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# Some variants of Ćirić's multi-valued contraction principle

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*Dedicated to the memory of Professor Ştefan Măruşter*

**Abstract.** In this article, a study of the fixed point problem for Ćirić type multi-valued operators is presented. More precisely, some variants of Ćirić's contraction principle for multi-valued operators, as well as a strict fixed point principle for Ćirić type multi-valued will be given.

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## 1 Introduction

The aim of this paper is to present a study on Ćirić type multi-valued operators. Following the approach given in [16], where the author considered some variants of the multivalued contraction principle given by Nadler [14], respectively a so-called strict multi-valued contraction principle, we will consider here the case of Ćirić type multi-valued operators, see [3].

We also notice that in [24] Reich developed some fixed point theorems for multi-valued generalized contractions. A fully comprehensive study on Reich operators was made in [12] by T. Lazăr et al. Also, qualitative properties, namely data dependence, Ulam-Hyers stability and so on, were studied for the case of multi-valued $\varphi$-contractions by V.L. Lazăr in [13]. Moreover, C. Chifu and G. Petruşel in [5] studied qualitative properties concerning Hardy-Rogers multi-valued operators (see [7] for the single-valued case) in the framework of b-metric spaces, while T. Lazăr, D. O'Regan et al. [11] studied the case of multi-valued operators of Ćirić type defined on a set endowed with two metrics. Finally, we point out that in [2], M. Boriceanu studied existence and uniqueness of the fixed point and data dependence for multi-valued Ćirić type operators in the context of b-metric spaces. At the same time, Ćirić type multi-valued operators were are studied in [17] and [19].

Regarding terminology and basic concepts for fixed point problems related to multi-valued operators, we will follow the works [1],[9], [18] and [23]. Furthermore, for the approximation of strict fixed points (also called end-points) of multi-valued mappings, we refer to [6], [8] and [22]. Finally, regarding data dependence, multi-valued fractal operators, selections and qualitative properties for the fixed point inclusion and for multi-valued fractals, we will refer to [4], [10] and [20].

Led $(X, d)$ be a metric space. Denote by $P(X)$ the family of all nonempty subsets of $X$. Also, $P_b(X)$ stands for the family of nonempty, bounded subsets of $X$ and $P_{cl}(X)$ the family of nonempty, closed subsets of $X$. In a similar manner, by $P_{cp}(X)$ we refer to the family of nonempty, compact subsets of $X$. From now on, $\overline{B}(x_0, r)$ means the closure in $(X, d)$ of the ball $B(x_0, r)$, where $B(x_0, r) := \{x \in X \mid d(x_0, x) < r\}$ is the open ball with radius $r > 0$ and the center $x_0 \in X$. By $\widetilde{B}(x_0; r) := \{x \in X | d(x_0, x) \leq r\}$ we denote the closed ball centered in $x_0$ with radius $r$. We recall now some important functionals which will be used through the paper:

• the gap functional $D : P(X) \times P(X) \to \mathbb{R}_+$, $D(A, B) := \inf_{a \in A,\ b \in B} \{d(a, b)\}$.

• the generalized Pompeiu-Haussdorf functional $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, where $H(A, B) = \max\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\}$.

Furthermore, if $T : X \to P(X)$ is a multi-valued operator, then an element $x \in X$ is a fixed point for $T$ if and only if $x \in T(x)$. We denote by $F_T$ the set of all fixed points of the operator $T$ and by $(SF)_T$ the set of all strict fixed points of $T$, where $x \in X$ is a strict fixed point of $T$ (or an endpoint, or a stationary point) if and only if $\{x\} = Tx$.

For a multi-valued operator $T : X \to P(Y)$ we can also define the following useful notions. The graph of the operator $T$, defined by $Graph(T) := \{(x, y) \in X \times Y \mid y \in T(x)\}$, and the image of the set $Y \in P(X)$ will be denoted by $T(Y) := \bigcup_{x \in Y} T(x)$. A single-valued mapping $t : X \to Y$ is called a selection of $T$ if for each $x \in X$, we have that $t(x) \in T(x)$.

We present now an important concept, which appears naturally by Nadler's contraction principle. By [21], we recall here the notion of multi-valued weakly Picard operator.

**Definition 1.1.** *Let $(X, d)$ be a metric space.*
*Consider $T : X \to P(X)$ be a multi-valued operator. By definition, $T$ is a multi-valued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and for each $y \in T(x)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$, satisfying the following*
*(i) $x_0 = x$ and $x_1 = y$,*
*(ii) $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$,*
*(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to a fixed point of $T$.*

**Remark 1.1.** A sequence $(x_n)_{n\in\mathbb{N}}$ satisfying conditions $(i)$ and $(ii)$ is called a sequence of successive approximations of $T$ starting from $(x, y) \in Graph(T)$. If $T : X \to P(X)$ is a MWP operator, then we define the operator $T^\infty : Graph(T) \to P(F_T)$, by $T^\infty(x, y) := \{z \in F_T \mid$ there exists a sequence of successive approximations of T starting from (x,y) that converges to z$\}$.

Furthermore, if $(X, d)$ is a metric space and $T : X \to P(X)$ a multi-valued operator, then $T$ is said to be closed if $Graph(T)$ is a closed set in $X \times X$. By $T^1(x) := T(x), ..., T^n(x) := T(T^{n-1}(x))$ we denote the iterates of the multi-valued mapping $T$, while the set $V^0(Y; \varepsilon) := \{x \in X \mid D(x, Y) < \varepsilon\}$ is called the (open) $\varepsilon$-neighborhood of $Y \in P(X)$.

From [14], we shall recall some important lemmas that are used throughout the article.

**Lemma 1.1.** *Let $A$ and $B$ from $P(X)$ and $q > 1$. Then, for each $a \in A$, there exists $b \in B$, such that $d(a, b) \leq qH(A, B)$.*

**Lemma 1.2.** *Let $A$ and $B$ from $P(X)$. Also, consider $\eta > 0$, such that*
*(i) for each $a \in A$, there exists $b \in B$, with $d(a, b) \leq \eta$,*
*(ii) for each $b \in B$, there exists $a \in A$, with $d(a, b) \leq \eta$.*
*Then $H(A, B) \leq \eta$.*

Now, we recall the basic concepts for the qualitative properties of the fixed point inclusion and of the fixed point iteration. The first two definitions are related to well-posedness of the fixed point problem. For the concept of well-posedness, we let the reader follow [12] and [19].

**Definition 1.2.** *Let $(X, d)$ be a metric space and $T : Y \to P_{cl}(X)$ be a multi-valued operator. Then the fixed point problem is well-posed for $T$ with*

*respect to the gap functional $D$ if and only if:*
*(i)* $F_T = \{x^*\}$*;*
*(ii) if* $(x_n) \in X$ *has the property that* $D(x_n, T(x_n)) \to 0$*, then* $x_n \to x^*$*.*

**Definition 1.3.** *Let* $(X, d)$ *be a metric space,* $Y \in P(X)$ *and* $T : Y \to P_{cl}(X)$ *be a multi-valued operator. Then the fixed point problem is well-posed for $T$ with respect to the Pompeiu-Haussdorf functional $H$ if and only if:*
*(i)* $(SF)_T = \{x^*\}$*;*
*(ii) if* $(x_n) \in X$ *is a sequence such that* $H(x_n, Tx_n) \to 0$*, then* $x_n \to x^*$*.*

Now, the second important concept related to the fixed point problem is limit shadowing or Ostrowski property, which can be found in [12] and [13].

**Definition 1.4.** *Let* $(X, d)$ *be a metric space and* $T : X \to P(X)$ *be a multi-valued operator. By definition, the multi-valued operator $T$ has the Ostrowski property, if* $F_T = \{x^*\}$ *and for any sequence* $(y_n)_{n\in\mathbb{N}} \subset X$*, such that* $D(y_{n+1}, Ty_n) \to 0$*, we have* $(y_n)_{n\in\mathbb{N}} \to x^*$*, as* $n \to \infty$*.*

We introduce now the notions of $\psi$-MWP operator and of generalized Ulam-Hyers stabilites. For the study of generalized Ulam-Hyers stability we refer to [15].

**Definition 1.5.** *Let* $(X, d)$ *be a metric space and* $T : X \to P(X)$ *be a MWP operator. Let* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *be continuous in 0, increasing, such that* $\psi(0) = 0$*. By definition, $T$ is $\psi$-MWP operator, if there exists a selection* $t^\infty : Graph(T) \to F_T$ *of* $T^\infty$*, such that* $d(x, t^\infty(x, y)) \le \psi(d(x, y))$*, for each* $(x, y) \in Graph(T)$*.*

**Definition 1.6.** *Let* $(X, d)$ *be a metric space and* $T : X \to P(X)$*. By definition, the fixed point inclusion*

$$x \in T(x) \tag{1.1}$$

*is called generalized Ulam-Hyers stable if and only if there exists an increasing, continuous in 0 function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$*, with* $\psi(0) = 0$*, such that for every* $\varepsilon > 0$ *and for each* $y^* \in X$ *for which* $D(y, T(y)) \le \varepsilon$*, there exists a solution* $x^*$ *a solution of the fixed point inclusion (1.1), such that* $d(y^*, x^*) \le \psi(\varepsilon)$*.*

**Definition 1.7.** *Let* $(X, d)$ *be a metric space and* $T : Y \to P(X)$*. By definition, the strict fixed point inclusion*

$$\{x\} = T(x) \tag{1.2}$$

*is called generalized Ulam-Hyers stable if and only if there exists an increasing, continuous in 0 function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$*, with* $\psi(0) = 0$*, such that for* *every $\varepsilon > 0$ and for each $y^* \in X$ for which $H(y, T(y)) \leq \varepsilon$, there exists a solution $x^*$ a solution of the strict fixed point inclusion (1.2), such that $d(y^*, x^*) \leq \psi(\varepsilon)$.*

Finally, following [6], [8] and [22], we recall the last important concepts.

**Definition 1.8.** *Let $X \neq \emptyset$ and $T : X \to P(X)$ be a multi-valued operator. Then, $T$ has the approximate endpoint property if $\inf\limits_{x \in X} \sup\limits_{y \in Tx} d(x, y) = 0$.*

## 2 Main results

The aim of this paper is to extend to the case of Ćirić type multi-valued generalized contractions, the results given in [16], where the author studied extended properties for the fixed point problem related to Nadler's multi-valued contractions through relevant metrical and topological properties.
In the present section some variants of the multi-valued Ćirić principle are given. We shall enhance the classical result of Ćirić [3] with additional metrical and topological conclusions with respect to the fixed point problem.

**Theorem 2.1** (An extended version of the Ćirić's multi-valued contraction principle)**.** *Let $(X, d)$ be a complete metric space and $T : X \to P_{cl}(X)$ be a multi-valued $\alpha$-Ćirić type operator, i.e., there exists $\alpha \in (0, 1)$, such that*

$$H(T(x), T(y)) \leq \alpha \cdot M(x, y), \text{ for each } x, y \in X,$$

*where*

$$M(x, y) := \{d(x, y), D(x, T(x)), D(y, T(y)), \frac{1}{2}\left[D(x, T(y)) + D(y, T(x))\right]\}.$$

*Then, the following conclusions hold:*
*(a) there exists $x^* \in F_T$;*
*(b) for each $(x, y) \in Graph(T)$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of successive approximations for $T$ starting from $(x, y)$, convergent to a fixed point of $T$;*
*(c) there exists a selection $t^\infty : Graph(T) \to F_T$ of $T^\infty$, such that*

$$d(x, t^\infty(x, y)) \leq \frac{1}{1-\alpha} d(x, y), \forall (x, y) \in Graph(T);$$

*(d) $F_T$ is closed in $(X, d)$;*
*(e) if $(x_n)_{n \in \mathbb{N}}$ is a sequence of successive approximations for $T$, starting from a pair $(x, y) \in Graph(T)$, which converges to a fixed point $x^*(x, y)$ of $T$, then*

$$d(x_n, x^*) \leq \frac{\alpha^n}{1-\alpha} d(x, y), \forall n \in \mathbb{N}^*;$$

*(f) if* $G : X \to P_{cl}(X)$ *is a Ćirić-type multi-valued operator with coefficient* $\beta$*, and there exists* $\eta > 0$*, such that* $H(T(x), G(x)) \leq \eta$*, for all* $x \in X$*, then* $H(F_T, F_G) \leq \eta \cdot \max\left\{\frac{1}{1-\alpha}, \frac{1}{1-\beta}\right\}$*;*
*(g) if* $T_n : X \to P_{cl}(X)$ *is a sequence of multi-valued* $\alpha$*-Ćirić-type operators, with* $T_n(x) \overset{H}{\to} T(x)$ *as* $n \to \infty$*, uniformly with respect to* $x \in X$*, then*

$$\lim_{n\to\infty} H(F_{T_n}, F_T) = 0;$$

*(h) if there exists* $x_0 \in X$ *and* $r > 0$*, such that* $D(x_0, T(x_0)) < (1-\alpha)r$*, then there exists* $x^* \in F_T \cap B(x_0, r)$*;*
*(i) if there exists* $x_0 \in X$ *and* $r > 0$ *such that* $\delta(x_0, T(x_0)) < (1-\alpha)r$*, then* $T : \tilde{B}(x_0, r) \to P\left(\tilde{B}\left(x_0, \frac{1}{1-\alpha}r\right)\right)$ *and there exists* $x^* \in F_T \cap B(x_0, r)$*;*
*(j) if* $X$ *is a Banach space,* $U$ *an open subset of* $X$ *and* $T : U \to P_{cl}(X)$ *is a Ćirić multi-valued operator, then the associated multi-valued field* $G : U \to P(X)$*,* $G(x) := x - T(x)$ *is open;*
*(k) there exists a Caristi selection of* $T$*;*
*(m) if, additionally,* $T : X \to P_{cp}(X)$*, then the fixed point inclusion* $x \in T(x)$ *is generalized Ulam-Hyers stable;*
*(n) the multi-valued operator* $T$ *has the approximate fixed point property;*
*(o) if the multi-valued operator* $T$ *is lower semicontinuous, then it has the approximate endpoint property if and only if it has a unique strict fixed point;*
*(p) if* $\alpha < \frac{1}{2}$*, then the fixed point set* $F_T$ *is compact.*
*(q) if* $T : X \to P_{b,cl}(X)$*, then for each* $p > 0$*, one has* $H(F_p^*, F_T) \leq \frac{p}{1-\alpha}$*, where* $F_p^* := \{x \in X \mid D(x, T(x)) < p\}$*.*

*Proof.* (a), (b), (c) and (e) (In fact (a) and (b) means that $T$ is a MWP operator, while (a), (b) and (c) can be concise represented by saying that $T$ is a $\psi$-MWP operator, with $\psi(t) = \frac{1}{1-\alpha}t$).
Let $x_0 \in X$ and $x_1 \in T(x_0)$ be arbitrary elements. Then $H(T(x_0), T(x_1)) \leq \alpha M(x_0, x_1)$. Furthermore, consider $q \in \left(1, \frac{1}{\alpha}\right)$.
Now, for $x_1$, there exists $x_2 \in T(x_1)$, such that $d(x_1, x_2) \leq qH(T(x_0), T(x_1))$, so $d(x_1, x_2) \leq q\alpha M(x_0, x_1)$. We consider the following cases :
If $M(x_0, x_1) = d(x_0, x_1)$, then $d(x_1, x_2) \leq (q\alpha)d(x_0, x_1)$.
If $M(x_0, x_1) = D(x_0, T(x_0)) \leq d(x_0), x_1$, then $d(x_1, x_2) \leq (q\alpha)d(x_0, x_1)$.
If $M(x_0, x_1) = D(x_1, T(x_1)) \leq d(x_1, x_2)$, then $d(x_1, x_2) \leq (q\alpha)d(x_1, x_2)$, which is a contradiction, So $M(x_0, x_1)$ can not be $d(x_1, x_2)$.

Finally, if $M(x_0,x_1) = \frac{1}{2}\left[D(x_1,T(x_0)) + D(x_0,T(x_1)),\right]$, then by using the fact that $D(x_1,T(x_0)) \le d(x_1,x_1) = 0$ and the fact that $D(x_0,T(x_1)) \le d(x_0,x_2) \le d(x_0,x_1) + d(x_1,x_2)$, it follows that $M(x_0,x_1) \le \frac{1}{2}d(x_0,x_1) + \frac{1}{2}d(x_1,x_2)$. So $d(x_1,x_2) \le \frac{q\alpha}{2}d(x_0,x_1) + \frac{q\alpha}{2}d(x_1,x_2)$. Then $d(x_1,x_2) \le \frac{q\alpha}{2-q\alpha}d(x_0,x_1)$.
Since $q \in \left(1, \frac{1}{\alpha}\right)$, we get that $d(x_1,x_2) \le (q\alpha)d(x_0,x_1)$.
Let us denote by $\lambda := q\alpha$. Then, by all the cases $d(x_1,x_2) \le \lambda d(x_0,x_1)$.
Also, denote by $\rho_n := d(x_n,x_{n+1})$, for each $n \in \mathbb{N}$. By induction, we can construct a sequence $(x_n)_{n\in\mathbb{N}}$, such that for $x_n \in T(x_{n-1})$, there exists $x_{n+1} \in T(x_n)$, for which $\rho_n \le \lambda\rho_{n-1}$, for each $n \in \mathbb{N}$. Then $\rho_n \le \lambda^n\rho_0$, so by triangle inequality $d(x_n,x_{n+p}) \le \lambda^n\frac{1-\lambda^p}{1-\lambda}\rho_0$. Taking $n \to \infty$, it follows up that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy, so there exists $x^* \in X$, such that $x_n \to x^*$. Furthermore, in the estimate $d(x_n,x_{n+p}) \le \lambda^n\frac{1-\lambda^p}{1-\lambda}\rho_0$. taking $p \to \infty$, it follows that $d(x_n,x^*) \le \frac{\lambda^n}{1-\lambda}d(x_0,x_1)$. Taking $n = 0$ and making $q \searrow 1$, it follows the estimate $d(x,x^*) = d(x,t^\infty(x,y)) \le \frac{1}{1-\alpha}d(x,y)$, with $y \in T(x)$. Here, we denoted by $x := x_0$ and $y := x_1 \in T(x_0)$.
The final step is to show that $x^* \in F_T$, i.e., to prove that $D(x^*,T(x^*)) = 0$. We have the following estimation:

$$D(x^*,T(x^*)) \le d(x^*,x_{n+1}) + H(T(x_n),T(x^*)) \le d(x^*,x_{n+1}) + \alpha M(x_n,x^*).$$

Moreover, since

$$M(x_n,x^*) \le \max\{d(x_n,x^*), d(x_n,x_{n+1}), D(x^*,T(x^*)), \frac{1}{2}\left[d(x^*,x_n) + d(x^*,x_{n+1}) + D(x^*,T(x^*))\right]\},$$

by letting $n \to \infty$, we obtain that $\lim_{n\to\infty} M(x_n,x^*) \le D(x^*,T(x^*))$.
This means that

$$D(x^*,T(x^*)) \le \alpha\max\{D(x^*,T(x^*)), \frac{1}{2}D(x^*,T(x^*))\} < D(x^*,T(x^*)),$$

and the conclusion follows.
(d) We know that $F_T \in P(X)$. We shall show that $F_T$ is closed in $(X,d)$. For this, let $x_n \in F_T$, such that $x_n \to x^*$. So, for each $n \in \mathbb{N}$, $D(x_n,T(x_n)) = 0$.

We shall show that $x^* \in F_T$, i.e. $x^* \in T(x^*)$. Also, since the operator $T$ has closed values, then it is enough to show that $D(x^*, T(x^*)) = 0$. We have the following inequalities :

$$D(x^*, T(x^*)) \leq d(x^*, x_n) + D(x_n, T(x^*)) \leq d(x^*, x_n) + H(T(x_n), T(x^*)) \\ \leq d(x^*, x_n) + \alpha M(x^*, x_n)$$

We have the following cases:
If $M(x^*, x_n) = d(x^*, x_n)$, then $D(x^*, T(x^*)) \leq (1 + \alpha)d(x_n, x^*) \to 0$.
Furthermore, if $M(x^*, x_n) = D(x_n, T(x_n)) = 0$, then $D(x^*, T(x^*)) \leq d(x^*, x_n) \to 0$. Moreover, if $M(x^*, x_n) = D(x^*, T(x^*))$, then we obtain that

$$D(x^*, T(x^*)) \leq \frac{1}{1-\alpha} d(x^*, x_n) \to 0.$$

Finally, if $M(x^*, x_n) = \frac{1}{2}[D(x_n, T(x^*)) + D(x^*, T(x_n))] \leq \frac{1}{2} D(x_n, T(x^*)) + \frac{1}{2} d(x^*, x_n)$.
Also, $D(x_n, T(x^*)) \leq H(T(x_n), T(x^*)) \leq \alpha M(x_n, x^*) \leq \frac{\alpha}{2} D(x_n, T(x^*)) + \frac{\alpha}{2} d(x^*, x_n)$, so $D(x_n, T(x^*)) \leq \frac{\alpha}{2-\alpha} d(x_n, x^*)$. This implies that

$$D(x^*, T(x^*)) \leq d(x^*, x_n) + \frac{\alpha}{2-\alpha} d(x_n, x^*) = \frac{2}{2-\alpha} d(x_n, x^*) \to 0, n \to \infty.$$

Thus, by all cases $x^* \in F_T$, so $F_T$ is closed.
(f) By (a),(b),(c) and (e), we have that $d(x, x^*) \leq \frac{1}{1-\alpha} d(x, y)$, where $x$ is an arbitrary element of $X$ and $y \in T(x)$, where $x^* \in F_T$. Taking $x = y^* \in F_G$, then we obtain that $d(x^*, y^*) \leq \frac{1}{1-\alpha} d(y, y^*)$, where $y \in T(y^*)$. Furthermore, since $y \in T(y^*)$ is arbitrary, we can make the following assertion: for $y^* \in F_G$, there exists $y \in T(y^*)$, such that $d(y, y^*) \leq H(G(y^*), T(y^*)) \leq \eta$, so $d(x^*, y^*) \leq \frac{\eta}{1-\alpha}$.
Now, also from the global principle of the existence of the fixed point of $G$, we get that $d(x, x^*) \leq \frac{1}{1-\beta} d(x, y)$, with $x^* \in F_G$, $x$ is an arbitrary element of $X$ and $y \in Gx$.
Taking $x = y^* \in F_T$, then we obtain that $d(x^*, y^*) \leq \frac{1}{1-\beta} d(y, y^*)$, where $y \in G(y^*)$.
As in the first case, since $y \in G(y^*)$ is arbitrary, then for $y^* \in F_T$, there exists

$y \in G(y^*)$, such that $d(y^*, y) \leq H(T(y^*), G(y^*)) \leq \eta$. So $d(x^*, y^*) \leq \frac{\eta}{1-\beta}$.
From the first case we get that for $y^* \in F_G$, there exists $x^* \in F_T$, such that

$$d(x^*, y^*) \leq \eta \cdot \max\left\{\frac{1}{1-\alpha}, \frac{1}{1-\beta}\right\},$$

while from the second case we infer that for $y^* \in F_T$, there exists $x^* \in F_G$, such that

$$d(x^*, y^*) \leq \eta \cdot \max\left\{\frac{1}{1-\alpha}, \frac{1}{1-\beta}\right\}.$$

By Lemma 1.2, we get the conclusion $H(F_T, F_G) \leq \eta \cdot \max\left\{\frac{1}{1-\alpha}, \frac{1}{1-\beta}\right\}$.

(g) Let $\varepsilon > 0$ be an arbitrary fixed element. Since $T_n(x) \overset{H}{\to} T(x)$ as $n \to \infty$, uniformly for each $x \in X$, then for all $x \in X$, we have that $\lim_{n\to\infty} H(T_n x, Tx) = 0$.
This means that for $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$, such that for each $n \geq N(\varepsilon)$, we have that $\sup_{x\in X} H(T_n(x), T(x)) < \varepsilon$. From the conclusion (f) of data dependence, we have that for $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$, such that for all $n \geq N(\varepsilon)$, one has $H(F_{T_n}, F_T) < \frac{1}{1-\alpha} \cdot \varepsilon$. So, the conclusion is valid.

(h) Let $s \in (0, r)$, such that $\tilde{B}(x_0, s) \subset B(x_0, r)$, with $D(x_0, T(x_0)) < (1-\alpha)s < (1-\alpha)r$. Since $D(x_0, T(x_0)) < (1-\alpha)s$, then there exists $x_1 \in T(x_0)$, such that $d(x_0, x_1) < (1-\alpha)s < s$, so $x_1 \in B(x_0, s) \subset \tilde{B}(x_0, s)$.
From the hypothesis, we have that $H(T(x_0), T(x_1)) \leq \alpha M(x_0, x_1)$, where:

$$\begin{aligned}
&M(x_0, x_1) = \\
&= \max\{d(x_0, x_1), D(x_0, T(x_0)), D(x_1, T(x_1)), \frac{1}{2}\left[D(x_0, T(x_1)) + D(x_1, T(x_0))\right]\} \\
&\leq \max\{d(x_0, x_1), D(x_0, T(x_0)), D(x_1, T(x_1)), \frac{1}{2}\left[d(x_0, x_1) + D(x_1, T(x_1))\right]\} \\
&= \max\{d(x_0, x_1), D(x_0, T(x_0)), D(x_1, T(x_1))\} \\
&\leq \max\{d(x_0, x_1), H(T(x_0), T(x_1))\}
\end{aligned}$$

We consider the following cases:
If the maximum is $d(x_0, x_1)$, then $H(T(x_0), T(x_1)) \leq \alpha d(x_0, x_1)$.
If the maximum is $H(T(x_0), T(x_1))$, then, since $\alpha < 1$, we obtain a contradiction.
From the above cases, it follows that $H(T(x_0), T(x_1)) \leq \alpha d(x_0, x_1)$. Since $D(x_1, T(x_1)) \leq H(T(x_0), T(x_1)) \leq \alpha d(x_0, x_1) < \alpha(1-\alpha)s$, then there exists $x_2 \in T(x_1)$ for which $d(x_1, x_2) < \alpha(1-\alpha)s$.

Furthermore, by triangle inequality one can obtain $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) < (1-\alpha)s + \alpha(1-\alpha)s = (1-\alpha^2)s < s$, so $x_2 \in T(x_1) \cap \tilde{B}(x_0, s)$. By induction, we can construct a sequence $(x_n)_{n\in\mathbb{N}}$, with $x_n$ from $T(x_{n-1}) \cap \tilde{B}(x_0, s)$, such that :

$$\begin{cases} x_{n+1} \in T(x_n), \text{ for each } n \in \mathbb{N}, \\ d(x_{n-1}, x_n) \leq \alpha^{n-1}(1-\alpha)s, \text{ for each } n \in \mathbb{N}^*, \\ d(x_0, x_n) \leq (1-\alpha^n)s, \text{ for all } n \in \mathbb{N}. \end{cases}$$

It follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy, so there exists $x^* \in \tilde{B}(x_0, s)$, such that $x_n \to x^*$.
As in the proof of (a),(b),(c) and (e), one can show that $x^* \in T(x^*)$. Moreover, since $x_n \in \tilde{B}(x_0, s)$ and $\tilde{B}(x_0, s)$ is closed in $X$, then $x^* \in \tilde{B}(x_0, s) \subset B(x_0, r)$.
(i) Let $u \in T(x_0)$. Then, by applying the triangle inequality, we get that $d(z, x_0) \leq d(z, u) + d(u, x_0)$, so $d(z, x_0) \leq d(z, u) + \delta(x_0, T(x_0))$. Now, taking $\inf\limits_{u\in T(x_0)}$, it follows that $d(z, x_0) \leq D(z, T(x_0)) + \delta(x_0, T(x_0))$.
We first show that $T(\tilde{B}(x_0, r)) \subset \tilde{B}\left(x_0, \frac{1}{1-\alpha}r\right)$.
Let $y \in \tilde{B}(x_0, r)$. We will show that $T(y) \subset \tilde{B}\left(x_0, \frac{1}{1-\alpha}r\right)$.
So, take $z \in T(y)$. The aim is to show that $z \in \tilde{B}\left(x_0, \frac{1}{1-\alpha}r\right)$, i.e., $d(z, x_0) \leq \frac{1}{1-\alpha}r$.
Then $d(z, x_0) \leq D(z, T(x_0)) + \delta(x_0, T(x_0)) < H(T(y), T(x_0)) + (1-\alpha)r$. So $d(z, x_0) < \alpha M(y, x_0) + (1-\alpha)r$.
We know that:

$$M(y, x_0) = \max\{d(y, x_0), D(x_0, T(x_0)), D(y, T(y)), \frac{1}{2}\left[D(y, T(x_0)) + D(x_0, T(y))\right]\}.$$

We also have $d(y, x_0) \leq r$ and $D(x_0, T(x_0)) \leq \delta(x_0, T(x_0)) < (1-\alpha)r \leq r$. So, we obtain:

$$M(y, x_0) \leq \max\{r, D(y, T(y)), \frac{1}{2}\left[D(y, T(x_0)) + D(x_0, T(y))\right]\}.$$

We employ an analysis on the following cases :
If the maximum from the right hand side is $r$, then $d(z, x_0) < \alpha r + (1-\alpha)r = r < \frac{1}{1-\alpha}r$.
If the maximum is $D(y, T(y))$, then $d(z, x_0) < \alpha D(y, T(y)) + (1-\alpha)r$. So

$d(z, x_0) < (1-\alpha)r + \alpha d(y, z) \leq (1-\alpha)r + \alpha d(y, x_0) + \alpha d(z, x_0)$. This means that $d(z, x_0) < \frac{1}{1-\alpha} r$.

Finally, if the maximum is $\frac{1}{2}[D(y, T(x_0)) + D(x_0, T(y))]$, then $d(z, x_0) < \frac{\alpha}{2} D(y, T(x_0)) + \frac{\alpha}{2} d(x_0, z) + (1-\alpha)r$. This implies that $(2-\alpha)d(z, x_0) < \alpha d(y, x_0) + \alpha\delta(x_0, Tx_0) + 2(1-\alpha)r$ and thus $d(z, x_0) \leq \frac{2-\alpha^2}{2-\alpha} r$.

From all the cases, it follows that $d(z, x_0) \leq \max\{\frac{1}{1-\alpha} r, \frac{2-\alpha^2}{2-\alpha} r\} = \frac{1}{1-\alpha} r$.

This means that $T\left(\tilde{B}(x_0, r)\right) \subset \tilde{B}\left(x_0, \frac{1}{1-\alpha} r\right)$. We have that

$$D(x_0, T(x_0)) \leq \delta(x_0, T(x_0)) < (1-\alpha)r.$$

Taking $X := \tilde{B}\left(x_0, \frac{1}{1-\alpha} r\right)$ and $T : \tilde{B}(x_0, r) \to P_{cl}(X)$, we apply the conclusion (i) for local version of the fixed point problem for the Ćirić operator on the closed ball. We mention that we have used the fact that $\tilde{B}\left(x_0, \frac{1}{1-\alpha} r\right)$ is closed in the complete metric space $(X, d)$. Then, there exists $x^* \in F_T \cap \tilde{B}(x_0, r)$. Using the fact that $d(x_0, x^*) \leq r$, we can show that $d(x_0, x^*) < r$.

Suppose to the contrary that $r = d(x_0, x^*)$. Then, we have the following inequalities: $r = d(x^*, x_0) \leq H(T(x^*), T(x_0)) + \delta(x_0, T(x_0)) < \alpha M(x^*, x_0) + (1-\alpha)r$, where

$$M(x^*, x_0) = \max\{d(x^*, x_0), D(x^*, T(x^*)), D(x_0, T(x_0)), \frac{1}{2}[D(x^*, T(x_0)) + D(x_0, T(x^*))]\}.$$

Notice that $D(x^*, T(x^*)) = 0$, $D(x_0, T(x_0)) \leq \delta(x_0, T(x_0)) \leq (1-\alpha)r < r$, $D(x_0, T(x^*)) \leq d(x_0, x^*)$ and $D(x^*, T(x_0)) \leq d(x^*, x_0) + D(x_0, T(x_0)) < d(x^*, x_0) + (1-\alpha)r$.

Then, we get the following cases :

If the maximum from the right hand side is $d(x^*, x_0)$, then $r < \alpha d(x^*, x_0) + (1-\alpha)r = \alpha r + (1-\alpha)r = r$, which is false.

If the maximum is $D(x^*, T(x^*))$, then $r < (1-\alpha)r < r$, which is also false.

If the maximum is $D(x_0, T(x_0)) < (1-\alpha)r$, then we get $r < (1-\alpha)r + \alpha(1-\alpha)r = (1-\alpha^2)r < r$, also false.

For the last case, if the maximum from the right hand side is $\frac{1}{2}[D(x^*, T(x_0))$

$+D(x_0,T(x^*))]$, then $M(x^*,x_0) \leq \frac{1}{2}H(T(x^*),T(x_0)) + (1-\alpha)r$. Furthermore, by the condition that the operator is of Ćirić-type, we have that $H(T(x^*),T(x_0)) \leq \alpha M(x^*,x_0)$. So $H(T(x^*),T(x_0)) \leq \frac{\alpha(1-\alpha)r}{2-\alpha}$.
It follows that $r = d(x^*,x_0) < H(T(x^*),T(x_0)) + (1-\alpha)r$, so $r < (1-\alpha^2)r < r$, which is false.
From all the cases from above, it follows that $d(x^*,x_0) < r$.
(j) We prove that, if $V$ is an open subset of $U$, then $G(V)$ is open in $X$. This means that for $x_0 \in U$ and $r_0 > r > 0$, with $B(x_0,r) \subset U$, then $V^0(G(x_0),(1-\alpha)r) \subset G(B(x_0,r))$.
So, let $y \in V^0(G(x_0),(1-\alpha)r)$, i.e. $D(y,G(x_0)) < (1-\alpha)r$. We shall show that $y \in G(B(x_0,r))$. In other words, we shall show that there exists $x^* \in B(x_0,r)$, such that $y \in G(x^*)$, i.e., $y \in x^* - T(x^*)$.
Let us consider the multi-valued operator $F : B(x_0,r) \to P_{cl}(X)$, defined by $F(x) := y + T(x)$.
If $F$ has a fixed point $x^*$, then $x^* \in y + T(x^*)$ or $y \in x^* - T(x^*)$. Now, for each $x,z \in B(x_0,r)$, we have that :
$H(F(x),F(z)) = H(y+T(x),y+T(z)) \leq H(T(x),T(z)) \leq \alpha M(x,z)$. Moreover, $D(x_0,F(x_0)) = D(x_0,y+T(x_0)) = D(y,x_0-T(x_0)) = D(y,G(x_0)) < (1-\alpha)r$. Then $F$ is a Ćirić operator defined on the open ball $B(x_0,r)$, where $D(x_0,F(x_0)) < (1-\alpha)r$. Applying the conclusion (h), i.e. the local version involving an open ball, it follows easily that $G$ is open.
(k) For the proof of the Caristi selection of the multi-valued Ćirić operator $T$, we refer to the work of A. Petruşel and G. Petruşel [20].
(m) Let $\varepsilon > 0$ and consider $y^* \in X$ that satisfies $D(y^*,T(y^*)) \leq \varepsilon$. Then, for each $(x,y) \in Graph(T)$, we have that $d(x,t^\infty(x,y)) \leq \frac{1}{1-\alpha}d(x,y)$.
Now, since there exists $(y^*,u^*) = D(y^*,T(y^*))$, we take $x^* := t^\infty(y^*,u^*)$. This implies that $d(y^*,x^*) = d(y^*,t^\infty(y^*,u^*)) \leq \frac{1}{1-\alpha}d(y^*,u^*) = \psi(\varepsilon)$, where $\psi(t) = \frac{t}{1-\alpha}$.
(n) For the proof of this, we refer to [2].
(o) Let $\varepsilon > 0$. Let's denote $E_\varepsilon(T) := \left\{x \in X \mid \sup_{z\in Tx} d(x,z) \leq \varepsilon\right\}$. Since $T$ is lower semicontinuous, by Lemma 3.3 from [8], we get that for each $\varepsilon > 0$, the set $E_\varepsilon(T)$ is nonempty. Now, let $x,y \in E_\varepsilon(T)$. It follows that:

$$d(x,y) \leq H(\{x\},T(x)) + H(T(x),T(y)) + H(\{y\},T(y)).$$

Now, since $x,y \in E_\varepsilon(T)$, then $H(\{x\},T(x)) \leq \varepsilon$ and $H(\{y\},T(y)) \leq \varepsilon$. So, we get that

$$d(x,y) \le 2\varepsilon + \alpha M(x,y).$$

Then, we have the following cases:
If $M(x,y) = d(x,y)$, then $d(x,y) \le 2\varepsilon + \alpha d(x,y)$, so $d(x,y) \le \dfrac{2\varepsilon}{1-\alpha}$.
If $M(x,y) = D(x,T(x)) = D(\{x\},T(x)) \le H(\{x\},T(x)) \le \varepsilon$, then $d(x,y) \le 2\varepsilon + \alpha\varepsilon = \varepsilon(2+\alpha)$.
Similarly, if $M(x,y) = D(y,T(y)) = D(\{y\},T(y)) \le H(\{y\},T(y)) \le \varepsilon$, then $d(x,y) \le 2\varepsilon + \alpha\varepsilon = \varepsilon(2+\alpha)$.
Finally, if $M(x,y) = \dfrac{1}{2}\left[D(x,T(y)) + D(y,T(x))\right]$, then we infer that:

$$\begin{aligned} D(x,T(y)) &\le d(x,y) + D(y,T(y)) \\ &= d(x,y) + \inf_{z\in T(y)} d(y,z) \le d(x,y) + \sup_{z\in T(y)} d(y,z) \\ &\le d(x,y) + \varepsilon, \end{aligned}$$

since $y \in E_\varepsilon(T)$.
Furthermore,

$$\begin{aligned} D(y,T(x)) &\le d(x,y) + D(x,T(x)) \\ &= d(x,y) + \inf_{z\in T(x)} d(x,z) \le d(x,y) + \sup_{z\in T(x)} d(x,z) \\ &\le d(x,y) + \varepsilon, \end{aligned}$$

since $x \in E_\varepsilon(T)$. Thus, $d(x,y) \le 2\varepsilon + \alpha\varepsilon + \alpha d(x,y)$.
From all the cases, it follows that

$$d(x,y) \le \varepsilon \max\left\{\frac{2+\alpha}{1-\alpha}, \frac{2}{1-\alpha}, 2+\alpha\right\} = \frac{2+\alpha}{1-\alpha}\varepsilon.$$

Now, if the multi-valued Ćirić-type operator $T$ has a strict fixed point, then $T$ has the approximate endpoint property. Let us suppose now that the multi-valued operator $T$ has the approximate endpoint property. We define $C_n := E_{\frac{1}{n}}(T) = \left\{x \in X \mid \sup_{y\in T(x)} d(x,y) \le \frac{1}{n}\right\}$. Then, by our hypothesis, for each $n \in \mathbb{N}$, $C_n$ is nonempty. Furthermore, for all $n \in \mathbb{N}$, $C_{n+1} \subseteq C_n$.
Also, since $T$ is lower semicontinuous, then $C_n$ are closed, for each $n \in \mathbb{N}$.
Also, we observe that :
$\delta(C_n) = \delta\left(E_{\frac{1}{n}}(T)\right) \le \dfrac{2+\alpha}{1-\alpha}\cdot\dfrac{1}{n}$, so $\lim_{n\to\infty}\delta(C_n) = 0$.
Then, by Cantor's intersection theorem, it follows that $\bigcap_{n\in\mathbb{N}} C_n = \{x_0\}$, so the conclusion follows easily.

(p) By (d), we have that $F_T$ is closed in $(X, d)$. Since $(X, d)$ is complete, then $F_T$ is complete with respect to $d$. Furthermore, let's suppose that $F_T$ is not compact. Then $F_T$ is not precompact. This means that there exist $\delta > 0$ and $(x_k)_{k\in\mathbb{N}} \subset F_T$, such that $d(x_i, x_j) \geq \delta$, for all $i \neq j$.
Denote $\rho := \inf\{R \mid \exists a \in X, \text{ such that } B(a, R) \text{ contains an infinity of } x_k's\}$. It is obvious that $\rho \geq \frac{\delta}{2}$, because for each $a \in X$, $B\left(a, \frac{\delta}{2}\right)$ contains at most one $x_k$.
Furthermore, consider $0 < \varepsilon < (1 - 2\alpha)\rho$ and take $a \in X$, such that the set $J := \{k \mid x_k \in B(a, \rho + \varepsilon)\}$ is infinite. Then, for each $k \in J$, we have

$$D(x_k, T(a)) \leq H(T(x_k), T(a)) \leq \alpha M(x_k, a).$$

Now, we have the following cases:
If $M(x_k, a) = d(x_k, a)$, then $D(x_k, T(a)) \leq \alpha d(x_k, a) \leq \alpha(\rho + \varepsilon)$.
Also, if $M(x_k, a) = D(a, T(a))$, then $D(x_k, T(a)) \leq \alpha d(a, y)$, for $y \in Ta$.
Now, if $M(x_k, a) = \frac{1}{2}D(x_k, T(a)) + \frac{1}{2}D(a, T(x_k))$, then

$$D(x_k, T(a)) \leq \frac{\alpha}{2}D(x_k, T(a)) + \frac{\alpha}{2}d(a, x_k),$$

so $D(x_k, T(a)) \leq \frac{\alpha}{2-\alpha}d(a, x_k)$. It implies that $D(x_k, T(a)) \leq \alpha(\rho + \varepsilon)$.
So, all the cases from above imply that $D(x_k, a) \leq \max\{\alpha(\rho + \varepsilon), \alpha d(a, y)\}$, where $y \in T(a)$. From all of this, we have two cases to consider :
In the first case, by $D(x_k, T(a)) \leq \alpha d(a, y)$, with $y \in T(a)$, we obtain that $D(x_k, T(a)) \leq \alpha d(a, x_k) + \alpha d(x_k, y)$. Taking $\inf_{y\in T(a)}$, we get, for each $k \in J$, that $D(x_k, T(a)) \leq \frac{\alpha}{1-\alpha} \cdot (\rho + \varepsilon)$.
Now, the second case is for $D(x_k, T(a)) \leq \alpha(\rho + \varepsilon)$. From these two cases, one can get $D(x_k, T(a)) \leq \max\left\{\alpha, \frac{\alpha}{1-\alpha}\right\} \cdot (\rho + \varepsilon) = \frac{\alpha}{1-\alpha} \cdot (\rho + \varepsilon)$. Then $D(x_k, T(a)) \leq \frac{\alpha}{1-\alpha} \cdot (\rho + \varepsilon)$, so since $T(a)$ is compact, there exists $y_k \in T(a)$, such that $d(x_k, y_k) \leq \frac{\alpha}{1-\alpha}(\rho + \varepsilon)$, for each $k \in J$.
Moreover, since $T(a)$ is compact, then there exists $b \in T(a)$, for which the set $J' := \{k \in J \mid d(y_k, b) < \varepsilon\}$ is infinite. This means that for each $k \in J'$ (since $\alpha < \frac{1}{2}$ and $\varepsilon$ was chosen such that $\varepsilon < \rho \cdot (1 - 2\alpha)$), we have that

$$d(x_k, b) \leq d(x_k, y_k) + d(y_k, b) < \frac{\alpha}{1-\alpha}(\rho + \varepsilon) + \varepsilon < \rho.$$

This contradicts the fact that the ball $B(b, R)$ contains an infinite number of elements $x_k's$, where $R = \frac{\alpha}{1-\alpha}\rho + \varepsilon\left(1 + \frac{\alpha}{1-\alpha}\right)$.
(q) Let $F_p^* := \{x \in X \mid D(x, T(x)) < p\}$, for each $p > 0$. Notice that if $x \in F_T$, then $D(x, T(x)) = 0 < p$, for each $p > 0$. So $F_T \subseteq F_p^*$. This implies that $H(F_p^*, F_T) = \rho(F_p^*, F_T) := \sup_{x \in F_p^*} D(x, F_T)$, for all $p > 0$, where $\rho$ denotes the excess functional.
Moreover, let $x \in F_p^*$ and $\varepsilon > 0$. Because $x \in F_p^*$, then $D(x, T(x)) < p$. So, for $x \in F_p^*$ there exists $x_1 \in T(x)$, for which $d(x, x_1) < (1 + \varepsilon)p$.
For $x_0 = x$ and $x_1 \in T(x) = T(x_0)$, following (b) there exists a sequence of successive approximations $(x_n)_{n\in\mathbb{N}}$, starting from $(x_0, x_1) \in Graph(T)$, such that $d(x_n, x^*) \leq \frac{L^n(q)}{1 - L(q)} d(x_0, x_1)$, for each $n \in \mathbb{N}$, where $L(q) := q\alpha$, with $q \in \left(1, \frac{1}{\alpha}\right)$ and with the property that $x_n \to x^* \in F_T$ as $n \to \infty$.
Taking $n = 0$, we obtain $d(x_0, x^*) \leq \frac{1}{1 - L(q)} d(x_0, x_1) \leq \frac{(1+\varepsilon)p}{1 - L(q)}$. So $d(x_0, x^*) \leq \frac{(1+\varepsilon)p}{1 - q\alpha}$. Taking $q \searrow 1$, respectively $\varepsilon \searrow 0$, it follows that $d(x_0, x^*) \leq \frac{p}{1-\alpha}$. So, the conclusion follows easily from this inequality. □

We will present now the second result of this article, which is an extended version of strict fixed point principle for multi-valued Ćirić operators. Since all the conclusion from Theorem 2.1 are valid even in the particular case when $(SF)_T \neq \emptyset$, for this case we shall present only the metrical conclusions that are new.

**Theorem 2.2** (An extended strict fixed point principle for multi-valued Ćirić operators)**.** *Let $(X, d)$ be a complete metric space and $T : X \to P_{cl}(X)$ be a multi-valued $\alpha$-Ćirić type operator. Suppose that $(SF)_T \neq \emptyset$. Then, the following conclusions hold :*
*(a) $(SF)_T = F_T = \{x^*\}$;*
*(b) if $\alpha < \frac{1}{2}$, then $T$ has the Ostrowski property;*
*(c) the fixed point inclusion $x \in T(x)$ is generalized Ulam-Hyers stable;*
*(d) the strict fixed point inclusion $\{x\} = T(x)$ is generalized Ulam-Hyers stable;*
*(e) the fixed point problem is well-posed for $T$, with respect to $D$ and, respectively, with respect to $H$;*
*(f) if $\alpha < \frac{1}{2}$, then $H(T(x), x^*) \leq \frac{\alpha}{1-alpha} d(x, x^*)$, for each $x \in X$;*

*(g)* $d(x, x^*) \leq \dfrac{1}{1-\alpha} H(x, T(x))$, *for each* $x \in X$;
*(h) if* $G : X \to P(X)$ *is a multi-valued operator with* $F_G \neq \emptyset$, *and there exists* $\eta > 0$, *such that* $H(T(x), G(x)) \leq \eta$, *for all* $x \in X$, *then* $H(F_T, F_G) \leq \eta \cdot \dfrac{1}{1-\alpha}$.

*Proof.* (a) Since $(SF)_T \neq \emptyset$, then there exists $x^* \in (SF)_T \subset F_T$. Suppose there exists $y^* \in F_T$. We show that $x^* = y^*$. For this, suppose the contrary that $x^* \neq y^*$. Then:

$$d(x^*, y^*) = D(T(x^*), y^*) \leq H(T(x^*), T(y^*)) \leq \alpha M(x^*, y^*).$$

Since $D(x^*, T(x^*)) = D(y^*, T(y^*)) = 0$, $D(x^*, T(y^*)) \leq d(x^*, y^*)$ and $D(y^*, T(x^*)) = d(x^*, y^*)$, it follows that $M(x^*, y^*) \leq d(x^*, y^*)$.
So $d(x^*, y^*) \leq \alpha d(x^*, y^*) < d(x^*, y^*)$. This implies that $d(x^*, y^*) = 0$, so we obtain a contradiction.
Finally, $x^* = y^*$, so $F_T = \{x^*\} = (SF)_T$.
(b) Let $(y_n)_{n\in\mathbb{N}}$ be a sequence, such that $D(y_{n+1}, T(y_n)) \to 0$. We shall show that $d(y_n, x^*) \to 0$. Then, we have $d(x^*, y_{n+1}) \leq H(T(x^*), T(y_n)) + D(y_{n+1}, T(y_n)) \leq \alpha M(x^*, y_n) + D(y_{n+1}, T(y_n))$, where

$$\begin{aligned} M(x^*, y_n) &= \\ &= \max\left\{d(x^*, y_n), D(x^*, T(x^*)), D(y_n, T(y_n)), \frac{1}{2}\left[D(x^*, T(y_n)) + D(y_n, T(x^*))\right]\right\} \\ &\leq \max\left\{d(x^*, y_n), D(y_n, T(y_n)), \frac{1}{2}\left[d(y_n, x^*) + D(x^*, T(y_n))\right]\right\}. \end{aligned}$$

Now, we have the following cases :
If the maximum from the right hand side is $d(x^*, y_n)$, then $d(x^*, y_{n+1}) \leq D(y_{n+1}, T(y_n)) + \alpha d(x^*, y_n)$.
If the maximum is $D(y_n, T(y_n)) \leq d(y_n, x^*) + D(x^*, T(y_n))$, then we have $H(x^*, T(y_n)) = H(T(x^*), T(y_n)) \leq \alpha M(x^*, y_n) \leq \alpha d(y_n, x^*) + \alpha H(x^*, T(y_n))$.
So, we get that $H(T(x^*), T(y_n)) \leq \dfrac{\alpha}{1-\alpha} d(y_n, x^*)$.
It implies that $d(y_{n+1}, x^*) \leq D(y_{n+1}, T(y_n)) + \dfrac{\alpha}{1-\alpha} d(y_n, x^*)$.
Consider now the case when the maximum is $\dfrac{1}{2}\left[d(y_n, x^*) + D(x^*, T(y_n))\right]$.
Then, we obtain $D(x^*, T(y_n)) \leq H(T(x^*), T(y_n)) \leq \alpha M(x^*, y_n)$. Thus $H(T(x^*), T(y_n)) \leq \dfrac{\alpha}{2}\left(d(y_n, x^*) + H(T(x^*), T(y_n))\right)$. This means that

$$H(T(x^*), T(y_n)) \leq \frac{\alpha}{2-\alpha} d(y_n, x^*).$$

Hence $d(y_{n+1}, x^*) \leq D(y_{n+1}, T(y_n)) + \frac{\alpha}{2-\alpha} d(y_n, x^*)$.
Now, since $\beta := \max\left\{\alpha, \frac{\alpha}{1-\alpha}, \frac{\alpha}{2-\alpha}\right\} = \frac{\alpha}{1-\alpha}$, then from all the cases from above, it follows that $d(y_{n+1}, x^*) \leq D(y_{n+1}, T(y_n)) + \beta d(y_n, x^*) \leq D(y_{n+1}, T(y_n)) + \beta D(y_n, T(y_{n-1})) + \beta^2 d(y_{n-1}, x^*) \leq \ldots \leq \beta^{n+1} d(y_0, x^*) + \sum_{k=0}^{n} \beta^{n-k} D(y_{k+1}, T(y_k))$. Now, since $\beta < 1$, using Cauchy's lemma, we get that $d(y_{n+1}, x^*) \to 0$.
(c) By (a) we know that $(SF)_T = F_T = \{x^*\}$.
Now, let us consider $x \in X$ and $y \in T(x)$. Then, we have the following: $d(x, x^*) \leq d(x, y) + H(T(x), T(x^*)) \leq d(x, y) + \alpha M(x, x^*) \leq d(x, y) + \alpha \max\left\{d(x, x^*), D(x, T(x)), \frac{1}{2} d(x^*, y) + \frac{1}{2} d(x, x^*)\right\}$. Moreover, we consider the following cases:
If $M(x, x^*) = d(x, x^*)$, then $d(x, x^*) \leq \frac{1}{1-\alpha} d(x, y)$.
If $M(x, x^*) = D(x, T(x))$, then

$$d(x, x^*) \leq d(x, y) + \alpha D(x, T(x)) \leq (1+\alpha) d(x, y)$$

Finally, if $M(x, x^*) \leq \frac{1}{2} d(x^*, y) + \frac{1}{2} d(x, x^*)$, then we have $d(x, x^*) \leq d(x, y) + \frac{\alpha}{2} d(x^*, y) + \frac{\alpha}{2} d(x, x^*)$. So, we get $d(x, x^*) \leq \frac{2+\alpha}{2(1-\alpha)} d(x, y)$. From all the cases we obtain that

$$d(x, x^*) \leq \max\left\{\frac{1}{1-\alpha}, 1+\alpha, \frac{2+\alpha}{2(1-\alpha)}\right\} d(x, y) = \frac{2+\alpha}{2(1-\alpha)} d(x, y).$$

Now, let us define $\psi(t) := \frac{2+\alpha}{2(1-\alpha)} t$, so $d(x, x^*) \leq \psi(d(x, y))$. We notice that $\psi$ is continuous in 0, increasing and with $\psi(0) = 0$.
Then, as in (m) of Theorem 2.1, we have the following:
Let $\varepsilon > 0$ and consider $y^* \in X$ that satisfies $D(y^*, T(y^*)) \leq \varepsilon$. Then, for each $(x, y) \in Graph(T)$, we have $d(x, t^\infty(x, y)) \leq \psi(d(x, y))$.
Now, since there exists $(y^*, u^*) = D(y^*, T(y^*))$, we take $x^* := t^\infty(y^*, u^*)$. This implies that $d(y^*, x^*) = d(y^*, t^\infty(y^*, u^*)) \leq \psi(d(y^*, u^*))$ and the conclusion follows.
(d) Let $\varepsilon > 0$ and $y^* \in X$, such that $H(y^*, T(y^*)) \leq \varepsilon$. Since $T$ is a Ćirić multi-valued operator, from (h) we have that $d(x, x^*) \leq \frac{1}{1-\alpha} H(x, T(x))$, for each $x \in X$. This implies that $d(y^*, x^*) \leq \frac{1}{1-\alpha} H(y^*, T(y^*)) \leq \psi(\varepsilon)$, where $\psi(t) := \frac{t}{1-\alpha}$ satisfies $\psi(0) = 0$ and it is an increasing and continuous

mapping in 0.
(e) The proof of this conclusion is given in [19].
(f) We know that

$$H(T(x),T(x^*)) \leq \alpha \max\{d(x,x^*), D(x,T(x)), \frac{1}{2}[D(x,T(x^*)) + D(x^*,T(x))]\}.$$

We have the following cases:
If the maximum is $d(x,x^*)$, then $H(T(x),T(x^*)) \leq \alpha d(x,x^*)$.
If the maximum is $\frac{1}{2}[D(x,T(x^*)) + D(x^*,T(x))]$, then $H(T(x),T(x^*)) = \frac{\alpha}{2}d(x,x^*) + \frac{\alpha}{2}H(T(x),T(x^*))$ and so $H(T(x),T(x^*)) \leq \frac{\alpha}{2-\alpha}d(x,x^*)$.
If the maximum is $D(x,T(x))$, then we obtain $H(T(x),T(x^*)) \leq \frac{\alpha}{1-\alpha}d(x,x^*)$.
Since $\max\left\{\frac{\alpha}{2-\alpha}, \alpha, \frac{\alpha}{1-\alpha}\right\} = \frac{\alpha}{1-\alpha}$, $H(T(x),x^*) = H(T(x),T(x^*)) \leq \frac{\alpha}{1-\alpha}d(x,x^*)$.
(g) We have the following chain of inequalities $d(x,x^*) \leq H(x,T(x)) + H(T(x),x^*) \leq H(x,T(x)) + \alpha d(x,x^*)$. Thus $d(x,x^*) \leq \frac{1}{1-\alpha}H(x,T(x))$.
(h) Let $x^* \in (SF)_T$ and $y^* \in F_G$. Then, we have

$$d(x^*,y^*) \leq H(G(y^*),x^*) \leq H(G(y^*),T(y^*)) + H(T(y^*),x^*) \leq \eta + \alpha M(y^*,x^*).$$

Now, we have the following cases for $M(y^*,x^*)$:
1) if $M(y^*,x^*) = d(y^*,x^*)$, then $d(y^*,x^*) \leq \frac{\eta}{1-\alpha}$.
2) if $M(y^*,x^*) = D(x^*,T(x^*))$, then $d(y^*,x^*) = 0$.
3) if $M(y^*,x^*) = D(y^*,T(y^*)) \leq H(G(y^*),T(y^*)) \leq \eta$, then $d(y^*,x^*) \leq (1+\alpha)\eta$.
4) finally, if $M(y^*,x^*) = \frac{1}{2}D(x^*,T(y^*)) + \frac{1}{2}D(y^*,T(x^*))$, then
$H(T(x^*),T(y^*)) \leq \alpha M(y^*,x^*) \leq \frac{\alpha}{2}H(T(y^*),T(x^*)) + \frac{\alpha}{2}d(x^*,y^*)$. Hence, we get that $H(T(x^*),T(y^*)) \leq \frac{\alpha}{2-\alpha}d(x^*,y^*)$. Then $d(x^*,y^*) \leq \eta + \frac{\alpha}{2-\alpha}d(x^*,y^*)$, which implies that $d(y^*,x^*) \leq \frac{2-\alpha}{2(1-\alpha)}\eta$. It follows that $d(y^*,x^*) \leq \eta \cdot \max\left\{(1+\alpha), \frac{1}{1-\alpha}, \frac{2-\alpha}{2(1-\alpha)}\right\} = \frac{1}{1-\alpha}\eta$. Using Lemma 1.2 the conclusion follows. $\square$

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