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SOONER AND LATER WAITING TIME PROBLEMS IN A TWO-STATE MARKOV CHAIN

MASAYUKI UCHIDA* AND SIGEO AKI

Department of Mathematical Science, Faculty of Engineering Science, Osaka University, Machikaneyama-cho, Toyonaka, Osaka 560, Japan

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Abstract. Let $X_1, X_2, \dots$ be a time-homogeneous $\{0,1\}$-valued Markov chain. Let F_0 be the event that l runs of "0" of length r occur and let F_1 be the event that m runs of "1" of length k occur in the sequence $X_1, X_2, \dots$. We obtained the recurrence relations of the probability generating functions of the distributions of the waiting time for the sooner and later occurring events between F_0 and F_1 by the non-overlapping way of counting and overlapping way of counting. We also obtained the recurrence relations of the probability generating functions of the distributions of the sooner and later waiting time by the non-overlapping way of counting of "0"-runs of length r or more and "1"-runs of length k or more.

Key words and phrases: Sooner and later problems, discrete distribution, Markov chain, recurrence relations, overlapping, non-overlapping, non-overlapping "0"-runs of length r or more, generalized probability generating function.

1. Introduction

Since the introduction of the geometric distribution of order k by Philippou *et al.* (1983), discrete distributions of order k and their generalizations have received a great deal of attention. Some of the results were applied to problems on the reliability of the consecutive-k-out-of-n:F system by many authors (Koutras and Papastavridis (1993), Godbole (1993), Fu and Koutras (1994*a*) and references therein). The geometric distribution of order k is one of the simplest waiting time distributions. Several waiting time problems have been studied by many authors in more general situations (Ebneshahrashoob and Sobel (1990), Aki (1992), Aki and Hirano (1989, 1993), Balasubramanian *et al.* (1993), Fu and Koutras (1994*b*), Mohanty (1994) and references therein).

An interesting class of waiting time problems was proposed by Ebneshahrashoob and Sobel (1990). They obtained the probability generating

* Now at the Institute of Statistical Mathematics.

functions (p.g.f.) of the waiting time distributions for a run of "0" of length r or (and) a run of "1" of length k whichever comes sooner (later) when the sequence $X_1, X_2, \dots$ is constructed from Bernoulli trials, that is, X's are i.i.d. and $\{0,1\}$-valued random variables. Aki and Hirano (1993) obtained the p.g.f.s of the distributions of the sooner and later waiting time for the same event as Ebneshahrashoob and Sobel (1990) in the following Markov chain.

Let $X_0, X_1, X_2, \dots$ be a time-homogeneous $\{0,1\}$-valued Markov chain defined by $P(X_0 = 0) = p_0$, $P(X_0 = 1) = p_1$ and for $i = 0, 1, 2, \dots$, $P(X_{i+1} = 0 \mid X_i = 0) = p_{00}$, $P(X_{i+1} = 1 \mid X_i = 0) = p_{01}$, $P(X_{i+1} = 0 \mid X_i = 1) = p_{10}$ and $P(X_{i+1} = 1 \mid X_i = 1) = p_{11}$, where $0 < p_{00} < 1$, $0 < p_{11} < 1$.

Let F_0 be the event that l runs of "0" of length r occur and let F_1 be the event that m runs of "1" of length k occur in the sequence $X_1, X_2, \dots$. In this paper, we consider the recurrence relations of the p.g.f.s of the distributions of the sooner and later waiting time between F_0 and F_1 by the non-overlapping way of counting and by the overlapping way of counting in the Markov chain. We also obtain the recurrence relations of the p.g.f.s of the distributions of the sooner and later waiting time by the non-overlapping way of counting of "0"-runs of length r or more and "1"-runs of length k or more in the Markov chain.

In Section 2, we firstly obtain the recurrence relations of the p.g.f.s of the sooner waiting time distributions between F_0 and F_1 by the non-overlapping way of counting (in the sense of Feller (1968), Chapter XIII) in the Markov chain. Next, we have the p.g.f.s of the distribution of the waiting time for the occurring event F_0 and F_1 by the non-overlapping way of counting in the Markov chain, respectively. At last, we give the recurrence relations of the p.g.f.s of the later waiting time distributions between F_0 and F_1 by the non-overlapping way of counting in the Markov chain. In Section 3, we consider the recurrence relations of the p.g.f.s of the sooner and the later waiting time distributions between F_0 and F_1 by the overlapping way of counting (in the sense of Ling (1988)) in the Markov chain. In Section 4, we consider the recurrence relations of the p.g.f.s of the distributions of the sooner and later waiting time by the non-overlapping way of counting of "0"-runs of length r or more and "1"-runs of length k or more (in the sense of Goldstein (1990)) in the Markov chain.

Our results in this paper are not only general and new but also suitable for numerical and symbolic calculations by means of the computer algebra system, for example, the REDUCE (Hearn (1984)) system. In particular, when r, s, l and m are given, we can obtain the probability mass function (p.m.f.), expectation and variance of the distribution of the sooner and later waiting time by using the computer algebra system. In the final section, we introduce an application of the sooner waiting time by the overlapping way of counting in a two-state Markov chain. For the derivation of the main part of the results, we used the method of generalized p.g.f. (cf. Ebneshahrashoob and Sobel (1990)).

2. Non-overlapping way of counting

2.1 *Sooner waiting time problem*

Let F_0 be the event that l runs of "0" of length r occur by the non-overlapping way of counting and let F_1 be the event that m runs of "1" of length k occur by the same way of counting in the sequence $X_1, X_2, \ldots$. Let τ be the waiting time for the sooner occurring event between F_0 and F_1. Let $\phi(t)$ be the p.g.f. of the distribution of the waiting time τ. Let $\xi(t)$ and $\psi(t)$ be the p.g.f. of the conditional distribution of the waiting time τ given that $X_0 = 0$ and $X_0 = 1$, respectively. Let t_0 be an arbitrary positive integer and let $x(a, b, c, d; t)$ be the p.g.f. of the conditional distribution of $\tau - t_0$ given that a non-overlapping runs of "0" of length r and b non-overlapping runs of "1" of length k have occurred until the t_0-th trial and at the t_0-th trial a run of "$c(= 0$ or $1)$" of length d has just occurred, where if a run of "0" of length r has just occurred at the t_0-th trial, we denote that $(c, d) = (0, 0)$ and if a run of "1" of length k has just occurred at the t_0-th trial, we denote that $(c, d) = (1, 0)$, that is, if $c = 0$ then $d = 0, 1, \ldots, r - 1$ and if $c = 1$ then $d = 0, 1, \ldots, k - 1$. We also note that $x(a, b, c, d; t)$ does not depend on t_0. Then, $\phi(t)$, $\xi(t)$, $\psi(t)$, $x(a, b, c, d; t)$ satisfy the following system of equations:

$$\phi(t) = p_0\xi(t) + p_1\psi(t), \tag{2.1}$$

$$\xi(t) = p_{00}tx(0,0,0,1) + p_{01}tx(0,0,1,1), \tag{2.2}$$

$$\psi(t) = p_{10}tx(0,0,0,1) + p_{11}tx(0,0,1,1), \tag{2.3}$$

$$\begin{cases} x(a,b,0,1) = p_{00}tx(a,b,0,2) + p_{01}tx(a,b,1,1), \\ x(a,b,0,2) = p_{00}tx(a,b,0,3) + p_{01}tx(a,b,1,1), \\ \vdots \\ x(a,b,0,r-1) = p_{00}tx(a+1,b,0,0) + p_{01}tx(a,b,1,1), \end{cases} \tag{2.4}$$
$$\text{for} \quad a = 0,1,\ldots,l-1, \; b = 0,1,\ldots,m-1,$$

$$\begin{cases} x(a,b,1,1) = p_{10}tx(a,b,0,1) + p_{11}tx(a,b,1,2), \\ x(a,b,1,2) = p_{10}tx(a,b,0,1) + p_{11}tx(a,b,1,3), \\ \vdots \\ x(a,b,1,k-1) = p_{10}tx(a,b,0,1) + p_{11}tx(a,b+1,1,0), \end{cases} \tag{2.5}$$
$$\text{for} \quad a = 0,1,\ldots,l-1, \; b = 0,1,\ldots,m-1,$$

$$x(a,b,0,0) = p_{00}tx(a,b,0,1) + p_{01}tx(a,b,1,1), \tag{2.6}$$
$$\text{for} \quad a = 1,2,\ldots,l-1, \; b = 0,1,\ldots,m-1,$$

$$x(a,b,1,0) = p_{10}tx(a,b,0,1) + p_{11}tx(a,b,1,1), \tag{2.7}$$
$$\text{for} \quad a = 0,1,\ldots,l-1, \; b = 1,2,\ldots,m-1,$$

$$x(l,b,0,0) = 1, \quad \text{for} \quad b = 0,1,\ldots,m-1, \tag{2.8}$$

$$x(a,m,1,0) = 1, \quad \text{for} \quad a = 0,1,\ldots,l-1. \tag{2.9}$$

From (2.2), (2.3), (2.4) and (2.5), we obtain

$$\xi(t) = A_1(t)x(1,0,0,0) + A_2(t)x(0,1,1,0),$$
$$\psi(t) = B_1(t)x(1,0,0,0) + B_2(t)x(0,1,1,0),$$

where

$$A_1(t) = C(t)\left[(p_{00}t)^r + (p_{01}t)\frac{1-(p_{11}t)^{k-1}}{1-p_{11}t}(p_{10}t)(p_{00}t)^{r-1}\right],$$

$$
\begin{aligned}
A_2(t) &= C(t)\left[\frac{1-(p_{00}t)^r}{1-p_{00}t}(p_{01}t)(p_{11}t)^{k-1}\right],\\
B_1(t) &= C(t)\left[\frac{1-(p_{11}t)^k}{1-p_{11}t}(p_{10}t)(p_{00}t)^{r-1}\right],\\
B_2(t) &= C(t)\left[(p_{10}t)\frac{1-(p_{00}t)^{r-1}}{1-p_{00}t}(p_{01}t)(p_{11}t)^{k-1}+(p_{11}t)^k\right],\\
C(t) &= \frac{1}{1-\dfrac{1-(p_{00}t)^{r-1}}{1-p_{00}t}(p_{01}t)\dfrac{1-(p_{11}t)^{k-1}}{1-p_{11}t}(p_{10}t)}.
\end{aligned}
$$

From (2.4), (2.5), (2.6) and (2.7), we obtain

$$
\begin{aligned}
x(a,b,0,0) &= A_1(t)x(a+1,b,0,0)+A_2(t)x(a,b+1,1,0),\\
&\qquad \text{for}\quad a=1,2,\dots,l-1,\ \ b=0,1,\dots,m-1,\\
x(a,b,1,0) &= B_1(t)x(a+1,b,0,0)+B_2(t)x(a,b+1,1,0),\\
&\qquad \text{for}\quad a=0,1,\dots,l-1,\ \ b=1,2,\dots,m-1.
\end{aligned}
$$

Hence, we obtain

THEOREM 2.1. *The p.g.f. $\phi(t)$ satisfies the following recurrence relation*

$$
\begin{aligned}
&\phi(t) = p_0\xi(t)+p_1\psi(t),\\
&\xi(t) = A_1(t)x(1,0,0,0)+A_2(t)x(0,1,1,0),\\
&\psi(t) = B_1(t)x(1,0,0,0)+B_2(t)x(0,1,1,0),\\
&x(a,b,0,0) = A_1(t)x(a+1,b,0,0)+A_2(t)x(a,b+1,1,0),\\
&\qquad\qquad \textit{for}\quad a=1,2,\dots,l-1,\ \ b=0,1,\dots,m-1,\\
&x(a,b,1,0) = B_1(t)x(a+1,b,0,0)+B_2(t)x(a,b+1,1,0),\\
&\qquad\qquad \textit{for}\quad a=0,1,\dots,l-1,\ \ b=1,2,\dots,m-1,\\
&x(l,b,0,0) = 1,\qquad \textit{for}\quad b=0,1,\dots,m-1,\\
&x(a,m,1,0) = 1,\qquad \textit{for}\quad a=0,1,\dots,l-1.
\end{aligned}
$$

Remark. In Theorem 2.1, setting $p_{00}=p_{10}=q$, $p_{01}=p_{11}=p$, $l=m=1$, $x(1,0,0,0)=x$, $x(0,1,1,0)=y$, we have the equation (2) of Ebneshahrashoob and Sobel (1990).

2.2 *Later waiting time problem*

In this subsection, firstly, we consider the p.g.f. of the waiting time for the occurrence of the event F_0. Let τ_0 be the waiting time for the occurrence of the event F_0. Let $\varphi_0(t)$ be the p.g.f. of the distribution of τ_0. Let $\xi(t)$ and $\psi(t)$ be the p.g.f. of the conditional distribution of the waiting time τ_0 given that $X_0=0$ and $X_0=1$, respectively. Let t_0 be an arbitrary positive integer and let $y(a,c,d;t)$ be the p.g.f. of the conditional distribution of τ_0-t_0 given that *a* non-overlapping runs of "0" of length r have occurred until the t_0-th trial and at the t_0-th trial a run of "$c(=0$ or $1)$" of length d has just occurred, where if a run of "0" of length r has just occurred at the t_0-th trial, we denote that $(c,d)=(0,0)$, that is, if $c=0$ then $d=0,1,\ldots,r-1$ and if $c=1$ then $d=1$. We also note that $y(a,c,d;t)$ does not depend on t_0. Then, $\varphi_0(t)$, $\xi(t)$, $\psi(t)$, $y(a,c,d;t)$ satisfy the following system of equations:

$$\varphi_0(t) = p_0\xi(t) + p_1\psi(t), \tag{2.10}$$

$$\xi(t) = p_{00}ty(0,0,1) + p_{01}ty(0,1,1), \tag{2.11}$$

$$\psi(t) = p_{10}ty(0,0,1) + p_{11}ty(0,1,1), \tag{2.12}$$

$$\begin{cases} y(a,0,1) = p_{00}ty(a,0,2) + p_{01}ty(a,1,1), \\ y(a,0,2) = p_{00}ty(a,0,3) + p_{01}ty(a,1,1), \\ \quad \vdots \\ y(a,0,r-1) = p_{00}ty(a+1,0,0) + p_{01}ty(a,1,1), \end{cases} \tag{2.13}$$
$$\text{for} \quad a=0,1,\ldots,l-1,$$

$$y(a,1,1) = p_{10}ty(a,0,1) + p_{11}ty(a,1,1), \quad \text{for} \quad a=0,1,\ldots,l-1, \tag{2.14}$$

$$y(a,0,0) = p_{00}ty(a,0,1) + p_{01}ty(a,1,1), \quad \text{for} \quad a=1,2,\ldots,l-1, \tag{2.15}$$

$$x(l,0,0) = 1. \tag{2.16}$$

From (2.11), (2.12), (2.13) and (2.14), we obtain

$$\xi(t) = C_1(t)\left(p_{00}t + p_{01}t\frac{p_{10}t}{1-p_{11}t}\right)(p_{00}t)^{r-1}y(1,0,0),$$
$$\psi(t) = C_1(t)\frac{p_{10}t}{1-p_{11}t}(p_{00}t)^{r-1}y(1,0,0),$$

where

$$C_1(t) = \frac{1}{1 - \dfrac{1-(p_{00}t)^{r-1}}{1-p_{00}t}(p_{01}t)\dfrac{(p_{10}t)}{1-p_{11}t}}.$$

From (2.13), (2.14) and (2.15), we obtain

$$y(a,0,0) = C_1(t)\left(p_{00}t + p_{01}t\frac{p_{10}t}{1-p_{11}t}\right)(p_{00}t)^{r-1}y(a+1,0,0) \quad \text{for} \quad a=1,2,\ldots,l-1. \tag{2.17}$$

From (2.16), (2.17), we obtain

$$y(1,0,0) = \left[C_1(t)\left(p_{00}t + p_{01}t\frac{p_{10}t}{1-p_{11}t}\right)(p_{00}t)^{r-1}\right]^{l-1}.$$

Hence, we obtain

PROPOSITION 2.1. *The p.g.f. $\varphi_0(t)$ is given by*

$$\varphi_0(t) = C_1(t)\left[p_0\left(p_{00}t + p_{01}t\frac{p_{10}t}{1-p_{11}t}\right) + p_1\frac{p_{10}t}{1-p_{11}t}\right](p_{00}t)^{r-1}$$
$$\times\left[C_1(t)\left(p_{00}t + p_{01}t\frac{p_{10}t}{1-p_{11}t}\right)(p_{00}t)^{r-1}\right]^{l-1}.$$

Similarly, we obtain

PROPOSITION 2.2. *The p.g.f. $\varphi_1(t)$ of the distribution of the waiting time for the occurrence of the event F_1 is given by*

$$\varphi_1(t) = C_2(t)\left[p_0\frac{p_{01}t}{1-p_{00}t} + p_1\left(p_{11}t + p_{10}t\frac{p_{01}t}{1-p_{00}t}\right)\right](p_{11}t)^{k-1}$$
$$\times\left[C_2(t)\left(p_{11}t + p_{10}t\frac{p_{01}t}{1-p_{00}t}\right)(p_{11}t)^{k-1}\right]^{m-1},$$

where

$$C_2(t) = \frac{1}{1 - \dfrac{1-(p_{11}t)^{k-1}}{1-p_{11}t}(p_{10}t)\dfrac{(p_{01}t)}{1-p_{00}t}}.$$

Remark. In Proposition 2.2, setting $p_{00} = p_{10} = q$, $p_{01} = p_{11} = p$, $m = 1$, we have Lemma 2.2 of Philippou *et al.* (1983).

Next, we consider the p.g.f. of the distribution of the waiting time for the later occurring event between F_0 and F_1. Let τ be the waiting time for the later occurring event between F_0 and F_1. Let $\phi(t)$ be the p.g.f. of the distribution of the waiting time τ. Let $\xi(t)$ and $\psi(t)$ be the p.g.f. of the conditional distribution of the waiting time τ given that $X_0 = 0$ and $X_0 = 1$, respectively. Then, we have

THEOREM 2.2. *The p.g.f. $\phi(t)$ satisfies the following recurrence relation*

$$\begin{aligned}
&\phi(t) = p_0\xi(t) + p_1\psi(t),\\
&\xi(t) = A_1(t)x(1,0,0,0) + A_2(t)x(0,1,1,0),\\
&\psi(t) = B_1(t)x(1,0,0,0) + B_2(t)x(0,1,1,0),\\
&x(a,b,0,0) = A_1(t)x(a+1,b,0,0) + A_2(t)x(a,b+1,1,0),\\
&\qquad\qquad\textit{for}\quad a = 1,2,\ldots,l-1,\ \ b = 0,1,\ldots,m-1,\\
&x(a,b,1,0) = B_1(t)x(a+1,b,0,0) + B_2(t)x(a,b+1,1,0),\\
&\qquad\qquad\textit{for}\quad a = 0,1,\ldots,l-1,\ \ b = 1,2,\ldots,m-1,\\
&x(l,b,0,0) = \frac{p_{01}t}{1-p_{00}t}C_2(t)(p_{11}t)^{k-1}
\end{aligned}$$

$$\cdot \left[\left(p_{11}t + p_{10}t \frac{p_{01}t}{1-p_{00}t} \right) C_2(t)(p_{11}t)^{k-1} \right]^{m-b-1},$$

$$\textit{for} \quad b = 0, 1, \ldots, m-1,$$

$$x(a, m, 1, 0) = \frac{p_{10}t}{1-p_{11}t} C_1(t)(p_{00}t)^{r-1}$$

$$\cdot \left[\left(p_{00}t + p_{01}t \frac{p_{10}t}{1-p_{11}t} \right) C_1(t)(p_{00}t)^{r-1} \right]^{l-a-1},$$

$$\textit{for} \quad a = 0, 1, \ldots, l-1.$$

PROOF. In Theorem 2.1, $x(l, b, 0, 0)$ can be regarded as a marker which means that the event F_0 occurs sooner and by that time b runs of "1" of length k have occurred. From the marker, we can see that the later occurring event is F_1 and that $m - b$ runs of "1" of length k must occur. In Proposition 2.2, setting $\varphi_1(t) = \varphi_1(t; p_0, m)$, we have

$$x(l, b, 0, 0) = \varphi_1(t; 1, m-b).$$

Similarly, in Proposition 2.1, setting $\varphi_0(t) = \varphi_0(t; p_1, l)$, we have

$$x(a, m, 1, 0) = \varphi_0(t; 1, l-a).$$

This completes the proof.

Remark. In Theorem 2.2, setting $l = m = 1$, we have Theorem 2.2 of Aki and Hirano (1993).

3. Overlapping way of counting

3.1 *Sooner waiting time problem*

Let F_0 be the event that l runs of "0" of length r occur by the overlapping way of counting and let F_1 be the event that m runs of "1" of length k occur by the same way of counting. Let τ be the waiting time for the sooner occurring event between F_0 and F_1. Let $\phi(t)$ be the p.g.f. of the distribution of the waiting time τ. Let $\xi(t)$ and $\psi(t)$ be the p.g.f. of the conditional distribution of the waiting time τ given that $X_0 = 0$ and $X_0 = 1$, respectively. Let t_0 be an arbitrary positive integer and let $x(a, b, c, d; t)$ be the p.g.f. of the conditional distribution of $\tau - t_0$ given that a overlapping runs of "0" of length r and b overlapping runs of "1" of length k have occurred until the t_0-th trial and at the t_0-th trial a run of "$c (= 0$ or $1)$" of length d has just occurred, where if $c = 0$ then $d = 1, 2, \ldots, r-1$ and if $c = 1$ then $d = 1, 2, \ldots, k-1$. We also note that $x(a, b, c, d; t)$ does not depend on t_0. Then, $\phi(t)$, $\xi(t)$, $\psi(t)$, $x(a, b, c, d; t)$ satisfy the following system of equations:

$$\phi(t) = p_0\xi(t) + p_1\psi(t), \tag{3.1}$$

$$\xi(t) = p_{00}tx(0, 0, 0, 1) + p_{01}tx(0, 0, 1, 1), \tag{3.2}$$

$$\psi(t) = p_{10}tx(0, 0, 0, 1) + p_{11}tx(0, 0, 1, 1), \tag{3.3}$$

$$(3.4)\quad \begin{cases} x(a,b,0,1) = p_{00}tx(a,b,0,2) + p_{01}tx(a,b,1,1), \\ x(a,b,0,2) = p_{00}tx(a,b,0,3) + p_{01}tx(a,b,1,1), \\ \quad\vdots \\ x(a,b,0,r-1) = p_{00}tx(a+1,b,0,r-1) + p_{01}tx(a,b,1,1), \end{cases}$$
$$\text{for} \quad a = 0,1,\dots,l-1, \ b = 0,1,\dots,m-1,$$

$$(3.5)\quad \begin{cases} x(a,b,1,1) = p_{10}tx(a,b,0,1) + p_{11}tx(a,b,1,2), \\ x(a,b,1,2) = p_{10}tx(a,b,0,1) + p_{11}tx(a,b,1,3), \\ \quad\vdots \\ x(a,b,1,k-1) = p_{10}tx(a,b,0,1) + p_{11}tx(a,b+1,1,k-1), \end{cases}$$
$$\text{for} \quad a = 0,1,\dots,l-1, \ b = 0,1,\dots,m-1,$$

$$(3.6)\quad x(a,b,0,r-1) = p_{00}tx(a+1,b,0,r-1) + p_{01}tx(a,b,1,1),$$
$$\text{for} \quad a = 1,2,\dots,l-1, \ b = 0,1,\dots,m-1,$$

$$(3.7)\quad x(a,b,1,k-1) = p_{10}tx(a,b,0,1) + p_{11}tx(a,b+1,1,k-1),$$
$$\text{for} \quad a = 0,1,\dots,l-1, \ b = 1,2,\dots,m-1,$$

$$(3.8)\quad x(l,b,0,r-1) = 1, \qquad \text{for} \quad b = 0,1,\dots,m-1,$$

$$(3.9)\quad x(a,m,1,k-1) = 1, \qquad \text{for} \quad a = 0,1,\dots,l-1.$$

From (3.2), (3.3), (3.4) and (3.5), we obtain

$$\xi(t) = A_1(t)x(1,0,0,r-1) + A_2(t)x(0,1,1,k-1),$$
$$\psi(t) = B_1(t)x(1,0,0,r-1) + B_2(t)x(0,1,1,k-1).$$

From (3.4), (3.5), (3.6) and (3.7), we obtain

$$x(a,b,0,r-1) = D_1(t)x(a+1,b,0,r-1) + D_2(t)x(a,b+1,1,k-1),$$
$$\text{for} \quad a = 1,2,\dots,l-1, \ b = 0,1,\dots,m-1,$$
$$x(a,b,1,k-1) = G_1(t)x(a+1,b,0,r-1) + G_2(t)x(a,b+1,1,k-1),$$
$$\text{for} \quad a = 0,1,\dots,l-1, \ b = 1,2,\dots,m-1,$$

where

$$D_1(t) = p_{00}t + C(t)p_{01}t\frac{1-(p_{11}t)^{k-1}}{1-p_{11}t}(p_{10}t)(p_{00}t)^{r-1},$$
$$D_2(t) = C(t)p_{01}t(p_{11}t)^{k-1},$$
$$G_1(t) = C(t)(p_{10}t)(p_{00}t)^{r-1},$$
$$G_2(t) = C(t)(p_{10}t)\frac{1-(p_{00}t)^{r-1}}{1-p_{00}t}(p_{01}t)(p_{11}t)^{k-1} + p_{11}t.$$

Hence, we obtain

THEOREM 3.1. *The p.g.f. $\phi(t)$ satisfies the following recurrence relation*

$$\phi(t) = p_0\xi(t) + p_1\psi(t),$$
$$\xi(t) = A_1(t)x(1,0,0,r-1) + A_2(t)x(0,1,1,k-1),$$
$$\psi(t) = B_1(t)x(1,0,0,r-1) + B_2(t)x(0,1,1,k-1),$$
$$x(a,b,0,r-1) = D_1(t)x(a+1,b,0,r-1) + D_2(t)x(a,b+1,1,k-1), \quad \text{for} \quad a = 1,2,\ldots,l-1, \ b = 0,1,\ldots,m-1,$$
$$x(a,b,1,k-1) = G_1(t)x(a+1,b,0,r-1) + G_2(t)x(a,b+1,1,k-1), \quad \text{for} \quad a = 0,1,\ldots,l-1, \ b = 1,2,\ldots,m-1,$$
$$x(l,b,0,r-1) = 1, \quad \text{for} \quad b = 0,1,\ldots,m-1,$$
$$x(a,m,1,k-1) = 1, \quad \text{for} \quad a = 0,1,\ldots,l-1.$$

3.2 *Later waiting time problem*

In this subsection, firstly, we consider the p.g.f. of the waiting time for the occurrence of the event F_0. Let τ_0 be the waiting time for the occurrence of the event F_0. Let $\varphi_0(t)$ be the p.g.f. of the distribution of τ_0. Let $\xi(t)$ and $\psi(t)$ be the p.g.f. of the conditional distribution of the waiting time τ_0 given that $X_0 = 0$ and $X_0 = 1$, respectively. Let t_0 be an arbitrary positive integer and let $y(a,c,d;t)$ be the p.g.f. of the conditional distribution of $\tau_0 - t_0$ given that a overlapping runs of "0" of length r have occurred until the t_0-th trial and at the t_0-th trial a run of "$c(= 0$ or $1)$" of length d has just occurred, where if $c = 0$ then $d = 0,1,\ldots,r-1$ and if $c = 1$ then $d = 1$. We also note that $y(a,c,d;t)$ does not depend on t_0. Then, $\varphi_0(t)$, $\xi(t)$, $\psi(t)$, $y(a,c,d;t)$ satisfy the following system of equations:

$$\varphi_0(t) = p_0\xi(t) + p_1\psi(t), \tag{3.10}$$
$$\xi(t) = p_{00}ty(0,0,1) + p_{01}ty(0,1,1), \tag{3.11}$$
$$\psi(t) = p_{10}ty(0,0,1) + p_{11}ty(0,1,1), \tag{3.12}$$
$$\begin{cases} y(a,0,1) = p_{00}ty(a,0,2) + p_{01}ty(a,1,1), \\ y(a,0,2) = p_{00}ty(a,0,3) + p_{01}ty(a,1,1), \\ \quad\vdots \\ y(a,0,r-1) = p_{00}ty(a+1,0,r-1) + p_{01}ty(a,1,1), \end{cases} \quad \text{for} \quad a = 0,1,\ldots,l-1, \tag{3.13}$$
$$y(a,1,1) = p_{10}ty(a,0,1) + p_{11}ty(a,1,1), \quad \text{for} \quad a = 0,1,\ldots,l-1, \tag{3.14}$$
$$y(a,0,r-1) = p_{00}ty(a+1,0,r-1) + p_{01}ty(a,1,1), \quad \text{for} \quad a = 1,2,\ldots,l-1, \tag{3.15}$$
$$y(l,0,0) = 1. \tag{3.16}$$

From (3.11), (3.12), (3.13) and (3.14), we obtain

$$\xi(t) = C_1(t)\left(p_{00}t + p_{01}t\frac{p_{10}t}{1-p_{11}t}\right)(p_{00}t)^{r-1}y(1,0,r-1),$$
$$\psi(t) = C_1(t)\frac{p_{10}t}{1-p_{11}t}(p_{00}t)^{r-1}y(1,0,r-1).$$

From (3.13), (3.14) and (3.15), we obtain

$$y(a,0,r-1) = \left[p_{00}t + C_1(t)p_{01}t\frac{p_{10}t}{1-p_{11}t}(p_{00}t)^{r-1}\right] y(a+1,0,r-1), \tag{3.17}$$
$$\text{for} \quad a = 1,2,\dots,l-1.$$

From (3.16), (3.17), we obtain

$$y(1,0,r-1) = \left[p_{00}t + C_1(t)p_{01}t\frac{p_{10}t}{1-p_{11}t}(p_{00}t)^{r-1}\right]^{l-1}.$$

Hence, we obtain

PROPOSITION 3.1. *The p.g.f. $\varphi_0(t)$ is given by*

$$\begin{aligned}\varphi_0(t) = C_1(t)&\left[p_0\left(p_{00}t + p_{01}t\frac{p_{10}t}{1-p_{11}t}\right) + p_1\frac{p_{10}t}{1-p_{11}t}\right](p_{00}t)^{r-1}\\ &\times\left[p_{00}t + C_1(t)\frac{p_{10}t}{1-p_{11}t}p_{01}t(p_{00}t)^{r-1}\right]^{l-1}.\end{aligned}$$

Similarly, we obtain

PROPOSITION 3.2. *The p.g.f. $\varphi_1(t)$ of the distribution of the waiting time for the occurrence of the event F_1 is given by*

$$\begin{aligned}\varphi_1(t) = C_2(t)&\left[p_0\frac{p_{01}t}{1-p_{00}t} + p_1\left(p_{11}t + \frac{p_{01}t}{1-p_{00}t}p_{10}t\right)\right](p_{11}t)^{k-1}\\ &\times\left[p_{11}t + C_2(t)\frac{p_{01}t}{1-p_{00}t}p_{10}t(p_{11}t)^{k-1}\right]^{m-1}.\end{aligned}$$

Remark. In Proposition 3.2, setting $p_{00} = p_{10} = q$, $p_{01} = p_{11} = p$, $m = r$, we have Theorem 4.1 (b) of Ling (1989) and Theorem 4.1 of Hirano *et al.* (1991).

Next, we consider the p.g.f. of the distribution of the waiting time for the later occurring event between F_0 and F_1. Let τ be the waiting time for the later occurring event between F_0 and F_1. Let $\phi(t)$ be the p.g.f. of the distribution of the waiting time τ. Let $\xi(t)$ and $\psi(t)$ be the p.g.f. of the conditional distribution of the waiting time τ given that $X_0 = 0$ and $X_0 = 1$, respectively. Then, we have

THEOREM 3.2. *The p.g.f. $\phi(t)$ satisfies the following recurrence relation*

$$\begin{aligned}\phi(t) &= p_0\xi(t) + p_1\psi(t),\\ \xi(t) &= A_1(t)x(1,0,0,r-1) + A_2(t)x(0,1,1,k-1),\\ \psi(t) &= B_1(t)x(1,0,0,r-1) + B_2(t)x(0,1,1,k-1),\end{aligned}$$

$$x(a,b,0,r-1) = D_1(t)x(a+1,b,0,r-1) + D_2(t)x(a,b+1,1,k-1),$$
$$\textit{for} \quad a = 1,2,\dots,l-1, \ \ b = 0,1,\dots,m-1,$$
$$x(a,b,1,k-1) = G_1(t)x(a+1,b,0,r-1) + G_2(t)x(a,b+1,1,k-1),$$
$$\textit{for} \quad a = 0,1,\dots,l-1, \ \ b = 1,2,\dots,m-1,$$
$$x(l,b,0,r-1) = \left[\frac{p_{01}t}{1-p_{00}t}C_2(t)(p_{11}t)^{k-1}\right] \cdot \left[p_{11}t + p_{10}t\frac{p_{01}t}{1-p_{00}t}C_2(t)(p_{11}t)^{k-1}\right]^{m-b-1},$$
$$\textit{for} \quad b = 1,2,\dots,m-1,$$
$$x(a,m,1,0) = \left[\frac{p_{10}t}{1-p_{11}t}C_1(t)(p_{00}t)^{r-1}\right] \cdot \left[p_{00}t + p_{01}t\frac{p_{10}t}{1-p_{11}t}C_1(t)(p_{00}t)^{r-1}\right]^{l-a-1},$$
$$\textit{for} \quad a = 0,1,\dots,l-1.$$

PROOF. In Theorem 3.1, $x(l,b,0,r-1)$ can be regarded as a marker which means that the event F_0 occurs sooner and by that time b runs of "1" of length k have occurred. From the marker, we can see that the later occurring event is F_1 and that $m-b$ runs of "1" of length k must occur. In Proposition 3.2, setting $\varphi_1(t) = \varphi_1(t; p_0, m)$, we have

$$x(l,b,0,r-1) = \varphi_1(t;1,m-b).$$

Similarly, in Proposition 3.1, setting $\varphi_0(t) = \varphi_0(t; p_1, l)$, we have

$$x(a,m,1,k-1) = \varphi_0(t;1,l-a).$$

This completes the proof.

4. Non-overlapping way of counting of "0"-runs of length r or more and "1"-runs of length k or more

4.1 *Sooner waiting time problem*

Let F_0 be the event that $l-1$ non-overlapping "0"-runs of length r or more occur and then a run of "0" of length r occurs and let F_1 be the event that $m-1$ non-overlapping "1"-runs of length k or more occur and then a run of "1" of length k occurs. Let τ be the waiting time for the sooner occurring event between F_0 and F_1. Let $\phi(t)$ be the p.g.f. of the distribution of the waiting time τ. Let $\xi(t)$ and $\psi(t)$ be the p.g.f. of the conditional distribution of the waiting time τ given that $X_0 = 0$ and $X_0 = 1$, respectively. Let t_0 be an arbitrary positive integer and let $x(a,b,c,d;t)$ be the p.g.f. of the conditional distribution of $\tau - t_0$ given that a non-overlapping "0"-runs of length r or more and b non-overlapping "1"-runs of length k or more have occurred until the t_0-th trial and at the t_0-th trial a run of

"$c(= 0 \text{ or } 1)$" of length d has just occurred, where if $c = 0$ then $d = 1, 2, \dots, r$ and if $c = 1$ then $d = 1, 2, \dots, k$. We also note that $x(a, b, c, d; t)$ does not depend on t_0. Then, $\phi(t)$, $\xi(t)$, $\psi(t)$, $x(a, b, c, d; t)$ satisfy the following system of equations:

$$\phi(t) = p_0\xi(t) + p_1\psi(t), \tag{4.1}$$

$$\xi(t) = p_{00}tx(0,0,0,1) + p_{01}tx(0,0,1,1), \tag{4.2}$$

$$\psi(t) = p_{10}tx(0,0,0,1) + p_{11}tx(0,0,1,1), \tag{4.3}$$

$$\begin{cases} x(a,b,0,1) = p_{00}tx(a,b,0,2) + p_{01}tx(a,b,1,1), \\ x(a,b,0,2) = p_{00}tx(a,b,0,3) + p_{01}tx(a,b,1,1), \\ \quad\vdots \\ x(a,b,0,r-1) = p_{00}tx(a,b,0,r) + p_{01}tx(a,b,1,1), \end{cases} \tag{4.4}$$
$$\text{for} \quad a = 0,1,\dots,l-1, \; b = 0,1,\dots,m-1,$$

$$\begin{cases} x(a,b,1,1) = p_{10}tx(a,b,0,1) + p_{11}tx(a,b,1,2), \\ x(a,b,1,2) = p_{10}tx(a,b,0,1) + p_{11}tx(a,b,1,3), \\ \quad\vdots \\ x(a,b,1,k-1) = p_{10}tx(a,b,0,1) + p_{11}tx(a,b,1,k), \end{cases} \tag{4.5}$$
$$\text{for} \quad a = 0,1,\dots,l-1, \; b = 0,1,\dots,m-1,$$

$$x(a,b,0,r) = p_{00}tx(a,b,0,r) + p_{01}tx(a+1,b,1,1), \tag{4.6}$$
$$\text{for} \quad a = 0,1,\dots,l-2, \; b = 0,1,\dots,m-1,$$

$$x(a,b,1,k) = p_{10}tx(a,b+1,0,1) + p_{11}tx(a,b,1,k), \tag{4.7}$$
$$\text{for} \quad a = 0,1,\dots,l-1, \; b = 0,1,\dots,m-2,$$

$$x(l-1,b,0,r) = 1, \quad \text{for} \quad b = 0,1,\dots,m-1, \tag{4.8}$$

$$x(a,m-1,1,k) = 1, \quad \text{for} \quad a = 0,1,\dots,l-1. \tag{4.9}$$

From (4.4), (4.5), (4.6) and (4.7), we obtain

$$x(a,b,0,1) = H_1(t)x(a+1,b,1,1) + H_2(t)x(a,b+1,0,1),$$
$$\text{for} \quad a = 0,1,\dots,l-2, \; b = 0,1,\dots,m-2,$$
$$x(a,b,1,1) = I_1(t)x(a+1,b,1,1) + I_2(t)x(a,b+1,0,1),$$
$$\text{for} \quad a = 0,1,\dots,l-2, \; b = 0,1,\dots,m-2,$$

where

$$H_1(t) = C(t)(p_{00}t)^{r-1}\frac{p_{01}t}{1-p_{00}t},$$
$$H_2(t) = C(t)\frac{1-(p_{00}t)^{r-1}}{1-p_{00}t}(p_{01}t)(p_{11}t)^{k-1}\frac{p_{10}t}{1-p_{11}t},$$
$$I_1(t) = C(t)\frac{1-(p_{11}t)^{k-1}}{1-p_{11}t}(p_{10}t)(p_{00}t)^{r-1}\frac{p_{01}t}{1-p_{00}t},$$
$$I_2(t) = C(t)(p_{11}t)^{k-1}\frac{p_{10}t}{1-p_{11}t}.$$

From (4.4), (4.5), we obtain

$$x(l-1,m-1,0,1) = \Gamma_1(t)x(l-1,m-1,0,r) + \Gamma_2(t)x(l-1,m-1,1,k),$$
$$x(l-1,m-1,1,1) = \Lambda_1(t)x(l-1,m-1,0,r) + \Lambda_2(t)x(l-1,m-1,1,k),$$

where

$$\begin{aligned}
\Gamma_1(t) &= C(t)(p_{00}t)^{r-1},\\
\Gamma_2(t) &= C(t)\frac{1-(p_{00}t)^{r-1}}{1-p_{00}t}(p_{01}t)(p_{11}t)^{k-1},\\
\Lambda_1(t) &= C(t)\frac{1-(p_{11}t)^{k-1}}{1-p_{11}t}(p_{10}t)(p_{00}t)^{r-1},\\
\Lambda_2(t) &= C(t)(p_{11}t)^{k-1}.
\end{aligned}$$

From (4.4), (4.5) and (4.7), we obtain

$$\begin{aligned}
x(l-1,b,0,1) &= \Gamma_1(t)x(l-1,b,0,r) + H_2(t)x(l-1,b+1,0,1),\\
x(l-1,b,1,1) &= \Lambda_1(t)x(l-1,b,0,r) + I_2(t)x(l-1,b+1,0,1),\\
&\qquad\qquad \text{for} \quad b = 0,1,\dots,m-2.
\end{aligned}$$

From (4.4), (4.5) and (4.6), we obtain

$$\begin{aligned}
x(a,m-1,0,1) &= H_1(t)x(a+1,m-1,1,1) + \Gamma_2(t)x(a,m-1,1,k),\\
x(a,m-1,1,1) &= I_1(t)x(a+1,m-1,1,1) + \Lambda_2(t)x(a,m-1,1,k),\\
&\qquad\qquad \text{for} \quad a = 0,1,\dots,l-2.
\end{aligned}$$

Hence, we obtain

THEOREM 4.1. *The p.g.f. $\phi(t)$ satisfies the following recurrence relation*

$$\begin{aligned}
\phi(t) &= p_0\xi(t) + p_1\psi(t),\\
\xi(t) &= p_{00}tx(0,0,0,1) + p_{01}tx(0,0,1,1),\\
\psi(t) &= p_{10}tx(0,0,0,1) + p_{11}tx(0,0,1,1),\\
x(a,b,0,1) &= H_1(t)x(a+1,b,1,1) + H_2(t)x(a,b+1,0,1),\\
x(a,b,1,1) &= I_1(t)x(a+1,b,1,1) + I_2(t)x(a,b+1,0,1),\\
&\qquad\qquad \textit{for} \quad a = 0,1,\dots,l-2, \quad b = 0,1,\dots,m-2,\\
x(l-1,b,0,1) &= \Gamma_1(t)x(l-1,b,0,r) + H_2(t)x(l-1,b+1,0,1),\\
x(l-1,b,1,1) &= \Lambda_1(t)x(l-1,b,0,r) + I_2(t)x(l-1,b+1,0,1),\\
&\qquad\qquad \textit{for} \quad b = 0,1,\dots,m-2,\\
x(a,m-1,0,1) &= H_1(t)x(a+1,m-1,1,1) + \Gamma_2(t)x(a,m-1,1,k),\\
x(a,m-1,1,1) &= I_1(t)x(a+1,m-1,1,1) + \Lambda_2(t)x(a,m-1,1,k),\\
&\qquad\qquad \textit{for} \quad a = 0,1,\dots,l-2,
\end{aligned}$$

$$
\begin{aligned}
x(l-1,m-1,0,1) &= \Gamma_1(t)x(l-1,m-1,0,r) \\
&\quad + \Gamma_2(t)x(l-1,m-1,1,k), \\
x(l-1,m-1,1,1) &= \Lambda_1(t)x(l-1,m-1,0,r) \\
&\quad + \Lambda_2(t)x(l-1,m-1,1,k), \\
x(l-1,b,0,r) &= 1, \qquad \textit{for} \quad b = 0,1,\ldots,m-1, \\
x(a,m-1,1,k) &= 1, \qquad \textit{for} \quad a = 0,1,\ldots,l-1.
\end{aligned}
$$

4.2 *Later waiting time problem*

In this subsection, firstly, we consider the p.g.f. of the waiting time for the occurrence of the event F_0. Let τ_0 be the waiting time for the occurrence of the event F_0. Let $\varphi_0(t)$ be the p.g.f. of the distribution of τ_0. Let $\xi(t)$ and $\psi(t)$ be the p.g.f. of the conditional distribution of the waiting time τ_0 given that $X_0 = 0$ and $X_0 = 1$, respectively. Let t_0 be an arbitrary positive integer and let $y(a,c,d;t)$ be the p.g.f. of the conditional distribution of $\tau_0 - t_0$ given that a non-overlapping "0"-runs of length r or more have occurred until the t_0-th trial and at the t_0-th trial a run of "$c(= 0$ or $1)$" of length d has just occurred, where if $c = 0$ then $d = 1,2,\ldots,r$ and if $c = 1$ then $d = 1$. We also note that $y(a,c,d;t)$ does not depend on t_0. Then, $\varphi_0(t)$, $\xi(t)$, $\psi(t)$, $y(a,c,d;t)$ satisfy the following system of equations:

$$
\varphi_0(t) = p_0\xi(t) + p_1\psi(t), \tag{4.10}
$$

$$
\xi(t) = p_{00}ty(0,0,1) + p_{01}ty(0,1,1), \tag{4.11}
$$

$$
\psi(t) = p_{10}ty(0,0,1) + p_{11}ty(0,1,1), \tag{4.12}
$$

$$
\begin{cases}
y(a,0,1) = p_{00}ty(a,0,2) + p_{01}ty(a,1,1), \\
y(a,0,2) = p_{00}ty(a,0,3) + p_{01}ty(a,1,1), \\
\qquad \vdots \\
y(a,0,r-1) = p_{00}ty(a,0,r) + p_{01}ty(a,1,1),
\end{cases}
\quad \text{for} \quad a = 0,1,\ldots,l-1, \tag{4.13}
$$

$$
y(a,1,1) = p_{10}ty(a,0,1) + p_{11}ty(a,1,1), \quad \text{for} \quad a = 0,1,\ldots,l-1, \tag{4.14}
$$

$$
y(a,0,r) = p_{00}ty(a,0,r) + p_{01}ty(a+1,1,1), \quad \text{for} \quad a = 0,1,\ldots,l-2, \tag{4.15}
$$

$$
y(l-1,0,r) = 1. \tag{4.16}
$$

From (4.13), (4.14) and (4.15), we obtain

$$
y(a,0,1) = C_1(t)(p_{00}t)^{r-1}\frac{p_{01}t}{1-p_{00}t}y(a+1,1,1), \quad \text{for} \quad a = 0,1,\ldots,l-2.
$$

$$
y(a,1,1) = C_1(t)\frac{p_{10}t}{1-p_{11}t}(p_{00}t)^{r-1}\frac{p_{01}t}{1-p_{00}t}y(a+1,1,1), \quad \text{for} \quad a = 0,1,\ldots,l-2.
$$

From (4.13), (4.14), we obtain

$$y(l-1,1,1) = C_1(t)\frac{p_{10}t}{1-p_{11}t}(p_{00}t)^{r-1}y(l-1,0,r).$$

Hence, we obtain

PROPOSITION 4.1. *The p.g.f.* $\varphi(t)$ *is given by*

$$\varphi_0(t) = C_1(t)\left[p_0\left(p_{00}t + p_{01}t\frac{p_{10}t}{1-p_{11}t}\right) + p_1\frac{p_{10}t}{1-p_{11}t}\right](p_{00}t)^{r-1}\frac{p_{01}t}{1-p_{00}t}$$
$$\times\left[C_1(t)\frac{p_{10}t}{1-p_{11}t}(p_{00}t)^{r-1}\frac{p_{01}t}{1-p_{00}t}\right]^{l-2} C_1(t)\frac{p_{10}t}{1-p_{11}t}(p_{00}t)^{r-1}.$$

Similarly, we obtain

PROPOSITION 4.2. *The p.g.f.* $\varphi_1(t)$ *of the distribution of the waiting time for the occurrence of the event* F_1 *is given by*

$$\varphi_1(t) = C_2(t)\left[p_0\frac{p_{01}t}{1-p_{00}t} + p_1\left(p_{11}t + p_{10}t\frac{p_{01}t}{1-p_{00}t}\right)\right](p_{11}t)^{k-1}\frac{p_{10}t}{1-p_{11}t}$$
$$\times\left[C_2(t)\frac{p_{01}t}{1-p_{00}t}(p_{11}t)^{k-1}\frac{p_{10}t}{1-p_{11}t}\right]^{m-2} C_2(t)\frac{p_{01}t}{1-p_{00}t}(p_{11}t)^{k-1}.$$

Remark. In Proposition 4.2, setting $p_{00} = p_{10} = q$, $p_{01} = p_{11} = p$, we have the negative binomial distribution of order k in the sense of Goldstein (1990).

Next, we consider the p.g.f. of the distribution of the waiting time for the later occurring event between F_0 and F_1. Let τ be the waiting time for the later occurring event between F_0 and F_1. Let $\phi(t)$ be the p.g.f. of the distribution of the waiting time τ. Let $\xi(t)$ and $\psi(t)$ be the p.g.f. of the conditional distribution of the waiting time τ given that $X_0 = 0$ and $X_0 = 1$, respectively. Then, we have

THEOREM 4.2. *The p.g.f.* $\phi(t)$ *satisfies the following recurrence relation*

$$\begin{aligned}
&\phi(t) = p_0\xi(t) + p_1\psi(t),\\
&\xi(t) = p_{00}tx(0,0,0,1) + p_{01}tx(0,0,1,1),\\
&\psi(t) = p_{10}tx(0,0,0,1) + p_{11}tx(0,0,1,1),\\
&x(a,b,0,1) = H_1(t)x(a+1,b,1,1) + H_2(t)x(a,b+1,0,1),\\
&x(a,b,1,1) = I_1(t)x(a+1,b,1,1) + I_2(t)x(a,b+1,0,1),\\
&\qquad\qquad\qquad\qquad \textit{for} \quad a = 0,1,\ldots,l-2, \;\; b = 0,1,\ldots,m-2,\\
&x(l-1,b,0,1) = \Gamma_1(t)x(l-1,b,0,r) + H_2(t)x(l-1,b+1,0,1),\\
&x(l-1,b,1,1) = \Lambda_1(t)x(l-1,b,0,r) + I_2(t)x(l-1,b+1,0,1),
\end{aligned}$$

$$\text{for} \quad b = 0, 1, \ldots, m-2,$$

$$x(a, m-1, 0, 1) = H_1(t)x(a+1, m-1, 1, 1) + \Gamma_2(t)x(a, m-1, 1, k),$$
$$x(a, m-1, 1, 1) = I_1(t)x(a+1, m-1, 1, 1) + \Lambda_2(t)x(a, m-1, 1, k),$$
$$\text{for} \quad a = 0, 1, \ldots, l-2,$$

$$\begin{aligned} x(l-1, m-1, 0, 1) &= \Gamma_1(t)x(l-1, m-1, 0, r) \\ &\quad + \Gamma_2(t)x(l-1, m-1, 1, k), \\ x(l-1, m-1, 1, 1) &= \Lambda_1(t)x(l-1, m-1, 0, r) \\ &\quad + \Lambda_2(t)x(l-1, m-1, 1, k), \\ x(l-1, b, 0, r) &= \left[C_2(t)\frac{p_{01}t}{1-p_{00}t}(p_{11}t)^{k-1}\frac{p_{10}t}{1-p_{11}t}\right]^{m-b-1} \\ &\quad \cdot C_2(t)\frac{p_{01}t}{1-p_{00}t}(p_{11}t)^{k-1}, \qquad \text{for} \quad b = 0, 1, \ldots, m-1, \\ x(a, m-1, 1, k) &= \left[C_1(t)\frac{p_{10}t}{1-p_{11}t}(p_{00}t)^{r-1}\frac{p_{01}t}{1-p_{00}t}\right]^{l-a-1} \\ &\quad \cdot C_1(t)\frac{p_{10}t}{1-p_{11}t}(p_{00}t)^{r-1}, \qquad \text{for} \quad a = 0, 1, \ldots, l-1. \end{aligned}$$

PROOF. In Theorem 4.1, $x(l-1, b, 0, r)$ can be regarded as a marker which means that the event F_0 occurs sooner and by that time b non-overlapping "1"-runs of length k or more have occurred. From the marker, we can see that the later occurring event is F_1 and that $m-b-1$ non-overlapping "1"-runs of length k or more must occur and then a run of "1" of length k must occur. In Proposition 4.2, setting $\varphi_1(t) = \varphi_1(t; p_0, m)$, we have

$$x(l-1, b, 0, r) = \varphi_1(t; 1, m-b).$$

Similarly, in Proposition 4.1, setting $\varphi_0(t) = \varphi_0(t; p_1, l)$, we have

$$x(a, m-1, 1, k) = \varphi_0(t; 1, l-a).$$

This completes the proof.

5. Application and computation

In this section, we introduce an application of the sooner waiting time problem by the overlapping way of counting in a two-state Markov chain.

We firstly consider a volleyball game between A team and B team. When one plays which starts with a serve and finishes by some actions, for example, by failing to serve, failing to receive or spiking, we say that one trial finishes. When A team and B team play a set of volleyball, we are interested in the number of trials (waiting time) to finish the game.

Let $X_0, X_1, X_2, \ldots$ be a time-homogeneous $\{0, 1\}$-valued Markov chain defined by $P(X_0 = 0) = p_0$, $P(X_0 = 1) = p_1$ and for $i = 0, 1, 2, \ldots$, $P(X_{i+1} = 0 \mid X_i =$

$0) = p_{00}$, $P(X_{i+1} = 1 \mid X_i = 0) = p_{01}$, $P(X_{i+1} = 0 \mid X_i = 1) = p_{10}$ and $P(X_{i+1} = 1 \mid X_i = 1) = p_{11}$, where $0 < p_{00} < 1$, $0 < p_{11} < 1$.

We denote $X_i = 0$ if A team wins the i-th trial (play) for $i = 1, 2, \dots,$ of course, the $i + 1$-th trial starts with the serve of A team and $X_0 = 0$ if A team gets the serve before the game. We denote $X_i = 1$ if B team wins the i-th trial for $i = 1, 2, \dots,$ and $X_0 = 1$ if B team gets the serve before the game.

The probability that A team will win the i-th trial when the i-th trial starts with the serve of A team is evidently different from the probability that A team will win the i-th trial when the i-th trial starts with the serve of B team. So it is natural to assume the above Markov chain in this waiting time problem.

For example, we can record a volleyball game between A team and B team as the following $\{0, 1\}$-valued sequence.

$$\begin{array}{l}\text{score of A team}\\ \text{score of B team}\end{array}\ \overset{0}{\underset{0}{0}}\;1\;0\;1\;0\;1\;0\;1\;0\;1\;0\;\overset{1}{0}\;1\;0\;\overset{2}{0}\;1\;\underset{1}{1}\;0\;1\;\underset{2}{1}\;0\;\overset{3}{0}\;\overset{4}{0}\;\overset{5}{0}\;1\;\underset{3}{1}\;0$$

$$1\;0\;1\;0\;1\;0\;\overset{6}{0}\;1\;0\;\overset{7}{0}\;1\;0\;1\;0\;1\;0\;1\;0\;1\;0\;\overset{8}{0}\;1\;\underset{4}{1}\;0\;1\;0\;\overset{9}{0}\;1$$

$$\underset{5}{1}\;0\;1\;\underset{6}{1}\;0\;1\;0\;\overset{10}{0}\;\overset{11}{0}\;1\;0\;1\;0\;\overset{12}{0}\;\overset{13}{0}\;1\;0\;1\;0\;1\;\underset{7}{1}\;0\;\overset{14}{0}\;1\;\underset{8}{1}\;\underset{9}{1}\;0\;\overset{15}{0}$$

In the above sequence, the number of trials (waiting time) of a set of volleyball between A team and B team is 82 (we do not count $X_0 = 0$ as the number of trials).

We note that the waiting time of the volleyball game is regarded as the sooner waiting time problem by the overlapping 15 "0"-runs of length 2 and the overlapping 15 "1"-runs of length 2 (except for a tie, 14 to 14) in the Markov chain $X_1, X_2, \dots$. In other words, the waiting time of the volleyball game is an application of the sooner problem of Section 3.

As the matter of fact, we can compute the p.g.f., expectation, variance of the waiting time of the volleyball game by means of the computer algebra system, for example, the REDUCE (Hearn (1984)) system. Moreover, we can compute the p.m.f. of the waiting time by using the feature which the p.g.f. of the waiting time is a rational generating function (cf. Stanley (1986) Chapter IV).

Stanley (1986) described a fast method for computing the coefficients of a rational function $P(x)/Q(x) = \sum_{n \ge 0} f(n) x^n$. Suppose (without loss of generality) that $Q(x) = 1 + \alpha_1 x + \cdots + \alpha_d x^d$, and let $P(x) = \beta_0 + \beta_1 x + \cdots + \beta_e x^e$ (possibly $e \ge d$). Equating coefficients of x^n in

$$Q(x) \sum_{n \ge 0} f(n) x^n = P(x)$$

yields

$$f(n) = -\alpha_1 f(n-1) - \cdots - \alpha_d f(n-d) + \beta_n,$$

where we set $f(k) = 0$ for $k < 0$ and $\beta_k = 0$ for $k > e$. The above recurrence relation can easily be implemented by means of computer algebra system.

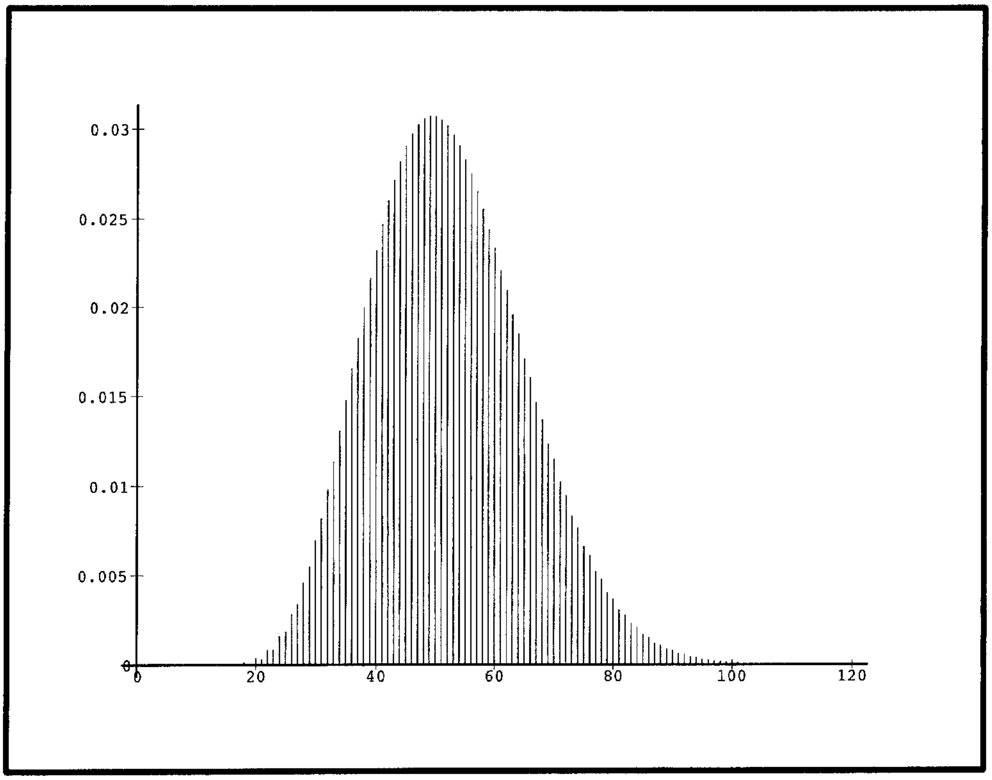

Fig. 1. (pmf of $p_{00} = 0.3$, $p_{11} = 0.5$).

We obtain the p.m.f. of the waiting time of the volleyball game by using the modification of Theorem 3.1 and the above method. Figure 1 is the bar graph of p.m.f.s of the waiting time τ ($\tau = 1, 2, \dots, 120$) of the volleyball game when $p_0 = 1$, $p_{00} = 0.3$ and $p_{11} = 0.5$.

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