

## *Some $Z_q$ -Equivariant Immersions*

Dedicated to Professor Kiiti Morita on his 60th birthday

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### § 1. Introduction

Let  $q$  be an integer and  $Z_q$  be the cyclic group of order  $q$ . A  $C^\infty$ -differentiable immersion  $f$  of a  $Z_q$ -manifold in another  $Z_q$ -manifold is called a  $Z_q$ -equivariant immersion (or simply a  $Z_q$ -immersion) if  $f$  is a  $Z_q$ -equivariant map. The purpose of this note is to study the conditions for the existence of some  $Z_q$ -equivariant immersions.

Let  $m$  and  $k$  be non-negative integers, and  $R^{m+2k}$  be Euclidean  $(m+2k)$ -space. Let  $R^{m,2k}$  be the  $Z_q$ -manifold  $(R^{m+2k}, Z_q)$  with the action

$$\mu: Z_q \times R^{m+2k} \longrightarrow R^{m+2k}$$

defined by

$$\mu(T, (t_1, \dots, t_m, z_{m+1}, \dots, z_{m+k})) = (t_1, \dots, t_m, Tz_{m+1}, \dots, Tz_{m+k}),$$

where  $T (= e^{2\pi\sqrt{-1}/q})$  is the generator of  $Z_q$ ,  $t_1, \dots, t_m$  are real numbers ( $\in R$ ), and  $z_{m+1}, \dots, z_{m+k}$  are complex numbers ( $\in C = R^2$ ).

Let  $S^{2n+1}$  be the unit  $(2n+1)$ -sphere in complex  $(n+1)$ -space  $C^{n+1}$ . Let  $(S^{2n+1}, Z_q)$  be the  $Z_q$ -manifold defined by the action

$$\nu: Z_q \times S^{2n+1} \longrightarrow S^{2n+1}; \nu(T, (z_0, \dots, z_n)) = (Tz_0, \dots, Tz_n),$$

where  $z_0, \dots, z_n$  are complex numbers with  $\sum_{j=0}^n |z_j|^2 = 1$ . The action  $\nu$  is free and differentiable of class  $C^\infty$ . The orbit manifold  $S^{2n+1}/Z_q$  is the standard lens space  $L^n(q) \bmod q$ .

A. Jankowski obtained in [7] some non-existence theorems for  $Z_2$ -immersions. In this note we consider  $Z_q$ -immersions  $(S^{n+1}, Z_q) \rightarrow R^{m,2k}$ , and study the bounds of  $m$  for fixed  $k$  and  $n$ .

As is easily seen, there is a  $Z_q$ -immersion of  $(S^{2n+1}, Z_q)$  in  $R^{m,0}$  if and only if there is an immersion of  $L^n(q)$  in  $R^m$ .

If  $k > n$ ,  $(S^{2n+1}, Z_q)$  is  $Z_q$ -immersionable in  $R^{m,2k}$  for any  $m$ , clearly. In case  $k \leq n$ , we have the following results.

**THEOREM 1.** *Let  $q$  be an integer  $> 1$ . Then  $(S^{2n+1}, Z_q)$  is not  $Z_q$ -immer-*

sible in  $R^{2\lfloor n/2\rfloor, 2n}$ .

THEOREM 2. Let  $p$  be an odd prime and  $r$  be a positive integer. Let  $n$  and  $k$  be integers with  $0 \leq k \leq n$ . Put

$$L = \max \left\{ i \mid 1 \leq i \leq \lfloor n/2 \rfloor, \binom{n-k+i}{i} \not\equiv 0 \pmod{p^{r+\lfloor (n-2i)/(p-1) \rfloor}} \right\}.$$

Then there does not exist a  $Z_{p^r}$ -immersion of  $(S^{2n+1}, Z_{p^r})$  in  $(R^{2n+2L}, Z_{p^r}) = R^{2n+2L-2k, 2k}$ .

If  $k=0$ , Theorem 2 is a consequence of Corollary 3.6 of T. Kawaguchi and M. Sugawara in [9]. A similar result for  $(S^{2n+1}, Z_4)$  is obtained from Corollary 6.7 in [12].

THEOREM 3. Let  $p$  be an odd prime and  $r$  be a positive integer. Let  $n$  and  $k$  be integers with  $0 \leq k \leq n$ . Assume that there is an integer  $m$  satisfying the following conditions:

- (i)  $0 < k+m \leq \lfloor n/2 \rfloor$ ,
- (ii)  $\binom{n+m}{n-k} \not\equiv (sp)^2 \pmod{p^r}$  for any integer  $s$  with  $0 \leq s < p^{r-1}$ ,
- (iii)  $n+m+1 \not\equiv 0 \pmod{p^{\lfloor (n-m-k-1)/(p-1) \rfloor}}$ .

Then there does not exist a  $Z_{p^r}$ -immersion of  $(S^{2n+1}, Z_{p^r})$  in  $(R^{2n+2m+2k+1}, Z_{p^r}) = R^{2n+2m+1, 2k}$ .

If  $k=0$ , we have a corollary which establishes the non-existence of an immersion of  $L^n(p^r)$  in  $R^{2n+2m+1}$  for some  $n$  and  $m$ . This corollary gives a generalization of Theorem C in [10] and Theorem 7.9 in [12].

There is an example of a  $Z_q$ -immersion of  $S^{2n+1}$  in  $R^{m, 2k}$  when both  $m$  and  $2k$  are less than  $2n+1$ . Let  $(re^{a\sqrt{-1}}, se^{b\sqrt{-1}})$  be a point of  $S^3$ , where  $a, b, r$  and  $s$  are real numbers with  $r^2 + s^2 = 1$ . Then the map  $f: S^3 \rightarrow R^4$ , defined by the equality

$$f(re^{a\sqrt{-1}}, se^{b\sqrt{-1}}) = ((r+s+2)e^{qa\sqrt{-1}}, (r-s-2)e^{b\sqrt{-1}}),$$

is a  $Z_q$ -immersion of  $(S^3, Z_q)$  in  $R^{2, 2}$ , where  $q$  is any integer  $> 1$ .

In § 2, the proofs of Theorems 1 and 2 are carried out by making use of the  $\gamma$ -operation in  $KO$ -theory (cf. [3]) and the fact that the generator  $\bar{\sigma}^i e \widetilde{KO}(L^n(p^r))$  is of order  $p^{r+\lfloor (n-2i)/(p-1) \rfloor}$  ([9, Theorem 1.1]). In § 3, we prove Propositions 3.1 and 3.2 which show the existence of some relations between the immersions of lens spaces and the stable homotopy types of the stunted lens spaces. The proof of Theorem 3 is based on these propositions. In § 4, the method in § 3 is applied to  $Z_2$ -equivariant immersions, and some results (Theorems 4.1 and 4.2) are obtained.

**§ 2.  $\gamma$ -operations in  $KO$ -theory**

For a  $Z_q$ -space  $(X, Z_q)$ , let  $\theta = \theta(X, Z_q)$  be a  $Z_q$ -vector bundle  $(X \times R^2, X, p_1, R^2)$  defined as follows.

(1)  $p_1: X \times R^2 \rightarrow X$  is the projection onto the first factor.

(2) The action of  $Z_q$  on  $X \times R^2$  is the diagonal action:  $T(x, z) = (Tx, Tz)$ , where  $x \in X, z \in R^2$ , and  $T (= e^{2\pi\sqrt{-1}/q})$  is the generator of  $Z_q$ .

Then we have immediately the following

LEMMA 2.1. *///:  $X \rightarrow Y$  is a  $Z_q$ -map between  $Z_q$ -spaces  $X$  and  $Y$ , we have  $f^*\theta(Y, Z_q) = \theta(X, Z_q)$ .*

A  $G$ -vector bundle  $E \rightarrow X$  determines naturally a vector bundle  $E/G \rightarrow X/G$ . This correspondence induces a homomorphism

$$\rho: KO_G(X) \longrightarrow KO(X/G).$$

It is well-known that  $\rho$  is an isomorphism if the  $G$ -action on  $X$  is free (cf. [4, Proposition 1.3.1]).

Let  $\eta$  be the real restriction of the canonical complex line bundle over  $L^n(q)$ . Considering the transition functions we see easily

$$\text{LEMMA 2.2.} \quad \rho(\theta(S^{2n+1}, Z_q)) = \eta.$$

Define the action of  $Z_q$  on the total space  $E_{m,2k}$  of the Whitney sum of the  $m$ -dimensional trivial bundle  $m$  over  $R^{m,2k}$  and  $k\theta(R^{m+2k}, Z_q)$  as follows:

$$\begin{aligned} T((u, t_1), \dots, (u, t_m), (u, z_{m+1}), \dots, (u, z_{m+k})) \\ = ((Tu, t_1), \dots, (Tu, t_m), (Tu, Tz_{m+1}), \dots, (Tu, Tz_{m+k})), \end{aligned}$$

where  $u \in R^{m,2k}, t_i \in R (i= 1, \dots, m), z_{m+j} \in R^2 (j= 1, \dots, k)$ , and  $T$  is the generator of  $Z_q$ . Then the following holds.

LEMMA 2.3. *There is a  $Z_q$ -bundle isomorphism of the tangent  $Z_q$ -bundle  $\tau(R^{m,2k})$  onto the  $Z_q$ -bundle  $m \oplus k\theta(R^{m+2k}, Z_q)$ .*

PROOF. Carrying a point  $(u, (t_1, \dots, t_m, z_{m+1}, \dots, z_{m+k}))$  of the total space of  $\tau(R^{m,2k})$  into a point  $((u, t_1), \dots, (M, t_m), (u, z_{m+1}), \dots, (M, z_{m+k})) \in E_{m,2k}$ , we obtain a desired  $Z_q$ -bundle isomorphism. q. e. d.

Now we are in a position to prove

PROPOSITION 2.4. *Let  $q$  be an integer  $> 1$ , and  $n$  and  $k$  be integers with  $0 \leq k \leq n$ . Put*

$$L = \max \left\{ j! \binom{n-k+j}{j} \bar{\sigma}^j \neq 0 \right\}, \text{ where } \sigma = \eta - 2 \in \widetilde{KO}(L^n(q)).$$

Then there does not exist 0  $Z_q$ -immersion of  $(S^{2n+1}, Z_q)$  into  $(R^{2n+2L}, Z_q) = R^{2n+2L-2k, 2k}$ .

PROOF. Suppose that there exists a  $Z_q$ -immersion  $f: (S^{2n+1}, Z_q) \rightarrow (R^{m+2k}, Z_q) = R^{m, 2k}$ . Let  $\nu_f$  be the normal  $Z_q$ -bundle of  $f$ . Then we have

$$\tau(S^{2n+1}, Z_q) \oplus \nu_f = f^* \tau(R^{m, 2k}).$$

Applying  $\rho$  to the equation, noting that  $\rho(\tau(S^{2n+1}, Z_q)) = \tau(L^n(q))$ , and using Lemmas 2.1-2.3, we get

$$\tau(L^n(q)) \oplus \rho \nu_f = \rho f^*(m \oplus k\theta(R^{m+2k}, Z_q)) = m \oplus k\rho\theta(S^{2n+1}, Z_q) = m \oplus k\eta.$$

Since  $\tau(L^n(q)) \oplus 1 = (n+1)\eta$ , we have

$$\rho \nu_f = m + 1 + (k - n - 1)\eta = m + 2k - 2n - 1 + (k - n - 1)\bar{\sigma} \text{ in } KO(L^n(q)).$$

Let  $\nu_0 = (k - n - 1)\bar{\sigma}$  be the stable class of  $\rho \nu_f$ . Then it follows that

$$g. \dim \nu_0 \leq m + 2k - 2n - 1,$$

where  $g. \dim \nu_0$  denotes the geometric dimension of  $\nu_0$ . According to [3, Proposition (2.3)],  $\gamma^j(\nu_0) = 0$  for  $j > g. \dim \nu_0$ , where  $\gamma^j: KO(X) \rightarrow KO(X)$  is the Grothendieck  $\gamma$ -operation. Hence we see

$$\gamma^j(\nu_0) = 0 \quad \text{for } j > m + 2k - 2n - 1.$$

On the other hand,

$$\begin{aligned} \sum_{j=0}^{\infty} \gamma^j(\nu_0) t^j &= \gamma_t(\nu_0) = \gamma_t((k-n-1)\bar{\sigma}) = \gamma_t(\bar{\sigma})^{k-n-1} \\ &= (1 + \bar{\sigma}t - \bar{\sigma}t^2)^{k-n-1} = \sum_{j=0}^{\infty} \binom{k-n-1}{j} \bar{\sigma}^j (t-t^2)^j \\ &= \sum_{j=0}^{\infty} (-1)^j \binom{n-k+j}{j} \bar{\sigma}^j (t-t^2)^j. \end{aligned}$$

(Here we use the fact that  $\text{fcrgn.}$ ) For the given  $L$ , we have

$$\gamma_t(\nu_0) = \sum_{j=0}^L (-1)^j \binom{n-k+j}{j} \bar{\sigma}^j (t-t^2)^j = \sum_{j=0}^L \gamma^j(\nu_0) t^j.$$

Therefore  $\gamma^{2L}(\nu_0) \neq 0$ . This implies that  $2L \neq m + 2k - 2n$ , as desired. **q. e. d.**

PROOF OF THEOREM 1. If  $\text{fc} = n$ , we have  $L = [n/2]$  by [9, Proposition 2.6]. Then Theorem 1 follows from Proposition 2.4. **q. e. d.**

PROOF OF THEOREM 2. If  $q = p^r$ , the order of  $\bar{\sigma}^j$  is equal to  $p^{r+[(n-2j)/(p-1)]}$  ([9, Theorem 1.1]). Hence Theorem 2 also follows from Proposition 2.4. **q. e. d.**

§3. **Mod  $p^r$  S-relation**

For a space  $X$  and an integer  $k \geq 0$ , we denote by  $S^k X$  the  $k$ -fold suspension of  $X$ . Two spaces  $X$  and  $Y$  are said to be **mod  $q$  S-related**, if there are non-negative integers  $a$  and  $b$ , and a map  $f: S^a X \rightarrow S^b Y$  which induces isomorphisms of all homology groups with  $Z_q$ -coefficients. We prepare the following two propositions for the proof of Theorem 3.

PROPOSITION 3.1. *Let  $p$  be an odd prime,  $r$  be a positive integer, and  $l$  and  $n$  be integers with  $0 < l \leq [n/2]$ . Suppose that there is a positive integer  $t$  satisfying the following conditions:*

(i)  *$(l+t)\eta$  has linearly independent  $2t$  cross-sections, where  $\eta$  is the real restriction of the canonical complex line bundle over  $L^n(p^r)$ .*

(ii)  *$\binom{l+t}{s} \not\equiv (sp)^2 \pmod{p}$  for any integer  $s$  with  $0 \leq s < p^{r-1}$ .*

*Then the stunted lens spaces  $L^n(p^r)/L^{l-1}(p^r)$  and  $L^{n+t}(p^r)/L^{l-1+t}(p^r)$  are mod  $p^r$  S-related.*

PROPOSITION 3.2. *Let  $p$  be a prime,  $r$  be a positive integer with  $p^r \neq 2$ , and  $l$  and  $n$  be integers with  $0 < l < n$ . // the stunted lens spaces  $L^n(p^r)/L^{l-1}(p^r)$  and  $L^{n+t}(p^r)/L^{l-1+t}(p^r)$  are mod  $p^r$  S-related, we have*

$$t \equiv 0 \pmod{p^{[(n-l-1)/(p-1)]}}$$

PROOF OF PROPOSITION 3.1. Let  $q = p^r$  and  $L^n = L^n(p^r)$ . It follows from the property (i) that there is a  $2t$ -dimensional vector bundle  $\alpha$  over  $L^n$  satisfying

$$(1) \quad (l+t)\eta = 2t \oplus \alpha.$$

Let  $\chi(\alpha)$  and  $p_j(\alpha)$  be the Euler class and the  $j$ -th Pontrjagin class of  $\alpha$ , respectively. Then it is well-known that

$$\chi(\alpha)^2 = p_l(\alpha) = \binom{l+t}{l} x^{2l},$$

where  $x$  is the generator of  $H^2(L^n; Z)$ . Now, our assumptions imply that

$$(2) \quad \chi(\alpha)_q = u(x^l)_q,$$

where  $z_q$  denotes the image of  $z$  by the mod  $q$  reduction, and  $u$  is some integer such that  $u \not\equiv 0 \pmod{p}$ .

Let  $D$  and  $E$  denote the total spaces of the disk and the sphere bundles associated with the bundle  $\alpha$ , respectively. Consider the commutative diagram (cf. [14])

$$\begin{array}{ccc} H^k(D, E; Z_q) & \xrightarrow{j^*} & H^k(D; Z_q) \\ \uparrow \phi & & \uparrow \pi^* \\ H^{k-2l}(L^n; Z_q) & \xrightarrow{h} & H^k(L^n; Z_q), \end{array}$$

where  $j^*$  is the homomorphism induced by the inclusion  $j: D \rightarrow (D, E)$ ,  $\pi^*$  is the isomorphism induced by the projection  $\pi: D \rightarrow L^n$ ,  $\phi$  is the Thom isomorphism, and  $h$  is the homomorphism defined by

$$AGO = y \cup \chi(\alpha)_q, \quad \text{for } y \in H^{k-2l}(L^n; Z_q).$$

It follows from (2) that  $A$  is isomorphic for each  $k$  with  $2l \leq k \leq 2n + 1$ . By  $(L^n)^\alpha$  we mean the Thom complex  $D/E$  of  $\alpha$ . Then we see from the above commutative diagram that the natural inclusion  $\lambda: L^n \rightarrow (L^n)^\alpha$  induces isomorphisms

$$\lambda^* : H^k((L^n)^\alpha; Z_q) \xrightarrow{\cong} H^k(L^n; Z_q) \text{ for } 2l \leq k \leq 2n + 1.$$

Since  $(L^n)^\alpha$  is  $(2l-1)$ -connected, there is a map  $/: L^n/L^{l-1} \rightarrow (L^n)^\alpha$  such that  $\lambda$  is homotopic to  $f/p$ , where  $p: L^n \rightarrow L^n/L^{l-1}$  is the projection. It is easy to see that  $/$  induces isomorphisms

$$f^* : H^k((L^n)^\alpha; Z_q) \xrightarrow{\cong} H^k(L^n/L^{l-1}; Z_q) \text{ for } 0 \leq k \leq 2n + 1.$$

According to [2, Lemma (2.4)] and [8, Theorem 1], there are natural homeomorphisms

$$S^{2t}(L^n)^\alpha \sim (L^n)^{2t \oplus \alpha}, \quad (L^n)^{(l+t)n} \sim L^{n+l+t}/L^{l-1+t}.$$

Thus we obtain, by (1), a natural homeomorphism

$$\psi : S^{2t}(L^n)^\alpha \xrightarrow{\cong} L^{n+l+t}/L^{l-1+t}.$$

Then there is a map

$$\text{flf} : S^{2t}(L^n/L^{l-1}) \xrightarrow{\cong} L^{n+t}/L^{l-1+t}$$

such that  $\psi S^{2t}f$  is homotopic to  $ig$ , where  $S^{2t}f$  is the  $2t$ -fold suspension of  $f$  and  $i: L^{n+t}/L^{l-1+t} \rightarrow L^{n+l+t}/L^{l-1+t}$  is the natural inclusion. We see that the map  $g$  induces isomorphisms of all cohomology groups with  $Z_q$ -coefficients. Hence  $g$  induces isomorphisms of all homology groups with  $Z_q$ -coefficients. q. e. d.

PROOF OF PROPOSITION 3.2. Put  $q = p^r$  and  $L^n = L^n(p^r)$ . By the assumption there are non-negative integers  $a$  and  $b$ , and a map

$$g: S^a(L^n/L^{l-1}) \longrightarrow S^b(L^{n+t}/L^{l-1+t})$$

which induces isomorphisms of all homology groups with  $Z_q$ -coefficients. We may assume that  $g$  is cellular. It is not difficult to prove that  $g$  induces a map

$$g_0: S^a(L_0^n/L_0^l) \longrightarrow S^b(L_0^{n+t}/L_0^{l+t})$$

which induces isomorphisms of all homology groups with  $Z_q$ -coefficients, where  $L_0^n$  denotes the  $2n$ -skeleton of  $L^n$ . Obviously  $g_0$  is a homotopy equivalence, and the result follows from [13, Theorem 1.1]. q. e. d.

PROOF OF THEOREM 3. Put  $q = p^r$ . Suppose that there exists a  $Z_q$ -immersion

$$f: (S^{2n+1}, Z_q) \longrightarrow R^{2n+2m+1, 2k}.$$

Let  $\nu_f$  be the normal  $Z_q$ -bundle of  $f$ . Then we have

$$\tau(L^n(q)) \oplus \rho\nu_f = (2n + 2m + 1) \oplus k\eta,$$

as in the proof of Proposition 2.4. Therefore

$$(n + 1 - k)\eta \oplus \rho\nu_f = 2n + 2m + 2.$$

Take a sufficiently large positive integer  $a$  so that  $A = ap^{r+[(n-2)/(p-1)]}$  satisfies  $2A + 2k - 2n - 2 > 2n + 1$ . Then  $A\bar{\sigma} = 0$ , because  $\sigma (= \eta - 2 \in \widetilde{KO}(L^n(q)))$  is of order  $p^{r+[(n-2)/(p-1)]}$  ([9, Theorem 1.1]), and hence we have

$$(A - n - 1 + k)\eta = (2A - 2n - 2m - 2) + \rho\nu_f = (2A - 2n - 2m - 2) \oplus \rho\nu_f.$$

Putting  $t = A - n - m - l$  and  $l = c + m$ , we see that the conditions (i) and (ii) of Proposition 3.1 are satisfied. (Notice that  $a$  can be taken sufficiently large.) Hence by Propositions 3.1 and 3.2 we get

$$t = A - n - m - 1 = -n - m - 1 \equiv 0 \pmod{p^{[(n-m-k-1)/(p-1)]}}.$$

But this contradicts the assumption (iii).

q. e. d.

#### § 4. $Z_2$ -equivariant immersions

Let  $(S^n, Z_2)$  be the  $Z_2$ -manifold whose  $Z_2$ -action on  $S^n$  is the antipodal involution, and  $R^{m+k}$  be the  $Z_2$ -manifold  $(R^{m+k}, Z_2)$  whose  $Z_2$ -action on  $R^{m+k}$  is given by the involution

$$\Gamma: R^{m+k} \longrightarrow R^{m+k}$$

$$T(t_1, \dots, t_m, t_{m+1}, \dots, t_{m+k}) = (t_1, \dots, t_m, -t_{m+1}, \dots, -t_{m+k}),$$

where  $T$  is the generator of  $\mathbf{Z}_2$  and  $t_1, \dots, t_{m+k}$  are real numbers.

For integers  $m$  and  $n$  with  $0 \leq m < n$ , denote by  $\phi(n, m)$  the number of integers  $s$  such that  $m < s \leq n$  and  $s \equiv 0, 1, 2$  or  $4 \pmod{8}$ . Let  $\psi(n, m)$  be the integer given by

$$\psi(n, m) = \begin{cases} \phi(n, m), & \text{if } m \equiv 0 \pmod{4}, \\ \phi(n, m-1), & \text{if } m \not\equiv 0 \pmod{4}. \end{cases}$$

THEOREM 4.1. *Let  $k$  and  $n$  be integers such that  $0 \leq k \leq n$  and*

(o)  $n+1-k \equiv 0 \pmod{8}$ .

*Assume that there is a non-negative integer  $m$  satisfying the following conditions:*

(i)  $n < m+k \leq 2n$ ,

(ii)  $\binom{m+1}{n+1-k} \not\equiv 0 \pmod{2}$ ,

(iii)  $n+1-k \not\equiv 0 \pmod{2^{\psi(n, m+k-n)-1}}$ .

*Then there does not exist a  $\mathbf{Z}_2$ -immersion of  $(\mathbf{S}^n, \mathbf{Z}_2)$  in  $R^{m, k} = (R^{m+k}, \mathbf{Z}_2)$ .*

THEOREM 4.2. *Suppose that the conditions (o) and (iii) in Theorem 4.1 are replaced by the following condition:*

$$n+1 \equiv 0 \pmod{2^{\phi(2n-m-k, 0)}} \text{ if and only if } 2n+2-k \not\equiv 0 \pmod{2^{\phi(2n-m-k, 0)}}.$$

*Then the same conclusion as Theorem 4.1 holds.*

PROOFS OF THEOREMS 4.1 AND 4.2. Assume that there is a  $\mathbf{Z}_2$ -immersion  $f: (\mathbf{S}^n, \mathbf{Z}_2) \rightarrow R^{m, k}$ . Let  $\nu_f$  be the normal  $\mathbf{Z}_2$ -bundle of  $f$ . Then it holds that

$$(n+1)\xi \oplus \rho\nu_f = (m+1)\xi \oplus k\xi \quad (\text{cf. the proof of Theorem 3.4 in [7]}),$$

where  $\xi$  is the canonical line bundle over the real projective space  $RP^n$ . Since  $\xi \otimes \xi = 1$ , we have

$$(1) \quad (m+1)\xi = (n+1-k)\xi \oplus \rho\nu_f \otimes \xi.$$

Applying Lemmas (4.1) and (4.2) in [11] to the equality (1), we obtain Theorem 4.1.

Applying Lemma (4.1) in [11] to the equality (1), and using the fact that  $RP^n/RP^{m-1}$  is  $\mathbf{S}$ -reducible if and only if  $n+1 \equiv 0 \pmod{2^{\phi(n-m, 0)}}$  (cf. [6, (3.1)], [5, (2.7)] and [1]), we have Theorem 5.2. q. e. d.



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