Sorting Networks Using L_p Mean Comparators for Signal Processing Applications

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***Abstract*—Digital implementations of sorting networks that rely on a digital signal processor core are not as efficient as their analog counterparts. This paper builds on the L_p comparators for which efficient analog implementations exist that employ operational amplifiers. From a statistical point of view, L_p comparators are based on nonlinear means. Their probability density function and the first- and second-order moments are derived for independent uniformly distributed inputs. L_p comparators provide estimates of the minimum and maximum of their inputs. A proper approach to compensate for the estimation errors is proposed. Applications of the L_p comparators in odd–even transposition networks, median approximation networks, and min/max networks are presented.**

***Index Terms*—L_p comparators, L_p means, moments of the comparators, sorting networks.**

I. Introduction

Sorting is a fundamental operation in data processing. Sorting operations are estimated to account for over 25% of processing time for all computations [1]. Sorting networks are special cases of sorting algorithms. Much work has been done on sorting networks since the original idea was conceived by Batcher [2]. A sorting network is a combinatorial circuit constructed from *comparators*, which are also known as *compare-swap* units that sort [3]. A comparator takes two values as input and outputs them in ascending order. Let us consider a sorting network of N inputs $x_1, \ldots, x_N$. The number of the inputs defines the *I/O size* of the sorting network [4]. A sorting network consists of N parallel channels, which can be thought of as wires carrying values, to which comparators are attached. The network is divided into a finite number of levels that consist of one or more comparators. The number of levels is a reasonable measure of parallel time and defines the *depth* of the network. The outputs of a sorting network are $x_{(1)}, \ldots, x_{(N)}$, where $x_{(i)}$ denotes the ith-order statistic of the set $\{x_1, \ldots, x_N\}$. That is, $x_{(1)}$ denotes the smallest element of the set, whereas $x_{(N)}$ denotes the largest element. Optimal sorting networks have *cost* (i.e., number of constant fan-in processing nodes) $\mathcal{O}(N \log_2 N)$ and depth $\mathcal{O}(\log_2 N)$[3]. Two of the most commonly used sorting networks are the odd-even transposition network shown in Fig. 1(a) and the Batcher's bitonic sorter depicted in Fig. 2. They have small throughput delay and a very uniform structure. For example, the Batcher's bitonic sorter merges two monotonic sequences into a single sorted sequence. The sorting network has a fast Fourier transform (FFT)-like structure that is recursive, e.g., it can be applied to design the $N/2$-element sorters [5]. Although sorting networks based on functional units other than comparators have been proposed, e.g., sorting networks based on a three-element median [6], the most common type of sorting network employs comparators. The analysis presented in this paper is also applicable to sorting networks based on three-element median units as well. Recently, a sorting network is shown to be a wave digital filter realization of an N-port memoryless nonlinear classical network [7]. In general, sorting networks of large I/O size M are implemented by employing sorting networks of fixed I/O size $N \ll M$[4], [8], [9]. The fixed I/O size sorting networks can be either odd-even transposition networks or bitonic sorters [10]. This modularity makes the architecture very suitable for VLSI applications. It has been shown that the complexity of sorting M values is between $(M \log_2 M)/(N \log_2 N)$ and $4(M \log_2 M)/(N \log_2 N)$ up to first-order terms in M and N when sorting networks of fixed I/O size N are used [11]. Common choices of N are 5 [9] and 8

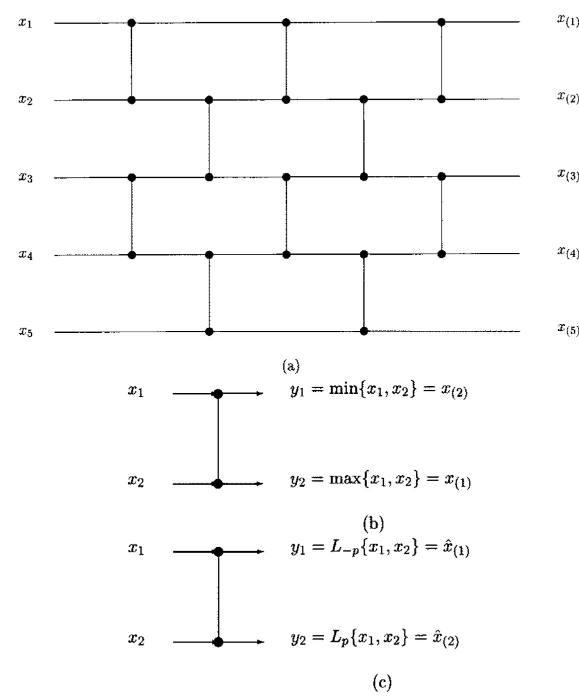

Fig. 1. (a) Odd-even transposition sorting network of $N = 5$ inputs. (b) Min/max comparator. (c) L_p comparator.

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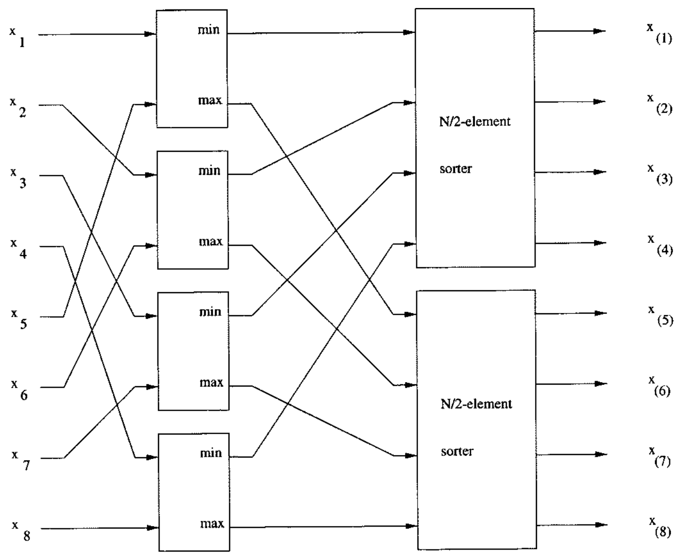


Fig. 2. Batcher's bitonic sorter for N numbers ($N = 8$).

[4], [8]. An efficient pipelined use of sorting networks of fixed I/O size N to sort $M \gg N$ numbers is studied in [4].

Sorting is the basic operation in order statistics filters that constitute effective techniques for image/signal processing due to their robustness properties. Order statistics filters employ usually a digital signal processor core. However, sorting is a computationally expensive operation, and a large area and power reduction can be obtained with simpler analog implementations [9]. Major reasons support the choice of analog implementation, namely [8]

i) The information/wiring ratio is better than in the digital approach.
ii) A smaller area is needed than in digital systems to obtain different functionality types.
iii) The basic cells are either bipolar junction transistors or operational amplifiers (op-amps) that are more compact and are faster than their digital counterparts.
iv) Analog implementations offer time complexity on the order of microseconds if they are based on op-amps (e.g., [12]) and on the order of tens of nanoseconds when they are based on bipolar junction transistors (e.g., [8]).

One class of implementations is based on op-amps. An op-amp-based circuit realization of a sorting network is described in [12]. It implements an analog sorting neural network that treats the sorting operation as an assignment problem. Another analog circuit based on op-amps suitable for the comparison and sorting of two input voltages is described in [13]. A nonlinear dynamic system that sorts N data is developed in [14]. It is based on the theory of completely integrable Hamiltonian systems, and its realization is based on op-amps as well. Another class of implementations of multiple-input min/max circuits is based on a common emitter/source configuration [15]. A novel configuration combining a voltage-mode common source circuit and a current-mode rank selector is proposed in [16]. This configuration implements a "winner takes all" circuit. A third class of implementations is based on current-mirroring [9], [17]–[19]. The design of sorting networks formed by sums of products or products of sums is reported in [8]. It is based on ADD and MUL gates realized by bipolar junction transistors. A closely related topic is the hardware/VLSI realization of median filters [20]–[23]. For a review of hardware median filters, see [24] and [25]. Existing architectures for median filters can be broadly classified into array-based and sorting network-based ones. The latter ones are of our interest in this paper. They are inherently pipelined, but they consist of a large number of compare-swap units. For example, median filter architectures based on bubble-sort [20], [23] require $N\lfloor N/2 \rfloor$ comparators to sort N numbers, whereas those based on Batcher's bitonic sort reduce the number of comparators to $\mathcal{O}(N \log_2 N)$[26]. Further shortcomings on the number of comparators are possible if the design is based on Batcher's odd-even merge sort [26]. In the latter case, $(\log_2^2 N - \log_2 N + 4)N/4$ comparators are needed [27]. However, the processing speed of the architecture in [26] is significantly lower than Batcher's odd-even sorting network [6]. Efficient parallel processing techniques, namely, pipelining and block processing, are employed in order to systematically determine shared merging networks and thus reduce the size of the maximum and minimum structures [27]. Sorting network-based architectures for nonrecursive and recursive weighted-order statistics filters are described in [28].

This paper builds on a new type of comparator (the L_p comparator) that can be materialized using op-amps [29]. Thus, the L_p comparator can be used for high-speed analog or hybrid signal processing. The L_p comparator is based on nonlinear mean filters [31], [32]. However, L_p comparators are "noisy" comparators. Therefore, we have to compensate for their errors before we replace the conventional comparators in a sorting network with them. To devise such an error compensation algorithm, first, the statistical properties of the L_p comparators are explored and compared against those of the min/max comparators. Then, we propose a simple error-compensation algorithm, and we derive theoretically the gain that is obtained when L_p comparators employing the proposed error compensation are used. Next, we develop a median approximation network that can be built using L_p comparators.

The major contributions of the paper are in

i) the derivation of the statistical properties of L_p comparators;
ii) the compensation for the errors that are introduced by L_p comparators;
iii) the generalization of L_p comparators to min/max networks of I/O size greater than 2;
iv) the concept of odd-even transposition sorting networks, median approximation networks, and min/max networks based on L_p mean comparators;
v) the experimental evidence that the aforementioned networks can be used within acceptable error levels for small values of p in the range [2, 5].

The paper focuses on the ideal performance of the L_p comparators, assuming that ideal op-amps are employed. It does not enter into the imperfections encountered when actual op-amps are used.

The outline of the paper is as follows. The definition of L_p comparators with two inputs is given in Section II. Their statistical properties are derived in this section as well. The compensation for the errors introduced by the L_p comparators, implementation issues, and the generalization of L_p comparators to min/max networks of I/O size greater than 2 are treated in Section III. Median approximation networks based on L_p comparators are described in Section IV. Experimental evidence is provided in Section V, and conclusions are drawn in Section VI.

II. L_p COMPARATORS

The L_p comparator employs nonlinear L_{-p} and L_p means with two inputs to estimate the minimum and maximum of two input samples, respectively [31], [32], i.e.,

$$\hat{x}_{(1)} = L_{-p}(x_1, x_2) = \left(\frac{x_1^{-p} + x_2^{-p}}{2}\right)^{-1/p} \quad (1)$$

$$\hat{x}_{(2)} = L_p(x_1, x_2) = \left(\frac{x_1^{p} + x_2^{p}}{2}\right)^{1/p} \quad (2)$$

where p is a positive real number different than 1, i.e., $p \in \mathbb{R}^+ - \{0, 1\}$. In contrast to classical min/max comparators, whose output is one of input samples, the L_p comparator provides estimates of the minimum and the maximum sample. L_p comparators can be treated as analog sorters in the sense that their outputs are not restricted to be one of their inputs. Such a performance can be tolerated in the case of order-statistics filters whose output is a linear combination of the rank-ordered input samples (e.g., α-trimmed mean, L-filters). If a fully analog implementation is pursued, the proposed L_p comparators could be used without requantization of the output so that it becomes one of the inputs. However, if such a quantization is required, then the procedures proposed in [33] can be employed, provided that the analog sorter preserves the ordering of input samples. Fig. 1(b) and (c) depict the min/max comparator and the L_p comparator. The comparator depicted in Fig. 1(b) or (c) is called *type 0 comparator* and is used in odd-even transposition networks. *Type 1* comparators that also support a swap function, i.e., they provide their outputs in descending order, can also be implemented by using L_p comparators. The latter comparators are used in bitonic sorting networks [4].

The proposed L_p comparators can easily be extended to min/max networks with I/O size $N > 2$ that estimate the minimum and maximum of N input samples by

$$\hat{x}_{(1)} = L_{-p}(x_1, x_2 \ldots, x_N) = \left(\frac{1}{N}\sum_{i=1}^{N} x_i^{-p}\right)^{-1/p} \quad (3)$$

$$\hat{x}_{(N)} = L_{p}(x_1, x_2 \ldots, x_N) = \left(\frac{1}{N}\sum_{i=1}^{N} x_i^{p}\right)^{1/p}. \quad (4)$$

The L_p comparators are "noisy" comparators. Indeed, the following well-known property [30] holds for $N = 2$:

$$x_{(1)} \le L_{-p}(x_1, x_2) \le \frac{1}{2}(x_1 + x_2) \le L_p(x_1, x_2) \le x_{(2)}. \quad (5)$$

In this section, the statistical properties of L_p comparators are derived for independent uniformly distributed input samples. The analysis will be confined to $N = 2$ to maintain mathematical tractability. Based on the results of this section, we modify the outputs of an min/max network of I/O size $N \ge 2$ in Section III as follows:

$$\tilde{x}_{(1)} = L_{-p}(x_1, x_2 \ldots, x_N) - \sum_{i=1}^{N-1} d_i |x_i - x_{i+1}| \quad (6)$$

$$\tilde{x}_{(N)} = L_{p}(x_1, x_2 \ldots, x_N) + \sum_{i=1}^{N-1} c_i |x_i - x_{i+1}| \quad (7)$$

where c_i and d_i are appropriate coefficients. It can be seen that for $N = 2$, (6) and (7) define the corrected outputs of an L_p comparator. Henceforth, N will be equal to 2.

If x_i, $i = 1, 2$ are independent random variables (RVs) uniformly distributed in the interval [0, L], the probability density function (pdf) of the RV $z = L_p(x_1, x_2)$ is given by (8), shown at the bottom of the page, where $B(\cdot)$ denotes the Beta function, and $I_x(a, b)$ is the incomplete Beta function defined as [36], [37]

$$I_x(a, b) = \frac{1}{B(a, b)} \int_0^x t^{a-1}\,(1-t)^{b-1}\,dt. \quad (9)$$

The derivation of (8) is given in Appendix A. For $p = 2$, we obtain

$$f(z) = \begin{cases} \frac{\pi}{L^2} z, & \text{if } 0 \le z < \frac{1}{\sqrt{2}}\,L \\ \frac{2}{L^2} \arcsin\left(\frac{L^2 - z^2}{z^2}\right), & \text{if } \frac{1}{\sqrt{2}}\,L \le z < L \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

The pdf of the RV z is plotted for p =2, 5, and 8 in Fig. 3. For completeness, the pdf of the RV $x_{(2)}$ for uniform parent distribution in the interval [0, L] and $N = 2$ is included $f_{(2)}(x) = (2/L)\,(x/L)$, $0 \le x \le L$.

The expected value and the mean square value of the RV z are given by

$$E\{z\} = \frac{2L}{3}\left(1 - \frac{2^{-1/p}}{p}\int_0^1 t^{1/p}(1+t)^{1/p-1}dt\right) \quad (11)$$

$$E\{z^2\} = \frac{L^2}{2}\left(1 - \frac{2^{-2/p}}{p}\int_0^1 t^{1/p}(1+t)^{2/p-1}dt\right) \quad (12)$$

$$f(z) = \begin{cases} \frac{2^{2/p}}{L^2 p}\, B\left(\frac{1}{p}, \frac{1}{p}\right) z, & \text{if } 0 \le z < 2^{-1/p}\,L \\ \frac{2^{2/p}}{L^2\, p}\, B\left(\frac{1}{p}, \frac{1}{p}\right) z\, \left(2\, I_{(L^p/2z^p)}\left(\frac{1}{p}, \frac{1}{p}\right) - 1\right), & \text{if } 2^{-1/p}\,L \le z < L \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

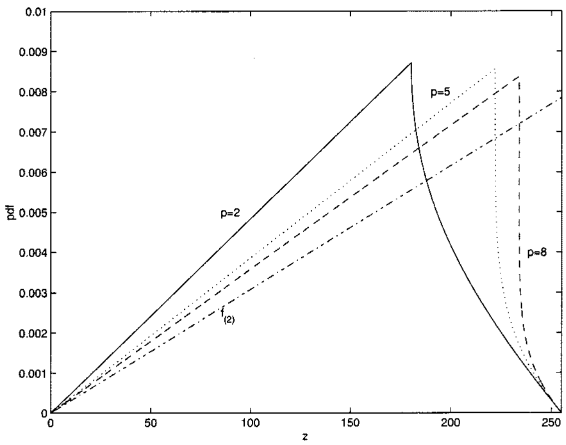


Fig. 3. Probability density function of the RV $z = L_p(x_1, x_2)$ for p = 2, 5, and 8 when x_1 and x_2 are independent RVs uniformly distributed in the interval [0, L].

respectively. The derivation of (11) is outlined in Appendix A. Equation (12) can be proven similarly. For $p = 2$, we obtain

$$E\{z\} = 2^{-1/2}\,\frac{L}{3}\left(\sqrt{2} - \ln\tan\left(\frac{\pi}{8}\right)\right),\; E\{z^2\} = \frac{1}{3}\,L^2. \tag{13}$$

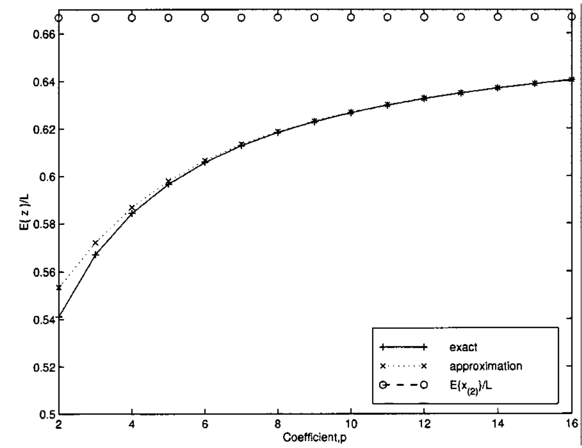
The following approximate expressions for the first and second moment of the RV z hold:

$$E\{z\} \approx \frac{L}{2}\left[(1 + 2^{-1/p}) - \frac{B\left(\frac{1}{p},\, \frac{1}{p}\right)}{6\,p}(3 - 2^{-1/p})\right] \tag{14}$$

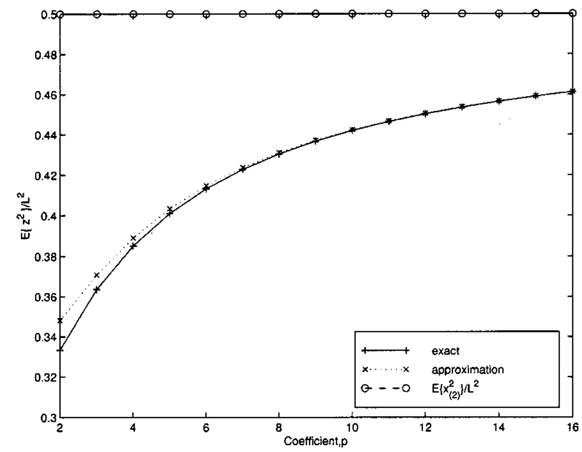
$$E\{z^2\} \approx 2^{-2/p}\,L^2\;\left[2^{1/p} + \frac{B\left(\frac{1}{p},\, \frac{1}{p}\right)}{2p}\;\left(\frac{1}{2} - 2^{1/p}\right)\right]. \tag{15}$$

The proofs of (14) and (15) are given in Appendix B. The expected value and the mean square value of the RV z for several values of the coefficient p are plotted in Fig. 4(a) and (b), respectively. The approximate values obtained by using (14) and (15) are overlaid for comparison purposes. It is seen that for $p > 8$, the values obtained by the approximate expressions are practically the same as those obtained by the numerical integration of (11) and (12). The expressions in (11) and (12) should be compared with those of the order statistics for $N = 2$ and uniform parent distribution that are given by [34] and [38]

$$E\{x_{(2)}\} = \frac{2L}{3}, \quad E\{x_{(2)}^2\} = \frac{L^2}{2}. \tag{16}$$



(a)



(b)

Fig. 4. First and second moment of the RV $z = L_p(x_1, x_2)$ for several values of the coefficient p. (a) Expected value. (b) Mean square value.

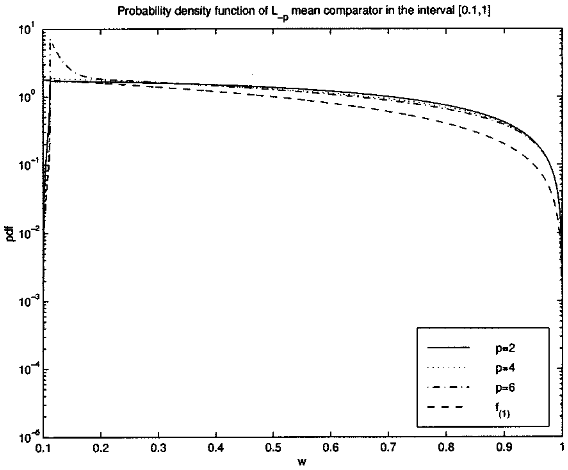
It is obvious that the first and second moments of the RV z tend to those of the RV $x_{(2)}$ for large p.

Similarly, if $x_i,\ i = 1,\ 2$ are independent RVs uniformly distributed in the interval $[\epsilon, L]$, the pdf of the RV $w = L_{-p}(x_1, x_2)$ is given by (17), shown at the bottom of the page. The proof of (17) is omitted due to lack of space. For $p = 2$, (17) is simplified to

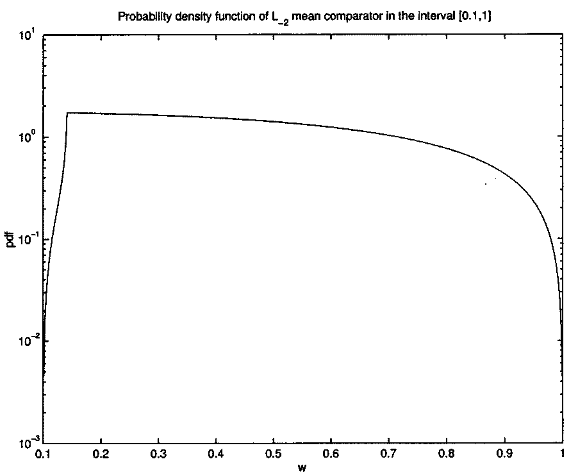
$$f_w(w) = \begin{cases} \frac{2}{(L-\epsilon)^2}\,\frac{w^2-\epsilon^2}{\sqrt{2\epsilon^2-w^2}}, & \text{if } \epsilon < w \le \sqrt{2}\frac{\epsilon\,L}{\sqrt{\epsilon^2+L^2}} \\ \frac{2}{(L-\epsilon)^2}\,\frac{L^2-w^2}{\sqrt{2L^2-w^2}}, & \text{if } \sqrt{2}\frac{\epsilon\,L}{\sqrt{\epsilon^2+L^2}} < w \le L \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

The pdf of the RV $x_{(1)}$ for uniform parent distribution in the interval [0, L] for $N = 2$ is given by $f_{(1)}(x) = 2/L\ (1 - x/L)$, $0 \le x \le L$. For $x_i,\ i = 1,\ 2$ independent RVs uniformly distributed in the interval [0.1,1], the pdf of the L_{-p} comparator output is found by employing numerical integration and is

$$f_w(w) = \begin{cases} \frac{2^{-2/p}}{p(L-\epsilon)^2}\;w\;\int_{1-w^p/2\,\epsilon^p}^{w^p/2\,\epsilon^p} t^{-(1+1/p)}(1-t)^{-(1+1/p)}\,dt, & \text{if } \epsilon < w \le 2^{1/p}\frac{\epsilon\,L}{(\epsilon^p+L^p)^{1/p}} \\ \frac{2^{-2/p}}{p\,(L-\epsilon)^2}\;w\;\int_{w^p/2L^p}^{1-w^p/2L^p} t^{-(1+1/p)}(1-t)^{-(1+1/p)}\,dt, & \text{if } 2^{1/p}\frac{\epsilon\,L}{(\epsilon^p+L^p)^{1/p}} < w \le L \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$



(a)



(b)

Fig. 5. (a) Probability density function of the RV $w = L_{-p}(x_1, x_2)$ obtained by numerical integration for $p = 2$, 4, and 6 when x_1 and x_2 are independent RVs uniformly distributed in the interval [0.1,1]. (b) Exact probability density function of the RV $w = L_{-2}(x_1, x_2)$ when x_1 and x_2 are independent RVs uniformly distributed in the interval [0.1,1].

plotted in Fig. 5(a) for $p = 2$, 4, and 6. The plot of (18) is shown in Fig. 5(b) for comparison purposes.

The limit of the expected value and the mean square value of the RV w for $\epsilon \to 0$ is given by

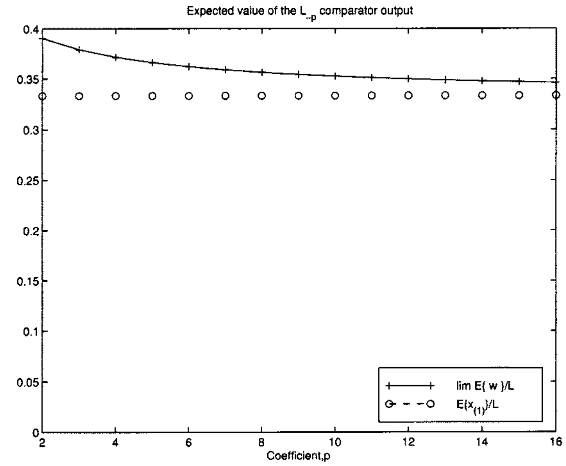
$$\lim_{\epsilon \to 0} E\{w\} = \frac{4}{3}\, 2^{1/p-1}\, L \int_0^{2^{-1/p}} \frac{t\, dt}{(1-t^p)^{1+1/p}} \quad (19)$$

$$\lim_{\epsilon \to 0} E\{w^2\} = 2^{2/p-1}\, L^2 \int_0^{2^{-1/p}} \frac{t^2\, dt}{(1-t^p)^{1+1/p}} \quad (20)$$

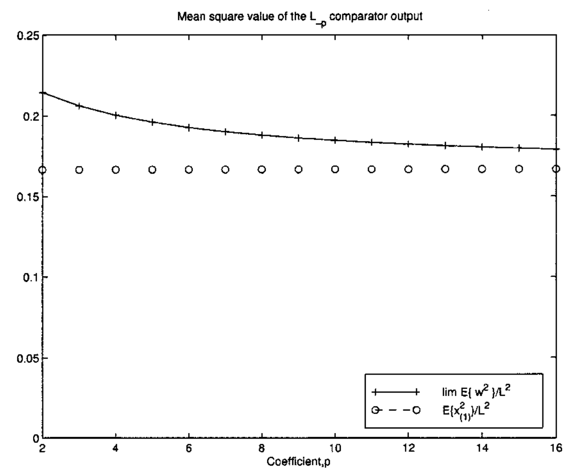
respectively. The derivation of (19) and (20) is outlined in Appendix C. For $p = 2$, we obtain

$$\lim_{\epsilon \to 0} E\{w\} = \frac{4}{3}\, L \left(1 - \frac{1}{\sqrt{2}}\right), \; \lim_{\epsilon \to 0} E\{w^2\} = L^2 \left(1 - \frac{\pi}{4}\right). \quad (21)$$

The expected value and the mean square value of the RV w for several values of the coefficient p are plotted in Fig. 6(a) and (b), respectively. It can easily be verified that for p large, the



(a)



(b)

Fig. 6. Limit of first and second moment of the RV $w = L_{-p}(x_1, x_2)$ for several values of the coefficient p when $\epsilon \to 0$. (a) Expected value. (b) Mean square value.

first and the second moment of the RV w approximate those of the RV $x_{(1)}$, i.e.,

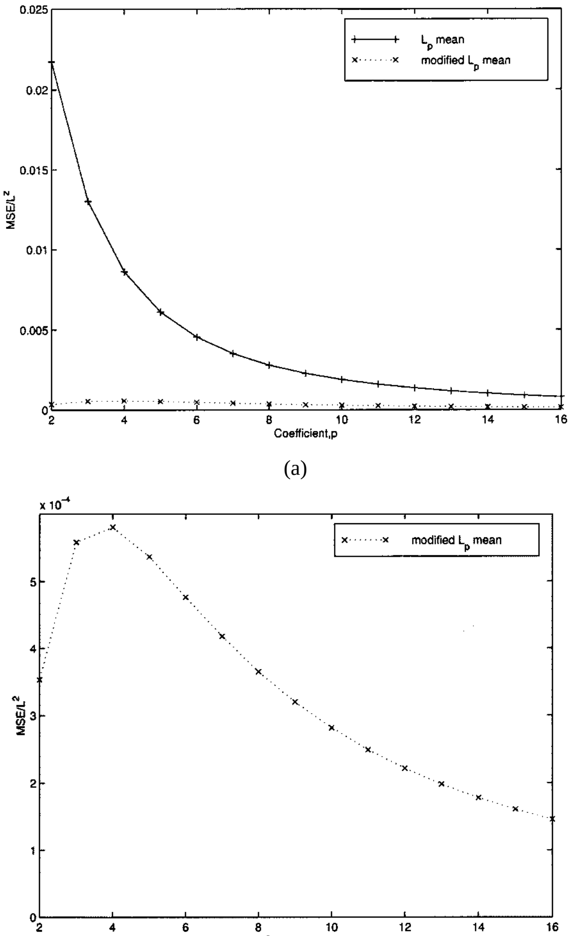
$$E\{x_{(1)}\} = \frac{L}{3}, \quad E\{x_{(1)}^2\} = \frac{L^2}{6}. \quad (22)$$

III. Error Compensation and Implementation Issues

Having derived the statistical properties of the L_p comparators in the previous section, we will estimate first the error introduced by a single L_p comparator and propose a method to compensate for it. Second, we discuss the overall error introduced by the L_p comparators in a sorting network when they replace the classical min/max comparators. Next, we describe the implementation of the L_p comparators studied in this paper, and finally, we generalize to the case of min/max networks of more than two inputs.

A. Error Introduced by a Single L_p Comparator

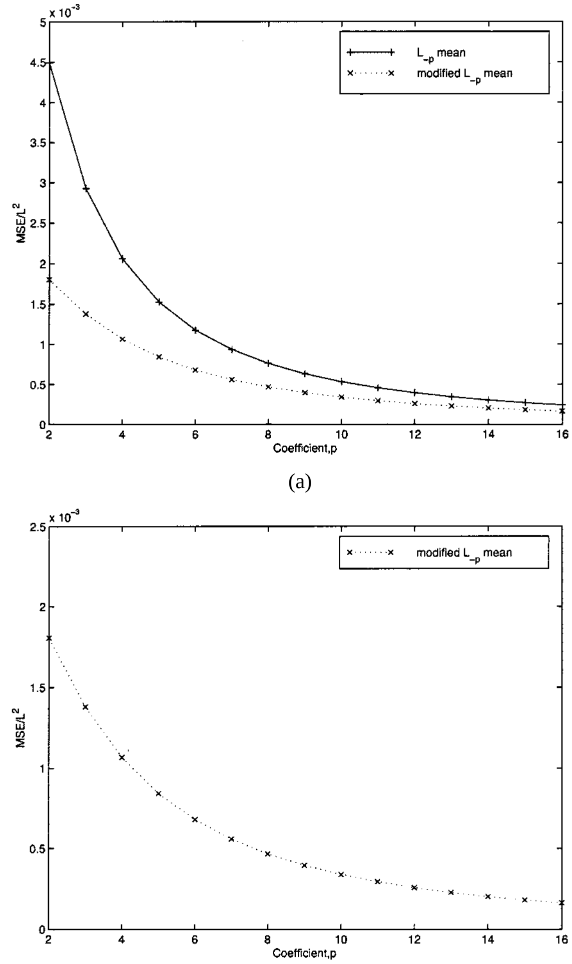
Let $e_{\max}(x_1, x_2) = x_{(2)} - \hat{x}_{(2)}$ denote the error introduced by the L_p comparator in the estimation of the maximum of two input samples. Similarly, let $e_{\min}(x_1, x_2) = x_{(1)} - \hat{x}_{(1)}$ denote the corresponding error in the estimation of the minimum of two



(a)

(b)

Fig. 7. (a) MSE of the L_p comparator and the modified L_p comparator in estimating the maximum of two independent input samples that are uniformly distributed in the interval [0, L]. (b) Zoom in the MSE introduced by the modified L_p comparator.



(a)

(b)

Fig. 8. (a) Limit of the MSE of the L_{-p} comparator and the modified L_{-p} comparator in estimating the minimum of two independent input samples that are uniformly distributed in the interval $[\epsilon, L]$ for $\epsilon \to 0$. (b) Zoom in the MSE introduced by the modified L_{-p} comparator.

input samples. Inequalities (5) imply that

$$\begin{aligned} 0 \le e_{\max}(x_1, x_2) \le& \frac{1}{2}\, |x_2 - x_1|, \\ -\frac{1}{2}\, |x_2 - x_1| \le& e_{\min}(x_1, x_2) \le 0. \end{aligned} \tag{23}$$

Moreover, if the absolute value of the difference between x_1 and x_2 is large, that is, $\max\{x_1, x_2\} \gg \min\{x_1, x_2\}$, then we obtain $e_{\max}(x_1, x_2) \simeq \left(1 - 1/2^{1/p}\right)\max\{x_1, x_2\}$. On the other hand, if $x_1 \simeq x_2$, then $e_{\max}(x_1, x_2) \simeq 0$. Similar results can also be found for $e_{\min}(x_1, x_2)$.

For x_i, $i = 1, 2$, independent RVs uniformly distributed in the interval [0, L], it is shown in Appendix D that the mean squared error (MSE) introduced by the L_p comparator is given by

$$\begin{aligned} E\{e^2_{\max}(x_1, x_2)\} =& \frac{L^2}{2} \Big\{ 1 + 2^{-2/p} \\ &\times \int_0^1 (1 + t^p)^{2/p} dt - 2^{1-1/p} \int_0^1 (1 + t^p)^{1/p}\, dt \Big\}. \end{aligned} \tag{24}$$

For $p = 2$, we obtain $E\{e^2_{\max}(x_1, x_2)\} = L^2/2 \left(2/3 - \ln(1 + \sqrt{2})/\sqrt{2}\right)$. The MSE of the L_p comparator is plotted for several values of the coefficient p in Fig. 7(a). It is seen that the larger the coefficient p is, the smaller the MSE introduced by the L_p comparator becomes. Accordingly, for large values of the coefficient p, the L_p comparator converges to the max operator, as expected.

If x_i, $i = 1, 2$, are independent RVs uniformly distributed in the interval $[\epsilon, L]$, for $\epsilon \to 0$, the limit of the MSE of the L_{-p} comparator is

$$\begin{aligned} \lim_{\epsilon \to 0} E\{e^2_{\min}(x_1, x_2)\} =& L^2 \Big\{ \frac{1}{6} + 2^{2/p-1} \\ &\times \int_0^1 \frac{t^2}{(1 + t^p)^{2/p}} dt - 2^{1/p} \int_0^1 \frac{t^2}{(1 + t^p)^{1/p}}\, dt \Big\}. \end{aligned} \tag{25}$$

For $p = 2$, (25) is simplified to $\lim_{\epsilon \to 0} E\{e^2_{\min}(x_1, x_2)\} = L^2 \left(1/6 - \pi/4 + \sqrt{2}/2 \,\ln(1 + \sqrt{2})\right)$. The MSE of the L_{-p} comparator is plotted for several values of the coefficient p in Fig. 8(a) as well. It is seen that the larger the coefficient p is, the smaller the MSE introduced by the L_{-p} comparator becomes. Accordingly, for large values of the coefficient p, the L_{-p} comparator converges to the min operator, as expected.

Next, we compensate for the MSE introduced by the L_p comparators for small p. We argue that the estimation error increases

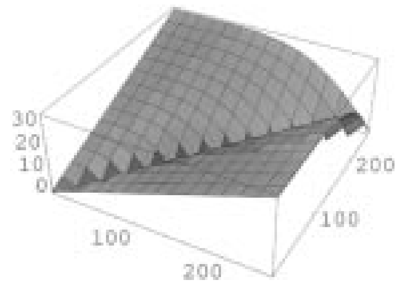


Fig. 9. Absolute value of $e_{\max}$ for $p = 5$.

almost linearly with the absolute value of the difference between x_1 and x_2. No error is introduced when $|x_2 - x_1| \simeq 0$. The inequalities (23) set an upper and lower bound on the estimation error that is a linear envelope in terms of $|x_1 - x_2|$. Indeed, for x_1 and x_2 independent RVs uniformly distributed in the interval (0, 255], the plot of the $e_{\max}$, as a function of the comparator inputs x_1, x_2 for $p = 5$ in Fig. 9, demonstrates that the error increases almost linearly with the distance from the line $x_2 = x_1$. Accordingly, we propose to modify the L_p comparator outputs as

$$\tilde{x}_{(1)} = L_{-p}(x_1, x_2) - d|s|, \quad d > 0 \tag{26}$$

$$\tilde{x}_{(2)} = L_p(x_1, x_2) + c\,|s|, \quad c > 0 \tag{27}$$

where $s = x_2 - x_1$ and c, d are constants. By doing so, the error between $\tilde{x}_{(1)}$ and $x_{(1)}$ and the corresponding error between $\tilde{x}_{(2)}$ and $x_{(2)}$ is bounded by

$$\left(d - \frac{1}{2}\right)|s| \le \tilde{e}_{\min} \le d\,|s|, \quad \tilde{e}_{\min} = x_{(1)} - \tilde{x}_{(1)} \tag{28}$$

$$-c\,|s| \le \tilde{e}_{\max} \le \left(\frac{1}{2} - c\right)|s|, \ \tilde{e}_{\max} = x_{(2)} - \tilde{x}_{(2)} \tag{29}$$

respectively. The dependence of $\tilde{x}_{(1)}$ and $\tilde{x}_{(2)}$ on $|s|$ is partially explained from the fact that [7]

$$x_{(1)} = \frac{x_1 + x_2}{2} - \frac{|s|}{2} \tag{30}$$

$$x_{(2)} = \frac{x_1 + x_2}{2} + \frac{|s|}{2}. \tag{31}$$

The constants c and d can be chosen so that $E\{\tilde{e}^2_{\max}\}$ and $\lim_{\epsilon \to 0} E\{\tilde{e}^2_{\min}\}$ is minimized, respectively. In Appendix D, it is shown that the optimal constants c and d are given by

$$c = \frac{3}{2} + 3 \cdot 2^{-1/p} \int_0^1 (s-1)\,(1+s^p)^{1/p}\,ds \tag{32}$$

$$d = -\frac{1}{2} - 3 \cdot 2^{1/p} \int_0^1 \frac{s\,(s-1)}{(1+s^p)^{1/p}}\,ds. \tag{33}$$

The optimal constants c and d are plotted for several values of the coefficient p in Fig. 10(a) and (b), respectively. As expected by inequalities (29) and (28), they are much smaller than 0.5. The MSE between the modified L_p comparator output (27) and the true maximum sample is given by $E\{\tilde{e}^2_{\max}\} = E\{e^2_{\max}\} - (c^2\ L^2/6)$. It is overlaid in Fig. 7(a) for comparison purposes. A zoom in the plot of the MSE versus p is depicted in Fig. 7(b). Similarly, the MSE between the modified L_{-p} comparator output (26) and the true minimum sample $E\{\tilde{e}^2_{\min}\} = E\{e^2_{\min}\} - (d^2\ L^2/6)$ is shown overlaid in Fig. 8. The details of the variation of MSE versus p are revealed in Fig. 8(b).

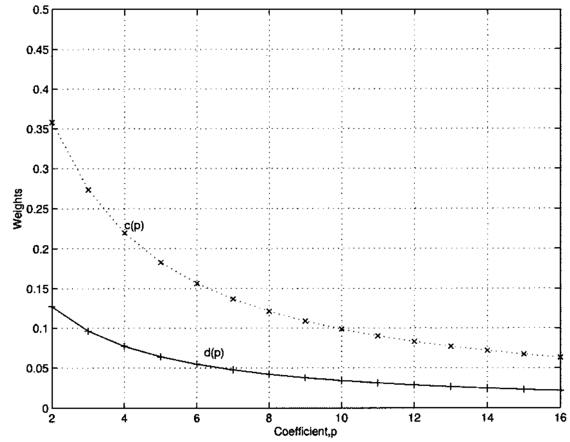


Fig. 10. Optimal constants c and d that minimize the MSE between the modified L_p comparator output and the true maximum and minimum of two independent uniformly distributed samples for several values of the coefficient p.

B. Overall Error

If L_p comparators are used in a sorting network, e.g., the odd-even transposition network of Fig. 1(a), it is obvious that the errors introduced by each L_p comparator propagate through the network and accumulate at the sorting network outputs. For large values of N or for small values of p, the ordering of sorting output samples may no longer hold. Quantitative results for the errors introduced at the several ranked-ordered samples are given in Section V, where it can be seen that the proposed modified L_p comparators reduce the accumulated error at the outputs of an odd-even transposition network or a median approximation network.

C. Implementation

The basic module in the analog implementation of the L_p comparator is the so-called *multifunction converter* [29, pp. 113–116]. A multifunction converter consists of four operational amplifiers, four logging transistors, and four resistors. The exponent p is determined by two external resistors. Raising to an arbitrary power p and computing the pth root can be achieved with the same module by controlling an external potentiometer together with two fixed resistors. Accuracy of 0.2% can be achieved for p ranging from 0.2 to 5 [29]. This module can be used to raise to a power of p, to compute pth roots, and as a divider. The latter operation is need in the implementation of an L_{-p} comparator. Moreover, the same module can be used to implement the correction term in (26) and (27). Clearly, the absolute value can be computed with a cascade of an adder and a multifunction converter that can be used first to raise to an even power (e.g., 2 or 4) and then to compute the pth root. Accordingly, efficient pipelined architectures for L_p comparators of two inputs can be developed to estimate the minimum and maximum by exploiting an adder, a multifunction converter used to raise to the power p and compute the pth root, and a divider.

D. Generalization to Min/Max Sorting Networks With Higher I/O Size

One may argue that (30) and (31) indicate that an ideal comparator can be implemented simply with a multifunction converter used to evaluate the absolute value and a few op-amp-based adders. The major benefit of L_p comparators stems from the fact that they can easily be extended to min/max networks with I/O size $N > 2$ that estimate the minimum and maximum of N input samples by $L_{-p}(x_1, x_2, \ldots, x_N)$ given by (3) and $L_{-p}(x_1, x_2, \ldots, x_N)$ and defined by (4), respectively. It can easily be verified that the min/max network employing (3) and (4) delivers an estimate of either min or max faster than any sorting network specialized for min/max calculations using regular comparators defined by (30) and (31). Moreover, the first estimates provided by (3) and (4) can be further corrected by generalizing (26) and (27) to (6) and (7), respectively. The expressions (6) and (7) are related to the so-called *piecewise-linear functions* (PWL), which are a widely used class of nonlinear approximate functions [39], [40]. In these equations, the coefficients c_i and d_i, $i = 1, 2, \ldots, N-1$ can then be derived arithmetically by a general least squares fit. The use of min/max network defined by (6) and (7) is found to reduce the estimation errors effectively, as can be seen in Section V. $L_{-p}(x_1, x_2, \ldots, x_N)$ and $L_p(x_1, x_2, \ldots, x_N)$ are two nonlinear functions whose nonlinearity increases with the increase of the exponent p. The compensation terms $-\sum_{i=1}^{N-1} d_i|x_i - x_{i+1}|$ and $\sum_{i=1}^{N-1} c_i|x_i - x_{i+1}|$ are, in fact, two piecewise-linear nonlinear functions whose approximation properties depend on the coefficients c_i and d_i. However, the aforementioned piecewise-linear nonlinear functions are not universal approximate functions, and they do not yield a canonical representation [39], [40]. The compensation effect can be enhanced further if a canonical piecewise-linear function is considered at the expense of an increase in the hardware complexity.

IV. Median Approximation Network

L_p mean comparators can also be used to estimate only the sample median. Let N be odd. The median of a set of N elements is the νth-order statistic $x_{(\nu)}$, where $\nu = (N + 1/2)$. Let us consider its subsets consisting of ν elements that are $C(N, \nu) = \binom{N}{\nu}$ in total. It can easily be proven that there is at least one ν-element subset $\mathcal{C}_i$ of the N-element set that satisfies

$$\max\{\mathcal{C}_i\} = x_{(\nu)},\ i \in \{1, \ldots, C(N, \nu)\} \tag{34}$$

while the remaining $(C(N, \nu) - 1)$ ν-element subsets satisfy

$$\max\{\mathcal{C}_j\} \geq \max\{\mathcal{C}_i\},\ i, j \in \{1, \ldots, C(N, \nu)\},\ j \neq i. \tag{35}$$

If the $C(N, \nu)$ maxima $\max\{\mathcal{C}_j\}$ are computed, then by (34) and (35)

$$x_{(\nu)} = \min\{\max\{\mathcal{C}_j\}, \quad 1 \leq j \leq C(N, \nu)\}. \tag{36}$$

Similarly, there is a dual form of (36), i.e.,

$$x_{(\nu)} = \max\{\min\{\mathcal{C}_j\},\ 1 \leq j \leq C(N, \nu)\}. \tag{37}$$

Since an L_{-p} comparator is a more costly operator because it employs dividers and yields generally larger errors than an L_p comparator of the same input size for small p values, we will comment on the implementation of (37). This implementation of median approximation network requires $C(N, \nu)$ L_{-p} comparators of ν inputs each and one L_p comparator of $C(N, \nu)$ inputs. Each L_{-p} unit operates in parallel with the rest of the $C(N, \nu) - 1$ units. The output of the L_{-p} comparator is an approximation of the subset minimum. Subsequently, the $C(N, \nu)$ minima are fed to an L_q comparator, which produces an estimate of the maximum of its inputs. The use of different symbols p and q manifest that there is no need for the coefficients to be the same in both comparators. Obviously, the above-described implementation has a cost, which comes in the form of the number of L_p comparators required. Clearly, for large values of the input size (i.e., ν or $C(N, \nu)$), the input set should be partitioned to subsets of smaller size. Min/max computations should be performed on these subsets, and the partial results should be combined to yield the final minimum or maximum. This is equivalent to performing efficient pipelining and block processing operations in a hardware implementation.

V. Simulation Results

Three sets of experiments have been conducted. The first set of experiments measures the approximation errors at the outputs of a sorting network when L_p comparators replace the classical min/max comparators. The second set of experiments measures the error introduced by the L_p comparators in a median selection network. The third set of experiments refers to the error introduced by the generalized min/max networks for I/O size $N > 2$.

A. Errors at the Outputs of a Sorting Network

The first set of experiments aims at measuring the errors at the outputs of an odd-even transposition sorting network using L_p comparators and demonstrating the error compensation achieved when the modified L_p comparators replace the L_p comparators. We created 5000 random samples that were uniformly distributed in the interval [0.0001, 255]. An odd-even transposition network of Fig. 1(a) was used where the min and max operators had been replaced by the L_p comparators defined by i) (1) and (2), and ii) (26) and (27), respectively. The root mean square (RMSE) error $RMSE_{(i)}$ between the actual ith rank-ordered output and its approximation, when either L_p comparators or modified L_p comparators were employed, was used as a quantitative criterion, i.e.,

$$\mathrm{RMSE}_{(i)} = \sqrt{E\{(x_{(i)} - \hat{x}_{(i)})^2\}} \simeq \sqrt{\frac{1}{K} \sum_{k=1}^{K} \left(x_{(i)}^{(k)} - \hat{x}_{(i)}^{(k)}\right)^2}. \tag{38}$$

In (38), K is the number of tests performed when a sliding window of length N is applied to the sequence of random samples. Since many nonlinear image filtering schemes employ three, five, or nine-point filter windows (one-dimensional and 3×3 two-dimensional, respectively), tests were performed on sorting networks of N inputs, where N took values in the set {3, 5, 9}.

In each case, parameter p took values in the set {2, 5, 8}. The RMSE for each sorting network output is plotted in Fig. 11(a) and (c) for $N = 5$ and $N = 9$ inputs, respectively, when the L_p

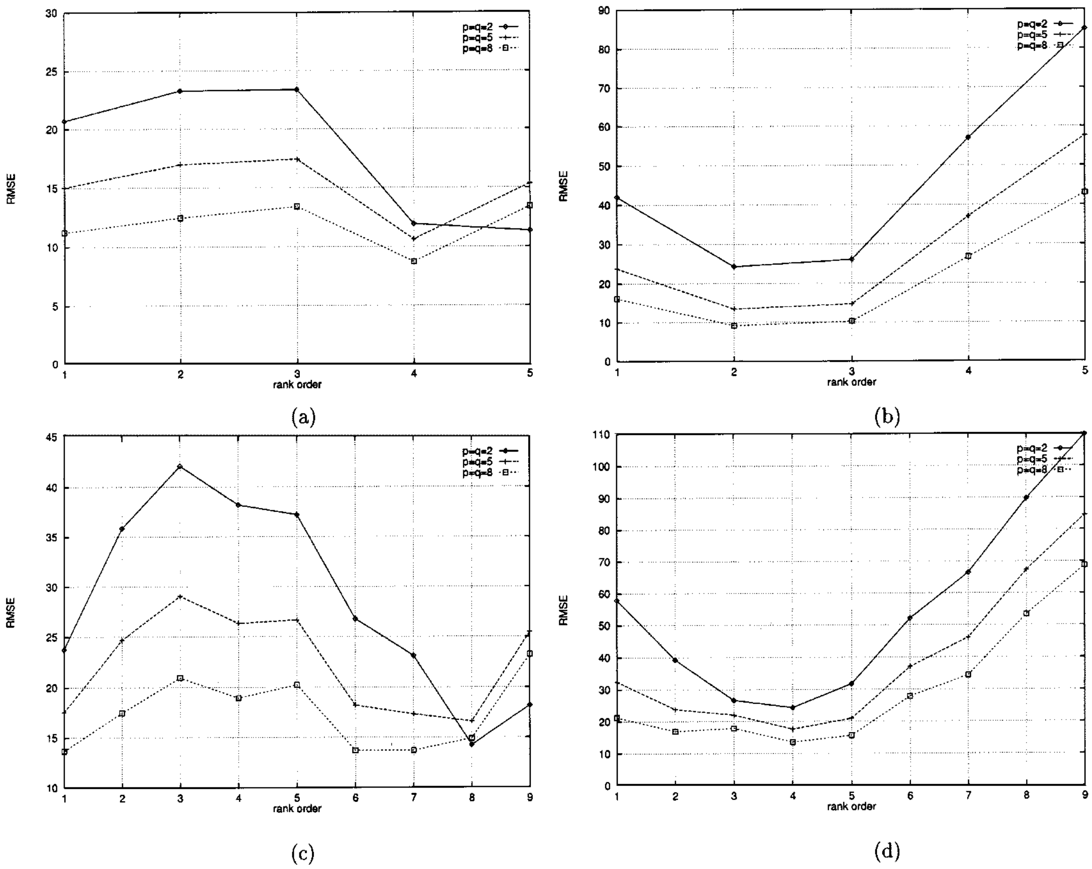


Fig. 11. Root mean square error at each output of an odd-even transposition network. (a) Sorting network of $N = 5$ inputs when L_p comparators (without compensation) are used. (b) Sorting network of $N = 5$ inputs when modified L_p comparators (with compensation) are used. (c) Sorting network of $N = 9$ inputs when L_p comparators are used. (d) Sorting network of $N = 9$ inputs when modified L_p comparators are used.

comparators defined in (1) and (2) were used. It is seen that the estimates of the higher ranked ordered samples deviate much from the actual values. The ranked ordered samples close to the sample median are estimated with fewer errors. As expected, the errors reduce as p increases. However, accurate analog implementations are available for small p values, i.e., $2 \leq p \leq 5$, as has been explained in Section III-C. For larger values of p, cascades of multifunction converters should be used, which is a fact that increases the hardware complexity. Small approximation errors were achieved when the modified L_p comparators defined in (26) and (27) were used. The $RMSE_{(i)}$ resulted are plotted in Fig. 11(b) and (d) for $N = 5$ and $N = 9$ inputs, respectively. The simulation results indicate that the approximation errors are within acceptable levels. In this case, the sample minimum and the sample maximum are more closely approximated than the sample median. To reduce the RMSE for the median estimate, an appropriate selection of the exponent p in L_{-p} comparators should be made. Such a selection is proposed in the next subsection.

B. Errors in Median Approximation Networks

The RMSE between the true median sample and its approximation, when three median approximation networks, namely

I) the odd-even transposition network of Fig. 1(a) with L_{-p} and L_q comparators having $p = 2q$;

II) median approximation network (37) with L_{-p} and L_q comparators having $p = q$;

III) median approximation network (37) with L_{-p} and L_q comparators having $q = 2p$;

were used for $N \in \{3, 5, 9\}$. The input samples were those of the previous set of experiments. For the network *I*, the output that corresponds to the median sample was taken into account only. The simulation results are summarized in Table I. It is seen that if q is chosen to be $2p$, the RMSE for the median estimate provided by the network *I* when compensation is used is always less than that without compensation. As a basis of comparison, the RMSE between the true median sample and the arithmetic mean for the same values of N is included in the column *IV* of this table. The best RMSE values are written boldface. It can be seen that when the proposed modified L_p comparators are employed in each median approximation network, the approximation errors are significantly reduced. Moreover, for $N = 3$ and 5, the median approximation network (37) outperforms the odd-even transposition network with respect to the RMSE. For $N = 9$, the situation is reversed due to the large sorting network of $C(N, \nu)$ inputs that need to be implemented. However, the RMSE achieved by the median approximation network is not far behind. From this point of view, all three median approximation networks are practically equivalent. Moreover, their performance is much better than that of the arithmetic mean in the estimation of the median sample.

TABLE I
ROOT MEAN SQUARE ERROR IN THE APPROXIMATION OF THE MEDIAN OF AN N-ELEMENT SET.

N	p	*I* without compensation	*I* with compensation	*II* without compensation	*II* with compensation	*III* without compensation	*III* with compensation	*IV*
3	2	30.958	9.847	33.002	11.721	20.156	10.441	26.703
	5	16.367	7.351	17.903	10.221	8.534	6.65	
	8	10.887	5.529	11.696	7.835	**5.097**	**4.664**	
5	2	30.157	20.393	46.137	19.980	32.319	17.809	24.345
	5	17.037	11.715	32.296	17.148	16.972	10.667	
	8	11.599	8.148	23.643	12.743	**10.834**	**7.303**	
9	2	39.953	33.556	54.367	39.874	40.370	35.555	20.840
	5	27.234	17.253	46.472	32.824	28.088	18.898	
	8	**20.135**	**11.330**	38.750	22.891	20.939	12.408	

C. Errors in Generalized Min/Max Networks

For $N = 9$, the generalized min/max networks discussed in Section III-D can be used to further reduce the RMSE. Such a network with L_{-p} and L_q comparators having $p = 8$ and $q = 16$ was tested in the same framework. The RMSE for the minimum, the median, and the maximum sample using (3) and (4) as well as using (6) and (7) are given in Table II. The computation of the median was done according to the analysis presented in Section IV. This analysis can be extended to any order statistic, as is demonstrated in [9]. A much lower RMSE for the median estimate than that of the network *III* (see Table I) is obtained. This is attributed to the more accurate estimates provided by the generalized min/max networks and the fewer accumulation errors that occurred.

VI. CONCLUSIONS

This paper has introduced the L_p comparator, which is a unit that can be utilized in analog implementations of sorting networks to improve performance. Analog implementations of L_p comparators employ multifunction converters that are based on operational amplifiers. The statistical properties of the L_p comparators have been derived for independent uniformly distributed inputs. Since L_p comparators provide only approximations of the minimum and the maximum of their inputs, an efficient compensation approach has been devised that succeeds in reducing significantly those errors. Generalizations of L_p comparators to min/max networks of I/O size $N > 2$ have also been proposed. Applications of the L_p comparators in odd–even transposition networks, median approximation networks, and min/max networks that employ the proposed modified L_p comparators have been demonstrated to operate within acceptable error levels.

TABLE II
ROOT MEAN SQUARE ERROR IN THE APPROXIMATION OF THE MINIMUM, THE MEDIAN, AND THE MAXIMUM OF NINE OBSERVATIONS USING GENERALIZED MIN/MAX NETWORKS WITH L_{-p} AND L_q COMPARATORS

p	q	order statistic	without compensation	with compensation
8	16	minimum	9.430	8.418
		median	17.291	5.06
		maximum	25.502	7.955

APPENDIX A

A. Proof of (8)

Let x_i, $i = 1, 2$ be independent RVs uniformly distributed in the interval [0, L]. Then, the pdf of the RV $\zeta = x^p$ is given by $f_\zeta(\zeta) = (1/pL)\ (1/\zeta^{1-1/p})$, $0 \le \zeta \le L^p$[35]. The pdf of the RV $\xi = \zeta_1 + \zeta_2$, where $\zeta_i = x_i^p$, $i = 1, 2$, is given simply by the convolution of the pdfs $f_{\zeta_1}(\xi)$ and $f_{\zeta_2}(\xi)$[35]. That is, we have (A.1), shown at the bottom of the page. In (A.1), it is easily seen that [36]

$$\int_0^\xi \lambda^{1/p-1}\,(\xi-\lambda)^{1/p-1}\,d\lambda = B\left(\frac{1}{p},\frac{1}{p}\right)\xi^{2/p-1} \qquad \text{(A.2)}$$

where $B(\cdot)$ denotes the Beta function. To calculate the remaining integral, we apply the change of variable $\lambda = \xi/2 + \psi$, i.e.,

$$\begin{aligned}\int_{\xi/2-L^p}^{L^p-\xi/2} &\left(\frac{\xi}{2}+\psi\right)^{1/p-1}\left(\frac{\xi}{2}-\psi\right)^{1/p-1} d\psi \\ &= \xi^{2/p-1}\left(B\left(\frac{1}{p},\frac{1}{p}\right)\right. \\ &\qquad \left. -2\int_0^{1-L^p/\xi}\psi^{1/p-1}\,(1-\psi)^{1/p-1}\,d\psi\right). \qquad \text{(A.3)}\end{aligned}$$

Let z denote the following function of the RV ξ, $z = 2^{-1/p}\ \xi^{1/p}$. Then, the pdf of the RV z is given by $f_z(z) = 2\ p\ z^{p-1}\ f_\xi(2\ z^p)$[35] for z such that $2z^p$ belongs to the domain of $f_\xi(\cdot)$, which completes the proof.

B. Proof of (11)

$$\begin{aligned}E\{z\} = &\frac{2^{2/p}}{p}B\left(\frac{1}{p},\frac{1}{p}\right)\frac{L}{3}(2^{1-3/p}-1) + \frac{2^{1+2/p}}{L^2\,p} \\ &\times\int_{2^{-1/p}L}^{L} z^2\left[\int_0^{2z^p} t^{1/p-1}(1-t)^{1/p-1}dt\right]dz. \qquad \text{(A.4)}\end{aligned}$$

$$f_\xi(\xi) = \begin{cases} 0, & \text{if } \xi < 0 \\ \frac{1}{L^2p^2}\int_0^\xi \lambda^{1/p-1}\,(\xi-\lambda)^{1/p-1}\,d\lambda, & \text{if } 0 \le \xi < L^p \\ \frac{1}{L^2p^2}\int_{z-L^p}^{L^p} \lambda^{1/p-1}\,(\xi-\lambda)^{1/p-1}\,d\lambda, & \text{if } L^p \le \xi < 2L^p \\ 0, & \text{if } \xi \ge 2L^p \end{cases} \qquad \text{(A.1)}$$

The integral in (A.4) can be calculated by integration by parts and by applying Leibnitz's rule for the differentiation of the inner integral, i.e.,

$$\begin{aligned}\int_{2^{-1/p}L}^{L} z^2 \left[\int_0^{2z^p} t^{1/p-1}(1-t)^{1/p-1}dt\right] dz =& \frac{1}{3}\left[L^3 B\left(\frac{1}{p},\frac{1}{p}\right) I_{1/2}\left(\frac{1}{p},\frac{1}{p}\right)\right. \\ & -2^{-3/p}L^3 B\left(\frac{1}{p},\frac{1}{p}\right) I_1\left(\frac{1}{p},\frac{1}{p}\right)+\frac{pL^p}{2} \\ & \left.\times\int_{2^{-1/p}L}^{L} z^{2-p}\left(\frac{L^p}{2z^p}\right)^{1/p-1}\left(1-\frac{L^p}{2z^p}\right)^{1/p-1} dz\right].\end{aligned} \tag{A.5}$$

It is well known that [37]

$$I_{1/2}\left(\frac{1}{p},\frac{1}{p}\right)=\frac{1}{2},\quad I_1\left(\frac{1}{p},\frac{1}{p}\right)=1 \quad \forall p>0. \tag{A.6}$$

By applying the change of variable $t=L^p/2z^p$, the last term in (A.5) is rewritten as

$$\frac{pL^p}{2}\int_{2^{-1/p}L}^{L} z^{2-p}\left(\frac{L^p}{2z^p}\right)^{1/p-1}\left(1-\frac{L^p}{2z^p}\right)^{1/p-1} dz = 2^{-3/p}L^3\int_{1/2}^{1} t^{-2/p-1}(1-t)^{1/p-1}\,dt. \tag{A.7}$$

The substitution of (A.5)–(A.7) in (A.4) yields

$$E\{z\}=\frac{2L}{3p}2^{-1/p}\int_{1/2}^{1} t^{-2/p-1}(1-t)^{1/p-1}\,dt \tag{A.8}$$

that can further be simplified by appropriate variable changes and integration by parts to (11).

Appendix B

A. Proof of (14)

Let us rewrite (8) as

$$f(z)=\begin{cases}\frac{2^{2/p}}{L^2 p}B\left(\frac{1}{p},\frac{1}{p}\right)z, & \text{if } 0\le z<2^{-1/p}L\\ g(z), & \text{if } 2^{-1/p}L\le z<L\\ 0, & \text{otherwise.}\end{cases} \tag{B.1}$$

The pdf $f(z)$ results from convolution, and accordingly, it is continuous. From its continuity, we know the exact values of function $g(z)$ at $z=2^{-1/p}L$ and $z=L$, i.e., $g(2^{-1/p}L)=(2^{-1/p}/L\,p)\;B(1/p,1/p)$ and $g(L)=0$, respectively. Moreover, at the same values of the argument z, the cumulative distribution $G(z)$ attains the following values:

$$G(2^{-1/p}L)=\frac{1}{2p}B\left(\frac{1}{p},\frac{1}{p}\right),\quad G(L)=1. \tag{B.2}$$

The expected value of the RV z is given by

$$E\{z\}=\int_0^{2^{-1/p}L}\frac{2^{2/p}}{L^2}2G(2^{-1/p}L)\,z^2dz+\int_{2^{-1/p}L}^{L} zg(z)dz. \tag{B.3}$$

Integrating by parts the second integral in (B.3), we obtain

$$E\{z\}=-2^{-1/p}\frac{L}{3}G(2^{-1/p}L)+L-\int_{2^{-1/p}L}^{L}G(z)\,dz. \tag{B.4}$$

By applying the trapezoidal rule, we obtain the following approximation for the integral in (B.4):

$$\int_{2^{-1/p}L}^{L}G(z)\,dz\simeq\frac{1+G(2^{-1/p}L)}{2}\quad(L-2^{-1/p}L). \tag{B.5}$$

The substitution of (B.5) in (B.4) yields

$$E\{z\}=\frac{L}{2}\left(1+2^{-1/p}\right)-\frac{G(2^{-1/p}L)\,L}{6}\left(3-2^{-1/p}\right). \tag{B.6}$$

By combining (B.2) and (B.6), we obtain (14).

B. Proof of (15)

By following the procedure outlined above, we rewrite $E\{z^2\}$ as

$$E\{z^2\}=\int_0^{2^{-1/p}L}\frac{2^{2/p}}{L^2}2G(2^{-1/p}L)\,z^3\,dz+\int_{2^{-1/p}L}^{L}z^2\,g(z)\,dz. \tag{B.7}$$

The integration by parts of the second integral in (B.7) and the application of the trapezoidal rule to approximate the integral

$$\int_{2^{-1/p}L}^{L}2z\,G(z)\,dz\simeq\Big(L+2^{-1/p}L\,G(2^{-1/p}L)\Big)(L-2^{-1/p}L) \tag{B.8}$$

yields (15).

Appendix C

A. Proof of (17)

Let x_i, $i=1, 2$, be independent RVs uniformly distributed in the interval $[\epsilon, L]$. Then, the pdf of the RV $\eta=x^{-p}$ is given by $f_\eta(\eta)=(1/p\,(L-\epsilon))\;(1/\eta^{1+1/p})$, $L^{-p}\le\eta\le\epsilon^{-p}$[35]. The pdf of the RV $\phi=\eta_1+\eta_2$, where $\eta_i=x_i^{-p}$, $i=1, 2$ is given by the convolution of the pdfs $f_{\eta_1}(\phi)$ and $f_{\eta_2}(\phi)$[35]. Let w denote the following function of the RV ϕ, $w=2^{1/p}\;\phi^{-1/p}$. Then, the pdf of the RV w is obtained by the transformation $f_w(w)=2\;p\;w^{-(p+1)}\;f_\phi(2\,w^{-p})$ [35] for w such that $2w^{-p}$ belongs to the domain of $f_\phi(\cdot)$, which completes the proof.

B. Proof of (19)

Let $\gamma=2^{1/p}(\epsilon\,L/(e^p+L^p)^{1/p})$. Then

$$\begin{aligned}E\{w\}=&\frac{2^{-2/p}}{p\,(L-\epsilon)^2}\left[\int_\epsilon^\gamma dw\,w^2\right.\\ &\times\int_{(2\epsilon^p-w^p)/2\epsilon^p}^{w^p/2\epsilon^p} t^{-(1+1/p)}\,(1-t)^{-(1+1/p)}\,dt\\ &+\int_\gamma^L dww^2\\ &\left.\int_{w^p/2L^p}^{(2L^p-w^p)/2L^p} t^{-(1+1/p)}\,(1-t)^{-(1+1/p)}\,dt\right].\end{aligned} \tag{C.1}$$

Integrating by parts in (C.1) and taking the limit for $\epsilon \to 0$, we obtain (19).

APPENDIX D

A. Proof of (24)

Let $x_i, i = 1, 2$, be independent uniformly distributed RVs in the interval $[0, L]$. In addition, let $s = x_2 - x_1$ denote their difference. Then, the MSE between the L_p comparator and the maximum sample $x_{(2)}$ among the input samples x_1 and x_2 is given by

$$\begin{aligned} &E\{e^2_{\max}(x_1,x_2)\} \\ &= \int_0^L dx_1\, f_{x_1}(x_1) \\ &\quad \times \int_{s<0} \left[x_1 - \frac{(x_1^p + (x_1+s)^p)^{1/p}}{2^{1/p}}\right]^2 f_s(s|x_1)\, ds \\ &\quad + \int_0^L dx_1\, f_{x_1}(x_1) \\ &\quad \times \int_{s\geq 0} \left[(x_1+s) - \frac{(x_1^p + (x_1+s)^p)^{1/p}}{2^{1/p}}\right]^2 f_s(s|x_1)\, ds. \end{aligned} \tag{D.1}$$

It can easily be proven that

$$f_s(s|x_1) = \begin{cases} \frac{1}{L}, & -x_1 \leq s \leq L - x_1 \\ 0, & \text{otherwise.} \end{cases} \tag{D.2}$$

By using (D.2), (D.1) is rewritten as

$$\begin{aligned} &E\{e^2_{\max}(x_1,x_2)\} \\ &= \frac{1}{L^2}\left(\int_0^L dx_1 \int_{-x_1}^0 \left[x_1 - \frac{(x_1^p + (x_1+s)^p)^{1/p}}{2^{1/p}}\right]^2 ds \right.\\ &\quad \left. + \int_0^L dx_1 \int_0^{L-x_1} \left[(x_1+s) - \frac{(x_1^p + (x_1+s)^p)^{1/p}}{2^{1/p}}\right]^2 ds\right). \end{aligned} \tag{D.3}$$

We then have (D.4), shown at the bottom of the page, where, by L' Hospital's rule

$$\lim_{x\to 0} x^4 \int_1^{L/x} t\,(1+t^p)\, dt = 0. \tag{D.5}$$

By substitution of (D.4) in (D.3), we obtain (24).

B. Proof of (25)

Let x_i, $i = 1, 2$, be independent uniformly distributed RVs in the interval $[\epsilon, L]$. In addition, let $s = x_2 - x_1$ denote their difference. Then, the MSE between the L_{-p} comparator and the minimum sample $x_{(1)}$ among the input samples x_1 and x_2 is given by

$$\begin{aligned} E\{e^2_{\min}(x_1,x_2)\} = &\frac{1}{(L-\epsilon)^2}\left(\frac{L^4-\epsilon^4}{12} - \frac{\epsilon^3}{3}(L-\epsilon)\right. \\ &+ \frac{(L-\epsilon)^4}{12} + 2^{2/p}\int_\epsilon^L dx_1 x_1^3 \\ &\times \int_{\epsilon/x_1}^{L/x_1} t^2(1+t^p)^{-2/p}\, dt - 2^{1+1/p} \\ &\times \int_\epsilon^L dx_1 x_1^3 \int_{\epsilon/x_1}^1 t^2(1+t^p)^{-1/p}\, dt \\ &- 2^{1+1/p}\int_\epsilon^L dx_1 x_1^3 \\ &\left.\times \int_1^{L/x_1} t\,(1+t^p)^{-1/p} dt\right). \end{aligned} \tag{D.6}$$

Integrating by parts and taking the limit of (D.6) for $\epsilon \to 0$ thanks to L' Hospital's rule, we obtain (25).

C. Proof of (32)

The error produced at the modified L_p comparator output (27) is given by

$$\tilde{e}_{\max}(x_1,x_2) = \begin{cases} x_1 - \frac{(x_1^p + x_2^p)^{1/p}}{2^{1/p}} + c\,s, & \text{if } s \leq 0 \\ x_1 - (c-1)\,s - \frac{(x_1^p + x_2^p)^{1/p}}{2^{1/p}}, & \text{if } s > 0. \end{cases} \tag{D.7}$$

Then, the MSE is given by (D.8), shown at the top of the next page. By setting the derivative of (D.8) with respect to c equal to zero and solving for c, we obtain

$$\begin{aligned} c = &\frac{3}{2}\left(1 + 2^{-1/p}\int_0^1 (t-1)\,(1+t^p)^{1/p}\, dt\right) \\ &- 6\,\frac{2^{-1/p}}{L^4}\int_0^L x_1^3 \int_1^{L/x_1} (t-1)(1+t^p)^{1/p}\, dt. \end{aligned} \tag{D.9}$$

The last integral in (D.9) can be integrating by parts. By taking into consideration that

$$\lim_{x_1\to 0} x_1^4 \int_1^{L/x_1} (t-1)\,(1+t^p)^{1/p}\, dt = 0 \tag{D.10}$$

we find that the last term is equal to

$$\frac{3}{2}\, 2^{-1/p}\int_0^1 (t-1)\,(1+t^p)^{1/p}\, dt. \tag{D.11}$$

$$\begin{aligned} \int_0^L dx_1 \int_{-x_1}^0 x_1^2 ds = \int_0^L dx_1 \int_0^{L-x_1} (x_1+s)^2 ds &= \frac{L^4}{4} \\ 2^{-2/p}\int_0^L dx_1 \int_{-x_1}^{L-x_1} [x_1^p + (x_1+s)^p]^{2/p}\ ds &= 2^{-2/p}\,\frac{L^4}{2}\int_0^1 (1+t^p)^{2/p}\, dt \\ 2^{1-1/p}\int_0^L dx_1 \int_{-x_1}^0 x_1\,[x_1^p + (x_1+s)^p]^{1/p}\ ds &= 2^{1-1/p}\,\frac{L^4}{4}\int_0^1 (1+t^p)^{1/p}\, dt \\ 2^{1-1/p}\int_0^L dx_1 \int_0^{L-x_1} (x_1+s)\,[x_1^p + (x_1+s)^p]^{1/p}\ ds &= 2^{1-1/p}\,\frac{L^4}{4}\int_0^1 (1+t^p)^{1/p} dt \end{aligned} \tag{D.4}$$

$$E\{\tilde{e}^2_{\max}(x_1,x_2)\} = \int_0^L dx_1\, f_{x_1}(x_1) \int_{s<0} \left[x_1 + c\,s - \frac{(x_1^p + (x_1+s)^p)^{1/p}}{2^{1/p}}\right]^2 f_s(s|x_1)\,ds + \int_0^L dx_1\, f_{x_1}(x_1) \int_{s\geq 0} \left[x_1 - (c-1)s - \frac{(x_1^p + (x_1+s)^p)^{1/p}}{2^{1/p}}\right]^2 f_s(s|x_1)\,ds. \tag{D.8}$$

By substitution of (D.11) in (D.9), (32) results. Equation (33) can be proven similarly.

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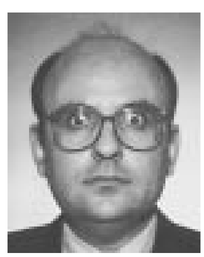

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