Ann. Inst. Statist. Math.
Vol. 48, No. 4, 773-787 (1996)

SOONER AND LATER WAITING TIME PROBLEMS FOR SUCCESS AND FAILURE RUNS IN HIGHER ORDER MARKOV DEPENDENT TRIALS*

S. AKI[1], N. BALAKRISHNAN[2] AND S. G. MOHANTY[2]

[1] *Department of Mathematical Science, Osaka University, Toyonaka, Osaka 560, Japan*
[2] *Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1*

(Received May 19, 1995; revised November 13, 1995)

Abstract. The probability generating functions of the waiting times for the first success run of length k and for the sooner run and the later run between a success run of length k and a failure run of length r in the second order Markov dependent trials are derived using the probability generating function method and the combinatorial method. Further, the systems of equations of 2^m conditional probability generating functions of the waiting times in the m-th order Markov dependent trials are given. Since the systems of equations are linear with respect to the conditional probability generating functions, they can be solved exactly, and hence the probability generating functions of the waiting time distributions are obtained. If m is large, some computer algebra systems are available to solve the linear systems of equations.

Key words and phrases: Probability generating function, discrete distributions, sooner and later problem, success and failure runs, Markov chain, waiting time.

1. Introduction

Exact distributions on runs in independent trials have been interested since De Moivre's works (cf. Johnson *et al.* ((1992), pp. 426–432)). Recently, distribution theory on runs has been developed by many authors (Rajarshi (1974), Schwager (1983), Philippou and Muwafi (1982), Philippou *et al.* (1983), Philippou (1986), Hirano (1986), Philippou and Makri (1986), Hirano and Aki (1993), Godbole (1993), Godbole and Papastavridis (1994), and Mohanty (1994)). Since Ebneshahrashoob and Sobel (1990) solved a sooner and later problem for success and failure runs, various extensions have been given (cf. Aki (1992), Balasubramanian *et al.* (1993), Aki and Hirano (1993), and Viveros *et al.* (1994)).

* This research was partially supported by the Natural Sciences and Engineering Research Council of Canada.

More generally, waiting times of appearance of a pattern among a given set of patterns have been investigated (Chrysaphinou and Papastavridis (1990) and Chrysaphinou *et al.* (1994)). Fu and Koutras (1994) have recently presented a simple unified approach for the distribution theory of runs based on a finite Markov chain imbedding technique.

As applications of the distributions of runs, we can mention start-up demonstration tests and reliability of consecutive-k-out-of-n:F systems (cf. Hahn and Gage (1983) and Derman *et al.* (1982)). In these practical situations, Markov dependence models are considered (cf. Viveros *et al.* (1994), Viveros and Balakrishnan (1993), Fu (1986), Lambiris and Papastavridis (1987) and Papastavridis and Lambiris (1987)). Sooner and later waiting times between a success run and a failure run of specified length respectively have meanings in practical situations. For example, we consider a start-up demonstration test such that a purchaser of power generation equipment requires k consecutive successful start-ups for each delivered unit to accept the unit and he rejects it if r consecutive failure start-ups occur. Then the number of attempted start-ups for a delivered unit becomes the sooner waiting time between a success run of length k and a failure run of length r. The distribution of the number of start-ups is main concern for researchers of a start-up demonstration test and it is very important.

In this paper, we deal with the waiting time problems for the first consecutive successes of a specified length and for the sooner and later runs between a success run and a failure run with specified length respectively. There are two standard approaches to these problems. One is to solve a system of equations of conditional probability generating functions. Then, some characteristics such as probability function and moments are derived from an expansion of the solution. The other is to give a typical sequence. And, by splitting it into subsequences which can be interpreted, we obtain the total characteristics of the distribution. By the former approach, Ebneshahrashoob and Sobel (1990) derived the p.g.f. of the sooner and later waiting time problems. Aki and Hirano (1993) studied the problem in the first order Markov dependent trials. By the latter approach, Balasubramanian *et al.* (1993) solved the sooner and later problems in the first order Markov dependent trials and Mohanty (1994) investigated the waiting time for success runs of length k in Markov dependent trials and some generalizations. Viveros *et al.* (1994) derived the binomial and negative binomial analogues under correlated Bernoulli trials.

The purpose of this paper is to unify various approaches which have been attempted and to extend the study of waiting time problems from the first order Markov dependent trials to the second order Markov dependent trials. We also give a method to handle higher order Markov dependent trials.

In Section 2 we study the distributions of the waiting time for the first success run of a specified length. The probability generating function (p.g.f.)'s are also given for the distributions of the sooner and later waiting times. In Section 3 we give systems of equations of the p.g.f.'s of the conditional distributions of the waiting times. Fortunately, they are linear with respect to the conditional p.g.f.'s and hence they can be solved, though some of them are rather messy. And since the solutions are rational functions, characteristics such as recurrence relations of probabilities and moments are derived by standard methods (cf. e.g. Stanley

((1986), Chapter 4)).

2. Waiting time problems in the second order Markov chain

Let $X_{-1}, X_0, X_1, X_2, \ldots$ be a sequence of $\{0,1\}$-valued second order Markov chain with the following probabilities: for $x, y = 0, 1$ and $i = 1, 2, \ldots$, $\pi_{xy} = P(X_{-1} = x, X_0 = y)$,

$$p_{xy} = P(X_i = 1 \mid X_{i-1} = y, X_{i-2} = x)$$

and

$$q_{xy} = 1 - p_{xy} = P(X_i = 0 \mid X_{i-1} = y, X_{i-2} = x).$$

For $x, y = 0, 1$, we assume that $0 < p_{xy}, q_{xy} < 1$. We also call X_n the n-th trial and we say success and failure for the outcomes "1" and "0", respectively. The outcomes "1" and "0" are often denoted by S and F, respectively. Let k be a given positive integer greater than 2.

First, we derive the p.g.f. of the distribution of the waiting time for the first "1"-run of length k in $X_{-1}, X_0, X_1, X_2, \ldots$. Let $\phi^{(x,y)}(t)$ be the p.g.f. of the conditional distribution of the waiting time for the first "1"-run of length k in $X_{-1}, X_0, X_1, X_2, \ldots$ given that $X_{-1} = x$ and $X_0 = y$. Suppose we have currently "1"-run of length i. Then, we denote by $\phi_i(t)$ the p.g.f. of the conditional distribution of the waiting time from this time for the first "1"-run of length k. From the definition, we see that $\phi_1(t) = \phi^{(0,1)}(t)$ and $\phi_2(t) = \phi^{(1,1)}(t)$.

Remark 2.1. We note that these distributions of waiting times are proper distributions. Let $p = \min\{p_{xy}, x, y = 0, 1\}$ and let $Y_1, Y_2, \ldots$ be a sequence of independent $\{0,1\}$-valued random variables with success probability p. From the assumption, $0 < p < 1$ holds. For $i = 1, 2, \ldots$, let $A_i = \{Y_{(i-1)k+1} = 1, Y_{(i-1)k+2} = 1, \ldots, Y_{ik} = 1\}$. From the definition, the events $\{A_i\}_{i=1}^{\infty}$ are independent and $\sum_{i=1}^{\infty} P(A_i) = \infty$, since for every i, $P(A_i) = p^k > 0$. Then, from the Borel-Cantelli lemma, $P(A_i$ occurs infinitely often$) = 1$. But,

$$\begin{aligned} &P(\text{the waiting time for the first "1"-run of length } k \text{ in } X_{-1}, X_0, X_1, X_2, \ldots < \infty) \\ &\quad \geq P(\text{the waiting time for the first "1"-run of length } k \text{ in } Y_1, Y_2, \ldots < \infty) \\ &\quad \geq P(A_i \text{occurs infinitely often}) = 1. \end{aligned}$$

This proves the assertion.

By considering the condition of one-step ahead from every condition, we have the following system of equations of conditional p.g.f.'s

$$\phi^{(0,0)} = p_{00} t \phi_1 + q_{00} t \phi^{(0,0)} \tag{2.1}$$

$$\begin{cases} \phi_1 = p_{01} t \phi_2 + q_{01} t \phi^{(1,0)} \\ \phi_2 = p_{11} t \phi_3 + q_{11} t \phi^{(1,0)} \\ \cdots \\ \phi_{k-1} = p_{11} t \cdot 1 + q_{11} t \phi^{(1,0)} \end{cases} \tag{2.2}$$

$$\phi^{(1,0)} = p_{10} t \phi_1 + q_{10} t \phi^{(0,0)}. \tag{2.3}$$

From (2.2), we have

$$\phi_1 = \left(q_{01}t + \frac{1-(p_{11}t)^{k-2}}{1-p_{11}t} \cdot p_{01}q_{11}t^2 \right) \phi^{(1,0)} + p_{01}{p_{11}}^{k-2}t^{k-1}. \tag{2.4}$$

From (2.4), (2.1) and (2.3) we obtain

$$\phi^{(0,1)} = \phi_1 = \frac{p_{01}{p_{11}}^{k-2}t^{k-1}}{1 - \left(p_{10}t + \frac{p_{00}q_{10}t^2}{1-q_{00}t} \right) \left(q_{01}t + \frac{1-(p_{11}t)^{k-2}}{1-p_{11}t} \cdot p_{01}q_{11}t^2 \right)}. \tag{2.5}$$

Then, $\phi^{(0,0)}$, $\phi^{(1,0)}$ and $\phi^{(1,1)}$ are given as

$$\phi^{(0,0)} = \frac{p_{01}{p_{11}}^{k-2}t^{k-1} \frac{p_{00}t}{1-q_{00}t}}{1 - \left(p_{10}t + \frac{p_{00}q_{10}t^2}{1-q_{00}t} \right) \left(q_{01}t + \frac{1-(p_{11}t)^{k-2}}{1-p_{11}t} \cdot p_{01}q_{11}t^2 \right)}, \tag{2.6}$$

$$\phi^{(1,0)} = \frac{p_{01}{p_{11}}^{k-2}t^{k-1} \left(p_{10}t + \frac{p_{00}q_{10}t^2}{1-q_{00}t} \right)}{1 - \left(p_{10}t + \frac{p_{00}q_{10}t^2}{1-q_{00}t} \right) \left(q_{01}t + \frac{1-(p_{11}t)^{k-2}}{1-p_{11}t} \cdot p_{01}q_{11}t^2 \right)}, \tag{2.7}$$

and

$$\phi^{(1,1)} = \phi_2 = \frac{{p_{11}}^{k-2}t^{k-2} \left(1 - \left(p_{10}q_{01}t^2 + \frac{q_{01}p_{00}q_{10}t^3}{1-q_{00}t} \right) \right)}{1 - \left(p_{10}t + \frac{p_{00}q_{10}t^2}{1-q_{00}t} \right) \left(q_{01}t + \frac{1-(p_{11}t)^{k-2}}{1-p_{11}t} \cdot p_{01}q_{11}t^2 \right)}, \tag{2.8}$$

where the last formula is derived from $\phi_1 = p_{01}t\phi_2 + q_{01}t\phi^{(1,0)}$.

Hence, we have

THEOREM 2.1. *The p.g.f. of the distribution of the waiting time for the first "1"-run of length k in $X_{-1}, X_0, X_1, X_2, \ldots$ is given as*

$$\pi_{00}\phi^{(0,0)}(t) + \pi_{01}\phi^{(0,1)}(t) + \pi_{10}\phi^{(1,0)}(t) + \pi_{11}\phi^{(1,1)}(t),$$

where $\phi^{(0,0)}$, $\phi^{(1,0)}$, $\phi^{(0,1)}$ and $\phi^{(1,1)}$ are given in (2.6), (2.7), (2.5) *and* (2.8), *respectively.*

Remark 2.2. (i) Expressions (2.5), (2.6), (2.7) and (2.8) for conditional p.g.f.'s can be alternatively derived by two different but well known approaches.

In the first approach we consider the outcomes at the beginning of a sequence and do not introduce ϕ_i. For $i = 0, 1, \ldots, k-1$, let us call the subsequence $\overbrace{S \cdots S}^{i} F$ the i-th type of subsequence and subsequence $\overbrace{S \cdots S}^{k}$ the k-th type of sequence.

Divide sequences in every ϕ into those starting with the i-th type of subsequence ($i = 0, 1, \ldots, k$). Then, we have the following equations:

$$\begin{aligned}
\phi^{(0,0)} &= q_{00}t\phi^{(0,0)} + p_{00}q_{01}t^2\phi^{(1,0)} + p_{00}p_{01}q_{11}t^3\phi^{(1,0)} + p_{00}p_{01}p_{11}q_{11}t^4\phi^{(1,0)} \\
&\quad + \cdots + p_{00}p_{01}{p_{11}}^{k-3}q_{11}t^k\phi^{(1,0)} + p_{00}p_{01}{p_{11}}^{k-2}t^k, \\
\phi^{(0,1)} &= q_{01}t\phi^{(1,0)} + p_{01}q_{11}t^2\phi^{(1,0)} + p_{01}p_{11}q_{11}t^3\phi^{(1,0)} + p_{01}p_{11}p_{11}q_{11}t^4\phi^{(1,0)} \\
&\quad + \cdots + p_{01}{p_{11}}^{k-3}q_{11}t^{k-1}\phi^{(1,0)} + p_{01}{p_{11}}^{k-2}t^{k-1}, \\
\phi^{(1,0)} &= q_{10}t\phi^{(0,0)} + p_{10}q_{01}t^2\phi^{(1,0)} + p_{10}p_{01}q_{11}t^3\phi^{(1,0)} + p_{10}p_{01}p_{11}q_{11}t^4\phi^{(1,0)} \\
&\quad + \cdots + p_{10}p_{01}{p_{11}}^{k-3}q_{11}t^k\phi^{(1,0)} + p_{10}p_{01}{p_{11}}^{k-2}t^k,
\end{aligned}$$

and

$$\begin{aligned}
\phi^{(1,1)} &= q_{11}t\phi^{(1,0)} + p_{11}q_{11}t^2\phi^{(1,0)} + p_{11}p_{11}q_{11}t^3\phi^{(1,0)} + p_{11}p_{11}p_{11}q_{11}t^4\phi^{(1,0)} \\
&\quad + \cdots + {p_{11}}^{k-3}q_{11}t^{k-2}\phi^{(1,0)} + {p_{11}}^{k-2}t^{k-2}.
\end{aligned}$$

By solving these equations, we obtain directly (2.5), (2.6), (2.7) and (2.8).

Another approach is to examine the structure of sequences in order to derive the p.g.f. A typical sequence in each of $\phi^{(0,1)}$, $\phi^{(0,0)}$, $\phi^{(1,0)}$ and $\phi^{(1,1)}$ is given below.

$$\phi^{(0,1)} : \underbrace{FS}_{\text{initial}} \mid \underbrace{\overbrace{S\cdots S}^{\le k-2}F \mid \overbrace{F\cdots F}^{\ge 0}S \mid}_{\text{repeat}\ge 0} \overbrace{SS\cdots S}^{k-1},$$

$$\phi^{(0,0)} : \underbrace{FF}_{\text{initial}} \mid \overbrace{F\cdots F}^{\ge 0}S \mid \underbrace{\overbrace{S\cdots S}^{\le k-2}F \mid \overbrace{F\cdots F}^{\ge 0}S \mid}_{\text{repeat}\ge 0} \overbrace{SS\cdots S}^{k-1},$$

$$\phi^{(1,0)} : \underbrace{SF}_{\text{initial}} \mid \overbrace{F\cdots F}^{\ge 0}S \mid \underbrace{\overbrace{S\cdots S}^{\le k-2}F \mid \overbrace{F\cdots F}^{\ge 0}S \mid}_{\text{repeat}\ge 0} \overbrace{SS\cdots S}^{k-1},$$

$$\phi^{(1,1)} : \begin{cases} \underbrace{SS}_{\text{initial}} \mid \overbrace{S\cdots S}^{k-2} \\[2ex] \underbrace{SS}_{\text{initial}} \mid \overbrace{S\cdots S}^{<k-2}\overbrace{F\cdots F}^{\ge 1}S \mid \underbrace{\overbrace{S\cdots S}^{\le k-2}F \mid \overbrace{F\cdots F}^{\ge 0}S \mid}_{\text{repeat}\ge 0} \overbrace{SS\cdots S}^{k-1}. \end{cases}$$

From these structures we can derive the p.g.f.'s. As an example, we do so for $\phi^{(0,0)}$ and have

$$\begin{aligned}
\phi^{(0,0)} &= (1 + q_{00}t + {q_{00}}^2t^2 + \cdots)p_{00}t \\
&\quad \cdot \left[\sum_{m=0}^{\infty} \{(q_{01}t + p_{01}q_{11}t^2 + p_{01}p_{11}q_{11}t^3 + \cdots + p_{01}p_{11}^{k-3}q_{11}t^{k-1}) \right. \\
&\qquad\qquad \left. \times (p_{10}t + q_{10}p_{00}t^2 + q_{10}q_{00}p_{00}t^3 + q_{10}{q_{00}}^2p_{00}t^4 + \cdots)\}^m \right]
\end{aligned}$$

$$
\begin{aligned}
&\times p_{01}{p_{11}}^{k-2}t^{k-1} \\
&= \frac{p_{00}t}{1-q_{00}t} \\
&\quad\times \frac{1}{1-\left(q_{01}t+p_{01}q_{11}t^2\left(\frac{1-(p_{11}t)^{k-2}}{1-p_{11}t}\right)\right)\left(p_{10}t+q_{10}p_{00}t^2\frac{1}{1-q_{00}t}\right)} \\
&\quad\times p_{01}{p_{11}}^{k-2}t^{k-1},
\end{aligned}
$$

which is the same as (2.6).

(ii) By expanding the p.g.f.'s we will be able to give an expression for the conditional distribution of the length of the waiting time. For instance consider $\phi^{(0,1)}$ and let X represent the length of the waiting time excluding the two initial observations. Then, we can obtain

$$
\begin{aligned}
P(X=x) = p_{01}{p_{11}}^{k-2} &\sum_{\{(x_1,x_2):x_1+x_2+k-1=x\}} \\
&\times\left[\sum_{\{(n_1,\dots,n_{k-1}):n_1+2n_2+\cdots+(k-1)n_{k-1}=x_1\}}\binom{n_1+\cdots+n_{k-1}}{n_1,\dots,n_{k-1}}\right. \\
&\qquad\left.\times {q_{01}}^{n_1}(p_{01}q_{11})^{n_2+\cdots+n_{k-1}}{p_{11}}^{n_3+2n_4+\cdots+(k-3)n_{k-1}}\right] \\
&\times\left[\sum_{\{(m_1,m_2,\dots):m_1+2m_2+\cdots=x_2\}}\binom{m_1+m_2+\cdots}{m_1,m_2,\dots}\right. \\
&\qquad\left.\times {p_{10}}^{m_1}(p_{00}q_{10})^{m_2+m_3+\cdots}{q_{00}}^{m_3+2m_4+\cdots}\right], \\
&\qquad\qquad x=k-1,k,\dots.
\end{aligned}
$$

Note that $n_1+\cdots+n_{k-1}=m_1+m_2+\cdots$. Denote by W, Y, N, M the number of S's, SF's, FSF's and SFS's respectively in a sequence (that includes the initial FS). Then, it is not difficult to derive from the above that

$$
\begin{aligned}
&P(X=x,Y=y,W=w,M=m,N=n) \\
&\quad= p_{01}{p_{11}}^{k-2}{q_{01}}^{n}(p_{01}q_{11})^{y-n}{p_{11}}^{w-k-2y+n}{p_{10}}^{m}(q_{10}p_{00})^{y-m}{q_{00}}^{x-w-2y+m+1} \\
&\quad\times\left[\sum_A\binom{y}{n_1,\dots,n_{k-1}}\right]\left[\sum_B\binom{y}{m_1,m_2,\dots}\right],
\end{aligned}
$$

where

$$A=\left\{(n_1,\dots,n_{k-1}):\sum n_j=y,\sum jn_j=w-k,n_1=n\right\}$$

and

$$B=\left\{(m_1,m_2,\dots):\sum m_j=y,\sum jm_j=x-w+1,m_1=m\right\}.$$

A combinatorial proof: We provide an alternative combinatorial proof. In the typical sequence of $\phi^{(0,1)}$ let the repeat portion be called a subsequence. In a subsequence call the portion $S\cdots SF$ the first pattern and the last portion $F\cdots FS$ the second pattern. Let n_{i+1} represent the number of subsequences of the i-th type $\overbrace{S\cdots S}^{i}F$ $(i=0,1,\ldots,k-2)$ in the first pattern and m_{j+1} represent the number of subsequences of the j-th type $\overbrace{F\cdots F}^{j}S$ $(j=0,1,\ldots)$ in the second pattern. We observe the following:

(a) Any sequence is an arrangement of n_{i+1} subsequences of the i-th type, $i=0,1,\ldots,k-2$, of the first pattern and an arrangement of m_{j+1} subsequences of the j-th type, $j=0,1,\ldots$, of the second pattern and is finally followed by $k-1$ S's.

(b) The number of repetitions of a subsequence is $n_1+\cdots+n_{k-1}$ which is also $m_1+m_2+\cdots$. Also each repetition creates exactly one SF in a sequence. Therefore $y=n_1+\cdots+n_{k-1}=m_1+m_2+\cdots$.

(c) FSF in a sequence $\Longleftrightarrow$ F in the first pattern (without S). SFS in a sequence $\Longleftrightarrow$ S in the second pattern (without F). SF in a sequence $\Longleftrightarrow$ Any subsequence of the i-th type $(i=0,1,\ldots,k-2)$ in the first pattern.

Thus $n_1=n$ and $m_1=m$.

(d) Every subsequence of the i-th type $(i=1,\ldots,k-2)$ in the first pattern contributes a factor $p_{01}q_{11}$ and every subsequence of the j-th type $(j=2,3,\ldots)$ in the second pattern contributes a factor $q_{10}p_{00}$.

In the probability expression, $p_{00}{p_{11}}^{k-2}$ comes from the last $k-1$ S's. From (b), (c) and (d) we get $q_{01}{}^{n}(p_{01}q_{11})^{y-n}$ and $p_{10}{}^{m}(q_{10}p_{00})^{y-m}$. Noting that the number of S's is w and the number of F's is $x-w$, the exponents of p_{11} and q_{00} are easily checked. The two multinomial coefficients come from the arrangements as described in (a). Finally, that $\sum jn_j$ and $\sum jm_j$ represent the number of S's and the number of F's in y subsequences gives rise to $\sum jn_j=w-k$ and $\sum jm_j=x-w-1$. This completes the proof.

Once we have an appreciation of some of the combinatorial structures of the problem, we do not intend to present these in further derivations.

(iii) In Theorem 2.1, we studied the waiting time for the first consecutive k successes in $X_{-1},X_0,X_1,X_2,\ldots$. However, if you want to know the waiting time for the first consecutive k successes in $X_1,X_2,X_3,\ldots$, you can modify Theorem 2.1 easily.

Let $\epsilon^{(x,y)}(t)$ be the p.g.f. of the conditional distribution of the waiting time for the first "1"-run of length k in $X_1,X_2,X_3,\ldots$ given that $X_{-1}=x$ and $X_0=y$. Let $\phi_1^*(t)$ be the p.g.f. of the conditional distribution of the waiting time from X_2 for the first "1"-run of length k in $X_1,X_2,X_3,\ldots$ given that $X_0=1$ and $X_1=1$.

It is easy to see that

$$\epsilon^{(0,0)}=\phi^{(0,0)},\qquad \epsilon^{(1,0)}=\phi^{(1,0)},$$

and

$$\epsilon^{(0,1)} = p_{01}t\phi_1^* + q_{01}t\phi^{(1,0)}, \qquad \epsilon^{(1,1)} = p_{11}t\phi_1^* + q_{11}t\phi^{(1,0)},$$
$$\phi_1^* = p_{11}t\phi^{(1,1)} + q_{11}t\phi^{(1,0)}.$$

Since $\phi^{(0,0)}$, $\phi^{(1,0)}$, $\phi^{(0,1)}$ and $\phi^{(1,1)}$ have already been given, then $\epsilon^{(0,0)}$, $\epsilon^{(1,0)}$, $\epsilon^{(0,1)}$ and $\epsilon^{(1,1)}$ can be solved easily. Therefore, we have

COROLLARY 2.1. *The p.g.f. of the distribution of the waiting time for the first "1"-run of length k in $X_1, X_2, X_3, \dots$ is given as*

$$\pi_{00}\epsilon^{(0,0)}(t) + \pi_{01}\epsilon^{(0,1)}(t) + \pi_{10}\epsilon^{(1,0)}(t) + \pi_{11}\epsilon^{(1,1)}(t).$$

Next, we consider the sooner waiting time problem. Let r be a given positive integer greater than 2. We denote by E_1 a "1"-run of length k and by E_0 a "0"-run of length r. We derive the p.g.f. of the waiting time for the sooner run between E_1 and E_0. The sooner and later waiting time problems were investigated by Ebneshahrashoob and Sobel (1990) in independent trials and were studied by Balasubramanian *et al.* (1993) and Aki and Hirano (1993) in the first order Markov chain.

Let $\xi^{(x,y)}(t)$ be the p.g.f. of the conditional distribution of the waiting time for the sooner run between E_1 and E_0 in $X_{-1}, X_0, X_1, X_2, \dots$ given that $X_{-1} = x$ and $X_0 = y$. Suppose we have currently "1"-run of length i. Then $\xi_i(t)$ denotes the p.g.f. of the conditional distribution of the waiting time from this time for the sooner run. Suppose we have currently "0"-run of length j. Then $\eta_j(t)$ denotes the p.g.f. of the conditional distribution of the waiting time from this time for the sooner run. From the definition, we see that $\xi_1 = \xi^{(0,1)}$, $\xi_2 = \xi^{(1,1)}$, $\eta_1 = \xi^{(1,0)}$, $\eta_2 = \xi^{(0,0)}$.

THEOREM 2.2. *The p.g.f. of the distribution of the waiting time for the sooner run between E_1 and E_0 in $X_{-1}, X_0, X_1, X_2, \dots$ is given as*

$$\pi_{00}\xi^{(0,0)}(t) + \pi_{01}\xi^{(0,1)}(t) + \pi_{10}\xi^{(1,0)}(t) + \pi_{11}\xi^{(1,1)}(t),$$

where

$$\begin{aligned}
\xi^{(0,1)} &= \xi_1 \\
&= \frac{\left(q_{01}t + \dfrac{1-(p_{11}t)^{k-2}}{1-p_{11}t}p_{01}q_{11}t^2\right) q_{10}{q_{00}}^{r-2}t^{r-1} + p_{01}{p_{11}}^{k-2}t^{k-1}}{1 - \left(q_{01}t + \dfrac{1-(p_{11}t)^{k-2}}{1-p_{11}t}p_{01}q_{11}t^2\right)\left(p_{10}t + q_{10}t \cdot \dfrac{(p_{00}t)(1-(q_{00}t)^{r-2})}{1-q_{00}t}\right)}, \\
\xi^{(0,0)} &= \eta_2 = \frac{(p_{00}t)(1-(q_{00}t)^{r-2})}{1-q_{00}t} \cdot \xi_1 + (q_{00}t)^{r-2}, \\
\xi^{(1,0)} &= \eta_1 = \left(p_{10}t + q_{10}t \cdot \frac{(p_{00}t)(1-(q_{00}t)^{r-2})}{1-q_{00}t}\right)\xi_1 + q_{10}{q_{00}}^{r-2}t^{r-1},
\end{aligned}$$

and

$$\xi^{(1,1)} = \xi_2 = \frac{q_{11}t(1-(p_{11}t)^{k-2})}{1-p_{11}t} \cdot \eta_1 + (p_{11}t)^{k-2}.$$

Since it is not so difficult to prove the theorem by solving the corresponding system of 4 equations given in Proposition 3.2 in Section 3, we omit the proof.

Similarly as Corollary 2.1, we consider the waiting time corresponding to the sequence $X_1, X_2, \dots$.

Let $\delta^{(x,y)}(t)$ be the p.g.f. of the conditional distribution of the waiting time for the sooner run in $X_1, X_2, \dots$ given that $X_{-1} = x$ and $X_0 = y$. Then we have

COROLLARY 2.2. *The p.g.f. of the distribution of the waiting time for the sooner run in $X_1, X_2, X_3, \dots$ is given as*

$$\pi_{00}\delta^{(0,0)}(t) + \pi_{01}\delta^{(0,1)}(t) + \pi_{10}\delta^{(1,0)}(t) + \pi_{11}\delta^{(1,1)}(t),$$

where

$$\delta^{(0,0)} = p_{00}t\xi^{(0,1)} + q_{00}t\eta_1^*, \qquad \delta^{(1,1)} = p_{11}t\xi_1^* + q_{11}t\xi^{(1,0)}$$
$$\delta^{(1,0)} = p_{10}t\xi^{(0,1)} + q_{10}t\eta_1^*, \qquad \delta^{(0,1)} = p_{01}t\xi_1^* + q_{01}t\xi^{(1,0)}$$

and

$$\xi_1^* = p_{11}t\xi^{(1,1)} + q_{11}t\xi^{(1,0)}, \qquad \eta_1^* = p_{00}t\xi^{(0,1)} + q_{00}t\xi^{(0,0)}.$$

In the last of this section, we study the later waiting time problem. Let $\psi^{(x,y)}(t)$ be the p.g.f. of the conditional distribution of the later run between E_1 and E_0 in $X_{-1}, X_0, X_1, X_2, \dots$ given that $X_{-1} = x$ and $X_0 = y$. Suppose we have currently "1"-run of length i. And the sooner run has not yet occurred. Then we denote by ψ_i the p.g.f. of the conditional distribution of the waiting time from this time for the later run. Suppose we have currently "0"-run of length j. And the sooner run has not yet occurred. Then we denote by ω_j the p.g.f. of the conditional distribution of the waiting time from this time for the later run. Note that $\psi_1 = \psi^{(0,1)}$, $\psi_2 = \psi^{(1,1)}$, $\omega_1 = \psi^{(1,0)}$ and $\omega_2 = \psi^{(0,0)}$.

Before giving the recurrence relations for ψ's, we need some preparations. Let $\alpha^{(x,y)}(t)$ be the conditional distribution of the waiting time for the first "0"-run of length r in $X_{-1}, X_0, X_1, X_2, \dots$ given that $X_{-1} = x$ and $X_0 = y$. Suppose we have currently "0"-run of length j. Then we denote by α_j the p.g.f. of the conditional distribution of the waiting time from this time for the first "0"-run of length r. By changing the roles of "1" and "0" in Theorem 2.1, we obtain

LEMMA 2.1. *The conditional p.g.f.'s are given as*

$$\alpha^{(0,0)} = \frac{{q_{00}}^{r-2}t^{r-2}\left(1-\left(q_{01}t+\dfrac{q_{11}p_{01}t^2}{1-p_{11}t}\right)\right)}{1-\left(q_{01}t+\dfrac{q_{11}p_{01}t^2}{1-p_{11}t}\right)\left(p_{10}t+\dfrac{1-(q_{00}t)^{r-2}}{1-q_{00}t}\cdot q_{10}p_{00}t^2\right)},$$

$$\alpha^{(1,0)} = \frac{q_{10}{q_{00}}^{r-2}t^{r-1}}{1-\left(q_{01}t+\dfrac{q_{11}p_{01}t^2}{1-p_{11}t}\right)\left(p_{10}t+\dfrac{1-(q_{00}t)^{r-2}}{1-q_{00}t}\cdot q_{10}p_{00}t^2\right)},$$

$$\alpha^{(0,1)} = \frac{q_{10}{q_{00}}^{r-2}t^{r-1}\left(q_{01}t+\dfrac{q_{11}p_{01}t^2}{1-p_{11}t}\right)}{1-\left(q_{01}t+\dfrac{q_{11}p_{01}t^2}{1-p_{11}t}\right)\left(p_{10}t+\dfrac{1-(q_{00}t)^{r-2}}{1-q_{00}t}\cdot q_{10}p_{00}t^2\right)},$$

and

$$\alpha^{(1,1)} = \frac{q_{10}{q_{00}}^{r-2}t^{r-1}\dfrac{q_{11}t}{1-p_{11}t}}{1-\left(q_{01}t+\dfrac{q_{11}p_{01}t^2}{1-p_{11}t}\right)\left(p_{10}t+\dfrac{1-(q_{00}t)^{r-2}}{1-q_{00}t}\cdot q_{10}p_{00}t^2\right)}.$$

THEOREM 2.3. *The p.g.f.'s of the conditional distributions of the waiting time for the later run,* $\psi^{(0,0)}$, $\psi^{(0,1)}$, $\psi^{(1,0)}$, $\psi^{(1,1)}$, ψ_i, $i=3,4,\dots,k-1$, *and* ω_j, $j=3,4,\dots,r-1$ *satisfy the following system of equations*:

$$\psi^{(0,0)} = p_{00}t\psi_1 + q_{00}t\omega_3$$

$$\begin{cases} \psi_1 = p_{01}t\psi_2 + q_{01}t\omega_1 \\ \psi_2 = p_{11}t\psi_3 + q_{11}t\omega_1 \\ \cdots \\ \psi_{k-1} = p_{11}t\alpha^{(1,1)} + q_{11}t\omega_1 \end{cases} \qquad \begin{cases} \omega_1 = p_{10}t\psi_1 + q_{10}t\omega_2 \\ \omega_2 = p_{00}t\psi_1 + q_{00}t\omega_3 \\ \cdots \\ \omega_{r-1} = p_{00}t\psi_1 + q_{00}t\phi^{(0,0)} \end{cases}.$$

PROOF. Though the sooner waiting time stops when either E_0 or E_1 comes first, the later waiting time continues after the sooner run comes. And, after the sooner waiting time, we wait for the run which did not come sooner. For example, if E_1 comes first, we wait for E_0 after the sooner waiting time. Then, we can see the result clearly. This completes the proof.

Remark 2.3. In independent trials, Ebneshahrashoob and Sobel (1990) introduced generalized p.g.f. with marker and showed the relationship between the sooner waiting time and the later waiting time. Theorem 2.3 is an extension of their result to a case of dependent trials.

We can solve the above equations by using Theorem 2.2, Lemma 2.1 and Theorem 2.1. We have

$$\psi^{(0,1)} = \psi_1$$
$$= \frac{\left(q_{01}t+\dfrac{1-(p_{11}t)^{k-2}}{1-p_{11}t}p_{01}q_{11}t^2\right)q_{10}{q_{00}}^{r-2}t^{r-1}\cdot\phi^{(0,0)} + p_{01}{p_{11}}^{k-2}t^{k-1}\cdot\alpha^{(1,1)}}{1-\left(q_{01}t+\dfrac{1-(p_{11}t)^{k-2}}{1-p_{11}t}p_{01}q_{11}t^2\right)\left(p_{10}t+q_{10}t\cdot\dfrac{(p_{00}t)(1-(q_{00}t)^{r-2})}{1-q_{00}t}\right)},$$

where

$$\phi^{(0,0)} = \frac{p_{01}{p_{11}}^{k-2}t^{k-1}\dfrac{p_{00}t}{1-q_{00}t}}{1-\left(p_{10}t+\dfrac{p_{00}q_{10}t^2}{1-q_{00}t}\right)\left(q_{01}t+\dfrac{1-(p_{11}t)^{k-2}}{1-p_{11}}\cdot p_{01}q_{11}t^2\right)},$$

and

$$\alpha^{(1,1)} = \frac{q_{10}{q_{00}}^{r-2}t^{r-1}\dfrac{q_{11}t}{1-p_{11}t}}{1-\left(q_{01}t+\dfrac{q_{11}p_{01}t^2}{1-p_{11}t}\right)\left(p_{10}t+\dfrac{1-(q_{00}t)^{r-2}}{1-q_{00}t}\cdot q_{10}p_{00}t^2\right)}.$$

The other ψ's are easily derived since they are liner function of ψ_1.

3. Waiting time problems in the higher order Markov chain

As we have seen in the previous section, final expressions of the p.g.f.'s of the distributions of the waiting times are rather messy even in the case of the second order dependency. As for the case of more higher order Markov chain (say the case of the m-th order Markov chain), we can give the systems of equations of the conditional p.g.f.'s of the waiting times. Fortunately, the systems are linear with respect to the conditional p.g.f.'s, and hence we can solve them. When m is large, number of linear equations become very large (2^m), so we recommend the use of computer algebra systems to obtain the final expressions of the conditional p.g.f.'s. The usages of computer algerbra is quite simple and straightforward.

Let m be a positive integer less than $\min(k,r)$. This assumption is only to avoid the case that the waiting time becomes negative.

Let $X_{-m+1}, X_{-m+2}, \ldots, X_0, X_1, X_2, \ldots$ be $\{0,1\}$-valued m-th order Markov chain with

$$\begin{aligned}
\pi_{x_1,\ldots,x_m} &= P(X_{-m+1}=x_1, X_{-m+2}=x_2, \ldots, X_0=x_m),\\
p_{x_1,\ldots,x_m} &= P(X_i=1 \mid X_{i-1}=x_m, X_{i-2}=x_{m-1}, \ldots, X_{i-m}=x_1),\\
q_{x_1,\ldots,x_m} &= 1-p_{x_1,\ldots,x_m},
\end{aligned}$$

for $x_1,\ldots,x_m = 0,1$ and $i=1,2,\ldots$. For $x_1,\ldots,x_m=0,1$, we assume that $0<p_{x_1,\ldots,x_m}, q_{x_1,\ldots,x_m}<1$.

Let $\phi^{(x_1,\ldots,x_m)}$ be the p.g.f. of the conditional distribution of the waiting time for the first "1"-run of length k given that $X_{-m+1}=x_1, X_{-m+2}=x_2, \ldots, X_0=x_m$. From similar arguments in Remark 2.1, we see that the distribution of the waiting time is proper distribution.

Similarly as the proof of Theorem 2.1, we have

PROPOSITION 3.1. *The p.g.f.'s of the conditional distributions of the waiting time for the first "1"-run of length k satisfy the following linear system of equations*:

$$\begin{cases}
\phi^{(x_1,\ldots,x_m)} = p_{x_1,\ldots,x_m}t\phi^{(x_2,\ldots,x_m,1)} + q_{x_1,\ldots,x_m}t\phi^{(x_2,\ldots,x_m,0)},\\
\qquad\qquad \textit{if } (x_1,\ldots,x_m)\neq(1,1,\ldots,1),\\
\phi^{(1,\ldots,1)} = A(t)\phi^{(1,\ldots,1,0)} + (p_{1,\ldots,1}t)^{k-m},
\end{cases}$$

where $A(t) = q_{1,\ldots,1}t(1-(p_{1,\ldots,1}t)^{k-m})/(1-p_{1,\ldots,1}t)$.

Remark 3.1. When m is small, the explicit solution of the system of equations given in Proposition 3.1 can be written. Indeed, the solution for $m = 3$ is not so messy.

Let $\alpha^{(x_1,\ldots,x_m)}$ be the p.g.f. of the conditional distribution of the waiting time for the first "0"-run of length r given that $X_{-m+1} = x_1, X_{-m+2} = x_2, \ldots, X_0 = x_m$. By changing the roles of "1" and "0" in Proposition 3.1, we have

COROLLARY 3.1. *The p.g.f.'s of the conditional distributions of the waiting time for the first "0"-run of length r satisfy the following linear system of equations*:

$$\begin{cases} \alpha^{(x_1,\ldots,x_m)} = p_{x_1,\ldots,x_m}t\alpha^{(x_2,\ldots,x_m,1)} + q_{x_1,\ldots,x_m}t\alpha^{(x_2,\ldots,x_m,0)}, \\ \qquad\qquad \textit{if } (x_1,\ldots,x_m) \neq (0,0,\ldots,0), \\ \alpha^{(0,\ldots,0)} = B(t)\alpha^{(0,\ldots,0,1)} + (q_{0,\ldots,0}t)^{r-m}, \end{cases}$$

where $B(t) = p_{0,\ldots,0}t(1-(q_{0,\ldots,0}t)^{r-m})/(1-q_{0,\ldots,0}t)$.

Next, we study the sooner waiting time problem. As in the previous section, E_1 and E_0 denote a "1"-run of length k and "0"-run of length r, respectively. Let $\xi^{(x_1,\ldots,x_m)}(t)$ be the p.g.f. of the conditional distribution of the waiting time for the sooner run between E_1 and E_0 in $X_{-m+1}, X_{-m+2}, \ldots, X_0, X_1, X_2, \ldots$ given that $X_{-m+1} = x_1, X_{-m+2} = x_2, \ldots, X_0 = x_m$.

PROPOSITION 3.2. *The conditional p.g.f.'s satisfy the following linear system of equations*:

$$\begin{cases} \xi^{(x_1,\ldots,x_m)} = p_{x_1,\ldots,x_m}t\xi^{(x_2,\ldots,x_m,1)} + q_{x_1,\ldots,x_m}t\xi^{(x_2,\ldots,x_m,0)}, \\ \qquad\qquad \textit{if } (x_1,\ldots,x_m) \neq (1,\ldots,1) \textit{ and } (x_1,\ldots,x_m) \neq (0,\ldots,0), \\ \xi^{(1,\ldots,1)} = A(t)\xi^{(1,\ldots,1,0)} + (p_{1,\ldots,1}t)^{k-m}, \\ \xi^{(0,\ldots,0)} = B(t)\xi^{(0,\ldots,0,1)} + (q_{0,\ldots,0}t)^{r-m}. \end{cases}$$

PROOF. Suppose that we have currently "1"-run of length i. $\xi_i(t)$ denotes the p.g.f. of the conditional distribution of the waiting time from this time for the sooner run. Suppose that we have currently "0"-run of length j. $\eta_j(t)$ denotes the p.g.f. of the conditional distribution of the waiting time from this time for the sooner run. From these definitions we see that for $i \le m$, $\xi_i = \xi^{(0,\ldots,0,\overbrace{1,\ldots,1}^{i})}$, and $\eta_i = \xi^{(1,\ldots,1,\overbrace{0,\ldots,0}^{i})}$. Then, by considering the condition of one-step ahead in time carefully, we see the following.

$$\begin{cases} \xi^{(x_1,\ldots,x_m)} = p_{x_1,\ldots,x_m}t\xi^{(x_2,\ldots,x_m,1)} + q_{x_1,\ldots,x_m}t\xi^{(x_2,\ldots,x_m,0)}, \\ \qquad\qquad \text{if } (x_1,\ldots,x_m) \neq (1,\ldots,1) \text{ and } (x_1,\ldots,x_m) \neq (0,\ldots,0), \end{cases}$$

$$(3.1)\quad \begin{cases} \xi^{(1,\dots,1)} = \xi_m = p_{1,\dots,1}t\xi_{m+1} + q_{1,\dots,1}t\xi^{(1,\dots,1,0)} \\ \xi_{m+1} = p_{1,\dots,1}t\xi_{m+2} + q_{1,\dots,1}t\xi^{(1,\dots,1,0)} \\ \dots \\ \xi_{k-1} = p_{1,\dots,1}t\cdot 1 + q_{1,\dots,1}t\xi^{(1,\dots,1,0)} \end{cases}$$

and

$$(3.2)\quad \begin{cases} \xi^{(0,\dots,0)} = \eta_m = q_{0,\dots,0}t\eta_{m+1} + p_{0,\dots,0}t\xi^{(0,\dots,0,1)} \\ \eta_{m+1} = q_{0,\dots,0}t\eta_{m+2} + p_{0,\dots,0}t\xi^{(0,\dots,0,1)} \\ \dots \\ \eta_{r-1} = q_{0,\dots,0}t\cdot 1 + p_{0,\dots,0}t\xi^{(0,\dots,0,1)} \end{cases}.$$

From (3.1) and (3.2), we have the desired result, which completes the proof.

Now, we study the later waiting time problem.

Let $\psi^{(x_1,\dots,x_m)}(t)$ be the p.g.f. of the conditional distribution of the waiting time for the later run between E_1 and E_0 in $X_{-m+1}, X_{-m+2}, \dots, X_0, X_1, X_2, \dots$ given that $X_{-m+1} = x_1, X_{-m+2} = x_2, \dots, X_0 = x_m$.

PROPOSITION 3.3. *The conditional p.g.f.'s satisfy the following linear system of equations*:

$$\begin{cases} \psi^{(x_1,\dots,x_m)} = p_{x_1,\dots,x_m}t\psi^{(x_2,\dots,x_m,1)} + q_{x_1,\dots,x_m}t\psi^{(x_2,\dots,x_m,0)}, \\ \qquad \textit{if } (x_1,\dots,x_m) \neq (1,\dots,1) \textit{ and } (x_1,\dots,x_m) \neq (0,\dots,0), \\ \psi^{(1,\dots,1)} = A(t)\xi^{(1,\dots,1,0)} + (p_{1,\dots,1}t)^{k-m}\alpha^{(1,\dots,1)}, \\ \psi^{(0,\dots,0)} = B(t)\xi^{(0,\dots,0,1)} + (q_{0,\dots,0}t)^{r-m}\phi^{(0,\dots,0)} \end{cases}.$$

PROOF. Suppose we have currently "1"-run of length i. And the sooner run has not yet occurred. Then, $\psi_i(t)$ denotes the p.g.f. of the conditional distribution of the waiting time from this time for the later run. Suppose we have currently "0"-run of length j. And the sooner run has not yet occurred. Then, $\omega_j(t)$ denotes the p.g.f. of the conditional distribution of the waiting time from this time for the later run. Note that for $i \leq m$, $\psi_i = \psi^{(0,\dots,0,\overbrace{1,\dots,1}^{i})}$, and $\omega_i = \psi^{(1,\dots,1,\overbrace{0,\dots,0}^{i})}$. Then, by considering the condition of one-step ahead in time carefully, we see the following.

$$\begin{cases} \psi^{(x_1,\dots,x_m)} = p_{x_1,\dots,x_m}t\psi^{(x_2,\dots,x_m,1)} + q_{x_1,\dots,x_m}t\psi^{(x_2,\dots,x_m,0)}, \\ \qquad \text{if } (x_1,\dots,x_m) \neq (1,\dots,1) \text{ and } (x_1,\dots,x_m) \neq (0,\dots,0), \end{cases}$$

$$(3.3)\quad \begin{cases} \psi^{(1,\dots,1)} = \psi_m = p_{1,\dots,1}t\psi_{m+1} + q_{1,\dots,1}t\psi^{(1,\dots,1,0)} \\ \psi_{m+1} = p_{1,\dots,1}t\psi_{m+2} + q_{1,\dots,1}t\psi^{(1,\dots,1,0)} \\ \dots \\ \psi_{k-1} = p_{1,\dots,1}t\cdot\alpha^{(1,\dots,1)} + q_{1,\dots,1}t\psi^{(1,\dots,1,0)} \end{cases}$$

and

$$
\left\{
\begin{array}{l}
\psi^{(0,\ldots,0)} = \omega_m = q_{0,\ldots,0}t\omega_{m+1} + p_{0,\ldots,0}t\psi^{(0,\ldots,0,1)} \\
\omega_{m+1} = q_{0,\ldots,0}t\omega_{m+2} + p_{0,\ldots,0}t\psi^{(0,\ldots,0,1)} \\
\cdots \\
\omega_{r-1} = q_{0,\ldots,0}t \cdot \phi^{(0,\ldots,0)} + p_{0,\ldots,0}t\psi^{(0,\ldots,0,1)}
\end{array}
\right. \tag{3.4}
$$

From (3.3) and (3.4), we obtain the desired result, which completes the proof.

References

Aki, S. (1992). Waiting time problems for a sequence of discrete random variables, *Ann. Inst. Statist. Math.*, **44**, 363–378.

Aki, S. and Hirano, K. (1993). Discrete distributions related to succession events in a two-state Markov chain, *Statistical Science & Data Analysis* (eds. K. Matusita, M. L. Puri and T. Hayakawa), 467–474, VSP Publishers, Amsterdam.

Balasubramanian, K., Viveros, R. and Balakrishnan, N. (1993). Sooner and later waiting time problem for Markovian Bernoulli trials, *Statist. Probab. Lett.*, **18**, 153–161.

Crysaphinou, O. and Papastavridis, S. (1990). The occurrence of sequence patterns in repeated dependent experiments, *Theory Probab. Appl.*, **35**, 145–152.

Crysaphinou, O., Papastavridis, S. and Tsapelas, T. (1994). On the waiting time appearance of given patterns, *Runs and Patterns in Probability: Selected Papers* (eds. A. P. Godbole and S. G. Papastavridis), 231–241, Kluwer, Dordrecht.

Derman, C., Lieberman, G. J. and Ross, S. M. (1982). On the consecutive-k-of-n:F system, *IEEE Transactions on Reliability*, **R-31**, 57–63.

Ebneshahrashoob, M. and Sobel, M. (1990). Sooner and later problems for Bernoulli trials: frequency and run quotas, *Statist. Probab. Lett.*, **9**, 5–11.

Fu, J. C. (1986). Reliability of consecutive-k-out-of-n:F systems with (k-1)-step Marcov dependence, *IEEE Transactions on Reliability*, **R-35**, 602–606.

Fu, J. C. and Koutras, M. V. (1994). Distribution theory of runs: A Markov chain approach, *J. Amer. Statist. Assoc.*, **89**, 1050–1058.

Godbole, A. P. (1993). Approximate reliabilities of m-consecutive-k-out-of-n:failure systems, *Statistica Sinica*, **3**, 321–327.

Godbole, A. P. and Papastavridis, S. G. (1994). *Runs and Patterns in Probability: Selected Papers*, Kluwer, Dordrecht.

Hahn, G. J. and Gage, J. B. (1983). Evaluation of a start-up demonstration test, *Journal of Quality Technology*, **15**, 103–106.

Hirano, K. (1986). Some properties of the distributions of order k, *Fibonacci Numbers and Their Applications* (eds. A. N. Philippou, G. E. Bergum and A. F. Horadam), 43–53, Reidel, Dordrecht.

Hirano, K. and Aki, S. (1993). On number of occurrences of success runs of specified length in a two-state Markov chain, *Statistica Sinica*, **3**, 313–320.

Johnson, N. L., Kotz, S. and Kemp, A. W. (1992). *Univariate Discrete Distributions*, Wiley, New York.

Lambiris, M. and Papastavridis, S. G. (1987). Reliability of a consecutive-k-out-of-n:F system for Markov dependent components, *IEEE Transactions on Reliability*, **R-36**, 78–79.

Mohanty, S. G. (1994). Success runs of length k in Markov dependent trials, *Ann. Inst. Statist. Math.*, **46**, 777–796.

Papastavridis, S. and Lambiris, M. (1987). Reliability of a consecutive-k-out-of-n:F system for Markov-dependent components, *IEEE Transactions on Reliability*, **R-36**, 78–79.

Philippou, A. N. (1986). Distributions and Fibonacci polynomials of order k, longest runs, and reliability of consecutive-k-out-of-n:F system, *Fibonacci Numbers and Their Applications* (eds. A. N. Philippou, G. E. Bergum and A. F. Horadam), 203–227, Reidel, Dordrecht.

Philippou, A. N. and Makri, F. S. (1986). Successes, runs, and longest runs, *Statist. Probab. Lett.*, **4**, 101–105.

Philippou, A. N. and Muwafi, A. A. (1982). Waiting for the k-th consecutive success and the Fibonacci sequence of order k, *Fibonacci Quart.*, **20**, 28–32.

Philippou, A. N., Georghiou, C. and Philippou, G. N. (1983). A generalized geometric distribution and some of its properties, *Statist. Probab. Lett.*, **1**, 171–175.

Rajarshi, M. B. (1974). Success runs in a two-state Markov chain, *J. Appl. Probab.*, **11**, 190–192.

Schwager, S. J. (1983). Run probability in sequences of Markov-dependent trials, *J. Amer. Statist. Assoc.*, **78**, 168–175.

Stanley, R. T. (1986). *Enumerative Combinatorics*, Wadsworth Publishers, Kentucky.

Viveros, R. and Balakrishnan, N. (1993). Statistical inference from start-up demonstration test data, *Journal of Quality Technology*, **25**, 119–130.

Viveros, R., Balasubramanian, K. and Balakrishnan, N. (1994). Binomial and negative binomial analogues under correlated Bernoulli trials, *Amer. Statist.*, **48**, 243–247.