SOME UNIFIED PRESENTATIONS OF THE VOIGT FUNCTIONS

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ABSTRACT

The principal object of this note is to provide a natural further step toward the unified presentations of the Voigt functions $K(x,y)$ and $L(x,y)$ which play a rather important rôle in such diverse fields of physics as astrophysical spectroscopy and the theory of neutron reactions. Explicit representations for these functions, given in terms of some relatively more familiar special functions of one and two variables, are potentially useful in finding many other needed (numerical or analytical) properties of the Voigt functions. Several erroneous recent contributions to the theory of Voigt functions, including (for example, the *main* result of A. Siddiqui (1990), are also corrected here.

1. INTRODUCTION

The familiar Voigt functions $K(x,y)$ and $L(x,y)$ occur rather frequently in a wide variety of problems in spectroscopy and neutron physics. These functions are more intensively investigated in astrophysical spectroscopy in which we need to consider the frequency dependence of spectral line profiles while computing opacities of hot stellar gases. On the other hand, the Doppler broadened Breit–Wigner resonances in neutron reactions are essentially the same as the Voigt functions. Furthermore, the function

$$K(x,y) + i\ L(x,y)$$

is, except for a numerical factor, identical to the so-called '*plasma dispersion function*', which is tabulated by Fried and Conte (1961) and by Fettis *et al.* (1972).

In any given physical problem, a numerical or analytical evaluation of the Voigt functions (or of their aforementioned variants) is required. For an excellent review of various mathematical properties and computational methods concerning the Voigt functions, see (for example) Armstrong and Nicholls (1972); see also Haubold and John (1979). We begin our present study by recalling here the following representations [due to Reiche (1913)] for the Voigt functions $K(x,y)$ and $L(x,y)$:

$$K(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp(-yt-\tfrac{1}{4}t^2) \cos(xt)\,dt \tag{1}$$

and

$$L(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp(-yt-\tfrac{1}{4}t^2) \sin(xt)\,dt \tag{2}$$

$$(-\infty < x \quad \infty; \;\; y > 0),$$

so that

$$K(x,y) + i\,L(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp[-(y-ix)t-\tfrac{1}{4}t^2]\,dt$$

$$= \exp[(y-ix)^2]\{1-erf(y-ix)\} \tag{3}$$

and

$$K(x,y) - i\,L(X,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp[-(y+ix)t-\tfrac{1}{4}t^2]\,dt$$

$$= \exp[(y+ix)^2]\{1-erf(y+ix)\}\;, \tag{4}$$

where use is made of an elementary integral given (among other places) in Gradshteyn and Ryzhik (1980, p. 307, Equation 3.322(2)). Since the error function (see, *e.g.*, Srivastava and Kashyap 1982, p. 17, Equation (71))

$$erf(z) = \frac{2z}{\sqrt{\pi}} \; {}_1F_1\left[\frac{1}{2};\frac{3}{2};-z^2\right]$$

$$= \frac{2z}{\sqrt{\pi}} \exp(-z^2) \; {}_1F_1\left[1;\frac{3}{2};z^2\right] \qquad (|z| < \infty), \tag{5}$$

by Kummer's transformation for the confluent hypergeometric function ${}_1F_1$ (*cf.*, *e.g.*, Erdélyi *et al.* 1953, p. 253, Equation (7); see also Srivastava and Kashyap 1982, p. 24, Equation (7)), substitution in (3) and (4) followed by separation of real and imaginary parts will readily yield the *corrected* versions of the ${}_1F_1$ representations for $K(x,y)$ and $L(x,y)$ due to Exton (1981), as was observed *independently* by Katriel (1982) and Fettis[1] (1983). Following Srivastava and Miller (1987, p. 112), we should like to mention here that, in view of (5), the corrected versions of Exton's ${}_1F_1$ representations for the Voigt functions $K(x,y)$ and $L(x,y)$ would follow *directly* from (1) and (2) by appealing to some known integral formulas (Erdélyi *et al.* 1954, p. 15, Equation (16); p. 74, Equation (27); see also Gradshteyn and Ryzhik 1980, p. 480, Equations 3.897(1) and (2)).

The main purpose of this note is to give a systematic account of the various attempts toward the unified presentations of the Voigt functions $K(x,y)$ and $L(x,y)$. We also derive some explicit representations for these functions in terms of certain relatively more familiar special functions of one and two variables. Each of these representations will naturally yield numerous other potentially useful (numerical or analytical) properties of the Voigt functions. Several erroneous recent contributions to the theory of Voigt functions, including (for example) the *main*

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[1]An obvious error in Fettis's expression for

$$K(x,y) + i\,L(x,y)$$

has been corrected in Equation (3) above (see also Srivastava and Miller 1987, p. 112).

result of Siddiqui (1990), are corrected during the course of our present investigation.

2. A NOVEL UNIFICATION OF $K(x,y)$ AND $L(x,y)$

For the Bessel function $J_\nu(z)$ defined by

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \; (\frac{1}{2}z)^{\nu+2m}}{m! \;\; \Gamma(\nu+m+1)} \qquad (|z| < \infty), \tag{6}$$

it is well known that

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z \quad \text{and} \quad J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z. \tag{7}$$

Motivated by these relationships, Srivastava and Miller (1987) introduced and studied rather systematically a unification (and generalization) of the Voigt functions $K(x,y)$ and $L(x,y)$ in the form:

$$V_{\mu,\nu}(x,y) = (\tfrac{1}{2}x)^{\frac{1}{2}} \int_0^\infty t^\mu \exp(-yt-\tfrac{1}{4}t^2) J_\nu(xt)\, dt, \tag{8}$$

so that [*cf.* Equations (1), (2), and (7)]

$$K(x,y) = V_{\frac{1}{2},-\frac{1}{2}}(x,y) \quad \text{and} \quad L(x,y) = V_{\frac{1}{2},\frac{1}{2}}(x,y). \tag{9}$$

Following Srivastava and Miller (1987), we make use of the series representation (6), expand the exponential function $\exp(-yt)$, and integrate the resulting (absolutely convergent) double series in (8) term–by–term; we thus obtain the explicit expression:

$$V_{\mu,\nu}(x,y) = 2^{\mu-\frac{1}{2}}\, x^{\nu+\frac{1}{2}} \sum_{m,n=0}^{\infty} \frac{(-x^2)^m(-2y)^n}{m!\,n!\;\Gamma(\nu+m+1)}\,\Gamma[\tfrac{1}{2}(\mu+\nu+2m+n+1)] \tag{10}$$

$$(\mathrm{Re}(\mu+\nu) > -1),$$

which, upon separating the n–series into its even and odd terms, yields

$$V_{\mu,\nu}(x,y) = \frac{2^{\mu-\frac{1}{2}}\, x^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \Big\{\Gamma[\tfrac{1}{2}(\mu+\nu+1)]\Psi_2\Big[\tfrac{1}{2}(\mu+\nu+1);\ \nu+1,\ \tfrac{1}{2};\ -x^2,\ y^2\Big]$$

$$-2y\ \Gamma[\tfrac{1}{2}(\mu+\nu+2)]\Psi_2\Big[\tfrac{1}{2}(\mu+\nu+2);\ \nu+1,\ \tfrac{3}{2};\ -x^2,\ y^2\Big]\Big\} \tag{11}$$

$$(\mathrm{Re}(\mu+\nu) > -1),$$

where Ψ_2 denotes one of Humbert's confluent hypergeometric functions of two variables, defined by (*cf.*, *e.g.*, Srivastava and Karlsson 1985, p. 26, Equation (22))

$$\Psi_2[\alpha;\gamma;\gamma';\ x,y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n}\,\frac{x^m}{m!}\,\frac{y^n}{n!} \tag{12}$$

$$(\max\{|x|,|y|\} < \infty),$$

with, as usual,

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if} \quad n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if} \quad n \in \{1,2,3,\cdots\}. \end{cases}$$

For $\mu = -\nu = \frac{1}{2}$, (11) would reduce immediately to the known representation (Exton 1981, p. L76, Equation (8))

$$K(x,y) = \Psi_2\left[\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}; -x^2, y^2\right] = \frac{2y}{\sqrt{\pi}}\, \Psi_2\left[1; \tfrac{1}{2}, \tfrac{3}{2}; -x^2, y^2\right], \tag{13}$$

while the special case $\mu = \nu = \frac{1}{2}$ of (11) yields the *corrected* version of another result due to Exton (1981, p. L76, Equation (9)):

$$L(x,y) = \frac{2x}{\sqrt{\pi}}\, \Psi_2\left[1; \tfrac{3}{2}, \tfrac{1}{2}; -x^2, y^2\right] - 2xy\, \Psi_2\left[\tfrac{3}{2}; \tfrac{3}{2}, \tfrac{3}{2}; -x^2, y^2\right]. \tag{14}$$

In terms of Meijer's G-function (*cf.*, *e.g.*, Srivastava and Manocha 1984, p. 45, Equation (1) *eq seq.*), it is easily observed that

$$J_\nu(xt) = (\tfrac{1}{2}xt)^\nu\, G^{1,0}_{0,2}\left[\tfrac{1}{4}x^2t^2 \middle| \begin{matrix} \text{---} \\ 0,-\nu \end{matrix}\right], \tag{15}$$

and

$$\exp(-yt) = G^{1,0}_{0,1}\left[yt \middle| \begin{matrix} - \\ 0 \end{matrix}\right]\; \pi^{-\frac{1}{2}}\; G^{2,0}_{0,2}\left[\tfrac{1}{4}y^2t^2 \middle| \begin{matrix} \text{---} \\ 0,\tfrac{1}{2} \end{matrix}\right], \tag{16}$$

where we have used a well-known duplication formula for the G-function (Srivastava and Manocha 1984, p. 47, Equation (8) with $N = 2$). Substituting from (15) and (16) into (8), we find that

$$V_{\mu,\nu}(x,y) = \frac{1}{\sqrt{\pi}} \left(\tfrac{1}{2}x\right)^{\nu+\frac{1}{2}} \int_0^\infty t^{\mu+\nu} \exp(-\tfrac{1}{4}t^2)$$

$$\cdot \; G^{1,0}_{0,2}\left[\tfrac{1}{4}x^2t^2 \middle| \begin{matrix} \text{---} \\ 0,-\nu \end{matrix}\right] G^{2,0}_{0,2}\left[\tfrac{1}{4}y^2t^2 \middle| \begin{matrix} \text{---} \\ 0,\frac{1}{2} \end{matrix}\right] dt. \tag{17}$$

Setting $t = 2\sqrt{u}$ in (17) and evaluating the resulting integral as a G-function of two variables by appealing to the Mellin–Barnes contour integral representing each of the G-functions involved (*cf.* Srivastava and Kashyap 1982, p. 37, Equation (1); see also Srivastava *et al.* 1982, p. 7, Equation (1.2.3) *et seq.*), we obtain (*cf.* Srivastava and Miller 1987, p. 114, Equation (18))

$$V_{\mu,\nu}(x,y) = \frac{2^{\mu-\frac{1}{2}}\, x^{\nu+\frac{1}{2}}}{\sqrt{\pi}}\; G^{0,1:1,0;2,0}_{1,0:0,2;0,2}\left[\begin{matrix} x^2 \\ y^2 \end{matrix} \middle| \begin{matrix} \frac{1}{2}(1-\mu-\nu):\ \text{---};\ \text{---} \\ \text{------}:\ 0,-\nu;\ 0,\frac{1}{2} \end{matrix}\right] \tag{18}$$

$$(\mathrm{Re}(\mu+\nu) > -1).$$

For $\mu = -\nu = \frac{1}{2}$ and $\mu = \nu = \frac{1}{2}$, (18) readily yields the representations (*cf.* Haubold and John 1979, p. 481)

$$K(x,y) = \frac{1}{\sqrt{\pi}}\; G^{0,1:1,0;2,0}_{1,0:0,2;0,2}\left[\begin{matrix} x^2 \\ y^2 \end{matrix} \middle| \begin{matrix} \frac{1}{2}:\ \text{---};\ \text{---} \\ -:\ 0,\frac{1}{2};\ 0,\frac{1}{2} \end{matrix}\right] \tag{19}$$

and[2]

[2]A notational error in Haubold and John (1979, p. 481, Equation (16a)) has been corrected here (see also Srivastava and Miller 1987, p. 115).

$$L(x,y) = \frac{1}{\sqrt{\pi}} \; G^{0,1:1,0;2,0}_{1,0:0,2;0,2}\left[\begin{matrix} x^2 \\ \\ y^2 \end{matrix}\middle|\begin{matrix} \tfrac{1}{2}: & -; & - \\ \\ -: & \tfrac{1}{2},0; & 0,\tfrac{1}{2} \end{matrix}\right]. \tag{20}$$

Each of the G-functions occurring in (19) and (20) can be rewritten as an H-function of two variables (Srivastava *et al.* 1982, p. 82, Equation (6.1.1) *et seq.*). We thus obtain the alternative (but equivalent) representations:

$$K(x,y) = \frac{1}{\sqrt{\pi}} \; H^{0,1:1,0;2,0}_{1,0:0,2;0,2}\left[\begin{matrix} x^2 \\ \\ y^2 \end{matrix}\middle|\begin{matrix} (\tfrac{1}{2};1,1): & \text{———}; & \text{———} \\ \\ \text{———}: & (0,1),(\tfrac{1}{2},1); & (0,1),(\tfrac{1}{2},1) \end{matrix}\right] \tag{21}$$

and

$$L(x,y) = \frac{1}{\sqrt{\pi}} \; H^{0,1:1,0;2,0}_{1,0:0,2;0,2}\left[\begin{matrix} x^2 \\ \\ y^2 \end{matrix}\middle|\begin{matrix} (\tfrac{1}{2};1,1): & \text{———}; & \text{———} \\ \\ \text{———}: & (\tfrac{1}{2},1),(0,1); & (0,1),(\tfrac{1}{2},1) \end{matrix}\right], \tag{22}$$

in terms of the H-functions of two variables.

Equations (21) and (22) are essentially the *corrected* versions of the corresponding representations given by Buschman (1982, p. 25, Equations (3.1) and (3.2)). In fact, as pointed out by Buschman (1982, p. 25, Equation (3.4)), the H-function representations (21) and (22) for the Voigt functions $K(x,y)$ and $L(x,y)$ are analytic in both variables x and y provided that

$$|\arg(x)| + |\arg(y)| < \frac{1}{2}\pi. \tag{23}$$

The G-function representation (18) can also be rewritten, in a straightforward manner, as an H-function representation for the generalized Voigt function $V_{\mu,\nu}(x,y)$. We are thus led to a unification (and generalization) of the H-function representations (21) and (22) in the form (*cf.* Srivastava and Miller 1987, p. 115, Equation (24)):

$$V_{\mu,\nu}(x,y) = \frac{2^{\mu-\frac{1}{2}}\, x^{\nu+\frac{1}{2}}}{\sqrt{\pi}}\; H^{0,1:1,0;2,0}_{1,0:0,2;0,2}\left[\begin{matrix} x^2 \\ \\ y^2 \end{matrix}\middle|\begin{matrix} (\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu;1,1): \\ \text{———}: \end{matrix}\right.$$

$$\left.\begin{matrix} \text{———};\ \text{———} \\ \\ (0,1),(-\nu,1);\ (0,1),(\frac{1}{2},1) \end{matrix}\right] \qquad (\mathrm{Re}(\mu+\nu) > -1), \tag{24}$$

in which the variables x and y are constrained, as also in (21) and (22), by (23).

If, in this last result (24), we employ a known series expansion of the H-function of two variables (*cf.* Srivastava *et al.* 1982, p. 84, Equation (6.2.1)), we shall arrive once again at the double confluent hypergeometric series representation (11) which we have already computed *directly*.

It may be of interest to observe here that the vast literature on the G- and H-functions of two variables (see Srivastava *et al.* 1982) can be appropriately used in order to derive many potentially useful (numerical or analytical) properties of the generalized Voigt function $V_{\mu,\nu}(x,y)$ and, in particular, of the Voigt functions $K(x,y)$ and $L(x,y)$ themselves.

3. FURTHER ATTEMPTS TOWARD UNIFIED PRESENTATIONS OF $K(x,y)$ AND $L(x,y)$

An interesting generalization of the (classical) Bessel function $J_\nu(z)$ is due to Wright (1935a) who studied the function $J^{\mu}_{\nu}(z)$ defined by

$$J_{\nu}^{\mu}(z) = \sum_{m=0}^{\infty} \frac{(-z)^{m}}{n!\ \Gamma(\nu+\mu m+1)} \qquad (\mu > 0;\ \ |z| < \infty), \tag{25}$$

so that, by comparing the definitions (6) and (25),

$$J_{\nu}(z) = (\tfrac{1}{2}z)^{\nu}\ J_{\nu}^{1}(\tfrac{1}{4}z^{2}). \tag{26}$$

In fact, *Wright's generalized Bessel function* $J_{\nu}^{\mu}(z)$ defined by (25) is contained, as a particular case, in the following class of *generalized hypergeometric functions* studied by Fox (1928) and Wright (1935b):

$${}_{p}\psi_{q}\left[\begin{matrix}(\alpha_1,A_1),\cdots,(\alpha_p,A_p);\\ \\ (\beta_1,B_1),\cdots,(\beta_q,B_q);\end{matrix}\ z\right] = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j+A_j m)}{\prod_{j=1}^{q} \Gamma(\beta_j+B_j m)}\ \frac{z^{m}}{m!} \tag{27}$$

$$[A_j > 0 \quad (j = 1,\cdots,p); \qquad B_j > 0 \quad (j = 1,\cdots,q)],$$

provided that the series converges, it being understood (as always) that no zeros appear in the denominator of (27). Clearly, we have

$${}_{p}\psi_{q}\left[\begin{matrix}(\alpha_1,1),\cdots,(\alpha_p,1);\\ \\ (\beta_1,1),\cdots,(\beta_q,1);\end{matrix}\ z\right] = {}_{p}F_{q}\left[\begin{matrix}\alpha_1,\cdots,\alpha_p;\\ \\ \beta_1,\cdots,\beta_q;\end{matrix}\ z\right] \tag{28}$$

and

$$J_{\nu}^{\mu}(z) = {}_{0}\psi_{1}[\text{---};\ (\nu+1,\mu);\ -z]. \tag{29}$$

More importantly, in terms of Fox's H-function, we can write (*cf.* Srivastava and Kashyap 1982, p. 42, Equation II.5(21))

$$
{}_p\psi_q\left[\begin{matrix}(\alpha_1,A_1),\cdots,(\alpha_p,A_p); \\ \\ (\beta_1,B_1),\cdots,(\beta_q,B_q);\end{matrix}\; z\right]
$$

$$
= H^{1,p}_{p,q+1}\left[-z\left|\begin{matrix}(1-\alpha_1,A_1),\cdots,(1-\alpha_p,A_p) \\ \\ (0,1),(1-\beta_1,B_1),\cdots,(1-\beta_q,B_q)\end{matrix}\right.\right]. \tag{30}
$$

In an attempt to generalize the work of Srivastava and Miller (1987), Siddiqui[3] (1990) chooses to replace the Bessel function occurring in the definition (8) by the following *very special case* of the (Fox–Wright) function ${}_p\psi_q$:

$$
J^{\mu}_{\nu,\lambda}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m\,(\tfrac{1}{2}z)^{\nu+2\lambda+2m}}{\Gamma(\lambda+m+1)\;\Gamma(\nu+\lambda+\mu m+1)}
$$

$$
= (\tfrac{1}{2}z)^{\nu+2\lambda}\;{}_1\psi_2\left[\begin{matrix}(1,1); \\ \\ (\lambda+1,1),(\nu+\lambda+1,\mu);\end{matrix}\; -\tfrac{1}{4}z^2\right] \tag{31}
$$

$$
(\mu > 0;\quad |z| < \infty),
$$

so that, obviously,

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[3]Siddiqui (1990) repeats *most* of what was already given by Srivastava and Miller (1987); unfortunately, as we shall observe here, his *only* notable contribution in his *entire* paper (Siddiqui 1990, p. 266, Equation (14)) is in *serious* error.

$$J^{1}_{\nu,0}(z) = J_{\nu}(z) \tag{32}$$

and

$$J^{\mu}_{\nu,0}(z) = (\tfrac{1}{2}z)^{\nu}\ J^{\mu}_{\nu}(\tfrac{1}{4}z^2) \qquad (\mu > 0). \tag{33}$$

Upon comparing the definition (31) with the relationship (30), we readily obtain

$$J^{\mu}_{\nu,\lambda}(z) = (\tfrac{1}{2}z)^{\nu+2\lambda}\ H^{1,1}_{1,3}\left[\tfrac{1}{4}z^2 \middle| \begin{matrix} (0,1) \\ (0,1),(-\lambda,1),(-\nu-\lambda,\mu) \end{matrix}\right] \qquad (\mu > 0), \tag{34}$$

which, for $\lambda = 0$, immediately yields [*cf.* Equations (29), (30), and (33)]

$$J^{\mu}_{\nu}(z) = (\tfrac{1}{2}z)^{\nu}\ H^{1,0}_{0,2}\left[\begin{matrix} \text{———} \\ (0,1),(-\nu,\mu) \end{matrix}\right] \qquad (\mu > 0). \tag{35}$$

Formula (34) was stated and used *incorrectly* by Siddiqui (1990, p. 265, Equation (11); p. 266, Equation (13)). Naturally, therefore, the *main* result of Siddiqui (1990, p. 266, Equation (14)), which he derived by using the obviously *erroneous*version of (34), does not hold true as claimed. With a view to finding the *correct* verstion of this only notable contribution in Siddiqui's paper, we begin by recalling his definition (Siddiqui 1990, p. 265, Equation (8)) in the *corrected* form:

$$\Omega^{\mu}_{\eta,\nu,\lambda}(x,y) = (\tfrac{1}{2}x)^{\frac{1}{2}} \int_0^{\infty} t^{\eta} \exp(-yt-\tfrac{1}{4}t^2)\ J^{\mu}_{\nu,\lambda}(xt)\,dt, \tag{36}$$

which, when compared with the Srivastava–Miller definition (8), yields the relationship:

$$\Omega^{1}_{\mu,\nu,0}(x,y) = V_{\mu,\nu}(x,y), \tag{37}$$

where we have made use of the reduction formula (32).

Next we apply the *correct* H-function representation (34), together with the elementary result:

$$\exp(-yt) = H_{0,1}^{1,0}\left[yt \,\middle|\, \begin{matrix} \text{———} \\ (0,1) \end{matrix}\right] = \frac{1}{\sqrt{\pi}} H_{0,2}^{2,0}\left[\tfrac{1}{4}y^2t^2 \,\middle|\, \begin{matrix} \text{———} \\ (0,1),(\tfrac{1}{2},1) \end{matrix}\right], \tag{38}$$

which is essentially the same as the result used earlier by Srivastava and Miller (1987, p. 114, Equation (16)), and we find from (36) that

$$\Omega_{\eta,\nu,\lambda}^{\mu}(x,y) = \frac{1}{\sqrt{\pi}} (\tfrac{1}{2}x)^{\nu+2\lambda+\frac{1}{2}} \int_0^{\infty} t^{\eta+\nu+2\lambda} \exp(-\tfrac{1}{4}t^2)$$
$$\cdot\, H_{1,3}^{1,1}\left[\tfrac{1}{4}x^2t^2 \,\middle|\, \begin{matrix} (0,1) \\ (0,1),(-\lambda,1),(-\nu-\lambda,\mu) \end{matrix}\right]$$
$$\cdot\, H_{0,2}^{2,0}\left[\tfrac{1}{4}x^2t^2 \,\middle|\, \begin{matrix} \text{———} \\ (0,1),(\tfrac{1}{2},1) \end{matrix}\right] dt, \tag{39}$$

provided that the integral converges.

Setting $t = 2\sqrt{\tau}$ in (39), if we evaluate the resulting integral in terms of an H-function of two variables by appealing to the Mellin–Barnes contour integral representing each of the H-function involved (*cf.* Srivastava *et al.* 1982, p. 3, Equation (1.1.4); p. 82, Equation (6.1.1) *et seq.*), we shall finally obtain the H-function representation:

$$\Omega^{\mu}_{\eta,\nu,\lambda}(x,y) = \frac{2^{\eta-\frac{1}{2}}\ x^{\nu+2\lambda+\frac{1}{2}}}{\sqrt{\pi}}$$

$$\cdot\ H^{0,1:1,1;2,0}_{1,0:1,3;0,2}\left[\begin{matrix} x^2 \\ y^2 \end{matrix}\middle| \begin{matrix} (\frac{1}{2}(1-\eta-\nu)-\lambda;1,1): & (0,1); & \text{———} \\ \text{————}: & (0,1),(-\lambda,1),(-\nu-\lambda,\mu); & (0,1),(\frac{1}{2},1) \end{matrix}\right] \tag{40}$$

$$(\mathrm{Re}(\eta+\nu+2\lambda) > -1; \quad \mu > 0),$$

in which the variables x and y are constrained, as also in (21), (22), and (24), by (23).

The H-function representation (40) for the generalized Voigt function defined by (36) provides the *corrected* version of the main (and, as we remarked above, the only significant) result of Siddiqui (1990, p. 266, Equation (14)). It is not difficult to observe also that, in its *special* case when

$$\lambda = \mu - 1 = 0,$$

(40) would correspond to the Srivastava–Miller result (24).

In view of the H-function representations (24) and (40), a natural further step toward the unified presentations of the Voigt functions $K(x,y)$ and $L(x,y)$ is provided by the definition [*cf.* Equation (39)]:

$$S^{P:m,n;r,s;\nu}_{Q:p,q;u,v;\zeta}\left[\begin{matrix} (\xi_j;e_j,f_j)_{1,P}: (\alpha_j,A_j)_{1,p};(\gamma_j,C_j)_{1,u}; \\ (\eta_j;g_j,h_j)_{1,Q}: (\beta_j,B_j)_{1,q};(\delta_j,D_j)_{1,v}; \end{matrix}\ x,y\right]$$

$$= \frac{1}{\sqrt{\pi}} \left(\tfrac{1}{2}x\right)^{\nu+\frac{1}{2}} \int_0^\infty t^{\zeta+\nu} \exp\left(-\tfrac{1}{4}t^2\right)$$

$$\cdot H^{0,0:m,n;r,s}_{P,Q:p,q;u,v} \left[\begin{matrix} \tfrac{1}{4}x^2t^2 \\ \tfrac{1}{4}y^2t^2 \end{matrix} \middle| \begin{matrix} (\xi_j;\, e_j,\, f_j)_{1,P} : (\alpha_j,\, A_j)_{1,p} ; (\gamma_j,\, C_j)_{1,u} ; \\ (\eta_j;\, g_j,\, h_j)_{1,Q} : (\beta_j,\, B_j)_{1,q} ; (\delta_j,\, D_j)_{1,v} ; \end{matrix} \quad x,y \right] dt, \tag{41}$$

where, as usual, $(\xi_j;\, e_j, f_j)_{1,P}$ abbreviates the P-member array:

$$(\xi_1; e_1, f_1), \cdots, (\xi_P;\, e_P, f_P),$$

with similar interpretations for $(\alpha_j, A_j)_{1,p}$, *et cetera* (see, for details, Srivastava *et al.* 1982, Chapter 6).

In particular, for $P = Q = 0$, the definition (41) reduces immediately to the form:

$$\mathscr{S}^{m,n;r,s;\nu}_{p,q;u,v;\zeta} \left[\begin{matrix} (\alpha_j, A_j)_{1,p} ;\ (\gamma_j, C_j)_{1,u} ; \\ (\beta_j, B_j)_{1,q} ;\ (\delta_j, D_j)_{1,v} ; \end{matrix} \quad x,y \right]$$

$$= \frac{1}{\sqrt{\pi}} \left(\tfrac{1}{2}x\right)^{\nu+\frac{1}{2}} \int_0^\infty t^{\zeta+\nu} \exp\left(-\tfrac{1}{4}t^2\right)$$

$$\cdot H^{m,n}_{p,q} \left[\tfrac{1}{4}x^2t^2 \middle| \begin{matrix} (\alpha_j, A_j)_{1,p} \\ \\ (\beta_j, B_j)_{1,q} \end{matrix} \right]$$

$$\cdot H^{r,s}_{u,v} \left[\tfrac{1}{4}y^2t^2 \middle| \begin{matrix} (\gamma_j, C_j)_{1,u} \\ \\ (\delta_j, D_j)_{1,v} \end{matrix} \right] dt \tag{42}$$

$$[A_j > 0 \quad (j = 1,\cdots,p); \quad B_j > 0 \quad (j = 1,\cdots,q); \quad C_j > 0 \quad (j = 1,\cdots,u); \quad D_j > 0 \quad (j = 1,\cdots,v)],$$

which may be compared with (39).

If we set $t = 2\sqrt{\tau}$ in (41) and evaluate the resulting integral as before, we obtain the following H-function representation for the generalized Voigt function defined by (41):

$$S^{P:m,n;r,s;\nu}_{Q:p,q;u,v;\zeta}\left[\begin{matrix}(\xi_j;e_j,f_j)_{1,P}:(\alpha_j,A_j)_{1,p};(\gamma_j,C_j)_{1,u};\\ \\ (\eta_j;g_j,h_j)_{1,Q}:(\beta_j,B_j)_{1,q};(\delta_j,D_j)_{1,v};\end{matrix}\quad x,y\right]$$

$$= \frac{2^{\zeta-\frac{1}{2}}\,x^{\nu+\frac{1}{2}}}{\sqrt{\pi}}\;H^{0,1:m,n;r,s}_{P+1,Q:p,q;u,v}\left[\begin{matrix}x^2\\ \\ y^2\end{matrix}\right|$$

$$\left.\begin{matrix}(\tfrac{1}{2}(1-\nu-\zeta);1,1),(\xi_j;e_j,f_j)_{1,P}:(\alpha_j,A_j)_{1,p};(\gamma_j,C_j)_{1,u}\\ (\eta_j;g_j,h_j)_{1,Q}:(\beta_j,B_j)_{1,q};(\delta_j,D_j)_{1,v}\end{matrix}\right] \tag{43}$$

$$\left[\mathrm{Re}\left[\nu + \zeta + \frac{\beta_j}{B_j} + \frac{\delta_k}{D_k}\right] > -1 \quad (j = 1,\cdots,m;\ k = 1,\cdots,r)\right].$$

For $P = Q = 0$, (43) yields the following H-function representation for the generalized Voigt function defined by (42):

$$\mathscr{S}^{m,n;r,s;\nu}_{p,q;u,v;\zeta}\left[\begin{matrix}(\alpha_j,A_j)_{1,p};(\gamma_j,C_j)_{1,u};\\ \\ (\beta_j,B_j)_{1,q};(\delta_j,D_j)_{1,v};\end{matrix}\quad x,y\right]$$

$$= \frac{2^{\zeta-\frac{1}{2}}\ x^{\nu+\frac{1}{2}}}{\sqrt{\pi}}\ H^{0,1:m,n;r,s}_{1,0:p,q;u,v}$$

$$\left[\begin{matrix} x^2 \\ \\ y^2 \end{matrix}\left|\begin{matrix} (\frac{1}{2}(1-\nu-\zeta);1,1):\ (\alpha_j, A_j)_{1,p};\ (\gamma_j, C_j)_{1,u} \\ \\ \text{———}:\ (\beta_j, B_j)_{1,q};\ (\delta_j, D_j)_{1,v} \end{matrix}\right.\right] \tag{44}$$

$$\left[\mathrm{Re}\left[\nu + \zeta + \frac{\beta_j}{B_j} + \frac{\delta_k}{D_k}\right] > -1 \qquad (j = 1,\cdots,m;\quad k = 1,\cdots,r)\right].$$

Upon comparing the definitions (36) and (42), we obtain the relationship:

$$\Omega^{\mu}_{\eta,\nu,\lambda}(x,y) = \mathscr{S}^{1,1;2,0;\nu+2\lambda}_{1,3;0,2;\eta}\left[\begin{matrix} (0,1);\ \text{———}; \\ \\ (0,1),(-\lambda,1),(-\nu-\lambda,\mu);\ (0,1),(\frac{1}{2},1); \end{matrix}\ x,y\right] \tag{45}$$

which, in view of (44), leads us immediately to the H-function representation (40). In particular, for the generalized Voigt function $V_{\mu,\nu}(x,y)$ of Srivastava and Miller (1987), we have [*cf.* Equation (37)]

$$V_{\mu,\nu}(x,y) = \mathscr{S}^{1,0;2,0;\nu}_{0,2;0,2;\mu}\left[\begin{matrix} \text{———};\ \text{———}; \\ \\ (0,1),(-\nu,1);\ (0,1),(\frac{1}{2},1); \end{matrix}\ x,y\right], \tag{46}$$

which, again in view of (44), leads us immediately to the H-function representation (24). More importantly, for the Voigt functions $K(x,y)$ and $L(x,y)$ *themselves*, we observe that

$$K(x,y) = \mathscr{S}^{1,0;2,0;-\frac{1}{2}}_{0,2;0,2;\ \frac{1}{2}}\left[\begin{matrix} \text{———};\ \text{———}; \\ \\ (0,1),(\frac{1}{2},1);\ (0,1),(\frac{1}{2},1); \end{matrix}\ x,y\right] \tag{47}$$

and

$$L(x,y) = \mathscr{S}^{1,0;2,0;\frac{1}{2}}_{0,2;0,2;\frac{1}{2}}\left[\begin{matrix} \text{———};\ \text{———}; \\ \\ (0,1),(-\frac{1}{2},1);\ (0,1),(\frac{1}{2},1); \end{matrix}\ x,y\right], \tag{48}$$

respectively.

These last relationships (47) and (48) can indeed be used to derive various needed (numerical or analytical) properties of the Voigt functions $K(x,y)$ and $L(x,y)$ by simply specializing the corresponding properties of the unifications proposed by (41) and (42). For example, the H-representations (21) and (22) would follow readily from (44) by appealing to the relationships (47) and (48), respectively.

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