SOME TRANSFORMATIONS ON MANIFOLDS WITH ALMOST CONTACT AND CONTACT METRIC STRUCTURES

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1. Introduction. Given a $(2n+1)$-dimensional differentiable manifold M, we denote by $F(M)$ the family of all real valued differentiable functions on M, and by $\mathfrak{X}(M)$ the totality of differentiable vector fields on M. Then $\mathfrak{X}(M)$ is an $F(M)$-module and a Lie algebra over R, R being a field of real numbers. An almost contact metric structure is a tetrad (ϕ, ξ, η, g), where ϕ is a linear operator $\phi : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ and η is a 1-form such that $\eta \cdot \phi = 0$, and ξ is a vector field such that $\eta(\xi) = 1$, satisfying the following relation :

$$(1.\ 1) \qquad \phi\cdot\phi(X) = -X + \eta(X)\cdot\xi, \qquad X \in \mathfrak{X}(M),$$

and finally g is a Riemannian metric which satisfies $\eta(X) = g(\xi, X)$ for $X \in \mathfrak{X}(M)$ and

$$(1.\ 2) \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\cdot\eta(Y), \qquad X, Y \in \mathfrak{X}(M).$$

Then we see that ϕ is of rank $2n$ and ξ is a characteristic unit vector field corresponding to characteristic value 0. Since it follows from (1. 1) and other relations that $\phi\cdot\xi = 0$ and that, at any point x of M, denoting by ϕ_η the restriction of ϕ to the tangent subspace $T_x(\eta)$ of M which is orthogonal to ξ_x, it has a property $\phi_\eta\cdot\phi_\eta = -$ Identity.

By virtue of (1. 2), we can define a differentiable 2-form w as follows:

$$w(X, Y) = g(X, \phi Y), \qquad X, Y \in \mathfrak{X}(M),$$

then the rank of w is $2n$. An almost contact metric structure is called a contact metric structure, if the relation $w = d\eta$ is valid. And a differentiable manifold with a (or an almost) contact metric structure is called to be a (or an almost) contact Riemannian manifold.

Suppose μ be a diffeomorphism of M, then μ is said to be an automorphism of an almost contact metric structure, if it leaves all of ϕ, ξ, η and g invariant. In the sequel, by a transformation on M we understand a diffeomorphism of M. In this report, we treat mainly transformations which leave ϕ invariant. Some propositions of this note are stated in [9] in terms of infinitesimal transformations. My hearty acknowledgement goes to Prof. S.Sasaki, Mr. Y.Hatakeyama and Mr. Y.Ogawa.

2. Transformations on almost contact Riemannian manifolds.

THEOREM 2-1. *Let M be a differentiable manifold with an almost contact metric structure. Then in order that a conformal transformation μ of the associated Riemannian metric g satisfies $\mu^* w = \alpha w$ for some positive scalar $\alpha \in F(M)$, it is necessary and sufficient that μ leaves ϕ invariant.*

PROOF. As μ is a conformal transformation, there exists a scalar field σ for which we have $\mu^* g = \sigma^2 g$ and hence for an arbitrary point x of M,

$$g_{\mu x}(\mu X, \mu\phi Y) = \sigma^2(x) g_x(X, \phi Y), \qquad X, Y \in \mathfrak{X}(M). \tag{2. 1}$$

And the relation $\mu^* w = \alpha w$ is written by definition as follows:

$$\begin{aligned}(\mu^* w)_x(X, Y) = w_{\mu x}(\mu X, \mu Y) &= g_{\mu x}(\mu X, \phi\mu Y) \\ &= \alpha(x) g_x(X, \phi Y).\end{aligned} \tag{2. 2}$$

From (2. 1) and (2. 2) it follows that

$$g_{\mu x}(\mu X, \mu\phi Y) = \frac{\sigma^2(x)}{\alpha(x)} g_{\mu x}(\mu X, \phi\mu Y).$$

Consequently, we have

$$\mu_x \phi_x Y_x = \frac{\sigma^2(x)}{\alpha(x)} \phi_{\mu x} \mu_x Y_x. \tag{2. 3}$$

Since ϕ satisfies $\phi\cdot\phi\cdot\phi = -\phi$ which follows from (1. 1), the left hand side of the last equation is

$$\begin{aligned}\mu_x \phi_x Y_x &= -\mu_x \phi_x (\phi_x \cdot \phi_x Y_x) = -\frac{\sigma^2(x)}{\alpha(x)} \phi_{\mu x} \mu_x (\phi_x \cdot \phi_x Y_x) \\ &= -\frac{\sigma^6(x)}{\alpha^3(x)} \phi_{\mu x} \cdot \phi_{\mu x} \cdot \phi_{\mu x} \mu_x Y_x = \frac{\sigma^6(x)}{\alpha^3(x)} \phi_{\mu x} \mu_x Y_x.\end{aligned}$$

And hence (2. 3) shows $\sigma^4(x) = \alpha^2(x)$. By assumption, α is positive and so we see that α is equal to σ^2, then (2. 3) turns to $\mu_x \phi_x = \phi_{\mu x} \mu_x$. Conversely, if a conformal transformation μ ($\mu^* g = \sigma^2 g$) leaves ϕ invariant, then we have

$$\begin{aligned}(\mu^* w)_x(X, Y) = g_{\mu x}(\mu X, \phi\mu Y) &= g_{\mu x}(\mu X, \mu\phi Y) \\ &= \sigma^2(x)\ w_x(X, Y), \qquad X, Y \in \mathfrak{X}(M). \qquad \text{(q. e. d.)}\end{aligned}$$

COROLLARY. *If a conformal transformation μ on an almost contact Riemannian manifold leaves w invariant, then μ leaves ϕ also invariant and μ is necessarily an isometry, therefore μ is an automorphism of this almost contact metric structure.*

In fact, by $\phi\mu = \mu\phi$ we have $\phi\cdot\mu\xi = 0$, and as μ is an isometry, we see that $\mu\xi = \xi$ and of course $\mu^*\eta = \eta$.

PROPOSITION 2-1. *Suppose μ be a conformal transformation ($\mu^*g = \sigma^2 g$) on an almost contact Riemannian manifold M. If μ satisfies the relation $\mu^*\eta = \alpha\eta$ ($\mu\xi = \beta\xi$ resp.) for some positive α (β resp.) $\in F(M)$, then we have $\alpha = \sigma$ ($\beta = \mu^*\sigma$ resp.) and $\mu\xi = (\mu^*\sigma)\xi$ ($\mu^*\eta = \sigma\eta$ resp.).*

Proof shall be omitted here.

Let H be a homogeneous holonomy group of a connected almost contact Riemannian manifold M. At an arbitrary but fixed point x of M, we consider the set $F(x,\xi) = \{\lambda\xi_x, \lambda \in H\}$ which may be identified with a subset of a $2n$-dimensional unit sphere. Further, for any point y of M, we join x and y by a piece-wise differentiable curve $l(x,y)$ and define $F_y(x,\xi) = \tau(l)F(x,\xi)$, where the notation $\tau(l)$ means the parallel displacement along the curve l. Clearly, $F_y(x,\xi)$ does not depend upon the choice of the curve joining x and y. Then we say temporarily that M has a F-property if at every point z, ξ_z belongs to $F_z(x,\xi)$. Of course, this property does not depend on x. It is equivalent to say that for any two points y and z, there exists a curve $l(y,z)$ such that $\xi_z = \tau(l)\xi_y$.

PROPOSITION 2-2. *Suppose that an almost contact Riemannian manifold M has a F-property. If an affine transformation μ preserves the direction of ξ and at one point p of M μ leaves η invariant, then μ leaves ξ and η globally invariant.*

PROOF. By virtue of $(\mu^*\eta)_p = \eta_p$, it is easy to see that $\mu\xi_p = \xi_{\mu p}$ is valid. We join p and an arbitrary point x of M by a curve $l(p,x)$ along which ξ_p is parallel to ξ_x and we have $\mu\xi_x = \mu\cdot\tau(l)\xi_p$. By the way, μ is an affine transformation and so it commutes with the parallel displacement and we see that $\mu\xi_x = \xi_{\mu x}$. In the next place, for any $X \in \mathfrak{X}(M)$, we have $g_x(\xi_x, \phi X) = 0$ and so $g_p(\xi_p, \tau^{-1}(l)\phi X) = 0$. Namely $\eta_p(\tau^{-1}(l)\phi X) = 0$ and hence $\eta_{\mu p}(\mu\cdot\tau^{-1}(l)\phi X) = 0$, or equivalently $g_{\mu p}(\xi_{\mu p}, \mu\cdot\tau^{-1}(l)\phi X) = 0$. And finally

$$g_{\mu x}(\xi_{\mu x}, \tau(\mu(l))\cdot\mu\cdot\tau^{-1}(l)\phi X) = g_{\mu x}(\xi_{\mu x}, \mu\phi X) = \eta_{\mu x}\cdot\mu\phi X = 0.$$

Consequently $\mu^*\eta = \alpha\eta$ for some $\alpha \in F(M)$ and necessarily $\alpha = 1$.

3. Transformations on contact Riemannian manifolds.

THEOREM 3-1. *If a transformation μ on a contact Riemannian manifold M leaves ϕ invariant, then there exists a positive constant α such that the relations $\mu^*\eta = \alpha\eta$, $\mu\xi = \alpha\xi$ and $\mu^*w = \alpha w$ hold good.*

PROOF. (i) From the equations $\eta\cdot\phi = 0$ and $\phi\cdot\mu = \mu\cdot\phi$, we get $\eta\cdot\mu\phi = 0$, or at any point x of M we have $(\mu^*\eta)_x\phi_x X_x = 0$, $X \in \mathfrak{X}(M)$. Thereby

(3. 1) $$(\mu^*\eta)_x = \alpha(x)\eta_x \text{ for some } \alpha \in F(M).$$

(ii) If we suppose $\phi\xi = 0$ and $\phi\cdot\mu = \mu\cdot\phi$, then we have $\phi\cdot\mu\xi = 0$. Hence, it follows that $(\mu\xi)_{\mu x} = \beta(\mu x)\xi_{\mu x}$ for some $\beta \in F(M)$. Combining (i) and this, we see that $\beta(\mu x) = \alpha(x)$.

(iii) We shall show that α is constant [9]. By operating the exterior differentiation to (3. 1), we get

(3. 2) $$d\mu^*\eta = d\alpha \wedge \eta + \alpha d\eta.$$

As d and μ^* commute, $d\mu^*\eta = \mu^* d\eta$. On the other hand, we have

$$(\mu^* d\eta)_x(\xi, Y) = d\eta_{\mu x}(\mu\xi, \mu Y) = 0, \qquad Y \in \mathfrak{X}(M),$$

since $(\mu\xi)_{\mu x} = \alpha(x)\xi_{\mu x}$ and $i(\xi)d\eta = i(\xi)w = 0$, where $i(\xi)$ is the interior product operator by ξ. Hence $i(\xi)_x(d\mu^*\eta) = 0$. Consequently, we have by virtue of (3. 2) $i(\xi)(d\alpha \wedge \eta) = 0$. Moreover,

$$i(\xi)(d\alpha \wedge \eta) = i(\xi)d\alpha \wedge \eta - d\alpha\cdot i(\xi)\eta = \mathfrak{L}(\xi)\alpha\cdot\eta - d\alpha,$$

where we have put $\mathfrak{L}(\xi)\alpha = i(\xi)d\alpha$. Thus, $\mathfrak{L}(\xi)\alpha\cdot\eta = d\alpha$. Therefore, $d\alpha \wedge \eta = 0$ and $d\alpha \wedge d\eta = 0$. Further $\mathfrak{L}(\xi)\alpha\cdot\eta \wedge d\eta = 0$. From this $\mathfrak{L}(\xi)\alpha$ must be zero and $d\alpha = 0$. This means that α is constant, and $\mu^* w = \alpha w$ is clear. The fact that α is positive will be proved in the next Proposition 3-1.

Several Propositions follow from this Theorem.

PROPOSITION 3-1. *Let M be a contact Riemannian manifold. If a transformation μ on M leaves ϕ invariant, then μ is conformal, precisely homothetic, relative to the η-plane $T_x(\eta)$, $x \in M$.*

PROOF. For an arbitrary point $x \in M$ and $X, Y \in \mathfrak{X}(M)$ we have

$$(\mu^* w)_x(X, Y) = w_{\mu x}(\mu X, \mu Y) = g_{\mu x}(\mu X, \phi\mu Y)$$
$$= g_{\mu x}(\mu X, \mu\phi Y) = (\mu^* g)_x(X, \phi Y).$$

On the other hand, by Theorem 3-1 the left hand side of the last equation is equal to

$$\alpha w_x(X, Y) = \alpha g_x(X, \phi Y),$$

for some constant α. Thus we have

(3. 3) $$(\mu^* g)_x(X, \phi Y) = \alpha g_x(X, \phi Y).$$

Here we assume that $X_x \neq 0$ and $X_x \in T_x(\eta)$ (i. e. $\eta_x(X) = 0$). And we define $Y = -\phi X$, then Y_x is also an element of the η-plane and we have

$$g_{\mu x}(\mu X, \mu X) = \alpha g_x(X, X), \qquad X_x \in T_x(\eta).$$

It follows from this that α is positive. Furthermore let Z be an arbitrary vector field such that $Z_x \in T_x(\eta)$ and Y be $-\phi Z$, then (3. 3) turns to

$$(\mu^* g)_x(X, Z) = \alpha g_x(X, Z), \qquad X_x,\ Z_x \in T_x(\eta).$$

PROPOSITION 3-2. *If a transformation μ on a contact Riemannian manifold M leaving ϕ invariant is conformal at some one point of M, then μ is an automorphism. Conversely, if a homothetic transformation μ leaves ϕ invariant in a small neighborhood of one point of M, then μ is an isometry.*

PROOF. By assumptions there exists a point p of M at which μ is conformal, that is $(\mu^* g)_p = \sigma^2 g_p$ holds good for some positive number σ. However, by Proposition 3-1, σ^2 must be equal to α corresponding to μ. On the other hand, by the relation $(\mu^* g)_p(\xi, \xi) = \sigma^2 g_p(\xi, \xi)$ and $(\mu\xi)_{\mu x} = \alpha\xi$, we have $\sigma^2 = \alpha^2$ and hence $\alpha^2 = \alpha = 1$. To see that μ leaves g invariant we rewrite (1. 2) as

(3. 4) $$g(X, Y) = w(\phi X, Y) + \eta(X)\cdot\eta(Y), \qquad X, Y \in \mathfrak{X}(M).$$

Two terms of the right hand side contain w, ϕ and η which are invariant by μ. This completes the proof of the first part of our statement. Conversely, suppose that we have a point q of M such that in a neighborhood $U(q)$ of it a homothetic transformation μ leaves ϕ invariant. Then, by applying the preceding result to $U(q)$,we see that μ is an isometry in $U(q)$ and hence on M.

PROPOSITION 3-3. *In a contact Riemannian manifold, if a conformal transformation μ satisfies $\mu^* w = \alpha w$ for some positive $\alpha \in F(M)$, then μ is an automorphism of the contact metric structure.*

This follows from Theorem 2-1 and Proposition 3-2.

PROPOSITION 3-4. *Let us denote by Φ the totality of transformations on a contact Riemannian manifold which leave ϕ invariant. If $\mu \in \Phi$ belongs either to the commutator subgroup $[\Phi, \Phi]$ or to some compact subgroup of Φ, then it is an isometry and so an automorphism of this structure.*

PROOF. In fact, the correspondence between a transformation μ and a constant α defines a homomorphism h of the group Φ into the multiplicative group of real positive numbers. That is, for μ and $\nu \in \Phi$, we have $\mu^*\eta = \alpha\eta$ and $\nu^*\eta = \beta\eta$ $(\alpha, \beta \in R)$, and then we see that

$$(\mu\cdot\nu)^*\eta = \nu^*(\mu^*\eta) = \alpha\beta\eta,$$

this permits us to define a homomorphism $h(\mu\cdot\nu) = \alpha\beta$.

PROPOSITION 3-5. *Let M be a compact manifold with a contact metric structure, if a transformation μ leaves ϕ invariant, then μ is an automorphism of this structure. Therefore all of such transformations constitutes a compact Lie group.*

PROOF. We notice that $\mu^*(\eta \wedge w^n) = \alpha^{n+1}\eta \wedge w^n$, $(\alpha = h(\mu))$. Integrating it over M we get

$$\alpha^{n+1} \int_M \eta \wedge w^n = \int_M \mu^*(\eta \wedge w^n) = \int_M \eta \wedge w^n.$$

From this we see that α is equal to 1. Therefore μ leaves ϕ, w and η invariant and so leaves g invariant too. (q. e. d.)

Now, if a conformal transformation μ on a contact Riemannian manifold leaves ξ or η invariant, it follows that μ leaves w invariant. Then, by Proposition 3-3, μ is an automorphism. However, we can prove the following

PROPOSITION 3-6. *If a conformal transformation μ on a contact Riemannian manifold M satisfies $\mu^*\eta = \alpha\eta$ for some (necessarily positive) $\alpha \in F(M)$ or preserves the direction of ξ, then μ is an automorphism.*

PROOF. By Proposition 2-1, we see that μ satisfies $\mu^*\eta = \alpha\eta$ and $\mu\xi = (\mu^*\alpha)\xi$. And we can verify that α is a positive constant by the similar argument just as in the proof of Theorem 3-1. Hence we have $\mu^* w = \alpha w$, therefore Proposition 3-6 is an immediate consequence of Proposition 3-3.

PROPOSITION 3-7. *If a transformation μ on a complete contact Riemannian manifold M leaves ϕ invariant and has no fixed point, then μ is an automorphism.*

PROOF. We see by Proposition 3-1 that μ is homothetic relative to the η-plane $T_x(\eta)$, $x \in M$, i. e.

$$(\mu^* g)_x(Y, Z) = \alpha g_x(Y, Z), \qquad Y_x, Z_x \in T_x(\eta), \tag{3. 5}$$

where $\alpha = h(\mu) > 0$. Here we assume that μ is not an automorphism, that is $\alpha \neq 1$, then α can be supposed to be smaller than 1. Since if α is greater than 1, we can replace μ by μ^{-1}. Next, we decompose any vector field $X \in \mathfrak{X}(M)$ $(X_x \neq 0)$ as $X = -\phi\cdot\phi\, X + \eta(X)\xi$. Operating μ to the both sides of the last equation

$$\mu_x X_x = -\mu_x \phi_x \cdot \phi_x\, X_x + \alpha\eta_x(X)\xi_{\mu x}, \tag{3. 6}$$

where we have utilized $\mu\xi = \alpha\xi$. As the both terms of the right hand side are orthogonal on account of $\mu\cdot\phi = \phi\cdot\mu$, we get

$$\begin{aligned} g_{\mu x}(\mu X, \mu X) &= \alpha^2\, \eta(X)^2 + g_{\mu x}(\mu\phi\cdot\phi X,\ \mu\phi\cdot\phi X) \\ &= \alpha^2 \eta(X)^2 + \alpha g_x(\phi\cdot\phi X, \phi\cdot\phi X), \end{aligned}$$

by virtue of (3. 5). Hence, we have the inequality

$$g_{\mu x}(\mu X, \mu X) \leqq \alpha g_x(X, X). \tag{3. 7}$$

If we denote by $d(x, y)$ the distance between two points x and y, and put $x_1 = \mu x$, $x_{k+1} = \mu x_k$, $k = 1,2,\cdots$, then (3. 7) means that $(dx_k, x_{k+1}) \to 0$ as $k \to \infty$ and $\{x_k\}$ constitutes a Cauchy sequence. By the completeness of M in consideration we see that there is a point x_∞ such that $\mu x_\infty = x_\infty$, this contradicts the hypotheses. (q. e. d.)

In the preceding Proposition 3-7, the condition that μ has no fixed point can be removed if the complete contact Riemannian manifold is not locally flat and μ leaving ϕ invariant is an affine transformation. This may be proved by the method of [3]. But we have the following

PROPOSITION 3-8. *If an affine transformation μ on a contact Riemannian manifold M leaves ϕ invariant, then μ is an automorphism.*

PROOF. By ∇ we denote the covariant differentiation which arises from the Riemannian connection defined by the associated metric g. An affine transformation commutes with the covariant differentiation and we have

$$\nabla(\mu\phi\mu^{-1})_{\mu x}(X, Y) = \mu\cdot(\nabla\phi)_x(\mu^{-1}X, \mu^{-1}Y), \qquad X, Y \in \mathfrak{X}(M).$$

By assumption $\mu\phi_x\mu^{-1} = \phi_{\mu x}$, so we have

$$\nabla\phi_{\mu x}(X, Y) = \mu\cdot(\nabla\phi)_x(\mu^{-1}X, \mu^{-1}Y). \tag{3. 8}$$

On the other hand, it is known [8] that $\delta w = n\eta$, where δ is the codifferentiation operator. Therefore, if we contract $\nabla\phi_x$ and $\nabla\phi_{\mu x}$ in both local coordinates at x and μx, we get $-n\eta_x$ and $-n\eta_{\mu x}$ respectively. It follows from (3. 8) that $n\eta_{\mu x}(X) = n\eta(\mu^{-1}X)$, namely $\eta_{\mu x} = \mu^{-1*}\eta_x$. Hence, our assertion is true.

PROPOSITION 3-9. *If a projective transformation μ on a contact Riemannian manifold M leaves ϕ invariant, then μ is an automorphism.*

PROOF. For any projective transformation μ, there exists a 1-form θ such that

$$\sum_{i=1}^{2n+1} ({}^{\mu}\Gamma - \Gamma)^i(X, Y)\frac{\partial}{\partial y^i} = \theta(X)\cdot Y + \theta(Y)\cdot X, \qquad X, Y \in \mathfrak{X}(M),$$

where Γ is the Christoffel's symbol and ${}^{\mu}\Gamma$ is the image by μ of Γ and (y^i)'s are local coordinates at $y = \mu x$, x being an arbitrary point of M. Then, by the similar way as above, we can derive the identity

$$n\eta_{\mu x} - (2n+1)\theta\cdot\phi_{\mu x} = n\mu^{-1*}\eta_x = \frac{n}{h(\mu)}\eta_{\mu x}.$$

Thus, if we operate $\xi_{\mu x}$ to the right of each term, we see that $h(\mu) = 1$ holds good. Hence, μ is an automorphism.

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