Some uniqueness and nonexistence theorems for embedded minimal surfaces

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1 Introduction.

Thanks to the works of Callahan, Costa, Hoffman, Karcher, Meeks, Rosenberg, Wei, etc.—see for example [1],[2],[3],[5],[8],[11],[12],[16],[20]—, we dispose now of a large number of properly embedded minimal surfaces in the euclidean space $\mathbb{R}^3$ other than the classical examples. All those surfaces can be viewed as minimal surfaces with finite total curvature properly embedded in complete flat three manifolds. The most basic invariants associated to a surface of this type are its topology and its periodicity. It is an interesting problem to decide if the simplest examples—like the catenoid, the helicoid, the Scherk's surfaces or the Riemann example—can be characterized in terms of some of these invariants. In the non-periodic case, there are two important uniqueness theorems in this direction: the first one, obtained by Schoen [17], characterizes the catenoid as the only complete minimal surface embedded

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in $I\!R^3$ with finite total curvature and two ends. The second one, proved two years ago by López and Ros [10], says that

> *"The plane and the catenoid are the only properly embedded minimal surfaces with finite total curvature and genus zero in $I\!R^3$".*

In this paper we first give a new and simpler proof of this fact. Our method of proof is based on the existence of a one-parameter deformation for minimal surfaces such that the flux along any of their closed curves is a vertical vector of $I\!R^3$. Although this deformation was used in [10], the arguments used here are different and can be adapted to other situations.

In fact, we will also apply the method to obtain some global theorems in the singly-periodic case, i.e., for minimal surfaces with finite topology properly embedded in $I\!R^3/S_\theta$, where S_θ is a screw motion obtained by rotation around the positive x_3-axis by angle $\theta \in [0, 2\pi[$ followed by a non trivial translation along the x_3-axis. In [13], Meeks and Rosenberg studied the geometry of these surfaces, classifying their possible end types: they are—all simultaneously—asymptotic to planes, flat vertical annuli or to ends of helicoids. The second Scherk's surface and its generalizations [8] are genus zero examples with $2k$ ends asymptotic to flat vertical annuli, properly embedded in $I\!R^3/T$, T being a non trivial vertical translation and k an integer greater than one. Moreover, Karcher [8] discovered genus zero surfaces minimally embedded in $I\!R^3/S_\theta$, $\theta \neq 0$, with helicoidal type ends, which are deformations of the above examples. In the case $\theta = 0$, the only known example of this type is the helicoid. In this paper we will prove a uniqueness theorem for this surface. More precisely, we obtain that

> *"The helicoid is the only properly embedded minimal surface of genus zero in $I\!R^3/T$, T being a non trivial vertical translation, with a finite number of helicoidal type ends."*

If the surface has only two ends, the above result was proved by Toubiana [18].

Another classic example of singly periodic properly embedded minimal surface is the Riemann's example [1]. It can be viewed as a genus zero surface with an infinite number of punctures in $I\!R^3$ or as a punctured torus with two planar ends in $I\!R^3/T$. We will prove that this surface has no equivalent in the ambient space $I\!R^3/S_\theta$, $\theta \neq 0$, that is

"There are no properly embedded minimal surfaces of genus one and a finite number of planar ends in $I\!R^3/S_\theta$, $\theta \neq 0$."

Furthermore, the method developed in this paper can be used to obtain information about properly embedded minimal surfaces with boundary: If Σ is a properly embedded minimal surface of genus zero contained in a halfspace of $I\!R^3$ with finite total curvature and whose boundary is a convex Jordan curve lying on the boundary of the halfspace then we prove Σ has only one end and so it is an annulus. We prove a similar result when the minimal surface lies in a slab of $I\!R^3$ and its boundary consists on two convex Jordan curves lying on the boundary of the slab.

The paper is organized as follows. In section 2 we review some basic facts about the flux of a surface along a curve, define the deformation in terms of the Weierstrass data of a minimal surface, and study its basic properties. The theorems in the non-periodic case are proved in section 3, and the singly-periodic ones are developed in section 4.

2 The deformation.

Let Σ be a Riemann surface and $\psi : \Sigma \longrightarrow I\!R^3$ a conformal minimal immersion of Σ into the three dimensional euclidean space. If Γ is a closed curve in Σ parametrized by $\gamma(s)$, s being the arc parameter of $\psi \circ \gamma$, we denote by $n(s)$ the conormal of ψ along the curve Γ, i.e. $n(s) = -d\psi\,(J\gamma'(s))$, where J is the complex structure of Σ. Recall that the *flux of* ψ *along* Γ is defined by

$$Flux(\Gamma) = \int_\Gamma n(s)\,ds.$$

The flux does not depend on the curve in a fixed homology class: in fact, it can be viewed as the $I\!R^3$-valued cohomology class on Σ determined by the closed one form $\quad -(d\psi)\circ J$.

Calling ψ^* the—in general not globally well-defined—conjugate minimal immersion of ψ, we have

$$\int_\Gamma d\psi^*\left(\gamma'(s)\right)\,ds = -\int_\Gamma (d\psi)\left(J\gamma'(s)\right)\,ds = Flux(\Gamma).$$

Hence, the flux of ψ along Γ is the period of the multivalued map ψ^*. Equivalently, using the $\mathbb{C}^3$-valued holomorphic one-form $(\phi_1, \phi_2, \phi_3) = \frac{\partial \psi}{\partial z}\, dz$, we have

$$\int_\Gamma (\phi_1, \phi_2, \phi_3) = i\, Flux(\Gamma).$$

Note that the real part of the above integral vanishes because ψ is well defined.

From the Weierstrass representation [15] we know that ψ is determinated by the meromorphic map g and the holomorphic one-form ω on Σ defined by the relations

$$\phi_1 = \frac{1}{2}(1 - g^2)\omega, \qquad \phi_2 = \frac{i}{2}(1 + g^2)\omega, \qquad \phi_3 = g\omega.$$

We recall that g is the stereographic projection from the north pole of the Gauss map of ψ. With this notation, the following assertions are equivalent:

i) For each closed curve Γ in Σ, the flux of ψ along Γ vanishes.

ii) The holomorphic one-forms ϕ_1, ϕ_2, ϕ_3 are exact.

iii) The holomorphic one-forms ω, $g\omega$, $g^2\omega$ are exact.

iv) The conjugate immersion ψ^* is globally well-defined on Σ.

For each positive number λ, we define on Σ the meromorphic map $g_\lambda = \lambda g$ and the holomorphic one-form $\omega_\lambda = \frac{1}{\lambda}\omega$. It follows that, via Weierstrass representation, g_λ and ω_λ determine a—generally multivalued—conformal minimal immersion $\psi_\lambda : \Sigma \longrightarrow \mathbb{R}^3$. As above, we obtain directly that the following assertions are equivalent:

i') For each closed curve Γ in Σ, the flux of ψ along Γ is a vertical vector.

ii') The holomorphic one-forms ϕ_1 and ϕ_2 are exact.

iii') The holomorphic one-forms ω and $g^2\omega$ are exact.

iv') For each $\lambda > 0$, the immersion ψ_λ is globally well-defined on Σ.

If the immersion satisfies *i')* we will say that *the flux of ψ is vertical.*

From now, we will assume in this section that $\psi : \Sigma \longrightarrow I\!\!R^3$ is a conformal minimal immersion with vertical flux and Weierstrass representation given by the meromorphic map g and the holomorphic one-form ω. Put

$$F = \int \frac{\omega}{2}, \qquad G = \int \frac{g^2\omega}{2}, \qquad \text{and} \qquad x_3 = \text{Re} \int g\omega.$$

Then, the immersion can be written as

$$\psi = \left(\overline{F} - G, x_3\right) : \Sigma \longrightarrow \mathbb{C} \times I\!\!R \equiv I\!\!R^3. \tag{1}$$

In the same way, for each $\lambda > 0$ the immersion $\psi_\lambda : \Sigma \longrightarrow I\!\!R^3$ is given by

$$\psi_\lambda = \left(\frac{1}{\lambda}\overline{F} - \lambda G, x_3\right). \tag{2}$$

Now, we will study some basic properties of the one-parameter family $\{\psi_\lambda\}_{\lambda>0}$. First, we remark that the third coordinate function of ψ_λ does not depend on λ.

Recall [15] that the induced metric and the Gauss curvature of ψ are respectively given by

$$ds^2 = \frac{1}{4}\left(1 + |g|^2\right)^2 |\omega|^2, \qquad K = -\left(\frac{4|dg|}{|\omega|\ (1 + |g|^2)^2}\right)^2.$$

From these formulae we obtain the following lemma:

Lemma 1 *Let ds_λ^2 and K_λ be the induced metric on Σ by ψ_λ and its Gauss curvature, respectively. Then, there exist real positive constants $c_1(\lambda), c_2(\lambda)$ such that*

$$c_1(\lambda)\, ds^2 \le ds_\lambda^2 \le c_2(\lambda)\, ds^2,$$
$$c_1(\lambda)\, |K| \le |K_\lambda| \le c_2(\lambda)\, |K|.$$

In particular, if ds^2 is complete the same holds for ds_λ^2. Also we obtain that $\int_\Sigma K\, dA$ is finite if and only if $\int_\Sigma K_\lambda\, dA_\lambda$ is finite.

Let Π be a horizontal plane in $\mathbb{R}^3$ such that its intersection with $\psi(\Sigma)$ is transversal. Then the intersection of $\psi_\lambda(\Sigma)$ with Π is also transversal. Denote by C a curve contained in $\psi(\Sigma) \cap \Pi$, and by C_λ the corresponding curve in $\psi_\lambda(\Sigma) \cap \Pi$. Let ν and ν_λ be the normal vectors to C and C_λ—as planar curves—, respectively. Both of them are horizontal vector fields pointing to the same directions as g and g_λ—viewed as plane vectors—. As $g_\lambda = \lambda g$, it follows that $\nu_\lambda = \nu$. In particular, we conclude the following lemma:

Lemma 2

i) If C is a segment, then C_λ is a segment parallel to C. Moreover, if C is a straight line, the same holds for C_λ.

ii) If C is a convex Jordan curve, then the same holds for C_λ.

The last assertion of *i)* follows from lemma 1.

Lemma 2 also holds if C lies on the boundary of the minimal surface. If this boundary is analytic, the above proof works. In the smooth category we can reason as follows: The $\overline{\mathbb{C}}$-valued Gauss map g and the coordinate function x_3 extend smoothly to the boundary. Hence the same holds for $\phi_3 = \frac{\partial x_3}{\partial z}\, dz$ and $\omega = \phi_3/g$. Then also $F = \int \omega/2$ and $G = \int g^2\omega/2$ extend and finally we conclude that $\psi_\lambda = \left(\frac{1}{\lambda}\overline{F} - \lambda G, x_3\right)$ is smooth at the boundary for each $\lambda > 0$. Now the arguments in the proof of lemma 2 are applied without changes.

The next lemma states some symmetry properties of $\{\psi_\lambda\}_{\lambda>0}$ induced by the symmetries of ψ. Note that the ψ_λ's are determinated up an additive constant.

Lemma 3 *Let ψ and $\{\psi_\lambda\}_{\lambda>0}$ as above.*

i) If ψ is invariant by a reflection in a vertical plane of $\mathbb{R}^3$ and non flat, then all the ψ_λ's can be chosen invariant by the same reflection.

ii) If ψ is invariant by a translation which induces a holomorphic transformation of Σ, then the same holds for all the ψ_λ's—actually, this translation depends on λ—.

iii) If ψ is invariant by a screw motion S_θ obtained by rotation around the x_3-axis by θ, $0 < \theta < 2\pi$, followed by a—possibly trivial—translation along the x_3-axis, and this screw motion induces a holomorphic transformation of Σ, then all the ψ_λ's can be chosen invariant by S_θ.

Proof: First assume that ψ is invariant by a reflection R_Π in a vertical plane Π of $I\!R^3$. Denote by S the induced conformal transformation on Σ. As the surface is not contained in Π, the intersection $\psi(\Sigma) \cap \Pi$ has dimension one. So S must be antiholomorphic. We can assume that Π is the (x_1, x_3)-plane. Hence, from (1) we have $\overline{F} \circ S - G \circ S = F - \overline{G}$. Grouping the holomorphic terms and the antihilomorphic ones in different sides of the equation, we find a complex constant a verifying

$$\overline{F} \circ S - F \equiv a \equiv G \circ S - \overline{G}.$$

Moreover a has zero real part—we can check this fact taking a point $p \in \Sigma$ such that $\psi(p)$ lies in Π—; as F and G are determined up to an additive constant, this fact allow us to choose the parameters of integration such that $\overline{F} \circ S - F \equiv 0 \equiv G \circ S - \overline{G}$. Thus $\left(\frac{1}{\lambda}\overline{F} - \lambda G\right) \circ S = \frac{1}{\lambda} F - \lambda \overline{G}$, and we have $\psi_\lambda \circ S = R_\Pi \circ \psi_\lambda$, for each $\lambda > 0$. This proves *i)*.

Take ψ invariant by a translation T_v of vector $v = (u, v_3) \in \mathbb{C} \times I\!R$. This translation induces a conformal transformation $S : \Sigma \longrightarrow \Sigma$, defined by the equation $\psi \circ S = T_v \circ \psi$. Assume that S is holomorphic; using (1), we have

$$\overline{F} \circ S - G \circ S = \overline{F} - G + u, \qquad x_3 \circ S = x_3 + v_3.$$

The left equation implies that there exists a complex constant a such that

$$\overline{F} \circ S - \overline{F} \equiv a \equiv G \circ S - G + u.$$

Hence, we have

$$\left(\frac{1}{\lambda}\overline{F} - \lambda G\right) \circ S = \frac{1}{\lambda}\overline{F} - \lambda G + a\left(\frac{1}{\lambda} - \lambda\right) + \lambda u,$$

so calling T_v^λ to the translation of vector $\left(a\left(\frac{1}{\lambda} - \lambda\right) + \lambda u, v_3\right) \in \mathbb{C} \times I\!R$, we obtain $\psi_\lambda \circ S = T_v^\lambda \circ \psi_\lambda$. This proves *ii)*.

Finally, if $\psi \circ S = S_\theta \circ \psi$ for $\theta \neq 0$, then (1) gives us

$$\overline{F} \circ S - G \circ S = e^{i\theta}\overline{F} - e^{i\theta} G, \qquad x_3 \circ S = x_3 + v_3,$$

where v_3 is the vertical component of the translation associated to S_θ. Grouping as above, we constrain two expressions to be constant and this constant can be chosen as zero, obtaining $\overline{F} \circ S - e^{i\theta}\overline{F} \equiv 0 \equiv G \circ S - e^{i\theta} G$. The assertion *iii)* is a direct consequence of this equation.

Next we will obtain two non-embeddedness properties of the deformation $\{\psi_\lambda\}_{\lambda>0}$, which will be key tools in further sections. Observe that the set of points in Σ where the normal is vertical is independent of λ. Suppose that ψ is not a plane, and let $p \in \Sigma$ be a point where the normal is vertical, say $N(p) = (0,0,-1)$. Take a conformal coordinate $(D(\varepsilon), z)$ centered at p satisfying

$$g(z) = z^k, \qquad \omega = (a + z\,h(z))\;dz,$$

where a is a non zero complex number, h is a holomorphic function in the disc $D(\varepsilon) = \{z \in \mathbb{C} : |z| < \varepsilon\}$, and k is a positive integer. In order to study the immersion ψ_λ around p we consider the new conformal coordinate $\left(D(\lambda^{\frac{1}{k}}\varepsilon), \xi\right)$, $\xi = \lambda^{\frac{1}{k}} z$. So, ψ_λ is determined by

$$g_\lambda(\xi) = \xi^k, \qquad \omega_\lambda = \frac{1}{\lambda^{1+\frac{1}{k}}}\left(a + \frac{\xi}{\lambda^{\frac{1}{k}}}\, h\left(\frac{\xi}{\lambda^{\frac{1}{k}}}\right)\right)\, d\xi.$$

Now we dilate ψ_λ by a homothety of ratio $\lambda^{1+\frac{1}{k}}$, obtaining a minimal surface $\widetilde{\psi}_\lambda = \lambda^{1+\frac{1}{k}}\,\psi_\lambda$. When λ goes to infinity, $\widetilde{\psi}_\lambda$ converges uniformly on compact subsets of $\mathbb{C}$ to the minimal surface $\psi_\infty : \mathbb{C} \longrightarrow \mathbb{R}^3$ determined by the Weierstrass data

$$g_\infty(\xi) = \xi^k, \qquad \omega_\infty = a\, d\xi.$$

This limit surface is complete, has finite total curvature and its end is not embedded—in fact it has transversal self-intersections—. So, the same holds for ψ_λ with λ sufficiently large.

If $N(p) = (0,0,1)$, the same fact is proved in a dual way.

From the above arguments, we conclude the following:

Lemma 4 *If $\psi : \Sigma \longrightarrow \mathbb{R}^3$ is a non flat minimal immersion with vertical flux and $p \in \Sigma$ is a point where the Gauss map of ψ is vertical, then for any neighbourhood D of p there exists a real number $\lambda > 0$ such that $\psi_\lambda\,|_D$ is not an embedding.*

Now take a properly embedded minimal end with finite total curvature ψ in $\mathbb{R}^3$, which is not a piece of a plane and such that the limiting normal vector is $(0, 0, -1)$. It is known [7] that we can parametrize this end in the punctured disc $D^*(\varepsilon) = D(\varepsilon) - \{0\}$, and the Weierstrass data of the end can be chosen as

$$g(z) = z^k, \qquad \omega = \left(\frac{a}{z^2} + h(z)\right) dz, \qquad z \in D^*(\varepsilon), \tag{3}$$

where $a \in \mathbb{C} - \{0\}$, h is holomorphic in $D(\varepsilon)$ and k is a positive integer. If $k = 1$, a must be real and the end is of *catenoid type*; in this case a is the logarithmic growth of the half-catenoid asymptotic to the end. If $k > 1$ the end is asymptotic to a horizontal plane, and is called a *planar type end*. The flux of ψ on this end vanishes for the planar case, and is vertical if it is catenoid type. Hence, ψ_λ is well-defined on $D^*(\varepsilon)$. Moreover, if ψ is a planar—resp. catenoid—type end, then ψ_λ is also a planar—resp. catenoid—type end. In the first case the height of ψ_λ is independent of λ and in the second one the same holds with the logarithmic growth of ψ_λ.

We obtain directly from (3) that

$$\psi_\lambda(z) = \phi_\lambda(z) + H(z, \lambda), \tag{4}$$

where $\phi_\lambda : D^*(\varepsilon) \longrightarrow \mathbb{R}^3$ denotes either a parametrization of the end of the (x_1, x_2)-plane or a parametrization of an end of the vertical catenoid symmetric respect to the origin with logarithmic growth a, and $H(z, \lambda)$ is a finite-valued smooth function on $D(\varepsilon)\times]0, \infty[$.

We consider the conformal coordinate $\left(D^*(\lambda^{\frac{1}{k}}\varepsilon), \xi\right)$, $\xi = \lambda^{\frac{1}{k}} z$. So the meromorphic data of ψ_λ are given by

$$g_\lambda(\xi) = \xi^k, \qquad \omega_\lambda = \frac{1}{\lambda^{1-\frac{1}{k}}} \left(\frac{a}{\xi^2} + \frac{1}{\lambda^{\frac{2}{k}}} h\left(\frac{\xi}{\lambda^{\frac{1}{k}}}\right)\right) d\xi.$$

After a homothety of ratio $\lambda^{1-\frac{1}{k}}$, we obtain a new minimal immersion $\widetilde{\psi}_\lambda(\xi) = \lambda^{1-\frac{1}{k}} \psi_\lambda(\xi)$. When λ goes to infinity, $\widetilde{\psi}_\lambda(\xi)$ converges uniformly on compact subsets of $\mathbb{C} - \{0\}$ to the minimal surface $\psi_\infty : \mathbb{C} - \{0\} \longrightarrow \mathbb{R}^3$ given by

$$g_\infty(\xi) = \xi^k, \qquad \omega_\infty = \frac{a}{\xi^2} d\xi, \qquad \xi \in \mathbb{C} - \{0\}.$$

This surface is a vertical catenoid if $k = 1$. If $k \geq 2$, it has an embedded end at the origin and a non embedded one at infinity. So, for λ large enough, ψ_λ is not embedded. The case $g(0) = \infty$ can be solved in a similar way. This completes the proof of the following assertion:

Lemma 5 *If $\psi : D^*(\varepsilon) \longrightarrow \mathbb{R}^3$ is a planar type end with vertical limiting normal and which is not a piece of a plane, then ψ_λ is not an embedding for some $\lambda > 0$.*

3 Minimal surfaces with finite total curvature in $\mathbb{R}^3$.

In this section and in the next one we will use the deformation studied above to prove some global theorems for properly embedded minimal surfaces. Another key ingredient in our method is the maximum principle at infinity of Meeks and Rosenberg [14] —see also Langevin and Rosenberg, [9]—, which states that the distance between two properly embedded disjoint minimal ends in a complete flat three manifold is non zero.

First we will give a new proof of the uniqueness theorem of López and Ros [10].

Theorem 1 *The only embedded complete minimal surfaces of finite total curvature and genus zero in $\mathbb{R}^3$ are the plane and the catenoid.*

Proof: Take a non flat complete minimal embedding $\psi : \Sigma \longrightarrow \mathbb{R}^3$ with genus zero and finite total curvature. Then Σ is conformally equivalent to a sphere with a finite number of punctures corresponding to the ends of the immersion, $\Sigma = \overline{\mathbb{C}} - \{p_1, \dots p_r\}$. These ends are planar or catenoid type ends, and we can assume that the normal vectors at the ends are vertical. As the flux along the curves around the ends is zero or a vertical vector, it follows that ψ has vertical flux and, so, we have a one-parameter family of complete minimal immersions with finite total curvature $\psi_\lambda : \Sigma \longrightarrow I\!R^3$, $\lambda > 0$. The following assertion and lemmae 4 and 5 prove that ψ has neither planar type ends nor points with vertical normal vector. As consequence, the third coordinate function of ψ is proper and has no critical points. So Σ is an annulus. It is a simple and well-known fact that the only embedded complete minimal annulus in $I\!R^3$ with finite total curvature is the catenoid [7], and the theorem is proved.

Assertion 1 *In the above conditions, all the ψ_λ's are embedded.*

Proof of the assertion 1: Denote by $B = \{\lambda > 0 : \psi_\lambda \,\text{is injective}\,\}$. If $\lambda_0 \in B$, then it follows from the maximum principle at infinity [14] that the distance between two fixed ends of ψ_{λ_0} must be non zero. We can easily conclude from (4) that this distance is infinity for each $\lambda > 0$ or it is a finite continuous function of λ. So, for λ near λ_0, ψ_λ is embedded, and therefore B is open.

Now take a sequence $\{\lambda_n\}_{n \in I\!N} \subset B$ converging to $\lambda_0 > 0$. If ψ_{λ_0} is not injective, we have $\psi_{\lambda_0}(x) = \psi_{\lambda_0}(y)$ for two distinct points $x, y \in \Sigma$. The convergence of $\{\psi_{\lambda_n}\}_n$ to ψ_{λ_0} uniformly over compact subsets of Σ and the maximum principle insure that there are neighbourhoods of x and y with the same image under ψ_{λ_0}. So the image point set of ψ_{λ_0} is a minimal surface with finite total curvature embedded in $I\!R^3$ and we have a finite covering $\psi_{\lambda_0} : \Sigma \longrightarrow \psi_{\lambda_0}(\Sigma)$. The maximum principle at infinity allows us to take an embedded ε-tubular neighbourhood U of $\psi_{\lambda_0}(\Sigma)$. Denote by $\pi : U \longrightarrow \psi_{\lambda_0}(\Sigma)$, and $l : U \longrightarrow I\!R$ the orthogonal projection of U onto $\psi_{\lambda_0}(\Sigma)$ and the oriented distance to $\psi_{\lambda_0}(\Sigma)$, respectively. It follows from (4) that for each n large enough, $\psi_{\lambda_n}(\Sigma) \subset U$ and that $\pi \circ \psi_{\lambda_n} : \Sigma \longrightarrow \psi_{\lambda_0}(\Sigma)$ is a proper local diffeomorphism and, hence, a finitely sheeted covering map. The embeddedness of ψ_{λ_n} implies that the continuous function $l \circ \psi_{\lambda_n}$ separates the points in the fibers of the covering $\pi \circ \psi_{\lambda_n}$ and therefore this covering has only one sheet. As $\pi \circ \psi_{\lambda_n}$ converges uniformly on compact subsets of Σ to $\pi \circ \psi_{\lambda_0} = \psi_{\lambda_0}$ it follows that also $\psi_{\lambda_0} : \Sigma \longrightarrow \psi_{\lambda_0}(\Sigma)$ has one sheet. This contradiction proves that B is closed in $]0, +\infty[$. As $1 \in B$, it follows that ψ_λ is embedded for each $\lambda > 0$, and the assertion is proved.

Now we prove two results for properly embedded minimal surfaces with smooth compact boundary. Denote by $H^+ = \{(x_1, x_2, x_3) \in I\!R^3 : x_3 \geq 0\}$ the upper halfspace in $I\!R^3$.

Theorem 2 *Let $\psi : \Sigma \longrightarrow H^+$ be a complete non flat minimal embedding with finite total curvature and boundary given by a convex Jordan curve in the (x_1, x_2)-plane. If Σ has genus zero, then it is an annulus.*

Proof: From the hypothesis, Σ is a finitely punctured closed disc, and the normal vector at each end must be vertical. As the flux of ψ is generated by the flux of its ends, that flux must be vertical. So the deformation $\{\psi_\lambda\}_{\lambda>0}$ is well defined on Σ. Now we claim that the assertion 1 holds in this situation. Calling B as above, the same arguments state the openness of B. If we take a sequence $\{\lambda_n\}_{n\in I\!N} \subset B$ converging to $\lambda_0 > 0$ and such that ψ_{λ_0} is not one-to-one, it follows as in assertion 1 that $\psi_{\lambda_0} : \Sigma \longrightarrow \psi_{\lambda_0}(\Sigma)$ is a finite covering. Note that from the maximum principle, the surface ψ_λ meets the (x_1, x_2)-plane only at its boundary. As, from lemma 2, ψ_{λ_0} is one-to-one at $\partial\Sigma$, the covering has only one sheet. This contradiction proves that B is closed in $]0, +\infty[$ and so, ψ_λ is an embedding for each $\lambda > 0$. Applying lemmae 4 and 5, ψ has neither planar type ends nor points with vertical normal vector. Consequently, the third coordinate function of ψ, which vanishes at the boundary of Σ, is proper and has no critical points. In particular, ψ has exactly one catenoid type end or, in other words, Σ is an annulus. This completes the proof of the theorem.

Denote by Π_1 and Π_2 two distinct horizontal planes in $I\!R^3$ and by E the slab bounded by these planes.

Theorem 3 *Let $\psi : \Sigma \longrightarrow E$ a complete minimal embedding with finite total curvature and boundary given by a pair of convex Jordan curves lying one in Π_1 and the other in Π_2. If ψ has vertical flux, then Σ is an annulus.*

Proof: From the flux hypothesis, the deformation $\{\psi_\lambda\}_{\lambda>0}$ is well-defined on Σ. Also from the finite total curvature assumption ψ—possibly—has a finite number of embedded planar type ends lying between Π_1 and Π_2. As in the proof of theorem 2 we can conclude that ψ_λ is embedded for each $\lambda > 0$, and so, from lemmae 4 and 5 we obtain that ψ has neither ends nor points where the normal is vertical. It follows directly that Σ is an annulus, and the theorem is proved.

Remark 1 *The hypothesis about the genus of* Σ *can be removed in theorems 1 and 2, putting the weaker one "*ψ *has vertical flux".*

The hypothesis "ψ *has vertical flux" in theorem 3 holds in some geometric situations, for example, when our surface has genus zero and either it is invariant by a non trivial rotation around the* x_3*-axis, or it is the intersection of a embedded complete minimal surface of finite total curvature in* $I\!R^3$ *with the slab.*

4 Singly-periodic minimal surfaces.

Firstly we recall some basic results about properly embedded singly-periodic minimal surfaces. Let S_θ, $0 \leq \theta < 2\pi$, be the screw motion of $I\!R^3$ obtained as the composition of the rotation around the x_3-axis by angle θ with a non trivial vertical translation. If $\theta = 0$ we will denote by T the translation S_0. Let $\widetilde{\psi} : \widetilde{\Sigma} \longrightarrow I\!R^3$ be a properly embedded minimal surface invariant by S_θ and denote by S the conformal transformation of Σ defined by $S_\theta \circ \widetilde{\psi} = \widetilde{\psi} \circ S$. Then the induced immersion $\psi : \Sigma \longrightarrow I\!R^3/S_\theta$, where $\Sigma = \widetilde{\Sigma}/S$, is a proper embedding. Reciprocally, any proper non flat minimal embedding $\psi : \Sigma \longrightarrow I\!R^3/S_\theta$ determines a connected singly-periodic minimal surface $\widetilde{\psi} : \widetilde{\Sigma} \longrightarrow I\!R^3$ related with ψ in the above way, see [6],[12]—Note that in the case $\theta = 0$, if the quotient surface Σ is orientable, the Weierstrass representation of $\widetilde{\psi}$ can be induced on Σ—. Suppose that Σ has finite topology. Then it follows from the results of Meeks and Rosenberg [13] that the proper embedding $\psi : \Sigma \longrightarrow I\!R^3/S_\theta$ has finite total curvature. Moreover it is proved in [13] that the behaviour of ψ at infinity is one of the followings:

1. All the ends of ψ are asymptotic to non vertical parallel planes, as in the Riemann example. These ends lift to planar type ends in $I\!R^3$. If $\theta \neq 0$ the ends are necessarily horizontal.

2. All the ends of ψ are asymptotic to flat vertical annuli, like the Scherk's second surface. This case occurs only if θ is a rational number.

3. All the ends of ψ are asymptotic to ends of helicoids. These helicoids have the same slope up to sign, the same winding number and their limit tangent planes at the ends are horizontal. In this case we will say that the ends are of *helicoidal type*.

In [8], Karcher gave examples of embedded minimal surfaces in $\mathbb{R}^3/T$ of genus zero and $2k$ ends asymptotic to flat vertical annuli, for any integer k greater than one. He also showed that these surfaces can be deformed into embedded minimal surfaces with helicoidal type ends in $\mathbb{R}^3/S_\theta$, $\theta \neq 0$.

An elementary lifting argument shows that the only properly embedded minimal surface of genus zero and a finite number of planar type ends in $\mathbb{R}^3/S_\theta$ is the plane. So the simplest non trivial classification problem for this type of ends appears in the case $genus(\Sigma) = 1$. The reasonable conjecture here is that the only surface of this kind is the Riemann example. Now we give a strong partial result in this direction:

Theorem 4 *There are no properly embedded minimal tori with a finite number of planar type ends in $\mathbb{R}^3/S_\theta$, $0 < \theta < 2\pi$.*

Proof: Consider a properly embedded minimal surface $\psi : \Sigma \longrightarrow \mathbb{R}^3/S_\theta$ with planar type ends, and assume that Σ is a finitely punctured torus. Using the notation above, let $\widetilde{\psi} : \widetilde{\Sigma} \longrightarrow \mathbb{R}^3$ be the corresponding singly-periodic minimal surface. Embeddedness implies that the ends are horizontal. The infinite cyclic covering map $\widetilde{\Sigma} \longrightarrow \Sigma$ extends in an unbranched way through the ends. So $\widetilde{\Sigma}$ is conformally the cylinder $\mathbb{C} - \{0\}$ with a infinite number of punctures corresponding to the planar type ends of $\widetilde{\psi}$. On the other hand, the holomorphic differential ϕ_3, which is globally well-defined on Σ [13], extends in a holomorphic way to the torus. So, up to scaling, the third coordinate function of $\widetilde{\psi}$ is given by $x_3 = \log |z|$. As the flux vanishes at the planar type ends, in order to insure that $\widetilde{\psi}$ has vertical flux we only need to consider the flux along the homology generator of the cylinder $\mathbb{C} - \{0\}$, which can be represented by any horizontal section of $\widetilde{\psi}$, $\Gamma = \{x_3 \circ \widetilde{\psi} = \text{constant}\} \subset \Sigma$, with height that does not coincide with any planar type end. As $S(\Gamma)$ is another planar section, we have

$$Flux(S(\Gamma)) = Flux(\Gamma).$$

As the conormal vector fields of the curves Γ and $S(\Gamma)$ differ by the rotation around the x_3-axis of angle θ, Rot_θ, using the definition of flux we obtain

$$Flux(S(\Gamma)) = Rot_\theta\left(Flux(\Gamma)\right).$$

The above two equalities and our assumption about θ insure that the flux along Γ is vertical, so the deformation $\{\widetilde{\psi}_\lambda\}_{\lambda>0}$ is well-defined on $\widetilde{\Sigma}$. From lemma 3 we have that all the $\widetilde{\psi}_\lambda$'s are invariant by the same screw motion. If we are able to prove that all the $\widetilde{\psi}_\lambda$'s or equivalently the ψ_λ's are embeddings, we will have a contradiction with lemma 5. But this last fact can be proved in the same way as in assertion 1. This completes the proof of the theorem 4.

Remark 2 *The Riemann's surface [1] demonstrates that the theorem does not hold for* $\theta = 0$.

Remark 3 *Let E be the slab determined by a pair of parallel planes Π_1 and Π_2. Consider a minimal annulus properly embedded in E whose boundary consists of a pair of straight lines, one lying on Π_1 and the other on Π_2. Then it is known that the annulus must be a piece of the Riemann example: this result was proved by Hoffman, Karcher and Rosenberg [4] if the boundary lines are parallel. Later Toubiana [19] proved that if they are not parallel the above minimal annulus can not exist. This last fact follows directly from theorem 4: If the annulus exists we can construct using the Schwarz reflection principle an embedded torus with two planar ends in $I\!R^3/S_\theta$, where $0 < \theta < 2\pi$, and this is impossible by our theorem.*

Now we consider a properly embedded minimal end of helicoidal type $\psi : A \longrightarrow I\!R^3/T$, T being a non trivial vertical translation, whose limiting normal vector is vertical, say $(0,0,-1)$. Meeks and Rosenberg [13] showed that this end type can be characterized in terms of its Weierstrass data as follows:

A is conformally a punctured disk. If g has a zero of order k at the origin, then ω has a pole of order $k+1$ at this point. Moreover ω has no residue at the origin and $g\omega = \left(\frac{-i\beta}{z} + f(z)\right) dz$, where β is a non zero real number and f is a bounded holomorphic function. The integer k coincides with the winding number of the end, and the parameter β gives us the slope of the helicoid asymptotic to the end A, and the period vector is given by $T = 2\pi\beta(0,0,1)$—for later use, we remark that if g has a pole at the origin, we have a symmetric situation. In particular, ω is holomorphic at the end and $g^2\omega$ has no residue at the origin—.

As a consequence of the above description, in a suitable conformal coordinate $(D^*(\varepsilon), z)$, the helicoidal type end can be represented by

$$g(z) = z^k\,(1 + z\,h(z))\,, \qquad \omega = \frac{-i\beta}{z^{k+1}}\,dz, \qquad z \in D^*(\varepsilon),$$

where k is a positive integer and h is holomorphic in $D(\varepsilon)$. This end can be lifted to a simply-connected minimal surface $\widetilde{\psi}$ in $I\!R^3$, and so, the deformation $\{\widetilde{\psi}_\lambda\}_{\lambda>0}$ is well-defined. This deformation can be determined by the multi-valued minimal surface defined on $D^*(\varepsilon)$ via the Weierstrass representation data

$$g_\lambda(z) = \lambda g(z) = \lambda z^k\,(1 + zh(z))\,, \qquad \omega_\lambda = \frac{1}{\lambda}\omega = -\frac{i\beta}{\lambda}\frac{dz}{z^{k+1}}. \tag{5}$$

Hence g_λ and ω_λ represent a helicoidal type end in $I\!R^3/T$ with horizontal limit tangent plane and slope β, $\psi_\lambda : D^*(\varepsilon) \longrightarrow I\!R^3/T$. In particular, all the $\widetilde{\psi}_\lambda$'s are T-invariant—note that in general this is not a consequence of the T-invariance of $\widetilde{\psi}$, see lemma 3—.

We conclude easily from (5) that

$$\psi_\lambda(z) = \phi_\lambda(z) + H(z,\lambda), \tag{6}$$

where $\phi_\lambda : D^*(\varepsilon) \longrightarrow I\!R^3/T$ is a parametrization of an end of a fixed helicoid with slope β and vertical limiting normal vector, and $H(z,\lambda)$ is a finite-valued smooth function on $D(\varepsilon)\times]0,\infty[$.

Now we will prove the following uniqueness theorem for the helicoid:

Theorem 5 *The only properly embedded minimal surface in $I\!R^3/T$ with genus zero and a finite number of helicoidal ends is the helicoid.*

Proof: Take a properly embedded minimal surface $\psi : \Sigma \longrightarrow I\!R^3/T$ and let $\widetilde{\psi} : \widetilde{\Sigma} \longrightarrow I\!R^3$ be the associated singly-periodic minimal surface in $I\!R^3$, where $\Sigma = \widetilde{\Sigma}/S$ and S is the holomorphic transformation on $\widetilde{\Sigma}$ induced by T. Assume that Σ is a finitely punctured sphere, $\Sigma = \overline{\mathbb{C}} - \{p_1, \ldots, p_r\}$ and that the ends of ψ are of helicoidal type. The embedding ψ is globally determined by its Weierstrass data (g, ω), which also determine the embedding $\widetilde{\psi}$. The above description of helicoidal ends in terms of the Weierstrass representation shows that ω is holomorphic on $\overline{\mathbb{C}}$ except at the ends p_i where $g(p_i) = 0$, where it has a pole without residue. Symmetrically, $g^2\omega$ has no residue at the ends p_i with $g(p_i) = \infty$ and is holomorphic otherwise. The genus zero hypothesis implies that ω and $g^2\omega$ are exact one-forms on Σ, and so on $\widetilde{\Sigma}$. Hence the flux of $\widetilde{\psi}$ is vertical, and the deformation $\{\widetilde{\psi}_\lambda\}_{\lambda>0}$ is well-defined. Our above study of helicoidal type ends show that $\widetilde{\psi}_\lambda$ is T-invariant, or in other words, g_λ and ω_λ determine a minimal immersion $\psi_\lambda : \Sigma \longrightarrow I\!R^3/T$ with horizontal helicoidal type ends. The following step is proving that the embeddedness is an invariant property throughout the deformation:

Assertion 2 *For each $\lambda > 0$, ψ_λ is a proper embedding.*

Proof: Denote by $B = \{\lambda > 0 : \psi_\lambda \text{ is injective}\}$. First note that the end types and their slope do not depend on λ. So, if $\lambda_0 \in B$, we know that two distinct ends of ψ_{λ_0} have the same slopes up to sign, and from the maximum principle at infinity [14] they are separated one from the other by a positive vertical distance. This distance is given by a finite continuous function of λ, as we can deduce from (6). So, ψ_λ is embedded for λ near λ_0 and B is open. Similarly, following the proof of assertion 1 but reasoning in $I\!R^3/T$ instead of in $I\!R^3$, we obtain that B is closed, and the assertion is proved.

As consequence of the above assertion, we conclude applying lemma 4 to the embedding $\widetilde{\psi}$ that there are no points of Σ where the normal of ψ is a vertical vector. So, the total winding number of the immersion, $W(\Sigma)$, which is defined as the sum of the winding numbers of the ends, is equal to $2\deg(g)$—recall that at each end, the winding number coincides with the order of the zero or pole of the Gauss map g at this point—.

On the other hand, the total curvature of ψ can be computed in terms of the winding numbers of its annular ends: Theorem 4 in [13] let us write

$$-4\pi \deg(g) = 2\pi \left(\chi(\Sigma) - W(\Sigma)\right),$$

where $\chi(\Sigma)$ denote the Euler characteristic of Σ. So in our situation, we have $\chi(\Sigma) = 2 - r$, where r is the number of ends of ψ. Therefore, the above formula yields

$$-2\deg(g) = 2 - r - 2\deg(g).$$

Hence, $r = 2$ and Toubiana's theorem [18] insures that ψ is the helicoid.

Remark 4 *The hypothesis "$\theta = 0$" is essential; consider the one-parameter family of examples obtained by Karcher in [8].*

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