*;

(b) *the recurrent trace vanishes on all nonidentity elements of G if and only if the action is essentially free.*

Proof. Let us first prove (a). In view of the previous proposition, it suffices to prove that the equality $\nu(\mathrm{Fix}(g)) = 0$ is equivalent to the fact that the set $\mathrm{Fix}(g)$ is nowhere dense. An initial automaton defines a continuous transformation of the boundary (in fact, it is an isometry with respect to an arbitrary ultrametric $\mathrm{dist}_{\bar{\lambda}}$ mentioned in Section 1). Therefore, the set $\mathrm{Fix}(g)$ is closed and, by virtue of the equality $\tau(g) = \nu(\mathrm{Fix}(g)) = 0$, cannot contain open subsets; hence, this set is nowhere dense.

Let us prove the converse statement. Let $\mathcal{B}_*$ be the minimal initial automaton described in the proof of Proposition 9.5. One of its states s defines the given element g, $g \in G$. Let Q be the set of states of the automaton $\mathcal{B}_*$. Since Q is finite and $\mathrm{Fix}(g)$ is nowhere dense, there exists a $k > 0$ such that, for every state $t \in Q$, the automaton $\mathcal{B}_{*t}$ acts nontrivially at least on one word of length k. Let us prove by induction on n that

$$|\mathrm{Fix}_{kn}(g)| \leq (d^k - 1)^n \tag{9.7}$$

for $n \geq 0$. Inequality (9.7) is obvious for $n = 0$. Suppose that it is valid for $n-1$, $n \geq 1$, and consider d^{kn} words of length kn. By the induction hypothesis, at most $(d^k - 1)^{n-1}$ prefixes of length $k(n-1)$ of such words, considered as vertices of the tree, are fixed by the element g. Suppose that a word u of length $k(n-1)$ is fixed under the action of g and t is a state into which the state s transforms under the action of the word u. Then t belongs to Q, and, in view of the choice of k, there exist at most $(r^k - 1)n - 1$ words fixed by the element corresponding to the state t. Thus, there are at most $(d^k - 1)^n$ words of length kn that are left fixed by the element g, which proves (9.7). So,

$$\frac{1}{d^{kn}} |\mathrm{Fix}_{kn}(g)| \leq \left(1 - \frac{1}{d^k}\right)^n.$$

Since the right-hand side of the last inequality tends to zero as $n \to \infty$, the application of (9.6) completes the proof.

Assertion (b) follows from (a). $\square$

As a corollary to the statement proved above and relation (9.6), we obtain the following important property of the actions of groups generated by finite automata, which we used in Section 5 for constructing examples of essentially free actions.

Corollary 9.8. *For self-similar groups defined by finite automata, the concepts of local nontriviality, topological freeness, and essential freeness of the action on the boundary of a tree are equivalent.*

In contrast to essentially free actions on the boundary of a tree, in the case of weakly branch actions the recurrent trace probably always takes a nonzero value at least on one element of the group (more precisely, on the operator $\pi(g)$). To justify this statement, recall some concepts related to representations and invoke some observations and results from the recent publication by Vershik [184].

A complex-valued function ϕ on a group G is said to be positive definite if the following inequality holds for any finite set of elements $\{g_1, g_2, \dots, g_n\}$, $g_i \in G$, $i = 1, \dots, n$:

$$\sum_{i,j=1}^{n} \phi(g_i g_j^{-1}) \geq 0.$$

If, in addition, the function ϕ is constant on conjugacy classes and is normalized by the condition $\phi(1) = 1$, then it is called a central character. Notice that the word "character" is often used in other senses as well, for example, as a homomorphism from a group into the unit circle group; however, here we will use precisely this definition, following the terminology of [184], which, however, is closely tied to the classical terminology (see, for example, [53]). Characters of this kind arise as restrictions to the operators $\pi(g)$ of the trace of factor representations of type II_1, i.e., representations that generate a von Neumann algebra that is a factor of type II_1. For an arbitrary action of a countable group on a space X with an invariant probability measure μ, the function

$$\phi(g) = \mu(\mathrm{Fix}(g))$$

is a character in view of the relation $\mu(\mathrm{Fix}(g)) = \langle \pi(g)1_\Delta, 1_\Delta \rangle$, where 1_Δ is the characteristic function of the diagonal $\Delta = \{(x,x) \colon x \in X\}$ and the inner product is taken in the space $L^2(\mathcal{R}, \rho)$; here $\mathcal{R}$ is a Borel subset in $X \times X$ consisting of pairs (x, gx): $x \in X$, $g \in G$, and the measure ρ with support in $\mathcal{R}$ is defined by the condition that its projection onto each of the factors in $X \times X$ is μ, while on every fiber over a point $x \in X$ the fiberwise measure corresponding to the measure ρ is a counting measure (i.e., the measure of each point is 1). Note that $\mathcal{R}$ is a countable Borel equivalence relation associated with the action of the group on X; some questions concerning such relations will be considered in the last Section 11.

The group G acts on $\mathcal{R}$ by left and right translations with respect to the first and second arguments. These two actions preserve the measure ρ and commute with each other. This defines two representations π_0 and π_1 in the space $L^2(\mathcal{R}, \rho)$ and two mutually commuting von Neumann algebras W_0 and W_1 generated by the operators of the above-mentioned representations and the operators of multiplication by functions in $L^\infty(\mathcal{R}, \rho)$. In the case of a finite measure μ and an ergodic action, these algebras are factors of type II_1 (we refer the reader to the fundamental monographs by Takesaki [179–181] for details related to this construction). Below, for definiteness, we suppose that $W = W_0$ is the first of the two algebras, and let V be the subalgebra of W generated by the operators of the group representation (i.e., by the operators $\pi_0(g)$, $g \in G$, while the closure, as is conventional in the theory of von Neumann algebras, is taken in the strong or, equivalently, in the weak operator topology). In the general case, the algebra V is a proper subalgebra of W. However, a remarkable observation made by Vershik consists in the validity of the following statement.

Theorem 9.9 [184, Theorem 10]**.** *For an ergodic action (G, Z, μ) with an invariant probability measure, the coincidence $V = W$ occurs if and only if the action is totally nonfree.*

Thus, for every totally nonfree group action, the described representation generates a factor of type II_1. To every character ϕ, there corresponds a representation π_ϕ defined up to quasi-equivalence of representations, and this representation is a factor representation if and only if ϕ is indecomposable (i.e., ϕ is an extreme point in the space of central characters on the group with the topology of pointwise convergence) [53].

In [184, Theorem 9], it is shown that for ergodic actions with invariant measure the function $\phi(g) = \mu(\mathrm{Fix}(g))$ is an indecomposable character if and only if V is a factor. Note also that when a group acts on the boundary of a tree, it follows from (9.6) that this character is a restriction of the recurrent trace to the group in view of the relations

$$\phi(g) = \mu(\mathrm{Fix}(g)) = \tau(g).$$

Theorem 9.10. *Let G be a weakly branch group and ϕ be a character defined by its action on a tree. Then the corresponding representation π_ϕ is a factor representation.*

Proof. By Theorem 2.4, the action of the group G on the boundary of a tree is totally nonfree, and it is ergodic by Proposition 4.1 and by virtue of the fact that, by definition, a weakly branch group acts transitively on the levels. It remains to apply Theorem 9 from [184]. $\square$

It is an interesting problem to calculate explicitly the trace on self-similar groups and examine its properties in greater detail. In our view, the following conjecture is plausible.

Conjecture 9.1. *Let G be a weakly branch group and τ be the self-similar trace on it. Then there exists an element $g \in G$ such that $\tau(g) \neq 0$.*

The recurrent trace can take values that densely fill the interval $[0,1]$. Let us demonstrate this using an example of the intermediate growth group $\mathcal{G}$ defined in Example 2.3. The relevant calculations have been performed by my student R. Kravchenko.

Example 9.1. For the group $\mathcal{G} = \langle a,b,c,d\rangle$, the set of values of the recurrent trace on the operators $\pi(g)$, $g \in \mathcal{G}$, is equal to the set

$$\tau(G) = \frac{1}{7}\mathbb{Q}_2 \cap [0,1], \tag{9.8}$$

where $\mathbb{Q}_2$ stands for the set of dyadic rationals.

For simplicity, we will write $\tau(g)$ instead of $\tau(\pi(g))$. If $g = (g_0, g_1) \in \mathrm{st}_G(1)$, then we have $\tau(g) = \frac{1}{2}(\tau(g_0) + \tau(g_1))$, while if $g = (g_0, g_1)\sigma$, then $\tau(g) = 0$ because the corresponding operator matrix has zero values on the diagonal. Taking into account the recurrent relations between the generators a, b, c, and d and the fact that $\tau(a)$ obviously vanishes, we arrive at the relations $\tau(b) = \tau(c)/2$, $\tau(c) = \tau(d)/2$, and $\tau(d) = (1+\tau(b))/2$, which imply $\tau(b) = 1/7$, $\tau(c) = 2/7$, and $\tau(d) = 4/7$. Since $\mathcal{G}$ is a contracting group with parameters $1/2$ and 1 (and hence the sections of any element of length ≥ 1 are shorter than the element itself) and, consequently, in the matrix representation of any of its elements g the lengths of nonzero elements of this matrix are not greater than the length of the element itself, we find that the value of the trace $\tau(g)$ belongs to the set $\frac{1}{7}\mathbb{Q}_2 \cap [0,1]$ for any $g \in \mathcal{G}$.

To prove the reverse inclusion, we will use the fact that $\mathcal{G}$ is a regularly branch group with respect to the subgroup $K = \langle [a,b]\rangle^{\mathcal{G}}$, the normal closure of the element $[a,b] = (ab)^2$ [80]. Since $\psi(K) > K \times K$, it follows that $(T+T)/2 \subset T$, where $T = \tau(K)$, i.e., $(x+y)/2 \in T$ if $x, y \in T$. Then, we can easily prove, by induction on $n \in \mathbb{N}$, that

$$\frac{m}{2^n}x + \frac{2^n - m}{2^n}y \in T$$

for an arbitrary m, $1 \leq m \leq 2^n$.

It remains to be noticed that since the element

$$(ab)^8 = (b, b, b, b, b, b, b, b)$$

in $\mathrm{st}_{\mathcal{G}}(3)$ belongs to K and since its trace is equal to $\tau(b) = 1/7$, it follows that $7T$ contains 0 and 1 and satisfies $(7T + 7T)/2 \subset 7T$, which immediately implies (9.8).

The possibility of similar calculations of the values of the recurrent trace on the elements of strongly self-similar contracting groups was pointed out in [81]. Calculations similar to the above can in principle be performed for other self-similar contracting groups that are regularly branch groups over one of their subgroups. Indeed, if G is a contracting group, then all sections g_u of any element $g \in G$ belong to the nucleus $\mathcal{N}$ starting from a certain level N, i.e., for $|u| > N$. Thus, the problem of calculating $\tau(g)$ for any element g reduces to calculating the trace on the set of elements of the nucleus; the latter problem is easily solved by compiling a closed system of linear relations between the values of the trace of the elements of the nucleus and the values of the trace on their projections (which are also elements of the nucleus). This system must be consistent. It is unlikely that it may be indeterminate, but this is still to be verified. If the group is regularly branch over one of its subgroups, then roughly the same calculations as those performed above should yield the full range of values of the trace on the elements of the group. At least if G is regularly branch with respect to P, then $\tau(P) \supseteq (\tau(P) + \ldots + \tau(P))/d$ (the sum contains d terms, where d is the cardinality of the alphabet). However, for the present there is no general statement that would describe the range of values of the trace for the class of groups in question; moreover, it is not clear how the knowledge of the structure of this range can be used to study the properties of a group (however, the results presented above make it obvious that the recurrent trace is certainly useful for solving problems of representation theory). It is worth noting that in the case of the group $\mathcal{G}$ the values of the recurrent trace coincide with the dimensions of the centralizers of elements of the group $\mathcal{G}$, which were defined and calculated by A. Rozhkov[4]; possibly, this points to a relationship between these quantities.

In the case of an irregular tree $T = T_{\overline{m}}$, provided that the sequence $\overline{m} = \{m_n\}_{n=1}^{\infty}$ defining the branch index is bounded, a statement similar to Proposition 9.5 is also valid for countable groups $G \le \mathrm{Aut}(T)$ that act essentially freely. It would be interesting to find conditions that guarantee relation (9.4) in the case of trees with unbounded branch index.

The following statement is undoubtedly known; however, we present its proof because we are not aware of an appropriate reference.

Theorem 9.11. *Let C be a unital C^*-algebra that has a faithful trace τ, $\tau(1) = 1$, let G be a countable subgroup of the group of unitary elements of the algebra C, and suppose that G generates C as a C^*-algebra. Suppose that $\tau(g) = 0$ for any nonidentity element $g \in G$. Then C is isomorphic to the reduced C^*-algebra $C^*_{\mathrm{r}}(G)$ of the group G.*

Proof. Using the trace τ, we construct a Gelfand–Naimark–Segal representation of the algebra C. Namely, consider an inner product on C defined by the relation $\langle x, y\rangle = \tau(y^*x)$, and let H be the Hilbert space obtained by completing C with respect to the norm defined by this inner product. Let $\rho\colon C \to B(H)$ be the representation of the algebra C in the space H by bounded operators of left multiplication: $\rho(c)(x) = cx$, $c, x \in C$. This representation extends to the whole space H by continuity. The representation ρ is faithful because

$$\langle \rho(x)1, x\rangle = \langle x, x\rangle = \tau(x^*x) > 0$$

if $x \ne 0$, $x \in C$. Let us index the elements of the group G in an arbitrary way, $G = \{g_1, \ldots, g_n, \ldots\}$. By the hypothesis, $\tau(g_i^* g_j) = \tau(g_i^{-1} g_j) = 0$ if $i \ne j$.

[4] A. V. Rozhkov, "Metric Relations in Groups of Tree Automorphisms," Algebra Logika **37** (3), 338–357 (1998) [Algebra Logic **37** (3), 192–203 (1998)].

Let K be a group isomorphic to G under an isomorphism φ and $k_i = \varphi(g_i) \in K$. Define an operator $U\colon H \to l^2(K)$ by the relation $U\bigl(\sum_i \alpha_i g_i\bigr) = \sum_i \alpha_i k_i$. This is an isometric isomorphism. Then

$$(U\rho(g_i)U^*)(\delta_{k_j}) = U\rho(g_i)(g_j) = U(g_i g_j) = \delta_{k_i k_j} = \lambda(k_i)\delta_{k_j},$$

where δ_k is a delta function with nonzero value on the element $k \in K$, and $\lambda\colon K \to \mathcal{U}(l^2(K))$ denotes the left regular representation of the group K. We have established that U intertwines the (faithful) representations of the algebras C and C^*_{r} and, hence, establishes an isomorphism between them. $\square$

Corollary 9.12. *Suppose that an action $(G, \partial T, \mu)$ of a strongly self-similar group is essentially free. Then there exists a natural $*$-homomorphism $C^*_\pi \to C^*_{\mathrm{r}}(G)$.*

Proof. Indeed, if the recurrent trace τ on C^*_π is faithful, then the algebras C_π and $C^*_{\mathrm{r}}(G)$ are isomorphic by the isomorphism described in the proof of the theorem. If the trace is not faithful, then the set of elements $x \in C_\pi$ satisfying the relation $\tau(x^*x) = 0$ forms a closed ideal I; on the quotient algebra C_π/I, the trace τ induces a faithful trace $\bar\tau$ that satisfies the hypotheses of the theorem; hence, $C_\pi/I \simeq C^*_{\mathrm{r}}(G)$. $\square$

It would be interesting to find out when the algebras C_π and $C^*_{\mathrm{r}}(G)$ are isomorphic for actions on rooted trees. Below we give a sufficient amenability-based condition for such an isomorphism. For strongly self-similar groups acting essentially freely, the question of when the algebras C_π and $C^*_{\mathrm{r}}(G)$ are isomorphic is solved easily (see Proposition 9.14). An isomorphism is certainly impossible if $C^*_{\mathrm{r}}(G)$ is not residually finite-dimensional. For example, the reduced C^*-algebra of the free group F_m of rank $m \geq 2$ is simple; hence, it does not belong to the RFD class. Moreover, the following theorem is valid.

Theorem 9.13 [25]. *The following assertions are equivalent for a group G:*

(i) *$C^*_{\mathrm{r}}(G)$ is residually finite-dimensional;*

(ii) *the group G is amenable and belongs to the class of maximally almost periodic (MAP) groups; i.e., its finite-dimensional representations separate the elements of the group (for finitely generated groups, this condition is equivalent to the condition of residual finiteness).*

Combining this statement with Corollary 9.12, we get the following result:

Proposition 9.14 [16]. *For a self-similar group G acting essentially freely, the isomorphism $C_\pi(G) \simeq C^*_{\mathrm{r}}(G)$ takes place if and only if the group is amenable.*

Recall that the full $C^*_{\mathrm{f}}(G)$-algebra is defined as the C^*-algebra generated by a representation that is the product of all irreducible unitary representations of the group. The reduced C^*-algebra of a group is isomorphic to the full C^*-algebra if and only if the group is amenable [53, 24].

Let us note a few other facts related to representations and algebras associated with groups acting on rooted trees. We begin with an irreducibility result for a class of quasiregular representations of weakly branch groups. Recall the following definition.

Definition 9.1. Let $H \leq G$. The *commensurator* $\mathrm{comm}_G(H)$ is a subgroup of G defined by the relation

$$\mathrm{comm}_G(H) = \bigl\{g \in G \mid H \cap H^g \text{ has finite index in } H \text{ and in } H^g\bigr\}.$$

The commensurator can also be defined by the relation

$$\mathrm{comm}_G(H) = \bigl\{g \in G \mid \text{the } H\text{-orbits } H(gH) \text{ and } H(g^{-1}H) \text{ are finite}\bigr\}.$$

The verification of the equivalence of these definitions is a useful exercise and is based on the following two facts.

1. If we denote by $\mathcal{T}$ a transversal (a set of representatives of the cosets) of H with respect to $H \cap H^g$, then the following decomposition into a disjoint union of left cosets with respect to the subgroup H holds for the double coset HgH:

$$HgH = \bigsqcup_{t \in \mathcal{T}} tgH.$$

2. Conversely, the above relation implies that $\mathcal{T}$ is a transversal; namely, the following decomposition holds:

$$H = \bigsqcup_{t \in \mathcal{T}} t(H \cap H^g).$$

To prove the irreducibility of quasiregular representations, we need the following criterion obtained by Mackey.

Theorem 9.15 [132]**.** *Let H be a subgroup of an infinite discrete countable group G. Then the quasiregular representation $\rho_{G/H}$ is irreducible if and only if* $\mathrm{comm}_G(H) = H$.

Proposition 9.16 [16; 17, Proposition 9.2]**.** *Let G be a weakly branch group acting on a rooted tree T and P be a parabolic subgroup* (*i.e., the stabilizer of some boundary point of the tree*)*. Then the representation $\rho_{G/P}$ is irreducible.*

Proof. To prove the proposition, we apply Lemma 2.3. Let $P = \mathrm{st}_G(\xi)$ and $g \in G \setminus P$. Let us show that the intersection $P \cap P^g$ has infinite index in P^g. Let u be the first vertex belonging to ξ such that $g^{-1}(u) \neq u$. Then $\mathrm{rist}_G(u) = \mathrm{rist}_{P^g}(u)$ because P^g stabilizes the vertex $g^{-1}(u) \neq u$, and $\mathrm{rist}_{P \cap P^g}(u) = \mathrm{rist}_P(u)$ for a similar reason. Since $\mathrm{rist}_P(u)$ leaves the point ξ fixed and, by Lemma 2.3, the orbit $\mathrm{rist}_G(u)(\xi)$ is infinite, we can apply the Mackey criterion, and the proposition is proved. $\square$

So, weakly branch groups have a continuum family of irreducible quasiregular representations parameterized by the boundary points. These representations separate the elements of the group because the intersection of all parabolic subgroups is trivial in view of the spherical transitivity of the actions of branch groups. All of them are pairwise nonisomorphic because have pairwise different kernels of representations (i.e., subgroups of elements acting by the identity operator).

Now, we will show that the class of weakly branch groups has one more good property from the viewpoint of representations and operator algebras; namely, weakly branch groups belong to the class of ICC groups that have infinite classes of conjugate elements (i.e., for any nonidentity element $g \in G$, the conjugacy class $C(g)$ of the element g is infinite). It is well known [181] that the von Neumann algebra generated by the left regular representation is a factor of type II_1 for an ICC group.

Theorem 9.17. *Let G be a weakly branch group. Then G belongs to the class* ICC.

Proof. Consider the action of G on itself by conjugations, and let $\mathrm{Stab}_G(g)$ be the stabilizer of an element $g \in G$, $g \neq 1$ (we have deliberately changed the notation for the stabilizer because now we consider another type of actions, namely, the adjoint action). The conjugacy class $C(g)$ is infinite if and only if $[G : \mathrm{Stab}_G(g)] = \infty$. For some n, there exists an nth-level vertex u that is not fixed under g. We will assume that the level n is chosen to be minimal among those on which the action of the element g is nontrivial. Then the predecessor w of the vertex u is g-invariant. Let $v = g(u)$, $v \neq u$, and $1 \neq h \in \mathrm{rist}_G(u)$. Suppose that the sections of the elements g and h at the vertex w have the form

$$g|_w = (g_1, \dots, g_{m_{n-1}})\sigma, \qquad h|_w = (1, \dots, 1, \xi, 1, \dots, 1),$$

where $1 \neq \sigma \in \mathrm{Sym}(m_n)$, $\xi \neq 1$, and the position of the component ξ corresponds to the position of the vertex u under the vertex w. Then

$$[h, g] = (1, \ldots, 1, \xi^{-1}, 1, \ldots, 1, \xi^{g_j}, 1, \ldots, 1),$$

where the nonidentity components of the latter decomposition correspond to the positions of the vertices v and u, with ξ^{g_j} being a component of the decomposition of $g|_w$ and corresponding to the position of the vertex u. Thus, we find that the inequality $g^h \neq g$ holds for any nonidentity element $h \in \mathrm{rist}_G(u)$. Hence, $\mathrm{rist}_G(u) \cap \mathrm{Stab}_G(g) = 1$, and since $\mathrm{rist}_G(u)$ is an infinite subgroup, $\mathrm{Stab}_G(g)$ has infinite index in G. The theorem is proved. $\square$

The following theorem, which gives a sufficient condition for an isomorphism between the C^*-algebra generated by the Koopman representation and C^*-algebras generated by quasiregular representations related to the actions on orbits, will be proved in the next section.

Theorem 9.18 [16]. *Suppose that G acts spherically transitively on a tree T.*

(i) *There exists a canonical $*$-homomorphism $C^*_\pi \to C^*_{G/P_\xi}$.*

(ii) *Suppose that the action of the group G on the orbit $G(\xi)$ of some point $\xi \in \partial T$ is amenable. Then $C^*_\pi \simeq C^*_{G/P_\xi}$, where $P_\xi = \mathrm{st}_G(\xi)$.*

10. RANDOM WALKS, SPECTRA, AND ASYMPTOTIC EXPANDERS

In this section, we touch upon some questions related to random walks on Schreier graphs and questions of the spectral theory of graphs, consider some spectral measures associated with graphs, and continue the discussion of the concept of amenability initiated in the previous section. Note that the spectral theory of graphs, some issues of which are discussed below, is being successfully developed for various generalizations of the class of graphs, for example, for *quantum* graphs (see [118] and references therein); however, we will not go into these generalizations. We begin with the definition of a Markov operator.

The adjacency matrix A of a nonoriented graph Γ is a matrix whose rows and columns are numbered by vertices and the intersection of a column and a row with labels u and v contains the number $a_{u,v}$ of edges that connect the vertices u and v. The adjacency matrix is symmetric and defines an operator $\bar{A}$ in the (complex) Hilbert space $l^2(V)$ (which will be also denoted by $l^2(\Gamma)$) of square summable functions defined on the vertex set V. The operator $\bar{A}$ acts as follows:

$$(\bar{A}f)(v) = \sum_{w \sim v} f(w).$$

We will prefer to deal with the normalized operator

$$(Mf)(v) = \frac{1}{\deg(v)} \sum_{w \sim v} f(w),$$

which is called a Markov operator. Since the sum of matrix elements of a Markov operator over each row is equal to 1, its norm is not greater than 1, and it is equal to 1 in the case of a finite graph (in this case the multiplicity of the eigenvalue 1 is equal to the number of connected components of the graph). Note also that in the case of a finite connected graph, constant functions are eigenfunctions corresponding to the eigenvalue 1. If the graph is infinite, then whether the norm of the operator M is equal to unity is a delicate question. In the case of graphs with uniformly bounded degree of vertices, there is a characterization of amenability in terms of the spectral radius of the operator M (namely, the amenability is equivalent to the equality of the spectral radius to unity). We will discuss this below.

In the orientable case, the Markov operator is defined by the formula

$$(Mf)(v) = \frac{1}{\deg(v)} \sum_{e\in E\colon\, \alpha(e)=v} f(\beta(e));$$

however, in what follows we will discuss only the spectral properties of nonoriented graphs.

The matrix of a Markov operator is a matrix of transition probabilities of a simple random walk on a graph when a wandering point passes from a current vertex to neighboring vertices along connecting edges with equal probabilities. If the edges of a graph are labeled by symbols of an alphabet $X = \{x_1, \dots, x_k\}$ whose elements are assigned probabilities $\{p_1, \dots, p_k\}$, $p_i \geq 0$, $\sum p_i = 1$, then one can consider a random walk in which the transition along an edge labeled by a symbol x_i occurs with probability p_i. In this case the entries of the matrix of the corresponding Markov operator belong to the set $0 \cup \{p_1, \dots, p_k\}$.

Let $a = \sum \alpha_g g \in \mathbb{R}[G]$ be an element of the group algebra with positive real coefficients that add up to unity and with the support (i.e., the set of elements g whose coefficient α_g is nonzero) generating the group (without using the operation of inversion; i.e., the generation is understood in the semigroup sense). With such an element, one can associate a random walk on any Schreier graph constructed for the group G with the use of the system of generators $\operatorname{supp}(a)$. If $\operatorname{supp}(a) = \{g_1, \dots, g_k\}$, then a transition along an edge labeled in the Schreier graph by a generator g_i occurs with probability α_{g_i}. If the element a is self-adjoint (or symmetric) in the sense that its support is invariant under the inversion (of the elements) and the relation $\alpha_g = \alpha_{g^{-1}}$ holds for every element g of the support, then the corresponding Markov operator is self-adjoint and the random walk is said to be symmetric. The study of such walks on noncommutative groups was initiated by Kesten [116], who tried to use these random walks in order to solve the Burnside problem for periodic groups.

The main questions related to random walks on infinite graphs are the recurrency of a random walk, the Liouville property of a graph (the absence of nonconstant bounded harmonic functions, i.e., functions $f(g)$ on the group that satisfy the relation $Mf = f$), the asymptotic behavior of the transition probabilities as time tends to infinity (in particular, the rate at which the probabilities of return to the initial point tend to zero), the behavior of the entropy of a random walk, etc.

The following dichotomy holds: the probabilities decrease either exponentially or subexponentially. For symmetric random walks, the probabilities have exponential asymptotics if and only if the norm (or, which is the same, the spectral radius) of the Markov operator is less than 1; this is equivalent to the nonamenability of the graph or of a finitely generated group represented by the Cayley graph (Kesten's theorem [115]). Amenable groups have already been defined above; now we will define amenable graphs. However, first we should agree on what we will mean by the boundary of a subset of vertices in a graph.

The boundary of a subset $F \subset V$ is the set ∂F of vertices in F that have a neighbor that does not belong to F. As pointed out in [41], there are four natural approaches to the definition of the boundary of a set of vertices in a graph. From the viewpoint of asymptotic graph theory, all these approaches are equivalent. In particular, any of the four definitions of the boundary can be used below in the definition of the amenability of a graph.

Definition 10.1. Suppose that a graph Γ has a uniformly bounded degree of vertices; i.e., there exists a constant C such that $\deg(v) \leq C$, $v \in V$. The graph Γ is said to be amenable if there exists a sequence $\{F_n\}_{n=1}^{\infty}$ of finite subsets of the vertex set such that $\frac{|\partial F_n|}{|F_n|} \to 0$.

The condition used in this definition is often called the Følner condition; it is interrelated with the condition appearing in Theorem 8.13 (in this case one naturally adds the condition that the sequence increases, i.e., $F_n \subset F_{n+1}$, $n \geq 1$, which can always be satisfied provided that there exists a sequence missing the monotonicity condition). This definition is consistent with the definition of amenability given above. In addition to the classical definition of the amenability of a group or the amenability of its action on a set, one can alternatively define a finitely generated group to be amenable if its Cayley graph is amenable (recall that a group is amenable if and only if each of its finitely generated subgroups is amenable). In exactly the same way, an action of a finitely generated group is amenable if the graph of the action (i.e., the Schreier graph associated with the action) is amenable.

Kesten's theorem [115] stating that a group is amenable if and only if the norm of the Markov operator (or, which is the same, the spectral radius) is equal to 1 has been generalized in many ways; one of these generalizations can be found in [41, Theorem 5.1], which, in particular, implies the following statement.

Theorem 10.1. *A graph Γ with a uniformly bounded degree of vertices is amenable if and only if the norm of the Markov operator M associated with the simple random walk is* 1.

In this theorem, the simple random walk can be replaced by an arbitrary nondegenerate symmetric random walk. Kaimanovich showed [106] that in the case of graphs with nonuniformly bounded degrees of vertices, the question of relationship between amenability and other asymptotic characteristics is rather delicate and requires a careful approach.

The above theorem is only one of many examples that demonstrate the importance of evaluating or estimating the norm of the Markov operator, as well as calculating its spectrum and spectral characteristics. A similar problem consists in solving the spectral problem for the elements of the group algebra that are represented as elements of the C^*-algebra generated by a quasiregular representation $\rho_{G/H}$. In other words, the matter concerns the spectrum of the corresponding operator acting in the space $l^2(\Gamma)$, where $\Gamma = \Gamma(G,H,A)$. When speaking of the spectrum of a graph, one usually means the spectrum of the Markov operator of a simple random walk on this graph. However, the study of the spectra of operators of the form $\rho_{G/H}(c)$, $c \in \mathbb{C}[G]$, is also of considerable interest. With these operators one can associate a colored Schreier graph $\Gamma(G,H,A)$, where A is the support of the element c (the support is assumed to generate the group), with each a-colored edge, $a \in A$, additionally labeled by the coefficient α_a, provided that $c = \sum_{a\in A} \alpha_a$. We will denote such a graph by $\Gamma(G,H,c)$; moreover, we will always assume that this is a rooted graph with the initial vertex represented by the coset $1H$ of the identity element.

Now we consider spectral measures. If M is a bounded self-adjoint operator in a Hilbert space $\mathcal{H}$, then its spectrum $\mathrm{sp}(M)$ is a closed bounded subset of the real axis, and the following decomposition is provided by the spectral theorem:

$$M = \int\limits_{\mathrm{sp}(M)} \lambda\, dE(\lambda),$$

where $E(\lambda)$ is a system of orthogonal projectors (which is often called a spectral function or a projector-valued measure). Each vector $\phi \in \mathcal{H}$ is assigned a measure μ_ϕ on $\mathrm{sp}(M)$ defined by the relation

$$\mu_\phi(B) = \langle e(B)\phi, \phi\rangle,$$

where B is any Borel subset of the real axis, $\langle\cdot,\cdot\rangle$ denotes the inner product, and $e(B)$ is the density of the spectral function: $e([a,b]) = E(b) - E(a)$. The choice of the vector ϕ depends on a specific problem, while the properties of the measure μ_ϕ usually depend on the choice of ϕ. For Markov operators, a natural choice for ϕ is the delta function with nonzero value at the root of the graph (which serves as the starting point of a random walk). Thus, for a graph Γ, one can define the measures μ_v^Γ, where v runs through the vertex set of the graph. These measures are called Kesten's spectral measures, because it was Kesten who first started to systematically analyze random walks on noncommutative groups and their Cayley graphs; in particular, he calculated the spectral measure of a random walk on free groups and obtained a probability criterion for amenability (Theorem 10.1).

Let us return to the Markov operator M. If we denote by $p_v^{(n)}$ the probability of return to the vertex v after n steps of a random walk on the graph with which this operator is associated, we can easily see that this probability is $\langle M^n\delta_v,\delta_v\rangle = \int_{\mathrm{sp}(M)} \lambda^n\,d\mu(\lambda)$. The integral in this relation is the nth moment of the measure μ_v. It is well known that a bounded measure is uniquely defined by its moments in view of the Stieltjes inversion formula.

In the case of finite graphs, a natural measure related to a graph is a *counting* measure ν (also called an *empirical* measure) whose value on a subset $B\subset\mathbb{R}$ is equal to the number of points of the spectrum of the Markov operator (counted with multiplicities) that fall into the set B, divided by the number of vertices of the graph. Below we will see that such a measure is the mean of the measures μ_v. Counting measures are widely used in probability theory, mathematical physics, and other fields of natural science. In different fields, different terms are used for these measures; for example, in physics, the term *density of states* is widely used.

The counting measure corresponds to a counting projector-valued spectral measure

$$e(B)=\sum_{\lambda\colon \lambda\in B\cap\mathrm{sp}(M)} E_\lambda,$$

where E_λ is the orthogonal projector onto the space of eigenfunctions corresponding to the eigenvalue λ. Hence we obtain the following relations:

$$\mu=\sum_{\lambda\in\mathrm{sp}(M)}\frac{\mathrm{Tr}(E_\lambda)}{|V|}\delta_\lambda=\sum_{\lambda\in\mathrm{sp}(M)}\frac{\#(\lambda)}{|V|}\delta_\lambda, \tag{10.1}$$

where V is the vertex set of the graph, Tr, as before, stands for the ordinary matrix trace, and $\#$ denotes the multiplicity of the eigenvalue.

Proposition 10.2. *Let Γ be a finite graph with vertex set V and M be the Markov operator of an arbitrary symmetric random walk on Γ. Then the counting measure μ associated with this operator is equal to the mean of the Kesten measures μ_v over the set of all vertices*; *i.e.*,

$$\mu=\frac{1}{|V|}\sum_{v\in V}\mu_v.$$

Proof. Let $|V|=m$ and P_{ij}^n be the probability of transition from vertex i to vertex j after n steps of the random walk defined by the matrix M (we assume that the vertices are numbered by positive integers from 1 to m). Since the random walk is symmetric (i.e., the matrix M is symmetric), M is similar to a diagonal matrix D with the eigenvalues λ_k of the matrix M situated on the diagonal. Let $M=\Phi^{-1}D\Phi$, where the columns of the matrix $\Phi=(\phi_{ij})$ are normalized eigenfunctions of the matrix M and $\Phi^{-1}=(\psi_{ij})$ is the inverse matrix. Then the transition probabilities of the random walk defined by the matrix M are given by

$$P_{ij}^n=\langle M^n\delta_i,\delta_j\rangle=\sum_{k=1}^m \phi_{ik}\lambda_k^n\psi_{kj}.$$

Let

$$\mu'=\frac{1}{|V|}\sum_{v\in V}\mu_v.$$

Consider the numbers

$$\widetilde{P}^n = \int \lambda^n \, d\mu'(\lambda) = \frac{1}{m}\sum_{i=1}^{m} P_{ii}^n = \frac{1}{m}\sum_{k=1}^{m} \lambda_k^n \sum_{i=1}^{m} \phi_{ik}\psi_{ki} = \frac{1}{m}\sum_{k=1}^{m} \lambda_k^n = \int \lambda^n \, d\mu(\lambda).$$

Since a finite measure is uniquely defined by its moments (the role of which in this case is played by the return probabilities) and since the averaging of measures corresponds to the averaging of momenta, the proposition is proved. □

As a bonus, we have found that the moments of the counting measure are equal to the mean of the moments of the Kesten measures over the vertex set.

Of great interest are the problems of approximating infinite graphs by finite ones and of finding conditions under which the asymptotic characteristics of infinite graphs are approximated, in one or another sense, by the corresponding characteristics of finite graphs. One of the simplest facts in this direction is the following theorem.

Theorem 10.3. *The Kesten spectral measure depends continuously on a point of the space $\mathcal{X}_{2m}^{\mathrm{Sch}}$ in the weak topology of the space of measures.*

In other words, if a sequence $(\Gamma_n, v_n)_{n=1}^{\infty}$ of rooted graphs converges in the space $\mathcal{X}_{2m}^{\mathrm{Sch}}$ to a graph (Γ, v), then the sequence of Kesten measures μ_{v_n} converges weakly to the Kesten measure μ_v.

Proof. Indeed, a bounded measure is uniquely defined by its moments, which coincide with the probabilities of return to the root of the graph (which is the starting point of a random walk). For any r and sufficiently large n, the neighborhoods of radius r in the graphs (Γ_n, v_n) and (Γ, v) are isomorphic as graphs. Hence, we have a convergence of the moments, which implies a weak convergence of the measures. The theorem is proved. □

In fact, there is no need to consider only simple random walks. The arguments are also valid when the random walks are defined by a symmetric probability distribution on the set of $2m$ generators and their inverses. Moreover, one can also define the Kesten measure for an arbitrary symmetric element $a \in \mathbb{C}[F_m]$ (F_m is a free group of rank m, which naturally covers all Schreier graphs in the space $\mathcal{X}_{2m}^{\mathrm{Sch}}$), and then Theorem 10.3 still remains valid.

One can go even further and admit nonpositive coefficients in the decomposition of the element c of the group algebra $\mathbb{C}[G]$ with respect to a basis consisting of elements of the group. However, in this case μ may turn out to be a *signed measure* (which is also called a *charge* in the Russian literature). In this case, one should speak of the coincidence of the sums of weights of closed paths of length n that start and end at the distinguished point of the graph, rather than of the coincidence of return probabilities. Here the weight of a path is the product of the labels of its edges (which in this case are given by the nonzero coefficients of the element c), and the associated graph on which the "random walk" occurs is the graph $\Gamma(G, P, c)$, which is a generalization of the Schreier graph dealt with above.

For example, Theorem 10.3 can be applied in the following situation. Let us return to the case when there is an infinite covering sequence of finite Schreier graphs $\{(\Gamma_n, v_n)\}_{n=1}^{\infty}$ that converges to an infinite Schreier graph (Γ, v). Such a sequence corresponds to a decreasing sequence $\{P_n\}_{n=1}^{\infty}$ of finite-index subgroups in the group G with intersection $P = \bigcap_{n=1}^{\infty} P_n$. In this case, the graphs Γ_n are the Schreier graphs constructed by means of the subgroups P_n, and Γ is the Schreier graph constructed by means of the subgroup P. A subgroup $P < G$ can be represented as the intersection of subgroups of finite index if and only if it is closed in the profinite topology.

Suppose, as usual, that the group G acts faithfully and spherically transitively on a spherically homogeneous tree T. Then the graphs Γ_n of the action on the nth, $n \geq 1$, level (which are also Schreier graphs $\Gamma(G, P_n, A)$, where $P_n = \mathrm{st}_G(u_n)$ and u_n is an nth-level vertex that belongs to a fixed path $\xi \in \partial T$) are connected. Let $a \in \mathbb{C}[G]$ be a symmetric element. Let π, π_n, and $\pi_n^{\perp}$

denote the same representations as in Section 9, and let $\rho_n = \rho_{G/P_n}$ and $\rho_{G/P}$ be the corresponding quasiregular representations, $P = \mathrm{st}_G(\xi)$. Recall that π_n is isomorphic to ρ_n.

To prove the next proposition, we need the concept of weak inclusion of representations (in the case of discrete groups).

Definition 10.2. Let ρ and ξ be two unitary representations of a group G that act in Hilbert spaces $\mathcal{H}_\rho$ and $\mathcal{H}_\xi$. Then ρ *is weakly contained* in ξ if, for any $\varepsilon > 0$, any finite subset $S \subset G$, and every orthonormal set of vectors $v_1, v_2, \dots, v_n$ in $\mathcal{H}_\rho$, there exists an orthonormal set of vectors $w_1, w_2, \dots, w_n$ in $\mathcal{H}_\xi$ such that the following inequality holds for any $s \in S$:

$$\sup_{s\in S}\bigl|\langle v_i, \rho(s)v_j\rangle - \langle v_i, \xi(s)w_j\rangle\bigr| < \varepsilon, \qquad 1 \le i,j \le n.$$

In other words, the matrix coefficients of one representation is approximated by the matrix coefficients of the other. In the topological language, a weak inclusion of ρ in ξ is equivalent to the fact that ξ belongs to the closure of the one-point set $\{\rho\}$ in the Fell topology on the dual space of unitary representations of the group (see [183]). In the language of C^*-algebras, this means that there exists a surjective homomorphism $C_\xi \to C_\rho$ that is constant on the elements of the group.

Proposition 10.4 [16]. 1. *The following relations hold*:

$$\mathrm{sp}(\pi(a)) = \overline{\bigcup_{n\ge 0} \mathrm{sp}(\pi_n(a))} = \overline{\bigcup_{n\ge 0} \mathrm{sp}(\pi_n^\perp)}. \tag{10.2}$$

2. *The spectrum of the operator* $\rho_{G/P}(a)$ *is contained in* $\mathrm{sp}(\pi(a))$, *and if the graph* Γ *is amenable* (*or, which is the same, the action* $(G, G/P)$ *is amenable*), *then* $\mathrm{sp}(\rho_{G/P}(a)) = \mathrm{sp}(\pi(a))$.

3. *If the subgroup* P *is amenable, then* $\mathrm{sp}(\rho_{G/P}(a)) \subset \mathrm{sp}(\rho_G(a))$, *where* ρ_G *is the left regular representation.*

Proof. Relations (10.2) are an obvious corollary to the fact that the representation π is decomposable into a direct sum of the representations $\pi_n^\perp$ and the fact that the spectrum of a symmetric matrix of block diagonal form with finite blocks is equal to the closure of the union of the spectra of the blocks.

Let us prove assertion 2. The inclusion $\mathrm{sp}(\rho(a)) \subset \bigcup_{n=1}^\infty \mathrm{sp}(\rho_n(a))$ is almost obvious (see [16]). To prove the reverse inclusion (assuming that the action on the orbit is amenable), suppose that $\lambda \in \mathrm{sp}(\rho_n(a))$ and that f is a corresponding eigenfunction (which can be assumed to be defined on the set of nth-level vertices of the tree), $\rho_n(a)f = \lambda f$. Let us identify the elements of the orbit $G(\xi)$ with the corresponding cosets with respect to the subgroup P. Since the action of G on the orbit $G(\xi)$ is amenable, it follows from Følner's criterion (Theorem 8.13) that there exists a Følner sequence $\{F_k\}_{k=1}^\infty$ of finite subsets of the orbit. Fix an n and define a function $f_{n,k}$ on $G(\xi)$ by the relation

$$f_{n,k}(gP) = \begin{cases} f(gP_n) & \text{if } \ gP \in F_k, \\ 0 & \text{if } \ gP \notin F_k. \end{cases} \tag{10.3}$$

The graph $\Gamma = \Gamma(G, P, a)$ covers the graph $\Gamma_n = \Gamma(G, P_n, a)$ in a natural way. Disregarding the labels of the edges, we apply the well-known path lifting theorem for the coverings of topological spaces [134]. Denote by $D = D_n$ the diameter of the graph Γ_n, which is not greater than the number $N_n = m_1 m_2 \dots m_n$ of nth-level vertices of the tree. For any two nth-level vertices u and v and any point $\zeta \in G(\xi)$ covering the vertex u (i.e., a point for which the corresponding path passes through the vertex u), there exists a point $\eta \in G(\xi)$ that covers v and is situated at a distance of at most D_n from ζ in the metric of the graph Γ. Let $\partial_D F$ denote the D-boundary of an arbitrary subset F of the vertex set of the graph Γ; i.e., $\partial_D F$ is the set of points situated at a distance of at most D from the complement of the set F. Let N be an upper bound for the cardinality of the set of vertices contained in balls of radius D_n in the graph Γ (as N, one can take m^{n+1}, where m is the degree of the vertices of the graph Γ). Then the following inequalities hold:

$$\|\rho(a)f_{n,k}-\lambda f_{n,k}\|\le(1+\lambda)|\partial F_k|\cdot\|f\|,$$

$$N\|f_{n,k}\|\ge\Biggl(\sum_{\zeta\in G(\xi)}\ \sum_{\eta\in G(\xi),\ d(\zeta,\eta)\le D_n}f_{n,k}^2(\eta)\Biggr)^{1/2}\ge(|F_k|-|\partial_{D_n}F_k|)\|f\|.$$

Indeed, the first inequality in the second relation is obvious because each term in the sum is repeated at most N times. To prove the second inequality, notice that each vertex ζ of the graph Γ can be surrounded by a set O_ζ that belongs to the neighborhood of radius D_n with center at this point and is bijectively projected to the vertices of the graph Γ_n. If this vertex belongs to the set F_k and lies at a distance greater than D_n from the boundary ∂F_k, then the terms that appear in the sum and correspond to this vertex add up to at least $\|f\|$, which implies the second inequality. Thus,

$$\left\|\rho(a)\frac{f_{n,k}}{\|f_{n,k}\|}-\lambda\frac{f_{n,k}}{\|f_{n,k}\|}\right\|\le\frac{2N|\partial F_k|}{|F_k|-|\partial_{D_n}F_k|}\to0$$

as $k\to\infty$. Hence, $\lambda\in\operatorname{sp}(\rho(a))$.

Finally, to prove assertion 3, we invoke the concept of weak inclusion of representations (the main facts concerning this concept are presented in detail in [53]). It is known (see, for example, [16, Proposition 3.5]) that a quasiregular representation $\rho_{G/P}$ is weakly contained in a regular representation ρ_G if and only if P is an amenable group. On the other hand, the weak inclusion under consideration is equivalent to the existence of a surjective homomorphism $C^*_{\mathrm r}\to C^*_{\rho_{G/P}}$ from the reduced C^*-algebra of the group G to the C^*-algebra generated by the representation $\rho_{G/P}$. Thus, if an element $a-\lambda I$ represented by an operator in $C^*_{\mathrm r}$ is invertible in this algebra, then the operator representing it in $C^*_{\rho_{G/P}}$ is also invertible. This implies the inclusion of the spectra. □

Now we can prove the earlier formulated Theorem 9.18.

Proof of Theorem 9.18. (i) Both algebras are the closures of the group algebra with respect to the norms defined by the operators of the corresponding representations. We have to prove that

$$\|x\|_\pi\ge\|x\|_{\rho_{G/P}}\tag{10.4}$$

for any $x\in\mathbb C[G]$.

In view of the relation $\|x\|^2=\|x^*x\|$, which is valid in an arbitrary C^*-algebra, we can assume that x is a symmetric element, and then the norm of the element x coincides with the upper bound of the points of the spectrum. Since, by assertion 2 of Proposition 10.4, the spectrum of the element x as an element of the algebra C^*_π contains the spectrum of the same element considered as an element of the algebra $C^*_{G/P}$, we obtain inequality (10.4).

(ii) Repeating the arguments used in the proof of assertion (i) and taking into account the second part of assertion 2 of Proposition 10.4 (which applies to the case of an amenable action on the orbit), we arrive at the conclusion that the spectra coincide and, hence, the norms of any element of the group algebra, considered as an element of each of the two C^*-algebras, are equal. This proves an isomorphism of the algebras. □

Note also the following fact, which is immediate corollary to the arguments used in the proof of Proposition 10.4.

Corollary 10.5. *The existence of a surjective $*$-homomorphism $C^*_{\mathrm r}(G)\to C^*_{G/P}$ that is constant on the elements of the group algebra $\mathbb Z[G]$ is equivalent to the amenability of the group P.*

Proof. This statement is an obvious consequence of the fact that a quasiregular representation is weakly contained in a regular representation if and only if there is a surjective homomorphism that is constant on the elements of the group algebra and of the fact that a subgroup $P \leq G$ is amenable if and only if the quasiregular representation $\rho_{G/P}$ is contained in the regular representation ρ_G. $\square$

Remark 10.1. The preceding statement is related to Kesten's theorem from [116], which states that if the support of a symmetric element x of the group algebra generates the group, then the norm of x increases under the factorization $G \to G/P$ with respect to a nonamenable subgroup P. Kesten considered the case when $P \trianglelefteq G$ is a normal subgroup. However, the proof of his theorem does not depend on the property of the subgroup P to be normal.

The following statement, which is based on a result of Serre [167], was first announced in [16]. For more details, see [98].

Theorem 10.6. *Suppose that a covering sequence $\{(\Gamma_n, v_n)\}_{n=1}^{\infty}$ of finite rooted graphs converges to a graph (Γ, v) in the sense of the topology of the space $\mathcal{X}_{2m}^{\mathrm{Sch}}$. Then the sequence $\{\mu_n\}_{n=1}^{\infty}$ of counting measures associated with the Markov operator of a simple random walk on the finite graphs converges weakly to some measure μ_*.*

In [98, 95], the measure μ_* was named the KNS spectral measure after Kesten, von Neumann, and Serre. It is intuitively obvious that the measure ν is in a sense the averaging of the Kesten measures μ_v over all vertices of the limit graph. The fact that this is indeed the case is confirmed by the argument given below, which I first heard from B. Steinberg and M. Abért in private communication (this fact is actually implicitly present in [95], namely, in the arguments preceding the proof of relation (4.7)). It seems that an analog of the KNS spectral measure exists not only for the Markov operator related to a simple random walk, but also in the case of arbitrary symmetric distributions on the set of generators of a group; however, this fact is yet to be proved.

The following statement allows one to calculate the Kesten spectral measure in some cases (for example, in the case of the lamplighter group). This statement is present implicitly in [95] and explicitly in [112].

Theorem 10.7. *Suppose that a self-similar group G defined by a finite automaton with a set of states A, which serves as a system of generators of G, acts on a tree T spherically transitively and essentially freely. Then the* KNS *spectral measure coincides with the Kesten measure of a simple random walk on the group G.*

Proof. Let θ be the Kesten measure of a simple random walk on the group, and

$$m = \frac{1}{2|A|} \sum_{a \in A \cup A^{-1}} a$$

be a group-algebra element that defines the Markov operator of a simple random walk on G and acts by means of left convolution (i.e., by means of the left regular representation) on the elements of $l^2(G)$. The moments $\theta^{(n)}$ of the measure θ satisfy the following relations:

$$\theta^{(n)} = \langle m^n \delta_1, \delta_1 \rangle = P_{1,1}^{(n)} = \mathrm{trace}\left(\sum_{a_{i_j} \in A \cup A^{-1}} a_{i_1} \dots a_{i_n} \right) = \sum_{a_{i_j} \in A \cup A^{-1}} \tau(a_{i_1} \dots a_{i_n}), \tag{10.5}$$

where trace is the von Neumann trace defined on the elements of the weak closure of the left regular representation by the relation $\mathrm{trace}(a) = \langle a\delta_1, \delta_1 \rangle$, and τ is the recurrent trace defined in Section 9. The last equality in (10.5) is valid because in the case of an essentially free action the recurrent trace behaves on the elements of the group in exactly the same way as the von Neumann trace (namely, it vanishes on nonidentity elements and is 1 on the identity element).

Let $\mu_k^{(n)}$ stand for the nth moment of the counting measure μ_k of the graph Γ_k. Then

$$\mu_k^{(n)}=\int\lambda^n\,d\mu_k(\lambda)=\langle m^n\delta_{v_k},\delta_{v_k}\rangle=\frac{1}{d^k}\sum_{k=1}^{d^k}\lambda_k^n=\operatorname{Tr}[m^n]_k=\sum_{a_{i_j}\in A\cup A^{-1}}\operatorname{tr}[a_{i_1}\dots a_{i_n}]_k,\tag{10.6}$$

where d^k is the cardinality of the set of kth-level vertices of the tree (we assume that the tree has the branch multiplicity d), $[m]_k$ and $[g]_k$ are the presentations of elements $m\in\mathbb{C}[G]$ and $g\in G$ by matrices of order d^k, Tr is the ordinary matrix trace, and tr is the normalized trace on the matrix algebra. Since $\operatorname{tr}[g]_k\to\tau[g]_k$ by the construction of the recurrent trace, we find that the moments $\mu_k^{(n)}$ converge to $\theta^{(n)}$ for any n. Thus, the sequence of counting measures converges to the Kesten measure, and, hence, the latter coincides with the KNS spectral measure. The theorem is proved. $\square$

Let (G,X,χ) be a dynamical system with invariant measure, where G is a finitely generated group with a system of generators A. For every point $x\in X$, the orbital graph Γ_x is isomorphic to the Schreier graph (Γ,P_x,A), where P_x is the stabilizer of the point x. Let $P_x^{(n)}$ be the probability of return to the point x after n steps of the simple random walk on the graph Γ_x. Recall that this probability coincides with the nth moment of the Kesten measure μ_x of the graph Γ_x with distinguished point (root) x. Define the numbers $\widetilde{P}^{(n)}$ by the relation

$$\widetilde{P}^{(n)}=\int\limits_X P_x^{(n)}\,d\chi(x),$$

and define the measure χ_* by the condition that its moments coincide with the numbers $\widetilde{P}^{(n)}$. Alternatively, the measure χ_* can be defined as the integral $\int_X\mu_x\,d\chi(x)$.

The measure χ_* and its characteristics (support, moments, etc.) are invariants of the dynamical system. We will call this measure the KNS spectral measure as well. This name is motivated by the following fact, which is implicitly presented in [112] and was independently noted by Abért (private communication).

Theorem 10.8. *The measure ν_* associated with a dynamical system $(G,\partial T,\nu)$, where T is a spherically homogeneous tree and $G\le\operatorname{Aut}(T)$ is a finitely generated group, coincides with the* KNS *spectral measure μ_*.*

Proof. Let Γ_k be the graph of the action on the kth level of the tree (this graph need not be connected because we do not assume that the actions are transitive on the levels of the tree), $\mu_v^{\Gamma_k}$ be the Kesten measure associated with a vertex v of the graph Γ_k, and $P_k^{(n)}$ denote the nth moment of the measure μ_k. The following relations hold:

$$P_k^{(n)}=\int\lambda^n\,d\mu_k(\lambda)=\frac{1}{|V_k|}\sum_{v\in V_k}\int\lambda^n\,d\mu_v^{\Gamma_k}(\lambda)=\int\limits_{V_k}\int\lambda^n\,d\mu_v^{\Gamma_k}(\lambda)\,d\nu_k(v),\tag{10.7}$$

where V_k is the vertex set of the graph Γ_k (i.e., the set of kth-level vertices of the tree) and ν_k is the uniform measure on V_k (here we used the relation from Proposition 10.2). The integral $\int\lambda^n\,d\mu_v^{\Gamma_k}$ defines the probability $P_{v,k}^{(n)}$ of return to the vertex $v\in V_k$ after n steps of the simple random walk on the graph Γ_k. If $x\in\partial T$ is a boundary point, n is fixed, and x_k denotes the kth-level vertex that belongs to the path x, then starting from a certain level k_x we have $P_{v,k}^{(n)}=P_x^{(n)}$, $k\ge k_x$. In fact, for a fixed n, one can find a K independent of x such that $P_{v,k}^{(n)}=P_x^{(n)}$ for all $k\ge K$, $x\in X$. Indeed, for every $x\in\partial T$, there exists a neighborhood U_x such that, for all of its points, the probabilities of return after n steps of the random walk on the graph Γ_x are equal. Choosing a finite subcovering

$\{U_{x_i}\}_{i\in I}$ from the open covering $\{U_x\}_{x\in X}$ of the boundary ∂T, fixing an appropriate k_i for every $i \in I$, and setting $K = \max_{i\in I} k_i$, we obtain a value for which the relations $P_{v,k}^{(n)} = P_x^{(n)}$ hold for $k \geq K$. Taking into account that $\lim_{k\to\infty} \mu_k = \mu_*$ and $\lim_{k\to\infty} \nu_k = \nu$ (recall that ν stands for the uniform measure) and passing to the limit under integral sign in (10.7), we arrive at the relations

$$\int \lambda^n \, d\mu_*(\lambda) = \int\limits_{\partial T} P_x^{(n)} \, d\nu(x) = \widetilde{P}^{(n)}, \tag{10.8}$$

which show that the moments of the measures μ_* and ν_* coincide. This completes the proof of the theorem. □

Now we apply our results to the construction of asymptotic expanders. Recall that a sequence $\{\Gamma_n\}_{n=1}^{\infty}$ consisting of connected m-regular graphs, $m \geq 3$, is called an expander if there exists an $\epsilon > 0$ such that, except for the eigenvalue 1, the spectra of the Markov operators M_n of the graphs Γ_n are contained in the interval $[-1+\epsilon, 1-\epsilon]$. The first examples of expanders were constructed by Margulis [133] with the use of groups with Kazhdan's T-property [24]. In spite of intensive investigations related to this concept (which are of both theoretical and practical interest; the point is that expanders have applied value [125]), at present there are only a few constructions of expanders, and new methods for constructing them are of undoubted interest.

The following heuristic reasoning shows how one can try to construct new examples of expanders. Namely, starting from a finite automaton $\mathcal{A}$ generating a nonamenable group $G = G(\mathcal{A})$, consider the covering sequence of Schreier graphs $\{\Gamma_n\}_{n=1}^{\infty}$ associated with the actions of the group on the levels of the tree, as described above. By Kesten's criterion, since the group is nonamenable, the norm of the Markov operator on this group is less than 1; therefore, this fact should imply the validity of the condition on the spectra that underlies the definition of an expander. However, there is something naive in this reasoning, because there are no arguments that would show that the nonamenability of G in the construction indeed implies the property of the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ to be an expander. It is only clear that the construction does not work in the case of amenable groups.

The hope to obtain expanders becomes greater if one additionally assumes that the action of a nonamenable group G is essentially free. Although we have no proof of the property of the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ to be an expander even under this additional condition, the results obtained above allow us to prove a weaker but still useful property, namely, the property to be an asymptotic expander.

Definition 10.3. A covering sequence $\{\Gamma_n\}_{n=1}^{\infty}$ of graphs is called an *asymptotic expander* if the support of the KNS-spectral measure μ_* associated with this sequence is contained in the interval $[-1+\epsilon, 1-\epsilon]$ with some positive ϵ.

The meaning of this definition is as follows. If the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ has "bad" (i.e., accumulating to ± 1) eigenvalues of the Markov operator, then their density tends to zero as n increases; thus, for large n, these eigenvalues become virtually invisible, and, hence, the corresponding graphs Γ_n behave almost as expanders. As P. Kuchment told me, these kinds of phenomena are observed in mathematical models of crystal physics. It seems quite probable that asymptotic expanders can be applied in practice with almost the same success as true expanders, because the effect of "bad" (i.e., close to ± 1) eigenvalues on telecommunication systems becomes negligible as $n \to \infty$.

Theorem 10.9. *Suppose that a strongly self-similar group $G = G(\mathcal{A})$ is nonamenable and acts essentially freely on the boundary of the corresponding tree. Then the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ is an asymptotic expander.*

Proof. This theorem is an obvious corollary to Theorem 10.7, because the support of the Kesten measure of a nonamenable subgroup does not contain the points -1 and 1. □

Now we give the "simplest" examples of asymptotic expanders.

Example 10.1. Let $\mathcal{A}$ be the Aleshin automaton represented by the Moore diagram in Fig. 5.1. It generates the free group F_3 of rank 3, which is nonamenable. The Kesten measure of the simple random walk on F_3 is concentrated on the interval $\left[-\frac{\sqrt{5}}{3}, \frac{\sqrt{5}}{3}\right]$. As already mentioned above in Section 5, the action of F_3 defined by the automaton $\mathcal{A}$ is essentially free. Therefore, the corresponding sequence $\{\Gamma_n\}_{n=1}^{\infty}$ is an asymptotic expander.

Example 10.2. Let $\mathcal{B}$ be the Bellaterra automaton represented by the diagram in Fig. 3.2. It generates the free product $C_2 * C_2 * C_2$ of three copies of an order 2 group. This group is nonamenable (it contains a noncommutative free subgroup of finite index) and acts essentially freely on the boundary of a binary tree. Thus, the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ is an asymptotic expander in this example as well. Moreover, the spectrum of the Markov operator of the simple random walk of the group $C_2 * C_2 * C_2$ fills the interval $\left[-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\right]$, and so the asymptotic spectrum of the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ is contained in this interval.

Recall that a finite m-regular graph Γ, $m \geq 3$, is called a Ramanujan graph if its spectrum, upon removal of the points -1 and 1, is contained in the Kesten interval $\left[-2\frac{\sqrt{m-1}}{m}, 2\frac{\sqrt{m-1}}{m}\right]$, i.e., in the interval that supports the spectrum of the Markov operator on a homogeneous (not rooted) tree with the branch degree m. Finding infinite sequences of m-regular finite Ramanujan graphs is a difficult and enthralling problem [127, 50]. These sequences possess the strongest expanding properties and, in this sense, are the most efficient expanders. One may have an impression that the two examples above are examples of sequences of Ramanujan graphs, because their asymptotic spectrum is precisely the Kesten interval. Nevertheless, a computer-aided verification shows that this is not the case, and, starting from not too large values of n, the graphs Γ_n cease to be Ramanujan. The same computer simulations show that probably the sequences $\{\Gamma_n\}_{n=1}^{\infty}$ are expanders in both examples. In this connection, we raise the following two questions.

Problem 10.1. Is it true that the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ in each of the two examples above (related to the Aleshin and Bellaterra automata) is an expander?

Problem 10.2. Is it true that the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ constructed by a finite automaton is never a sequence of Ramanujan graphs?

In our view, the answer to the second question is more likely to be positive rather than negative, and this is associated with the following fact: For the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ to be a sequence of Ramanujan graphs, it is necessary that the group G generated by an automaton be either a free group of rank equal to the number of states of the automaton, or a free product of groups of order 2 with the number of factors equal to the number of states of the automaton, or a free product of such groups. Moreover, the action on the boundary should be essentially free. However, it is most likely that the kernel of the homomorphism of C^*-algebras dealt with in Corollary 9.12 is always nontrivial, which points to the fact that the norm of the Markov operator in $L^2(\partial T, \nu)$ is greater than the Kesten threshold $2\frac{\sqrt{m-1}}{m}$. However, these arguments need substantiation.

We call the distance from the support of the KNS spectral measure of an asymptotic expander to the set $\{-1, 1\}$ Kazhdan's asymptotic constant (by analogy with a similar definition for ordinary expanders). Thus, in the above Examples 10.1 and 10.2, Kazhdan's asymptotic constants are $1-\frac{\sqrt{5}}{3}$ and $1-\frac{\sqrt{2}}{3}$, respectively.

In our view, the sequences of graphs $\{\Gamma_n\}_{n=1}^{\infty}$ constructed by means of a finite automaton have the most constructive definition, and the practical implementation of such sequences should be much easier than the implementation of existing expanders or sequences of graphs with other interesting properties. Therefore, we expect them to have good prospects in theoretical investigations and practical applications.

Let us mention another range of questions that arise in the study of covering sequences $\{\Gamma_n\}_{n=1}^{\infty}$ of Schreier graphs. We will assume that these graphs are connected (which corresponds to the case when the group generated by an automaton acts transitively on the levels). Denote by $\delta(n)$ the difference $1-\lambda_1$ between the eigenvalue 1 and the next smaller eigenvalue λ_1 of the Markov operator M_n on the graph Γ_n. The quantity $\delta(n)$ (which is often called a *spectral gap*) can alternatively be defined as the first nonzero eigenvalue of the Laplacian $\Delta_n = I - M_n$, where I is the identity operator. If the sequence $\{\delta(n)\}_{n=1}^{\infty}$ is bounded away from zero by a positive constant, then $\{\Gamma_n\}_{n=1}^{\infty}$ is an expander. Otherwise, the sequence $\{\delta(n)\}_{n=1}^{\infty}$ tends to zero; then the question of the rate of this convergence arises. The decrease may be either power, exponential, or, possibly, intermediate between power and exponential; however, it seems that there are no examples of such behavior at present. For the lamplighter group $\delta(n) \sim 1/n^2$ [95], while for the Hanoi Towers group $\mathcal{H}^3$ the sequence $\delta(n)$ decreases exponentially with $\delta(n) \sim (1/5)^n$ [93]. From the asymptotic solution obtained by Szegedy [178] in the Tower of Hanoi problem with k pegs, which shows that the distance between the vertices 0^n and 1^n in $\Gamma_n^{(k)}$ increases as $e^{n^{1/(k-2)}}$, one can easily derive that the diameter of the graphs $\Gamma_n^{(k)}$ increases with n as $e^{n^{1/(k-2)}}$. Applying Chung's inequality

$$d(n) \le \frac{\log|\Gamma_n| - 1}{-\log\delta(n)},$$

which connects $\delta(n)$ with the diameter $d(n)$ of the graph Γ_n, one can easily obtain the following upper estimate:

$$\delta(n) \preceq ne^{-n^{\frac{1}{k-2}}};$$

this estimate possibly indicates that the decay of the spectral hole for $k \ge 4$ is of intermediate character between power and exponential. However, this fact needs to be supported by an appropriate lower estimate.

Another important asymptotic characteristic of the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ is the growth of the diameters $d(n) = \operatorname{diam}(\Gamma_n)$. For an expander, it is linear, which follows from Chung's inequality. A straightforward verification shows that the growth of the diameters of the Schreier graphs associated with the realization of the lamplighter group by the automaton shown in Fig. 3.1 is linear. Since the lamplighter, being a solvable group, is amenable, it is clear that the linear growth of the diameters does not imply the property to be an expander. The examples above show that for many self-similar groups generated by automata, the growth of the diameters of the graphs Γ_n is exponential. In fact, this growth is exponential for contracting self-similar groups.

Proposition 10.10. *Let G be a self-similar contracting group and $\{\Gamma_n\}_{n=1}^{\infty}$ be the sequence of Schreier graphs associated with this group. Then* $\liminf_{n\to\infty}\sqrt[n]{d(n)} = \chi > 1$.

Proof. In [16], it is proved that the growth of the infinite Schreier graphs Γ_ξ for contracting groups is polynomial; i.e., the growth function $\gamma_\xi(r)$ of a graph Γ_ξ, which counts the number of vertices situated at a distance of at most r from ξ, is bounded from above by a function of the form Cr^α, where C and α are constants, $C > 0$ and $\alpha > 0$. Let u_n be the nth-level vertex of the path ξ in the tree. Then, for any n, the growth function $\gamma_n(r)$ of the graph Γ_n, which counts the number of vertices in the graph Γ_n that are situated at distance $\le r$ from the vertex u_n, is bounded from above by a power function Cr^α. Since the number of vertices in the graph Γ_n is d^n, where d is the regularity of the tree, this obviously implies that the growth of $d(n)$ is exponential. $\square$

Recall that in Definition 7.2 we introduced Sidki classes of polynomially growing automata and the corresponding classes of groups. Since the groups generated by bounded automata are contracting groups, the growth of diameters for them is always exponential. In [31], Bondarenko developed a method for calculating the order of exponential growth χ for this case.

The groups generated by automata that are polynomially growing but not bounded are not necessarily contracting, and the diameters for these groups may have intermediate growth, as shown in Example 7.2 above. The graphs generated by exponentially growing automata may have all three types of growth of the diameters. For example, the automaton shown in Fig. 7.7, which represents the Hanoi Towers group $\mathcal{H}^4$, has exponential growth in the sense of Sidki, whereas the diameters of the corresponding Schreier graphs $\Gamma_n^{(4)}$ grow in an intermediate way. Similarly, the automata defining the higher rank Hanoi Towers groups $\mathcal{H}^k$, $k \geq 4$, have exponential growth, whereas the diameters of the graphs $\Gamma_n^{(k)}$ associated with them grow in an intermediate way, more precisely, as $e^{n^{1/(k-2)}}$, which follows from the above-mentioned result of Szegedy.

It seems that the Bellaterra and Aleshin automata, which have exponential growth, also give series of graphs $\{\Gamma_n\}_{n=1}^{\infty}$ with linear growth of the diameters; however, this fact is still to be proved (this will be so if our conjecture that these automata generate expanders is confirmed).

For the sequences of finite graphs $\{\Gamma_n\}_{n=1}^{\infty}$, the other asymptotic characteristics related to random walks [140] and to models of statistical physics, such as the Ising model or the dimer model [49], should also be studied, including the asymptotic calculation of the number of spanning subtrees, the Euler characteristic, the chromatic number, the Jacobian of the graphs [11], the *sandpile* model [135], etc.

On the other hand, one may examine the asymptotic properties of the infinite Schreier graphs Γ_ξ, $\xi \in \partial T$: growth, amenability, spectral properties, asymptotic properties of random walks, models of statistical physics, sandpile model [135], etc. For example, we have already mentioned that the infinite graph associated with the linear automaton shown in Fig. 7.13 has growth of order $n^{\log_4 n}$. On the other hand, it is shown in [91] that the growth of the infinite graphs $\Gamma^{(k)}$ of the Hanoi Towers groups $\mathcal{H}^k$ satisfies the following estimates:

$$a^{(\log n)^{k-2}} \prec \gamma^{(k)}(n) \prec b^{(\log n)^{k-2}}$$

for some positive constants a and b, $a \leq b$. Thus, for $k \geq 4$, the growth is intermediate between polynomial and exponential. Note that the growth order of type $c^{(\log n)^d}$ is quite rare in combinatorial problems and is likely to be impossible for Cayley graphs of groups, because it contradicts the following conjecture, which was put forward by the author in many of his works (and was mainly formulated as a question rather than a conjecture; however, now we place emphasis on the conjecture).

Conjecture 10.1. *If the growth of a finitely generated group is slower than the growth of the function $e^{\sqrt{n}}$, then it is polynomial, and, hence, the group is virtually nilpotent.*

This conjecture was proved for groups approximated by finite p-groups [76] and is also valid for a more general class of groups approximated by nilpotent groups, with essentially the same proof. Although the problem of describing all possible growth orders of groups seems to be hopeless, we nevertheless think that this problem, just as other problems of asymptotic character, has a chance to be solved for groups, graphs, and other objects related to finite automata.

Quite recently, Bondarenko has proved that the growth of infinite Schreier graphs associated with polynomially growing automata is subexponential and that there is an upper bound of the form $n^{(\log n)^m}$ on the growth, with some positive constant m [33].

Problem 10.3. (a) What growth orders are possible for the diameter $d(n)$ in the sequences $\{\Gamma_n\}_{n=1}^{\infty}$ associated with finite automata?

(b) The same question for the sequence $\delta(n)$ of values of the first nonzero eigenvalue of the Laplacian.

(c) The same question for the growth degrees of infinite Schreier graphs Γ_ξ, $\xi \in \partial T$.

(d) All the previous questions, but now formulated for the Schreier graphs associated with polynomially growing automata.

Problem 10.4. Does there exist an algorithm that would allow one to determine the type of growth of a certain class of objects for a given automaton (for example, of the classes of objects mentioned in questions (a), (b), and (c))? A similar question for the property of the graphs Γ_ξ, $\xi \in \partial T$, to be amenable.

We conclude this section with the discussion of another range of interesting questions related to Schreier graphs. The matter concerns an analog of the Ruziewicz problem for actions on rooted trees. Suppose that a group G acts spherically transitively on a rooted (not necessarily regular) tree T. Then, on ∂T, the action is minimal, ergodic, and uniquely ergodic, i.e., has a unique invariant probability measure (namely, the uniform measure ν). By a generalized Ruziewicz problem we mean the question of uniqueness of a finitely additive G-invariant measure μ with the following properties: μ is defined on the σ-algebra $\mathcal{B}$ of Lebesgue measurable sets on the boundary of the tree, μ takes values in the interval $[0,1]$, and $\mu(\partial T) = 1$. Such measures are in one-to-one correspondence with G-invariant means in $L^\infty(\partial T, \mathcal{B}, \nu)$. The uniqueness problem for such a measure in the case of the rotation group of the sphere $\mathbb{S}^n$ of dimension $n \geq 1$ was raised by Ruziewicz in the early 1920s and was studied by Banach, who showed that there are many such measures in dimension 1. In dimension $n \geq 2$, Ruziewicz's problem was solved positively only in the early 1980s by Sullivan, Margulis, and Drinfeld, who used the above-mentioned Kazhdan's T-property to this end. Recall that a group is said to possess Kazhdan's T-property if any of its unitary representations that has an almost invariant vector contains in fact a nonzero invariant vector (i.e., if the trivial representation is weakly contained in a given one, then the trivial representation is a subrepresentation; in terms of the Fell topology, this means that the trivial representation is an isolated point in the dual space). A representation ρ of a group G in a Hilbert space H has an almost invariant vector if, for any $\epsilon > 0$ and any finite subset $F \subset G$, there exists a unit vector $v \in H$ such that

$$\|\rho(s)v - v\| < \epsilon$$

for all $s \in F$.

Rosenblatt showed that the nonuniqueness of such a measure is equivalent to the existence of an asymptotically invariant nontrivial net of subsets $E_\alpha \subset \partial T$, i.e., a net of measurable subsets of the boundary whose measures are separated from 1 (i.e., $\nu(E_\alpha) \leq c < 1$), such that

$$\lim_\alpha \frac{\nu(gE_\alpha \triangle E_\alpha)}{\nu(E_\alpha)} = 0$$

for any $g \in G$.

An action with invariant probability measure is said to be *strongly* ergodic if there do not exist nontrivial asymptotically invariant sequences. Such actions have been studied by K. Schmidt, A. Connes, and B. Weiss, who gave a characterization of the Kazhdan property in terms of this concept: a group G possesses the Kazhdan property if and only if an arbitrary action of the group with invariant probability measure is strongly ergodic [24, Theorem 6.3.4]. In an unpublished preprint,[5] following the ideas of Kaimanovich expressed in [106], we, together with F. Paulin, defined *fiberwise* amenable actions (G, X, μ) of a finitely generated group as actions such that, for an arbitrary $\epsilon > 0$, there exists a measurable map from X into the set of finite subsets of vertices of the Schreier graph Γ_x, $x \in X$, $x \to A_x$, that satisfies the condition

$$\frac{|\partial A_x|}{|A_x|} < \epsilon$$

for μ-almost every x.

[5]R. Grigorchuk and F. Paulin, "Spectral Characterization of Amenable Group Actions," Preprint (1999).

The following theorem was proved in the same preprint, and a larger part of the proof can be found in Sections 6.3 and 6.4 of the book by Bekka, de la Harpe, and Valette [24], where the authors also discuss many other questions related to the Ruziewicz problem.

Theorem 10.11. *Let G be a finitely generated group with a set of generators A. The generalized Ruziewicz problem has a negative solution* (*i.e., an invariant measure is not unique*) *if and only if any of the following equivalent conditions holds*:

(i) *the Hecke-type operator $\mathcal{M}$ associated with the representation π in $L^2(\partial T,\nu)$ and restricted to the orthogonal complement $L^2_0(\partial T,\nu)$ of constant functions contains* 1 *in its spectrum*;

(ii) *the representation π, restricted to $L^2_0(\partial T,\nu)$, contains an almost invariant vector*;

(iii) *the action of G on ∂T has a nontrivial asymptotically invariant sequence*;

(iv) *the action of G on ∂T is fiberwise amenable.*

11. COST OF ACTIONS AND RANK GRADIENT

The concept of cost of a group action by measure-preserving transformations was introduced by Levitt [124]. Later on, this concept was developed largely due to the remarkable studies by Gaboriau [63, 64]. The concept of rank gradient was introduced by Lackenby [121] in connection with the research on three-dimensional topology [121, 122]. Investigations around this concept received a new impetus from the study by Abért and Nikolov [5] and the ensuing study by Osin [151]. At present, both concepts (which, as we will soon see, are closely related) play a significant role in asymptotic group theory.

We begin with the rank gradient. Let G be a finitely generated residually finite group (we will keep this condition on the group until we begin the discussion of the cost of actions) and $\{H_n\}_{n=1}^{\infty}$ be a decreasing sequence of finite-index subgroups. The upper rank gradient of this sequence is

$$\mathrm{RG}(G,\{H_n\}) = \varlimsup_{n\to\infty}\frac{d(H_n)-1}{|G:H_n|} \tag{11.1}$$

($d(H)$ denotes the minimum number of generators of the group H, i.e., its rank). The lower rank gradient and the rank gradient (if the limit exists) are defined in a similar way. The study of the growth rate of the ranks of subgroups in decreasing chains of finite-index subgroups is of interest both for group theory itself and for applied questions. In this case, it is expedient to consider only chains $\{H_n\}_{n=1}^{\infty}$ with trivial core, i.e., such that the intersection $H=\bigcap_{n=1}^{\infty}H_n$ does not contain a nontrivial normal subgroup. An important particular case is that when this intersection is trivial. We also distinguish the case when H_n are normal subgroups.

The absolute rank gradient of a group G is defined as

$$\mathrm{RG}(G) = \inf_{H\le G,\,|G:H|<\infty}\frac{d(H)-1}{|G:H|}. \tag{11.2}$$

As already mentioned, this concept was introduced by Lackenby in [121, 122] and was motivated by the studies in the field of the theory of expanders and three-dimensional manifolds, in particular, by questions related to the conjecture on the relationship between the Heegaard genus and the rank of the fundamental group of a three-dimensional manifold.

An example of a group with positive rank gradient is given by the free group F_m with $m\ge 2$ generators. According to the classical Schreier theorem (see [128]), which connects the index $|F_m\!:\!H|$ of a subgroup H of the free group F_m with the number $d(H)$ of its generators by the formula

$$d(H)-1=|F_m:H|(m-1),$$

the rank gradient of any decreasing chain of finite-index subgroups is $m-1$, so $\mathrm{RG}(F_m)=m-1$.

Below we present conditions that guarantee the vanishing of the rank gradient of a sequence of subgroups, or even of the whole group. These conditions are based on the notion of amenability.

In Section 2, with a decreasing sequence of finite-index subgroups of an arbitrary group we associated a rooted tree (coset tree) $T = T(G, \{H_n\})$, whose vertices are left cosets with respect to the subgroups and the action is defined by left multiplication. If the core of the chain of subgroups is trivial, then the action of the group on this tree is faithful. We say that a decreasing chain of finite-index subgroups $\{H_n\}_{n=1}^{\infty}$ is essentially free if the action on the boundary of the associated tree is essentially free with respect to the uniform measure. Sometimes this condition is called Farber's condition, who applied it in order to prove approximation results for L^2-invariants of groups [57].

The following conjecture was put forward by M. Abért and N. Nikolov.

Conjecture 11.1. *If a chain $\{H_n\}_{n=1}^{\infty}$ is essentially free, then its rank gradient coincides with the absolute rank gradient of the group G,*

$$\mathrm{RG}(G, \{H_n\}) = \mathrm{RG}(G). \tag{11.3}$$

For sequences $\{H_n\}_{n=1}^{\infty}$ whose rank gradient vanishes, we can consider a function $\mathrm{gr}(n)$ defined by the relation

$$\mathrm{gr}(n) = \frac{d(H_n)}{|G : H_n|}.$$

We will call this function the *relative decrease rank gradient function* of the sequence $\{H_n\}_{n=1}^{\infty}$ (or, for short, the "relative rank") and consider the rate of its decrease as $n \to \infty$. If $\mathrm{gr}(n) \to 0$, then we can make the rate of decrease arbitrarily large by removing some terms from this sequence; therefore, of special interest are *noncondensable* sequences. For example, restricting ourselves to groups approximated by finite p-groups (p is a prime number), we can consider only chains of subgroups satisfying the condition $|H_{n+1} : H_n| = p$ for any n. The decrease of the rank in such chains—we call them *condensed p-chains*—is of particular interest.

Problem 11.1. 1. What decrease orders can the function $\mathrm{gr}(n)$ have for condensed p-chains with a trivial core in finitely generated groups approximated by finite p-groups?

2. The same question for normal chains.

3. The same question for chains with trivial intersection.

In addition to these questions, it is interesting to study within what limits the relative decrease of the rank gradient can vary within a single group. What conditions on a group guarantee that the asymptotic behavior of the relative decrease does not depend on the choice of one or another type of decreasing chain of subgroups?

If a group G acts on a rooted tree T, then, as we repeatedly pointed out, a natural choice for a decreasing sequence of finite-index subgroups is given by a sequence of stabilizers $P_n = \mathrm{st}_G(u_n)$, where $\{u_n\}$ is a sequence of vertices (the index n denotes the level to which the vertex belongs) that belong to some geodesic path connecting the root of the tree with infinity (i.e., to a boundary point of the tree). If the tree T is p-regular and the action is spherically transitive, then $|G : P_n| = p^n$, and such a chain of subgroups is p-condensed.

As an example, consider the intermediate growth group $\mathcal{G} = \langle a, b, c, d\rangle$ from Example 2.3, which acts spherically transitively on a binary tree, and let $P_n = \mathrm{st}_G(1^n)$, $n = 1, 2, \dots$. This sequence decreases, its intersection is equal to the stabilizer $\mathrm{st}_{\mathcal{G}}(1^\infty)$, and the core is trivial (because $\mathcal{G}$ is just-infinite). In [17], the stabilizers P_n were described recurrently; using this description, we can conclude that the sequence $d(P_n)$ grows no faster than $10n$ (the constant 10 can be replaced by a smaller one, possibly by 3). Thus, in this case the relative rank decreases no slower than $\frac{10n}{2^n}$. It seems that the relative decrease of the rank gradient in this case is asymptotically of order $\frac{n}{2^n}$;

however, this is yet to be verified. The relative rank of the sequence $\{\mathrm{st}_{\mathcal{G}}(n)\}$ of the level stabilizers (which is not 2-condensed, but can be condensed) exhibits similar asymptotic behavior; this follows from the explicit description of the stabilizers that was obtained in [160], as well as from the calculation of the indices of the stabilizers of levels [81]. Indeed, starting from the fourth level,

$$\mathrm{st}_{\mathcal{G}}(n) \simeq \mathrm{st}_{\mathcal{G}}(3) \times \ldots \times \mathrm{st}_{\mathcal{G}}(3)$$

(2^{n-3} factors); i.e., the number of generators grows exponentially with n, while the index

$$[\mathcal{G} : \mathrm{st}_{\mathcal{G}}(n)] = 2^{5\cdot 2^{n-3}+2}$$

grows as a double exponential.

Consider another example. Using the representation of the lamplighter group $\mathcal{L}$ by the automaton shown in Fig. 3.1 and taking an arbitrary boundary point corresponding to a dyadic irrational, we obtain a sequence of stabilizers that have trivial intersection and each of which is isomorphic to $\mathcal{L}$ (if the boundary point is not assumed to correspond to a dyadic irrational, then the intersection may turn out to be an infinite cyclic group; however, the core of the sequence will nevertheless be trivial). This follows from the results of [95] and was explicitly mentioned by Nekrashevych and Pete in [145], who gave numerous examples and even constructed series of groups that possess decreasing chains of finite-index subgroups with trivial intersection that consist of groups isomorphic to the group itself. The question of existence of such groups (which are said to be scale-invariant in [145]) was raised by Benjamini. For such sequences, the relative rank gradient is equal to $\mathrm{gr}(n) = \frac{C}{d^n}$ for some constants C and d. At the same time, in $\mathcal{L}$ there exist condenced 2-chains of subgroups for which the decrease of the relative rank gradient can vary from an exponential decay to a constant function (i.e., the rank gradient may be positive, which looks intriguing in the light of the theorems formulated below). This results from the following proposition proved in [8] (the structure of subgroups of the lamplighter is studied in more detail in a preprint[6] by the present author and R. Kravchenko, in which it is also shown that a faithful action of the lamplighter on the boundary of a tree obtained from a decreasing chain of finite-index subgroups is always topologically free).

Proposition 11.1. *Let $\mathcal{L}_n$ denote the group $(\mathbb{Z}_2^n) \wr \mathbb{Z}$. Then the index 2 subgroups of the group $\mathcal{L}_n$ are exhausted by groups isomorphic to $\mathcal{L}_n$ and $\mathcal{L}_{2n}$.*

An important role in studying the rank gradient (as well as the cost of actions) is played by the property of amenability.

Theorem 11.2. (1) [5]. *Let G be a residually finite group containing an amenable normal subgroup, and let a chain $\{H_n\}_{n=1}^{\infty}$ be essentially free. Then its rank gradient is zero.*

(2) [4]. *Let G be a finitely presented amenable residually finite group. Then the rank gradient vanishes for an arbitrary decreasing chain of finite-index subgroups with trivial intersection.*

(3) [4]. *Let G contain an infinite solvable normal subgroup. Then the rank gradient vanishes for an arbitrary decreasing chain of finite-index subgroups with trivial intersection.*

The above-presented information about the types of decrease of the relative rank gradient of the lamplighter group shows that even for metabelian (i.e., solvable of derived length 2) groups (including the lamplighter group), assertion (1) of Theorem 11.2 fails unless we assume that the action associated with the chain is essentially free. Assertions (2) and (3) of this theorem also fail to hold unless we assume that the intersection is trivial.

Now we consider the cost of a group action (and the concept of cost of an equivalence relation). Let (G, X, μ) be a dynamical system for which X is a standard Borel space [67], G is a countable

[6] R. Grigorchuk and R. Kravchenko, "The Lattice of Subgroups of the Lamplighter Group and Topological Freeness of Boundary Actions," Preprint (2011).

group, and the measure μ is invariant. The action generates an equivalence relation E, $E \subset X \times X$, on X (decomposition into orbits) for which x and y are in the same class if there exists a $g \in G$ such that $y = g(x)$ (this is expressed as xEy or $(x, y) \in E$). Obviously, every equivalence class is either finite or countable (these relations are called countable). E is a Borel subset of $X \times X$ (only measurable actions are considered). One can also study arbitrary Borel equivalence relations with at most countable classes (again defined by Borel subsets $E \subset X \times X$); however, in view of the following Feldman–Moore theorem from [58], there is no difference between considering countable equivalence relations and considering decompositions into orbits of actions of countable groups.

Theorem 11.3. *Let E be a countable Borel equivalence relation on a standard Borel space X. Then there exist a countable group G and a Borel action of G on X such that the decomposition into orbits coincides with E. Moreover, the group G and its action can be chosen so that*

$$xEy \quad \Leftrightarrow \quad \textit{There exists a } g \in G \textit{ such that } g^2 = 1 \textit{ and } g(x) = y.$$

The question raised by Feldman and Moore as to whether any equivalence relation can be generated by an essentially free action of a group when the decomposition has a quasi-invariant measure was solved in the negative (see [61] and Subsection 4.3.1 in [62]).

A measure μ defined on X is E-(quasi)invariant if it is G-(quasi)invariant for some group acting in a Borel fashion on X for which the decomposition into orbits coincides with the relation E (Propositions 2.1 and 16.1 in [113]). The ergodicity of an equivalence relation with respect to a quasi-invariant finite measure means the ergodicity of the corresponding group action that generates this equivalence relation, namely, the absence of nontrivial (i.e., with measure different from 0 and 1) invariant subsets.

Two equivalence relations (X, E) and (Y, F) are isomorphic if there exists a Borel bijection $\phi\colon X \to Y$ that maps E to F. Relations E and F are measure equivalent (more precisely, equivalent with respect to quasi-invariant measures μ and ν) if a Borel isomorphism can be established after removing invariant subsets of measure zero from X and Y. This corresponds to the definition of the orbital equivalence of group actions on measure spaces.

An equivalence relation can be considered as an edge set of an oriented graph (with the vertex set in X). A Borel graph is a symmetric (i.e., $(x, y) \in E$ implies $(y, x) \in E$) Borel subset $E \subset X$. The pairs (x, y) serve as edges with the initial vertex x and terminal vertex y. In this case, the vertex set V of the graph, which is the projection of the set E onto the first coordinate, is also a Borel set by virtue of Kuratowski's theorem [120] (the fact that the cardinality of equivalence classes is at most countable plays an important role in the proof of this statement). In the same way as for an ordinary graph, one naturally defines the following concepts for a Borel graph S: a path connecting two vertices; its combinatorial length; a neighborhood $D(n)$ of radius n of the diagonal, which consists of pairs (x, y) such that there exists a path of length $\le n$ that connects x and y ($D(n)$ is a Borel set); and the degree $\deg(x)$ of a vertex x, which means the number of points y such that $(x, y) \in S$. A graph is locally finite if the degrees of all of its vertices are finite. A subgraph $S \subset E$ generates E if, for any pair $(x, y) \in E$ with $x \ne y$, there exists a path from x to y that lies completely in S; in other words, a subgraph $S \subset E$ generates E if $\bigcup_n D_S(n) = E$.

For a subset $Y \subset X \times X$, denote $\{y\colon (x, y) \in Y\}$ by Y_x. Define a measure on the Borel subsets of the set E by the relation

$$\kappa(Y) = \int |Y_x| \, d\mu(x). \tag{11.4}$$

Note that this integral may also take infinite values.

Definition 11.1. (a) The *cost* $\mathrm{cost}(E)$ *of an equivalence relation* E is the quantity $\inf \kappa(S)$, where the infimum is taken over all Borel subgraphs $S \subset E$ that generate E.

(b) The cost of an action (G, X, μ) with invariant measure is the quantity $\mathrm{cost}(E)$, where E is the decomposition into orbits.

(c) The *cost of a group* is the quantity

$$\mathrm{cost}(G) = \inf \mathrm{cost}(G, X, \mu), \tag{11.5}$$

where the infimum is taken over all ergodic essentially free actions (G, X, μ) of the group G by transformations that preserve the probability measure μ.

(d) The group G has a *fixed cost* if the cost of all of its essentially free actions with invariant measure on a probability space is the same.

(e) An equivalence relation E on X that possesses an invariant probability measure is said to be *cheap* if $\mathrm{cost}(E) = 1$.

(f) A group G is said to be *cheap* if $\mathrm{cost}(G) = 1$.

For actions on probability spaces for which almost all orbits are infinite, the cost is not less than 1. For actions of amenable groups on probability spaces, the cost is 1, so such actions and groups are the "cheapest" from the viewpoint of the cost [63]. However, there are many nonamenable cheap groups; for example, the product of two infinite groups is a cheap group according to one of the results of Gaboriau [64, 63]. At the moment no group that has a fixed cost is known.

The cost of an equivalence relation depends on the measure μ. For example, the multiplication of a measure by a positive scalar proportionally increases the cost. When it is important to stress what measure is meant, we will write $\mathrm{cost}_\mu(E)$ or $c_\mu(E)$.

Until recently, the cost has been studied only for almost free actions. However, it is also interesting to study it for actions that are not almost free (but faithful). The first step in this direction was made in [5]. The following theorem demonstrates the relationship between the rank gradient and the cost of actions.

Theorem 11.4 (Abért and Nikolov [5]). *Let a sequence $\{H_n\}_{n=1}^\infty$ of subgroups of a group G be essentially free. Then*

$$\mathrm{RG}(G, \{H_n\}) = \mathrm{cost}_\nu(E) - 1, \tag{11.6}$$

where E is the decomposition into orbits of the action of G on the boundary of the tree associated with $\{H_n\}_{n=1}^\infty$.

We are interested in the information on possible values of the cost of self-similar and strongly self-similar groups and the cost of the equivalence relations arising when such a group is defined by a finite automaton and hence acts on the boundary of the corresponding tree. The following theorem is the first observation in this direction.

Theorem 11.5. *Suppose that a group G acting on a tree T is self-replicating and acts essentially freely on the boundary of the tree. Then $c_\nu(E) = 1$, where E is the decomposition into orbits of the action of the group G on the boundary of the tree T and ν is the uniform measure on the boundary.*

The first proof. Consider the decomposition $X = \bigsqcup_{i=1}^d X_i$ of the boundary $X = \partial T$ of the tree, where X_i are cylindrical sets corresponding to the first-level vertices. Let $E_i = E|X_i$ be the restriction to the subset X_i. Note that a natural identification of X_i with X, based on the self-similarity of a regular rooted tree, maps the relation E_i to a subrelation of the relation E. This is true for the action of any self-similar group. However, when a group is self-replicating, E_i is mapped under this identification to E. From a result of Gaboriau (see [113, Theorem 21]), we can derive the following relations:

$$c_\nu(E) = c_{\nu|X_1}(E|X_1) + \nu(\partial T \setminus X_1) = c_{\nu|X_1}(E|X_1) + \frac{d-1}{d}, \tag{11.7}$$

where d is the branching multiplicity of the tree. Using the similarity between the relations E and E_1 and the obvious homothety property of the uniform measure on the boundary, we find that $c_{\nu|X_1}(E_1)=\frac{1}{d}c_\nu(E)$. This, combined with (11.7), leads to the relation

$$c_\nu(E)\left(1-\frac{1}{d}\right)=\frac{d-1}{d},$$

which implies $c_\nu(E)=1$. $\square$

The second proof. Let $\xi\in\partial T$ be an arbitrary boundary point, $\{u_n\}_{n=1}^\infty$ be the sequence of vertices of the path ξ, and $P_n=\mathrm{st}_G(u_n)$. Then $|G:P_n|=d^n$. Since G is a self-replicating group, the restriction $G_n=P_n|T_{u_n}$ to the subtree T_{u_n} is a group isomorphic to the group G. The kernel of the restriction homomorphism $P_n\to P_n|T_{u_n}$ is trivial (otherwise the action would not be essentially free). Therefore, $P_n\simeq G$, $n=1,2,\dots$, and hence the rank gradient of the sequence $\{P_n\}$ vanishes. It remains to apply the Abért–Nikolov Theorem 11.4. $\square$

While proving this theorem with the second method, we have simultaneously proved the following proposition.

Proposition 11.6. *A self-replicating group acting freely on the boundary of a tree belongs to the class of scale-invariant groups, i.e., groups that possess a decreasing chain of finite-index subgroups with trivial core that are isomorphic to the group itself.*

In fact, for almost all boundary points of the tree, the intersection of the terms of the corresponding sequence is trivial. As already pointed out above, nontrivial examples of scale-invariant groups were constructed in [145].

We should stress that self-replicating groups represent an interesting class of groups of which very little is known, although they are rather frequently encountered among self-similar groups. For example, most of the 115 groups generated by three-state automata over a two-letter alphabet, whose (incomplete) classification is given in [164, 139, 35], are self-replicating groups. Many of them are branch groups. However, there are some that do not belong to this type, and a few of them, as already mentioned (for example, the lamplighter group and the Baumslag–Solitar group $\mathrm{BS}(1,3)$), act almost freely on the boundary of a tree.

Some concepts and methods of the theory of self-similar groups can be extended to the theory of Borel equivalence relations. Consider an example of this kind. Let $X=[0,1]$, μ be the Lebesgue measure, $d\ge 2$ be a positive integer, and $X=\bigsqcup_{i=1}^d X_i$ be a partition of the interval into d pieces of equal length. It is well known that the pair (X,μ) serves as a universal model of a measure space (a Lebesgue space, see [159]). Therefore, the whole theory of measurable equivalence relations can be presented using this space. Here it is important that the interval has a natural system of self-similarities: every subinterval X_i is mapped isomorphically onto the entire interval X by an appropriate (orientation-preserving) affine transformation φ_i. This self-similarity of the interval underlies the following definition.

Definition 11.2. Let E be a Borel equivalence relation on X.

(a) The relation E is said to be self-similar if there exists a $d\ge 2$ such that the relation $E|X_i$ turns into a subrelation $E_i\subset E$ under the affine transformation φ_i that maps X_i to X, $i=1,\dots,d$.

(b) The relation E is said to be self-replicating if there exists a $d\ge 2$ such that the relation $E|X_i$ turns into the relation E under the affine transformation φ_i that maps X_i to X, $i=1,\dots,d$.

In this definition, instead of the interval $[0,1]$, one can use the boundary ∂T of a d-regular tree and its natural decomposition $\partial T=\bigsqcup_{i=1}^d \partial T_i$ into cylindrical sets corresponding to the first-level vertices (in this case, the affine transformations φ_i are replaced by natural isomorphisms between the subtrees T_{u_i} and the tree T, and u_i, $i = 1, \dots, d$, are the first-level vertices numbered in a natural order). Another natural model is given by the Cantor set represented as the space of sequences over a d-letter alphabet and endowed with the Tikhonov topology.

The decomposition into orbits of an action of a self-similar group on the boundary of a tree is self-similar, while the decomposition into orbits of an essentially free action of a self-replicating group is self-replicating. Possibly, every self-similar or self-replicating decomposition can be realized by an action of a self-similar or, respectively, self-replicating group; however, this fact is yet to be proved or disproved.

Problem 11.2. Is an analog of the Feldman–Moore theorem valid for self-similar and self-replicating equivalence relations? In other words, is it true that for any self-similar Borel equivalence relation on the space X, there exists a group acting self-similarly on X for which the decomposition into orbits coincides with the given equivalence relation? A similar question for self-replicating equivalence relations.

The proof of the following theorem is analogous to the proof of Theorem 11.5.

Theorem 11.7. *The cost of any self-replicating Borel equivalence relation that preserves an invariant measure is* 1 *with respect to this measure.*

Problem 11.3. (i) How many pairwise nonisomorphic Borel self-similar equivalence relations do there exist?

(ii) The same question for self-replicating equivalence relations.

A special position among equivalence relations is occupied by hyperfinite relations.

Definition 11.3. (a) A countable equivalence relation E is said to be *hyperfinite* if there exists an increasing sequence $\{F_n\}_{n=1}^{\infty}$ of finite (i.e., with finite equivalence classes) Borel equivalence subrelations $F_n \subset E$ that exhaust E, i.e., $F_1 \subset F_2 \subset \dots$ and $\bigcup_n F_n = E$.

(b) A relation E having an invariant measure μ is *hyperfinite almost everywhere* if there exists a Borel subset $B \subset X$, $\mu(X \setminus B) = 0$, such that the restriction $E|B$ is hyperfinite.

The concept of hyperfiniteness can be regarded as an analog of the concept of amenability in view of the following classical theorems.

Theorem 11.8. (i) (Dye [55]) *Any two ergodic systems $(\mathbb{Z}, X, \mu)$ and $(\mathbb{Z}, Y, \nu)$ with invariant probability measures and nonatomic spaces X and Y are orbitally equivalent.*

(ii) (Ornstein–Weiss [150]) *Any ergodic action of an amenable group by transformations that preserve a probability measure is orbitally equivalent to an ergodic action of the group $\mathbb{Z}$ by transformations that preserve the probability measure.*

There exists a definition, going back to R. Zimmer, of an amenable (more precisely, μ-amenable) equivalence relation that has a quasi-invariant probability measure μ. We omit this definition and refer the reader, say, to Zimmer's book [197], to paper [106], or to Section 9 of book [113]. We present another statement that shows the relationship between amenability and hyperfiniteness and generalizes the previous statement.

Theorem 11.9 (Connes–Feldman–Weiss [47])**.** *Let E be a countable equivalence relation on X and μ be an E-quasi-invariant probability measure. If E is μ-amenable, then E is hyperfinite μ-almost everywhere.*

So, any two nonatomic ergodic actions of amenable groups that preserve a probability measure are orbitally equivalent, and the corresponding decompositions into orbits are equivalent (with respect to the corresponding measures) to a hyperfinite equivalence relation. Moreover, any two ergodic hyperfinite equivalence relations that preserve a probability measure are isomorphic. Hjorth showed [103] that the property of a group to have only one (up to orbital equivalence) ergodic action that preserves a probability measure is in fact equivalent to the property of being amenable.

The examples presented in Kaimanovich's note [106] show that all orbital Schreier graphs of a group acting on a probability space with invariant measure may be amenable, while the action is not amenable in the sense of Zimmer (i.e., the corresponding equivalence relation is not hyperfinite). Ceccherini-Silberstein and Elek [40] constructed an action of the free product of four copies of the group $\mathbb{Z}/2\mathbb{Z}$ (which is virtually free and, hence, nonamenable) on a compact metric space, such that all orbital Schreier graphs are amenable. For this action, they constructed two measures μ_1 and μ_2 such that the decomposition into orbits of the action is hyperfinite with respect to the first measure and is not hyperfinite with respect to the second; here the proof of nonhyperfiniteness uses a lower bound for the cost of actions.

Remark 11.1. The above-described situation with the hyperfiniteness of equivalence relations is similar to the situation with hyperfiniteness in von Neumann algebras, where it is proved that there exists only one hyperfinite factor of type II_1 and where the hyperfiniteness of algebras is also associated with their amenability.

In the last decade, owing largely to the studies by R. Zimmer, M. Gromov, A. Vershik, V. Kaimanovich, D. Gaboriau, A. Furman, N. Monod, and Y. Shalom (see [62]), a direction tentatively called "measured group theory" has been successfully developed. One of the central concepts in this theory is the concept of measurable equivalence of groups. In this language, the result formulated above concerning the orbital equivalence of the actions of amenable groups means that amenable groups make up a class of measurable equivalence.

A relation E is said to be aperiodic if all of its equivalence classes are infinite. For such relations, the following theorem of Levitt [124] holds, which was historically the first statement that demonstrated the importance of the concept of cost.

Theorem 11.10. *Let E be a countable aperiodic Borel equivalence relation on X and a measure μ be an E-invariant probability measure. Then the relation E is hyperfinite μ-almost surely if and only if* $\mathrm{cost}_\mu(E) = 1$ *and this value is achievable* (*i.e.,* $\mathrm{cost}_\mu(E)$ *achieves its value* 1 *on some Borel subgraph that generates this equivalence relation*).

The following important fact in the theory of equivalence relations correlates with some statements formulated above.

Theorem 11.11. *Let G be an amenable group acting on a space X by transformations that preserve a measure μ.*

(a) (Ornstein–Weiss [150]) *Then the decomposition into orbits is hyperfinite μ-almost everywhere.*

(b) (Zimmer [197]) *If the action is essentially free and measure-preserving and the decomposition into orbits E is hyperfinite μ-almost everywhere, then the group G is amenable.*

The following proposition is one of the numerous corollaries to these results.

Proposition 11.12. *Let G be a self-replicating group acting essentially freely on the boundary of a tree. Then G is amenable if and only if the cost of the decomposition E into orbits with respect to the uniform measure* (*which, recall, is* 1 *in view of Theorem* 11.5) *is achieved on some Borel graph that generates the decomposition E.*

Proof. Indeed, since the uniform measure ν is invariant, the amenability of G is equivalent to the hyperfiniteness of the decomposition into orbits. Therefore, it follows from Levitt's Theorem 11.10 that the value of $\mathrm{cost}_\nu(E) = 1$ is achieved. $\square$

The equivalence relations generated by finite automata have not yet been essentially studied, and our goal is to attract the reader's attention to this interesting direction. The following problem is one of many problems that can be set up in this connection (in this problem, it is assumed that the group $G(\mathcal{A})$ generated by an automaton $\mathcal{A}$ acts on the boundary ∂T of a tree by transformations that preserve the uniform measure ν).

Problem 11.4. Does there exist an algorithm that, given a finite invertible Mealy automaton $\mathcal{A}$, determines

(i) whether the dynamical system $(G(\mathcal{A}), \partial T, \nu)$ is ergodic?

(ii) whether the action of the group $G(\mathcal{A})$ on the boundary ∂T is almost free with respect to the uniform measure ν?

(iii) whether the decomposition into orbits of the dynamical system $(G(\mathcal{A}), \partial T, \nu)$ is hyperfinite?

The Borel actions of the group $\mathbb{Z}$, of the free abelian groups $\mathbb{Z}^n$ (an unpublished result by B. Weiss), and even of finitely generated nilpotent groups [113, Theorem 11.1] have hyperfinite decompositions into orbits in the pure form (i.e., irrespective of any measure; in other words, in the sense of Definition 11.3(a) given above).

By the Slaman–Steel [173] and Weiss [190] theorems, the pure hyperfiniteness of an equivalence relation is equivalent to the fact that this relation is generated by an action of the group $\mathbb{Z}$ (irrespective of any measure). As just mentioned, the decomposition into orbits of a Borel action of a polynomial growth group (i.e., of a finitely generated virtually nilpotent group) is hyperfinite (Jackson–Kechris–Louveau [105]). It is not known whether this result extends to groups of subexponential growth or, maybe, even to amenable groups.

Weiss [190] raised the following question.

Problem 11.5. Is it true that an arbitrary action of an amenable countable group by Borel automorphisms always has a hyperfinite decomposition into orbits?

It is not clear whether any ν-hyperfinite equivalence relation generated by a finite automaton is hyperfinite in the pure form.

In particular, it would be interesting to verify the hyperfiniteness of the decompositions into orbits for the actions of the lamplighter group and the Baumslag–Solitar group $\mathrm{BS}(1,3)$ that are described in Examples 2.2 and 5.4. Maybe some of them are not hyperfinite?

There are a number of statements, due mainly to Gaboriau [63, 64], on the behavior of the cost of group constructions, in particular, of free products with a union and HNN extensions. A survey of this information can be found in the last sections of book [113], which also contains an extensive list of problems (which are also mainly due to Gaboriau). As an interesting example of calculating a cost, note that the cost of the hereditary just-infinite group $\mathrm{SL}_3(\mathbb{Z})$ is zero because this group has a system of generators in which each next term commutes with the preceding term and the orders of all generators are infinite [63]. Note that the quotient group G/N may have a larger cost than G. Indeed, in the example

$$F_4 \to F_2 \times F_2 \to F_2,$$

which consists of groups and surjective homomorphisms, the costs of the groups F_4 and F_2 are 4 and 2, respectively, whereas the cost of the direct product of two copies of the group F_2 is zero. Anyway, the cost of just-infinite groups deserves very careful examination. However, for branch groups the answer is known; therefore, it remains to investigate the cost of simple groups and hereditary just-infinite groups.

Theorem 11.13. *Let G be a branch group. Then* $\mathrm{cost}(G) = 1$.

Proof. Indeed, the product of two infinite groups is a cheap group, and the property of a group to be cheap is preserved under passage to finite-index subgroups or, conversely, under a finite-index extension [113, Proposition 35.1]. Applying this statement to $\mathrm{rist}_G(1)$, we obtain the statement formulated above. ☐

Problem 11.6. Is it true that every just-infinite group has a fixed cost of 1 and this cost is achieved?

Unfortunately, there have not yet been developed any methods that would allow one to determine whether or not a given self-similar decomposition (defined, for example, by a finite automaton) is hyperfinite (recall that a problem regarding this issue was formulated above). It would be interesting to prove the amenability of some self-replicating groups (for which one fails to prove the amenability with other methods) by establishing, in one or another way, the hyperfiniteness (with respect to the uniform measure) of the decomposition into orbits of their actions on the boundary of a tree and the essential freeness of these actions. Recall that we already know that the cost of these actions is 1, which possibly points to their hyperfiniteness.

In addition, it is desirable to develop methods for calculating the L^2-Betti numbers of the equivalence relations generated by finite automata. The L^2-Betti numbers of equivalence relations were introduced by Gaboriau in [64], who showed that these numbers are zero for many groups and actions (in particular, for amenable groups). Thus, the fact that at least one of the L^2-Betti numbers is different from zero implies that the equivalence relation is not hyperfinite (and, accordingly, the group generating this relation is not amenable). The L^2-Betti numbers are invariants of orbital equivalence, just as various operator algebras (first of all, von Neumann algebras) that are associated with an action or with an equivalence relation. First of all, this is the classical von Neumann construction for the case of essentially free actions and its generalization obtained by Krieger in the case of nonfree actions. In addition to the original publications, one can learn about this issue in the books by Takesaki [181], Connes [46], and in many other sources. One of the topical questions is the problem of classification of von Neumann algebras associated with the actions of self-similar groups generated by finite automata (I mean the algebras mentioned in Section 9 in connection with a result of A.M. Vershik). It is also worthwhile to find conditions on an automaton under which these algebras are hyperfinite factors of type II_1. Does there exist an algorithm that would allow one to do this? It is interesting that Mealy-type finite automata appear in the study of some operator algebras, in particular, the Cuntz algebra, as demonstrated in [86].

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