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Some Typical Properties of the Spatial Preferred Attachment Model

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Abstract. We investigate a stochastic model for complex networks, based on a spatial embedding of the nodes, called the spatial preferred attachment (SPA) model. In the SPA model, nodes have spheres of influence of varying sizes, and a new node may link to a node only if it falls within its region of influence. The spatial embedding of the nodes models the background knowledge or identity of the node, which influences its link environment. In this paper, we focus on the (directed) diameter, small separators, and the (weak) giant component of the model.

1. Introduction

Discrete random graph processes exhibiting power-law properties have been studied by many researchers and in many contexts. The study of such processes dates back at least to [Yule 24]. Recent interest in preferential attachment models follows from [Barabási and Albert 99], in which the authors observed a power-law degree sequence for a subgraph of the World Wide Web, and from [Faloutsos et

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al. 99], whose authors observed power-law behavior for the Internet graph. Many models of such processes exist. For details see, for example, the surveys [Bollobás and Riordan 02, Mitzenmacher 01] and the monographs [Bonato 08, Chung and Lu 06].

In networked information spaces, vertices are defined not only by their link environment, but also by the information entity they represent. More recently, attempts have been made to model this alternative view of the vertices through *spatial models*. In a spatial model, vertices are embedded in a metric space, and link formation is influenced by the metric distance between vertices. The metric space is meant to be like a feature space, so that the coordinates of a vertex in this space represent the information associated with the vertex. For example, in text mining, documents are commonly represented as vectors in a word space. The metric is chosen so that metric distance represents similarity, i.e., vertices whose information entities are closely related will be at a short distance from each other in the metric space. A number of spatial models have been proposed to date [Bonato et al. 12, Bradonjic et al. 09, Flaxman et al. 06, Flaxman et al. 08, Higham et al. 08, Masuda et al. 05]. We direct the reader to the recent survey [Janssen 10] for more details.

We focus on the spatial preferred attachment (SPA) model, proposed in [Aiello et al. 07, Aiello et al. 09]. The SPA model generates directed graphs according to the following principle. Vertices are points in a given metric space. Each vertex v has a *sphere of influence*. The volume of the sphere of influence of a vertex is a function of its in-degree. A new vertex u can link to an existing vertex v only if u falls inside the sphere of influence of v. In the latter case, u links to v with probability p. The SPA model incorporates the principle of preferential attachment, since vertices with a higher in-degree will have a larger sphere of influence. The SPA model gives a power-law in-degree distribution, with exponent in $[2, \infty)$ depending on the parameters, and with concentration for a wide range of in-degree values [Aiello et al. 07, Aiello et al. 09]. In [Janssen et al. 10, Janssen et al. 13], it was shown, through theoretical analysis and simulation, that for graphs formed according to the SPA model, it is possible to infer the metric distance between vertices from the link structure of the graph.

In this paper, we investigate the (directed) diameter, small separators, and the (weak) giant component of the model. This is an extended version of a paper presented at the 9th Workshop on Algorithms and Models for the Web Graph (*WAW 2012*) [Cooper et al. 12].

2. The SPA Model

We begin by giving a precise description of the SPA model, presenting some known properties and deriving some facts about the model that we will need to prove our results. In [Aiello et al. 09] (see also [Aiello et al. 07] for an earlier version of this paper), the model is defined for a variety of metric spaces S. In this paper, we let S be the unit hypercube in $\mathbb{R}^m$, equipped with the torus metric derived from any of the L_p norms. This means that for all points x and y in S,

$$d(x,y) = \min\left\{\|x - y + u\|_p : u \in \{-1,0,1\}^m\right\}.$$

The torus metric thus "wraps around" the boundaries of the unit square; this metric was chosen to eliminate boundary effects. Let c_m be the constant of proportionality of volume used with the mth power of the radius in m dimensions, so the volume of a ball of radius r in m-dimensional space with the given metric equals $c_m r^m$. For example, for the Euclidean metric, $c_2 = \pi$, and for the product metric derived from L_∞, $c_m = 2^m$.

The parameters of the model consist of the *link probability* $p \in [0,1]$ and two positive constants A_1 and A_2, which, in order to avoid the resulting graph becoming too dense, must be chosen such that $pA_1 < 1$. The original model as presented in [Aiello et al. 09] has a third parameter, A_3, which is assumed to be zero here. This causes no loss of generality, since all asymptotic results presented here are unaffected by A_3.

The SPA model generates stochastic sequences of directed graphs $(G_t : t \geq 0)$, where $G_t = (V_t, E_t)$ and $V_t \subseteq S$. Let $\deg^-(v,t)$ be the in-degree of vertex v in G_t, and $\deg^+(v,t)$ its out-degree. We define the *sphere of influence* $S(v,t)$ of vertex v at time $t \geq 1$ to be the ball centered at v with volume $|S(v,t)|$ defined as follows:

$$|S(v,t)| = \frac{A_1 \deg^-(v,t) + A_2}{t}, \tag{2.1}$$

or $S(v,t) = S$ and $|S(v,t)| = 1$ if the right-hand-side of (2.1) is greater than 1.

The process begins at $t = 0$, with G_0 being the null graph. Time step t, $t \geq 1$, is defined to be the transition between G_{t-1} and G_t. At the beginning of each time step t, a new vertex v_t is chosen *uniformly at random* from S and added to V_{t-1} to create V_t. Next, independently, for each vertex $u \in V_{t-1}$ such that $v_t \in S(u, t-1)$, a directed link (v_t, u) is created with probability p. Thus, the probability that a link (v_t, u) is added at time step t equals $p\,|S(u, t-1)|$.

We say that an event holds asymptotically almost surely (a.a.s.) if the probability that it holds tends to 1 as t goes to infinity. It was shown in [Aiello et al. 09] that a.a.s., the SPA model produces graphs with a power-law degree distribution with exponent $1 + 1/(pA_1)$. Moreover, a precise expression for the probability distribution of the in-degree of the individual vertex v_i born at time i was given in that work. In [Janssen et al. 13] (see also [Janssen et al. 10]), the relationship between the link structure of graphs produced by the model

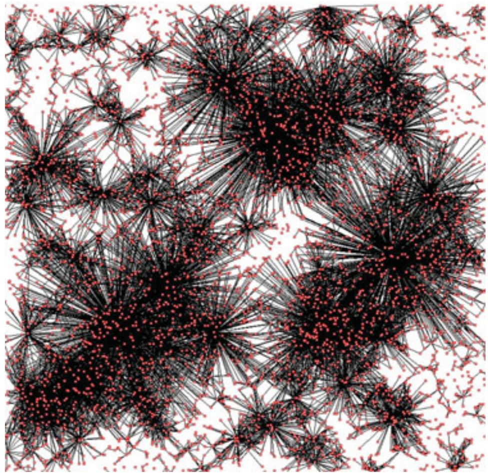


Figure 1. A simulation on the unit square with $t = 5000$ and $p = A_1 = A_2 = 1$.

and the relative positions of the vertices in the metric space was analyzed. See Figure 1 for a drawing of a simulation of the SPA model.

Now let us discuss a few simple new facts about the model. Knowing the expected in-degree of a node, given its age, will help us to analyze geometric properties of the SPA model. Let us note that the result for $i \gg 1$ ($f(t) \gg g(t)$ is used to indicate that $f(t)/g(t)$ tends to infinity together with t) was proved in [Janssen et al. 13]; see (2.2). We extend it here to all $i \geq 1$; see (2.3). As before, let v_i be the node added at time i.

Theorem 2.1. *Suppose that $i = i(t) \gg 1$ as $t \to \infty$. Then*

$$\begin{aligned} \mathbb{E}\left(\deg^-(v_i, t)\right) &= (1 + o(1)) \frac{A_2}{A_1} \left(\frac{t}{i}\right)^{pA_1} - \frac{A_2}{A_1}, \\ \mathbb{E}\left(|S(v_i, t)|\right) &= (1 + o(1)) A_2 t^{pA_1 - 1} i^{-pA_1}. \end{aligned} \tag{2.2}$$

Moreover, for all $i \geq 1$,

$$\begin{aligned} \mathbb{E}\left(\deg^-(v_i, t)\right) &\leq \frac{eA_2}{A_1} \left(\frac{t}{i}\right)^{pA_1} - \frac{A_2}{A_1}, \\ \mathbb{E}\left(|S(v_i, t)|\right) &\leq (1 + o(1)) eA_2 t^{pA_1 - 1} i^{-pA_1}. \end{aligned} \tag{2.3}$$

Proof. In order to simplify calculations, we make the following substitution:

$$X(v_i,t) = \deg^-(v_i,t) + \frac{A_2}{A_1}. \tag{2.4}$$

It follows from the definition of the process that

$$X(v_i,t+1) = \begin{cases} X(v_i,t)+1, & \text{with probability } pA_1X(v_i,t)/t, \\ X(v_i,t), & \text{otherwise.} \end{cases}$$

We obtain the conditional expectation

$$\begin{aligned} &\mathbb{E}\left(X(v_i,t+1) \mid X(v_i,t)\right) \\ &\quad = (X(v_i,t)+1)\frac{pA_1X(v_i,t)}{t} + X(v_i,t)\left(1 - \frac{pA_1X(v_i,t)}{t}\right) \\ &\quad = X(v_i,t)\left(1+\frac{pA_1}{t}\right). \end{aligned}$$

Taking expectations again, we get

$$\mathbb{E}\left(X(v_i,t+1)\right) = \mathbb{E}\left(X(v_i,t)\right)\left(1+\frac{pA_1}{t}\right).$$

Since all nodes start with in-degree zero, we have $X(v_i,i) = A_2/A_1$. Note that for $0<x<1$, $\log(1+x) = x - O(x^2)$. If $i \gg 1$, one can use this to obtain

$$\mathbb{E}\left(X(v_i,t)\right) = \frac{A_2}{A_1}\prod_{j=i}^{t-1}\left(1+\frac{pA_1}{j}\right) = (1+o(1))\frac{A_2}{A_1}\exp\left(\sum_{j=i}^{t-1}\frac{pA_1}{j}\right),$$

but in all cases $i \geq 1$,

$$\mathbb{E}\left(X(v_i,t)\right) \leq \frac{A_2}{A_1}\exp\left(\sum_{j=i}^{t-1}\frac{pA_1}{j}\right).$$

Therefore, when $i \gg 1$,

$$\mathbb{E}\left(X(v_i,t)\right) = (1+o(1))\frac{A_2}{A_1}\exp\left(pA_1\log\left(\frac{t}{i}\right)\right) = (1+o(1))\frac{A_2}{A_1}\left(\frac{t}{i}\right)^{pA_1},$$

and (2.2) follows from (2.4) and (2.1). Moreover, for every $i \geq 1$,

$$\mathbb{E}\left(X(v_i,t)\right) \leq \frac{A_2}{A_1}\exp\left(pA_1\left(\log\left(\frac{t}{i}\right)+\frac{1}{i}\right)\right) \leq \frac{eA_2}{A_1}\left(\frac{t}{i}\right)^{pA_1},$$

and (2.3) follows from (2.4) and (2.1) as before, which completes the proof. □

Another fact that we will need follows directly from the following result proved in [Janssen et al. 13]. The degree of an individual vertex is not concentrated, due to variation happening shortly after birth. (That is, a.a.s., there are vertices that have smaller or larger degrees than what we would expect.) However, provided that the degree of the vertex at end time t is large enough (that is, is tending to infinity faster than $\log t$), sharp bounds on the degree of the vertex during most of the process can be obtained. This is expressed in the following theorem. First, define an injective function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(i) = \frac{A_2}{A_1} \left(\frac{t}{i} \right)^{pA_1},$$

so $f(i)$ is the expected degree at time t of a vertex born at time i (up to a factor of $(1 + o(1))$). Thus, $f^{-1}(k)$ is the birth time of a vertex of final degree k, assuming that the degree of the vertex is close to the expected value during its lifetime. Hence, if a vertex of final degree k has behavior close to its expected degree, then

$$t_a = f^{-1} \left(\frac{A_2 k}{A_1 a} \right)$$

will be the time when that vertex has degree a. Indeed, for a vertex born at time $f^{-1}(k)$, the expected degree at time t_a is equal to

$$\begin{aligned} \frac{A_2}{A_1} \left(\frac{t_a}{f^{-1}(k)} \right)^{pA_1} &= \frac{A_2}{A_1} \left(\frac{A_2}{A_1} \left(\frac{t}{f^{-1}(k)} \right)^{pA_1} \right) \Bigg/ \left(\frac{A_2}{A_1} \left(\frac{t}{t_a} \right)^{pA_1} \right) \\ &= \frac{A_2 k}{A_1} \Bigg/ \left(\frac{A_2 k}{A_1 a} \right) = a. \end{aligned}$$

Theorem 2.2. [Janssen et al. 13] *Let $\omega = \omega(t)$ be any function tending to infinity together with t. The following statement holds a.a.s. for every vertex v for which* $\deg^-(v,t) = k = k(t) \geq \omega \log t$. *Let $i = f^{-1}(k)$, and let t_k be*

$$t_k = f^{-1} \left(\frac{A_2 k}{A_1 \omega \log t} \right).$$

Then for all values of s such that $t_k \leq s \leq t$,

$$\deg^-(v,s) = (1 + o(1)) \frac{A_2}{A_1} \left(\frac{s}{i} \right)^{pA_1} = (1 + o(1)) k \left(\frac{s}{t} \right)^{pA_1}. \qquad (2.5)$$

The theorem implies that once a given vertex accumulates $\omega \log t$ neighbors, the rest of the process (until time step t) can be predicted with high probability; in fact, a.a.s., we get a concentration around the expected value.

With Theorem 2.2 in hand, we immediately get the following.

Theorem 2.3. *Let $\omega = \omega(t)$ be a function that goes to infinity together with t. The following holds a.a.s. for every vertex v_i added at time i. For all $i \le s \le t$, we have*

$$\deg^-(v_i, s) = O\left((\omega \log t)\left(\frac{s}{i}\right)^{pA_1}\right), \quad |S(v_i, s)| = O\left(\frac{\omega \log t}{i}\right).$$

Proof. For a contradiction, suppose that $k = \deg^-(v_i, s) \ge (2\omega \log t)(s/i)^{pA_1}$ for some value of s ($i \le s \le t$). Since $k \ge \omega \log t$, Theorem 2.2 can be applied to obtain

$$\begin{aligned}
\deg^-(v_i, i) &= (1 + o(1))\frac{A_2}{A_1}\left(\frac{i}{f^{-1}(k)}\right)^{pA_1} \\
&= (1 + o(1))\frac{A_2}{A_1}\left(\frac{s}{f^{-1}(k)}\right)^{pA_1}\left(\frac{s}{i}\right)^{-pA_1} \\
&= (1 + o(1))k\left(\frac{s}{i}\right)^{-pA_1} \ge (2 + o(1))\omega \log t,
\end{aligned}$$

which is clearly a contradiction (in fact, $\deg^-(v_i, i) = 0$). □

3. Directed Diameter

The small-world property, introduced in [Watts and Strogatz 98], is a central notion in the study of complex networks (see also [Kleinberg 00]). The small-world property demands a low diameter of $O(\log t)$ and a higher clustering coefficient than found in a binomial random graph with the same number of nodes and same average degree. An early study of a social network at Stanford University was provided in [Adamic et al. 03], in which the authors found that the network has the small-world property. Similar results were found in [Ahn et al. 07], which studied Cyworld, MySpace, and Orkut, and in [Mislove et al. 07], which examined data collected from Flickr, YouTube, LiveJournal, and Orkut. Low diameter (of 6) and high clustering coefficient were reported in the Twitter network in both [Java et al. 07] and [Kwak et al. 10]. Many well-known models for complex networks, including the preferential attachment model of [Barabási and Albert 99], have diameters growing at most logarithmically with time. (In fact, in [Bollobás and Riordan 04], the authors showed that a.a.s., the diameter of the preferential attachment model is asymptotic to $\log t / \log \log t$.)

Consider a graph G_t produced by the SPA model. For a given pair of vertices $v_i, v_j \in V_t$ ($1 \le i < j \le t$), let $l(v_i, v_j)$ denote the length of the shortest directed path from v_j to v_i if such a path exists, and let $l(v_i, v_j) = 0$ otherwise. The directed diameter of a graph G_t is defined as

$$D(G_t) = \max_{1 \le i < j \le t} l(v_i, v_j).$$

The next subsection is devoted to proving the following result on the upper bound of $D(G_t)$.

Theorem 3.1. *Consider the SPA model. There exists an absolute constant $c_1 > 0$ such that a.a.s.,*

$$D(G_t) \le c_1 \log t.$$

Analyzing the lower bound appears to be more challenging and more technical. In order to avoid some additional technicalities, we will focus on the 2-dimensional Euclidean metric and assume that some extra condition holds (namely, that $A_1 < 3A_2$). Generalizing the result and removing the condition seems to be possible, but since it is not clear at the moment whether the upper or the lower bound (or neither) is correct, we do not do so. The proof of the following result can be found in Section 3.2.

Theorem 3.2. *Consider the SPA model for the 2-dimensional Euclidean metric, and assume that $A_1 < 3A_2$. There exists an absolute constant $c_2 > 0$ such that a.a.s.,*

$$D(G_t) \ge \frac{c_2 \log t}{\log \log t}.$$

3.1. Upper Bound

An $O(\log t)$ upper bound on the directed diameter is obtained as follows.

Theorem 3.3. *Let $C = 18 \max(A_2, 1)$. With probability $1 - o(t^{-2})$, we have that for all $1 \le i < j \le t$, G_t does not contain a directed (v_i, v_j)-path of length at least $k^* = C \log t$.*

Since there are at most t^2 pairs v_i, v_j, Theorem 3.1 will follow as well.

Proof. In order to simplify the notation, we use v to denote the vertex added at step $v \le t$. Let vPu be a directed (v, u)-path of length given by $vPu = (v, t_{k-1}, t_{k-2}, \ldots, t_1, u)$, and let $t_0 = u, t_k = v$. Then

$$\Pr(vPu) = \prod_{i=1}^{k} p\left(\frac{A_1 \deg^-(t_{i-1}, t_i) + A_2}{t_i}\right).$$

Let $N(v, u, k)$ be the number of directed (v, u)-paths of length k. Then

$$\mathbb{E}\, N(v, u, k) = \sum_{u<t_1<\cdots<t_{k-1}<v} p^k \,\mathbb{E}\left(\prod_{i=1}^{k}\left(\frac{A_1 \deg^-(t_{i-1}, t_i) + A_2}{t_i}\right)\right).$$

However,

$$\mathbb{E}\left(\deg^-(t_i, t_{i+1}) \mid \deg^-(t_{j-1}, t_j)\text{and}(t_{j-1}, t_j) \in E_t,\ j \le i\right) = \mathbb{E}\left(\deg^-(t_i, t_{i+1})\right).$$

We first consider the case that u tends to infinity together with t. From Theorem 2.1, it follows that

$$\mathbb{E}\left(\deg^-(t_{i-1}, t_i)\right) = (1 + o(1))\frac{A_2}{A_1}\left(\frac{t_i}{t_{i-1}}\right)^{pA_1} - \frac{A_2}{A_1}.$$

Thus

$$\begin{aligned}
\mathbb{E}\, N(v, u, k) &= \sum_{u<t_1<\cdots<t_{k-1}<v} p^k \prod_{i=1}^{k} \frac{1}{t_i}\left(A_1 \mathbb{E}\left(\deg^-(t_{i-1}, t_i)\right) + A_2\right) \\
&= \sum_{u<t_1<\cdots<t_{k-1}<v} (1 + o(1))^k (A_2 p)^k \prod_{i=1}^{k} \frac{1}{t_i}\left(\frac{t_i}{t_{i-1}}\right)^{pA_1} \\
&= (1 + o(1))^k (A_2 p)^k \left(\frac{v}{u}\right)^{pA_1} \frac{1}{v} \sum_{u<t_1<\cdots<t_{k-1}<v} \prod_{i=1}^{k-1} \frac{1}{t_i}.
\end{aligned}$$

However,

$$\begin{aligned}
\sum_{u<t_1<\cdots<t_{k-1}<v} \prod_{i=1}^{k-1} \frac{1}{t_i} &\le \frac{1}{(k-1)!}\left(\sum_{u<s<v} \frac{1}{s}\right)^{k-1} \\
&\le \frac{1}{(k-1)!}\left(\log v/u + 1/u\right)^{k-1} \le \left(\frac{e(\log v/u + 1/u)}{k-1}\right)^{k-1}.
\end{aligned}$$

Let $k^* = C \log t$, where $C = 18 \max(1, A_2)$. Assuming t sufficiently large and recalling that $pA_1 < 1$, we have

$$\begin{aligned}
\sum_{k>k^*} \mathbb{E}\, N(v, u, k) &\le 2A_2 \sum_{k>k^*}\left(\frac{(1 + o(1))A_2 pe(\log v/u + 1/u)}{k-1}\right)^{k-1} \\
&\le 2A_2\left(\frac{(1 + o(1))A_2 e(\log v/u + 1/u)}{C \log t}\right)^{k^*} \frac{1}{1 - 3A_2/C} \\
&= O(6^{-18 \log t}) = o\left(t^{-4}\right).
\end{aligned}$$

The result follows for u tending to infinity. In the case that u is a constant, it follows from Theorem 2.1 that a multiplicative correction of e can be used in

$\mathbb{E}\left(\deg^-(t_{i-1}, t_i)\right)$, leading to an error term of $O\left(t^{-18\log 2}\right) = o\left(t^{-4}\right)$, as before. This finishes the proof of the upper bound. □

3.2. Lower Bound

In this subsection, we provide an $\Omega(\log t/\log\log t)$ lower bound on the directed diameter. We use $C(u, r)$ to denote the disk of radius r centered at vertex u, where $C(u, r(t))$ and $S(u, t)$ are related through the equations above. Let $C_a(u, r)$ denote a cap of area a relative to the area πr^2 of $C(u, r)$. To form the cap of the disk $C(u_0, r)$ centered at $u_0 = (0, 0) \in \mathbb{R}^2$, we take the points $\{(x, y) : \rho r \le x \le r, x^2 + y^2 \le r\} \subseteq C(u_0, r)$. Here $0 < \rho < 1$ is taken to be an absolute constant sufficiently close to 1 to make some claimed inequalities below valid. The absolute area $\widehat{a}(\rho)$ of this cap is given by $\widehat{a}(\rho) = r^2\left(\pi/2 - \rho\sqrt{1-\rho^2} - \sin^{-1}\rho\right)$. We note that the relative area $a = \widehat{a}/\pi r^2$ of the cap is not a function of r.

3.3. Construction of a Good Sequence of Disks

We use the notation $r = r(t) = \sqrt{A_2/\pi t}$ and $r' = r'(t) = \sqrt{(A_1 + A_2)/\pi t}$ to indicate the radius of disks (at time t) with vertices of in-degree zero and of in-degree one, respectively. The condition that $r' < 2r$ (used below) is equivalent to

$$\sqrt{A_1 + A_2} < 2\sqrt{A_2}, \tag{3.1}$$

which is equivalent to $A_1 < 3A_2$.

As before, in order to simplify the notation, we use v to denote the vertex added at step $v \le t$. An important condition in our construction is that if at step v, a vertex v falls in $C_a(u, r(v))$, then $C(u, r'(v)) \cap C_a(v, r(v)) = \varnothing$. Thus if v attaches to u, so that $\deg^-(u, v) = 1$, there is still a cap of $C(v, r(v))$ [namely $C_a(v, r(v))$] that u does not reach. This condition holds, provided (3.1) is true. In this way, we can construct a series of events

$$u_1 \in C_a(u_0, r(u_1)),\ \ u_2 \in C_a(u_1, r(u_2)),\ \ \ldots,\ \ u_k \in C_a(u_{k-1}, r(u_k)). \tag{3.2}$$

Our construction further requires that no vertex v fall inside $C(u_0, r(v))$ at any step $u_0 < v < u_1$ or within $C(u_0, r'(v))$ and $u_1 < v < t$, and the same for each u_j, $1 \le j \le k$. As a consequence, $\deg^-(u_j, t) = 1$ for $0 \le j \le k - 1$. In this way, the areas of the disks are controlled at all times. Furthermore, under these circumstances, the path $u_k, u_{k-1}, \ldots, u_0$ will be a shortest path from u_k to u_0.

The next part of the construction is as follows. At step s, we divide the unit square into horizontal strips $R(1), R(2), \ldots, R(M)$ of height h and width w. Here

$$M = \frac{1}{wh}, \quad h = 4r, \quad w = 4(k+1)r, \quad r = r(s) = \sqrt{\frac{A_2}{\pi s}}.$$

Inside a strip $R = R(i)$, there is centered a strip $R' = R'(i)$ of height $2r$ and width $(4k+2)r$, thus placing a boundary of depth r around R' inside R. Note that the area of R is by a factor of $(2+o(1))$ larger than the area of R' (provided that $k \to \infty$). The purpose of this construction is that every disk of radius r centered in R' must be contained within R. Therefore, if two paths $u_k^1, u_{k-1}^1, \ldots, u_0^1$ and $u_k^2, u_{k-1}^2, \ldots, u_0^2$ are constructed such that $u_j^i \in R'(i)$, $i = 1, 2$, $j = 0, 1, \ldots, k$, then the events corresponding to the two strips are independent. Moreover, if $u \in R'(i)$, then at least half of the cap $C_a(u, r)$ falls in $R'(i)$.

Let

$$k = \frac{\beta \log t}{\log \log t}, \quad s = \frac{t}{\log t}, \quad \ell = 2(k+1)$$

for some small constant $\beta > 0$, and for convenience, pretend that k, s are integers.

We will argue that a.a.s., at least one strip will contain a sequence satisfying (3.2). The rectangle of size $L = 2r\ell$ is used to initialize the process, using some point $u = u_0$; and the sequence of k squares of side $2r$ will be enough to contain the subsequent vertices in the construction (3.2) above.

3.4. Probability Estimates for Good Sequences

We suppose that the construction of a good sequence of disks occurs in some R and that they are centered in R'. Given some set of steps $s = u_0 < u_1 < \cdots < u_k < t$, let $\mathcal{E}(u_0, u_1, \ldots, u_k)$ be the event that the construction occurred at these steps and that u_j attaches to u_{j-1}, $j = 1, 2, \ldots, k$. This forms a directed path from u_k to u_0 with the property that there are no shortcuts, i.e., no u_j attaches to any u_i where $i < j-1$.

These events are disjoint. Suppose we have another sequence $s = u_0' < u_1' < \cdots < u_k' < t$. Suppose that i is the first index such that $u_i' \neq u_i$. Then we have the contradiction that both u_i' and u_i are the first vertices in $C(u_{i-1}, r(u_i))$:

$$\begin{aligned}
\Pr(\mathcal{E}(u_0, u_1, \ldots, u_k)) &\geq \Pr(u_0 \in R') \\
&\times \prod_{\tau = u_0+1}^{u_1 - 1} \left(1 - \frac{A_2}{\tau}\right) \frac{p(a/2)A_2}{u_1} \\
&\times \prod_{\tau = u_1+1}^{u_2 - 1} \left(1 - \frac{A_1 + 2A_2}{\tau}\right) \frac{p(a/2)A_2}{u_2} \times \cdots \times \\
&\times \prod_{\tau = u_{k-1}+1}^{u_k - 1} \left(1 - \frac{(k-2)A_1 + (k-1)A_2}{\tau}\right) \frac{p(a/2)A_2}{u_k} \\
&\times \prod_{\tau = u_k+1}^{t} \left(1 - \frac{(k-1)A_1 + kA_2}{\tau}\right).
\end{aligned}$$

Let

$$q = A_1 + A_2.$$

If $x = o(1)$, then $1 - x = e^{-x-O(x^2)}$, and so, assuming $s \to \infty$,

$$\begin{aligned}
&\Pr(\mathcal{E}(u_0, u_1, \ldots, u_k)) \\
&\quad \geq \frac{1}{(2+o(1))M} (apA_2/2)^k \frac{1}{u_1} \cdots \frac{1}{u_k} \\
&\quad\quad \times \exp\left\{ -\sum_{i=0}^{k} (iA_1 + (i+1)A_2) \sum_{\tau=u_i+1}^{u_{i+1}} \left(\frac{1}{\tau} + O\left(\frac{i}{\tau^2}\right)\right)\right\} \\
&\quad \geq \frac{1}{3M} e^{-O(tk^2/s^2)} (apA_2/2)^k \frac{1}{u_1} \cdots \frac{1}{u_k} \left(\frac{u_0}{u_1}\right)^{A_2} \left(\frac{u_1}{u_2}\right)^{A_1+2A_2} \cdots \\
&\quad\quad \times \left(\frac{u_k}{t}\right)^{(k-1)A_1 + kA_2} \\
&\quad \geq \frac{1}{4M} (apA_2/2)^k \frac{u_0^{A_2} t^{A_1}}{t^{kq}} \, {u_1}^{q-1} {u_2}^{q-1} \cdots {u_k}^{q-1},
\end{aligned}$$

where the last line depends on the fact that $tk^2 = o(s^2)$.

We note that for an arbitrary function f,

$$\begin{aligned}
&\sum_{s \leq u_0 < u_1 < \cdots < u_k \leq t} f(u_1) f(u_2) \cdots f(u_k) \\
&\quad \geq \frac{1}{k!} \left(\sum_{\tau=s}^{t} f(\tau) \right)^k - \binom{k}{2} \frac{1}{k-2!} \left(\sum_{\tau=s}^{t} f^2(\tau) \right) \left(\sum_{\tau=s}^{t} f(\tau) \right)^{k-2}.
\end{aligned} \tag{3.3}$$

Indeed, to get the desired bound, we need to subtract any product in which a u_i is repeated. Thus we choose two terms from the product of k sums in $\binom{k}{2}$ ways. We choose the squared term in $\sum_{\tau=s}^{t} f^2(\tau)$ ways. Then we multiply by a bound on the number of completions $\left(\sum_{\tau=s}^{t} f(\tau)\right)^{k-2}$.

Thus

$$\begin{aligned}
&\sum_{s = u_0 < u_1 < \cdots < u_k \leq t} \prod_{j=1}^{k} {u_j}^{q-1} \\
&\quad \geq (1+o(1)) \frac{1}{k!} \left(\left(\frac{1}{q}(t^q - u_0^q) \right)^k - O(k^4) \Psi(u_0, t) \left(\frac{1}{q}(t^q - u_0^q) \right)^{k-2} \right),
\end{aligned}$$

where

$$\Psi(u_0,t)=\begin{cases}\dfrac{1}{1-2q}\left(\dfrac{1}{{u_0}^{1-2q}}-\dfrac{1}{t^{1-2q}}\right) & \text{if } 2q<1,\\ \log t-\log u_0 & \text{if } 2q=1,\\ \dfrac{1}{2q-1}\left(t^{2q-1}-{u_0}^{2q-1}\right) & \text{if } 2q>1.\end{cases}$$

(Note that s equals u_0, unlike the other u_i, and so the lower bound of the sum is a function of u_0 and t only.) Let

$$\mathcal{E}(s,t)=\bigcup_{s=u_0<u_1<\cdots<u_k<t}\mathcal{E}(u_0,u_1,\ldots,u_k).$$

From the above, it follows that

$$\begin{aligned}&\Pr(\mathcal{E}(s,t))\\ &\geq\frac{1}{5M}\frac{(apA_2/2)^k}{q^k k!}\frac{u_0^{A_2}t^{A_1}}{t^{kq}}\left((t^q-u_0^q)^k-O(k^4)\Psi(u_0,t)\left(t^q-u_0^q\right)^{k-2}\right)\\ &=\frac{1}{5Mk!}\left(\frac{apA_2}{2q}\right)^k t^q\times\left(\frac{u_0}{t}\right)^{A_2}\left(\left(1-\left(\frac{u_0}{t}\right)^q\right)^k-O(k^4t^{-2q})\Psi(u_0,t)\right)\\ &=\Omega\left(M^{-1}t^{q-\beta-o(1)}\right).\end{aligned}$$

Provided $s=o(t)$, we have

$$\begin{aligned}\int_{s/t}^1 z^\alpha(1-z^q)^k\,dz&=\frac{1}{q}\int_{(s/t)^q}^1 y^{1/q+\alpha/q-1}(1-y)^k\,dy\\ &=(1+o(1))\frac{1}{q}\frac{\Gamma((\alpha+1)/q)\Gamma(k+1)}{\Gamma(k+(\alpha+1)/q+1)}\geq ck^{-(\alpha+1)/q},\end{aligned}$$

for some absolute constant $c>0$.

Let

$$\mathcal{E}(s,t)=\bigcup_{s<u_0<t}\mathcal{E}(s,u_0,t). \tag{3.4}$$

Then

$$\Pr(\mathcal{E}(s,t))\geq\frac{c}{6Mk!}\left(\frac{apA_2}{2q}\right)^k\frac{t^q}{k^{(A_2+1)/q}}.$$

Thus $\mathcal{E}(s,t)$ is the event that the construction of an isolated directed path of length k *succeeds* in a particular strip R.

The expected number $N(s,t)$ of strips where our construction succeeds is

$$\mathbb{E}\left[N(s,t)\right]\geq ct^{q-\beta-o(1)}$$

for some absolute constant $c > 0$.

Suppose first that $q = A_1 + A_2 > 1/2$. As long as $\beta < q - 1/2$, then $\mathbb{E}[N(s,t)] = \Omega(t^{1/2+\epsilon})$ for some $\epsilon > 0$. The concentration of $N(s,t)$ follows from a standard martingale argument. All positions of the points v, $1 \le v \le t$, in the unit square are equally likely. Changing the location of a given point alters the value of $N(s,t)$ by at most 2. So

$$\Pr(N(s,t) = 0) \le \exp\left\{ -\Omega\left(\frac{(t^{1/2+\epsilon})^2}{t} \right) \right\} = o(1),$$

and the proof of the lower bound is complete.

Suppose now that $q = A_1 + A_2 \le 1/2$. In this case, the argument for a concentration of $N(s,t)$ is slightly more technical but standard as well. Unfortunately, since the events corresponding to horizontal strips $R(1), R(2), \dots, R(M)$ are not independent, we cannot use the Chernoff bound to get the result. However, the main effect on conditioning on a given strip (which yields a path $(w_1, w_2, \dots, w_k)$) is to make the sum (3.3) slightly smaller, by not allowing, u_i for every i to be equal to w_j for some j. This reduces the sum by an amount of order at most $kt^{q(k-1)}/k!$, which is negligible compared to the expression without the deletions. (Other effects are in our favor.) In particular, for two disjoint vectors $(u_0, u_1, \dots, u_k)$ and $(w_0, w_1, \dots, w_k)$ and the events $\mathcal{E}(u_0, u_1, \dots, u_k)$ and $\mathcal{E}'(w_0, w_1, \dots, w_k)$ that correspond to different strips, we have

$$\begin{aligned}\Pr(\mathcal{E}(u_0,&u_1, \dots, u_k) \wedge \mathcal{E}'(w_0, w_1, \dots, w_k)) \\ &= (1 + o(1)) \Pr(\mathcal{E}(u_0, u_1, \dots, u_k)) \Pr(\mathcal{E}'(w_0, w_1, \dots, w_k)).\end{aligned}$$

It follows that

$$\begin{aligned}\Pr(\mathcal{E}(s,t) \wedge \mathcal{E}'(s,t)) &= (1 + o(1)) \Pr(\mathcal{E}(s,t))^2, \\ \mathrm{Var}[N(s,t)] &= o(\mathbb{E}[N(s,t)]^2),\end{aligned}$$

and the concentration follows by Chebyshev's inequality.

4. Small Separators

Let us note that there are some significant differences between graphs generated by the preferential attachment model and those found in the real world. One major difference is found in their expansion properties. It was shown in [Mihail et al. 03] that a.a.s., the preferential attachment model has conductance bounded below by a constant. On the other hand, [Blandford et al. 03] shows that some WWW related graphs have smaller separators than what the preferential attachment model predicts. This observation is consistent with observations from

[Estrada 06], in which the author found that half of the real-world networks at which he looked were good expanders and the other half were not so good. In this subsection, we show that the SPA model has small separators.

Let us recall that $V_t \subseteq S$, where S is the unit hypercube $[0,1]^m$. We use the geometry of the model to obtain a sparse cut.

Theorem 4.1. *Let*

$$S' = \left\{ s = (s_1, s_2, \dots, s_m) \in S : s_1 < \frac{1}{2} \right\}.$$

Let us partition the vertex set V_t as follows: $V_t' = V_t \cap S'$, $V_t'' = V_t \cap (S \setminus S') = V_t \setminus V_t'$. Then, a.a.s., the following properties hold:

(1) $|V_t'| = (1+o(1))t/2$,

(2) $|V_t''| = (1+o(1))t/2$,

(3) $|E(V_t', V_t'')| = O\left(t^{\max\{1-1/m, pA_1\}} \log^5 t\right) = o(t)$.

Proof. Clearly, we expect $t/2$ vertices in each set V_t' and V_t''. The concentration follows immediately from the Chernoff bound. It remains to show that an upper bound for the size of the cut holds a.a.s.

It follows from Theorem 2.3 (by taking $\omega = \log t$) that a.a.s., for every $i \in [t]$, the maximum sphere of influence of a vertex v_i added at time i is $O(i^{-1} \log^2 t)$ (during the whole process). Since we aim for a result that holds a.a.s., we may assume that this property holds for all i. Therefore, the maximum radius of influence of v_i is $O\left((\log^2 t/i)^{1/m}\right)$.

We will investigate how many edges are in the cut by counting (independently) edges in this cut directed to vertices of similar age. For a given integer k such that $0 \le k < \log t$, let

$$\begin{aligned} V^{(k)} &= \left\{ v_i \in V_t : e^k \le i < \min\left\{e^{k+1}, t\right\}\right\}, \\ E^{(k)} &= \left\{ (v_i, v_j) \in E_t : v_i \in V^{(k)} \text{ and } i < j \le t \right\} \\ C^{(k)} &= E^{(k)} \cap E(V_t', V_t''). \end{aligned}$$

It is clear that $\{E^{(k)} : 0 \le k < \log t\}$ is a partition of the edge set, and so $\{C^{(k)} : 0 \le k < \log t\}$ is a partition of the cut $E(V_t', V_t'')$. It remains to estimate the size of $C^{(k)}$ for a given value of k.

Fix $0 \le k < \log t$, and let $v_i \in V^{(k)}$. Note that the maximum radius of influence of v_i is $O((e^{-k} \log^2 t)^{1/m})$. Therefore, if there is an edge in the cut directed to $v_i = (s_1, s_2, \dots, s_m)$, then v_i must fall into a strip within distance $O((e^{-k} \log^2 t)^{1/m})$

of the cutting hyperplane; that is, $|s_1 - 1/2| = O((e^{-k}\log^2 t)^{1/m})$. Since $|V^{(k)}| = O(e^k)$, we get that

$$O\left((e^{-k}\log^2 t)^{1/m}\right)\cdot|V^{(k)}| = O\left(e^{k(1-1/m)}(\log t)^{2/m}\right)$$

vertices of $V^{(k)}$ are expected to appear in this strip during the whole process. Hence, it follows from the Chernoff bound that with probability at least $1 - \exp(-\Theta(\log^2 t))$, there are $O(e^{k(1-1/m)}\log^2 t)$ vertices in this strip at the end of the process. (Note that the exponent of $\log t$ has changed from $2/m$ to 2 in order to guarantee that the value is at least $\log^2 t$, which is required for a bound to hold with the desired probability.) By Theorem 2.3 (again, by taking $\omega = \log t$), a.a.s., all vertices introduced in this time period have (final) in-degree at most $(t/e^k)^{pA_1}\log^2 t$, and we get that there are

$$|C^{(k)}| = O\left(e^{k(1-1/m)}\log^2 t\right)\cdot\left(t/e^k\right)^{pA_1}\log^2 t = O\left(t^{pA_1}e^{k(1-1/m-pA_1)}\log^4 t\right)$$

edges in the cut a.a.s.

Finally, we get that a.a.s.,

$$\begin{aligned}|E(V_t', V_t'')| &= \sum_{k=0}^{\lceil\log t\rceil-1}|C^{(k)}| = \sum_{k=0}^{\lceil\log t\rceil-1} O\left(t^{pA_1}e^{k(1-1/m-pA_1)}\log^4 t\right)\\ &\le \begin{cases}\log t\cdot O\left(t^{pA_1}t^{1-1/m-pA_1}\log^4 t\right) = O\left(t^{1-1/m}\log^5 t\right), & \text{if } pA_1 < 1-1/m;\\ \log t\cdot O\left(t^{pA_1}\log^4 t\right) = O\left(t^{pA_1}\log^5 t\right), & \text{otherwise,}\end{cases}\end{aligned}$$

which completes the proof. □

As we already mentioned, it is believed that a large fraction of real-world networks possess bad spectral expansion properties realized by relatively large gaps between the first and second eigenvalues of their adjacency matrices. The fact that the SPA model has sparse cuts easily implies bad spectral expansion properties.

The normalized Laplacian of a graph relates to important graph properties; see [Chung 97]. Let A denote the adjacency matrix and D the diagonal degree matrix of a graph G. Then the normalized Laplacian of G is

$$\mathcal{L} = I - D^{-1/2}AD^{-1/2}.$$

Let $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1} \le 2$ denote the eigenvalues of $\mathcal{L}$. The *spectral gap* of the normalized Laplacian is

$$\lambda = \max\{|\lambda_1 - 1|, |\lambda_{n-1} - 1|\}.$$

A spectral gap bounded away from zero is an indication of bad expansion properties. The next theorem represents a drastic departure from the good expansion found in binomial random graphs, where $\lambda = o(1)$ [Chung 97, Chung and Lu 06].

Theorem 4.2. *Consider the SPA model. Let $\lambda = \lambda(t)$ be the spectral gap of the normalized Laplacian of G_t. Then a.a.s.,*

$$\lambda(t) = 1 + o(1).$$

In order to prove this result, we use the expander mixing lemma for the normalized Laplacian (see [Chung 97] for a proof). For two sets of vertices X and Y, we use the notation $\mathrm{vol}(X)$ for the volume of the subgraph induced by X, $\bar{X}$ for the complement of X, and $e(X,Y)$ for the number of edges with one end in each of X and Y. (Note that $X \cap Y$ does not have to be empty; in general, $e(X,Y)$ is defined to be the number of edges between $X \setminus Y$ and Y plus twice the number of edges that contain only vertices of $X \cap Y$.)

Lemma 4.3. *Let λ be the spectral gap of the normalized Laplacian of G. For all sets $X \subseteq G$,*

$$\left| e(X,X) - \frac{(\mathrm{vol}(X))^2}{\mathrm{vol}(G)} \right| \le \lambda \frac{\mathrm{vol}(X)\mathrm{vol}(\bar{X})}{\mathrm{vol}(G)}.$$

We are ready now to return to the proof of Theorem 4.2.

Proof of Theorem 4.2. Using the notation introduced before Theorem 4.1 and the theorem itself, we get that a.a.s.,

$$\begin{aligned} \mathrm{vol}(V_t') &= (1+o(1))\,\mathrm{vol}(\bar{V_t'}) = \Theta(t), \\ \mathrm{vol}(G) &= (2+o(1))\,\mathrm{vol}(V_t'), \\ e(V_t', V_t') &= \mathrm{vol}(V_t') - e(V_t', V_t'') = (1+o(1))\,\mathrm{vol}(V_t'). \end{aligned}$$

It follows from Lemma 4.3 (applied to $X = V_t'$) that a.a.s., $\lambda(t) \ge 1 + o(1)$. By definition, $\lambda(t) \le 1$, so $\lambda(t) = 1 + o(1)$. □

5. Emergence of the Giant Component

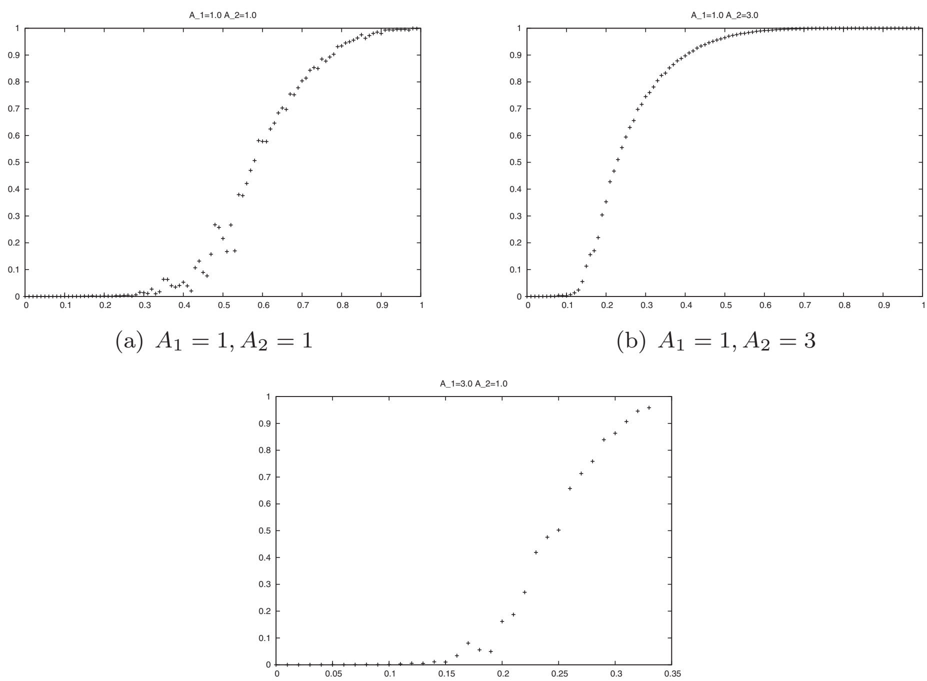
Let us note that all edges in G_t are from younger vertices to older ones; that is, denoting by v_i the vertex added at time i, we get that if $(v_j, v_i) \in E_t$, then $j > i$. This implies that G_t has t strongly connected components, each of which consists of one vertex.

On the other hand, it seems that investigating the size of the largest weak connected component is a nontrivial task. Let $\hat{G}_t = (V_t, \hat{E}_t)$ be the underlying graph of G_t; that is, $\hat{G}$ is an undirected graph on the vertex set V_t, and $\{v_j, v_i\} \in \hat{E}_t$ if and only if $(v_j, v_i) \in E_t$. We wish to know the size of the largest component in $\hat{G}_t$.

One can show that the expected number of edges added at time t of the process is

$$\deg^+(v_t, t) = \frac{pA_2}{1 - pA_1}.$$

Therefore, if $p > p_1 := (A_1 + A_2)^{-1}$, then the expected out-degree in G_t is larger than 1, and so is the expected degree in $\hat{G}_t$. By looking at the "branching factor" of the breadth-first search process, it is natural to conjecture that a.a.s., there exists a giant component if $p > p_1$. On the other hand, if $p < p_1$, then the expected out-degree in G_t is less than 1, but this fact does not in itself guar-



(a) $A_1 = 1, A_2 = 1$ (b) $A_1 = 1, A_2 = 3$

(c) $A_1 = 3, A_2 = 1$

Figure 2. A simulation of the SPA model on the unit 2-dimensional torus with $t = 100\,000$. (The horizontal axis is p; the vertical axis is the fraction of vertices in the largest component of $\hat{G}_t$.)

antee absence of the giant component in $\hat{G}_t$. Is p_1 the threshold we seek? If $p < p_2 := (A_1 + 2A_2)^{-1}$, then $\deg^+(v_t, t) < 1/2$, and so the average degree in $\hat{G}_t$ is less than 1. Perhaps p_2 is the threshold for the giant component.

We performed a number of simulations to make a better prediction (see Figure 2). For a given set of parameters A_1, A_2, we performed a number of simulations ($p = i/100$, $0 \leq i < 1/A_1$). Unfortunately, it seems that $t = 100\,000$ is still too small to observe a clear trend. However, based on these numerical results, one can conjecture that $p_3 := (2A_1 + 2A_2)^{-1}$ is the threshold for the giant component. It remains an open problem, but it seems that the answer lies between p_3 and p_1.

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