

Sound Attenuation in a Condensing Vapor

FRANK E. MARBLE AND DAVID C. WOOTEN*

*Daniel and Florence Guggenheim Jet Propulsion Center
California Institute of Technology, Pasadena, California 91109*

(Received 24 April 1970)

The process of acoustic attenuation in a condensing medium is investigated using a continuumlike formulation that allows for the phase-exchange process. The liquid phase is assumed sufficiently disperse so that the field may be treated as a continuum. The elementary relaxation processes associated with droplet velocity, temperature, and vapor pressure equilibration are equally important in determining the attenuation when vapor, liquid, and inert gas mass fractions are of the same order. When the liquid mass fraction is small, however, a strong attenuation band appears at low frequencies. This attenuation process involves a coupled relaxation process in which heat transfer and vaporization processes combine to change the temperature of the relatively large gas mass. This attenuation band (i) centers on a frequency that is proportional to the concentration of liquid, and (ii) has a maximum value that varies directly as the concentration of condensable vapor and roughly as the square of the latent heat of vaporization. When the concentrations of liquid and condensable vapor are both small, the low-frequency attenuation band is nearly isolated and may be described in a convenient analytical manner.

I. INTRODUCTION

As the result of extensive investigations, the attenuation and dispersion of plane acoustic waves by a suspension of solid particles may be considered well understood. The early treatment by Sewell¹ permitted no motion of the particles, and this restriction was subsequently removed in the elaborate analysis of Epstein.² Epstein and Carhart³ returned to the problem to include the thermal exchange between gas and particles. Recently these results have been confirmed experimentally by Temkin and Dobbins⁴ and the validity of a continuumlike treatment was demonstrated.

When the problem involves small droplets of liquid suspended in their vapor, the process of phase exchange may also be important. If the droplets and their vapor are accompanied by a substantial concentration of relatively inert gas, atmospheric fog for example, the mass exchange between phases constitutes a damping mechanism of significant magnitude. Although this phenomenon is explicitly excluded by Epstein and Carhart, some effects of phase exchange have been considered by Oswatitsch⁵ while excluding other damping mechanisms.

It is our purpose here to present the results of a unified treatment of the problem⁶ and to show that, under some important circumstances, the effects of condensation and vaporization dominate the damping mechanism.

II. COUPLING PROCESSES AND RELAXATION TIMES

When the wavelength of sound is large compared with the droplet spacing, then a continuumlike

description⁷ of both the vapor and liquid phases is applicable. In this representation, the mass per unit volume ρ_p of the droplet, the droplet temperature T_p , and the droplet velocity u_p , appear as continuous variables over space. The volume occupied by the liquid phase is a negligible fraction of that available for the gas. The vapor and inert gaseous components are assumed to have a common temperature T ; the partial pressure p_v of the vapor is a small fraction of the total pressure p . Under these assumptions, a set of equations representing continuity, momentum, and thermodynamic laws may be written separately for the gas phase and the liquid phase. These equations are coupled through the exchange of mass, momentum, and heat between phases; it is these transfer processes that determine the response of this medium to acoustic waves.

Assume that the Reynolds number of the droplets is small enough so that Stokes law is applicable; then, the viscous force acting on a droplet of radius σ is $6\pi\sigma\mu(u - u_p)$. The momentum transfer associated with mass transfer is treated separately. If a droplet of mass m is moving through the gas phase, then the characteristic time with which the droplet velocity approaches that of the gas is

$$\tau_s = \frac{m}{6\pi\sigma\mu}. \quad (1)$$

Physically, τ_s is the time required for the relative velocity of the droplet and vapor to relax to e^{-1} of its initial value. If the velocity relaxation time is very short, the droplets closely follow the gas motion; if it is long, the droplets are relatively independent of the gas motion. The total force

F_p , per unit volume exerted by the gas phase upon the particle cloud may also be expressed in terms of the velocity relaxation time,

$$F_p = \frac{1}{\tau_v} \rho_p (u - u_p), \quad (2)$$

where $nm \equiv \rho_p$, the mass of liquid per unit volume.

Again, under the assumption of very low droplet Reynolds number, the heat transferred per unit volume from liquid to vapor phase is

$$Q = \frac{1}{\tau_T} \rho_p c_p (T - T_p). \quad (3)$$

The temperature relaxation time is defined, analogous to Eq. (1),

$$\tau_T = \frac{mc_p}{4\pi\sigma k}, \quad (4)$$

where c_p is the specific heat of the gas at constant pressure and k is its thermal conductivity. This temperature relaxation time may be written $\tau_T = \frac{3}{2}(c_p\mu/k)\tau_v$, so that when the Prandtl number $c_p\mu/k$ has a value in the normal range for gases, the thermal and velocity relaxation times are nearly equal. The physical significance of τ_T is entirely similar to that of τ_v .

Finally, the mass transfer rate may be formulated as a diffusion controlled process. This is evident simply because the small mass fraction of condensible vapor must move through a relatively large mass fraction of inert gas in order to pass to or from a droplet surface. The vapor moves, of course, because of the gradient in its molecular concentration or partial pressure p_v . In a diffusion controlled process, the partial pressure of condensible vapor at the droplet surface is essentially the saturation pressure corresponding to the droplet temperature. In our calculations, this equilibrium saturation pressure, $p_v^*(T_p)$, will be given by the Clausius-Clapeyron relationship

$$\frac{p_v^*(T_p)}{p_0} = \exp \left[\frac{h_l}{RT_0} \left(1 - \frac{T_0}{T_p} \right) \right], \quad (5)$$

where p_0 , T_0 is a known reference saturation state, h_l is the latent heat of vaporization of the liquid, and R is the gas constant. The diffusion-controlled mass production rate of vapor per unit volume is therefore

$$\mu_v = \frac{\rho_p}{\tau_D} \left(\frac{p_v^*(T_p)}{p} - p_v \right), \quad (6)$$

where the mass transfer or diffusion relaxation

time is, utilizing the binary diffusion coefficient D and the gas density ρ ,

$$\tau_D = \frac{m}{4\pi\sigma(\rho D)}. \quad (7)$$

The value of τ_D may be related to that of the velocity relaxation time by noting that $\tau_D = \frac{3}{2}(\mu/\rho D)\tau_v$. Since the Schmidt number is usually very nearly unity, it is clear that τ_D is very nearly equal to τ_v . In fact, for many gas-vapor-liquid systems, the three time constants τ_v , τ_T , and τ_D that control the coupling processes between phases are all of the same order of magnitude.

The manner in which the exchange processes described by Eqs. (2), (3), and (6) enter into the equations describing the gas and droplet phases is fairly obvious. The vapor production rate appears as a mass-source term in the continuity equation for the gas phase and a mass-sink term in the continuity equation for the droplet cloud. The force F_p enters as a body force in the equation of motion for the gas and similarly, but with opposite sign, in the equations of motion for the droplet cloud. There is a momentum transfer caused by mass transfer between phases which arises because droplets and gas move with different velocities. Because this quantity is proportional to the mass exchange rate μ_v and the velocity difference $u_p - u$ between the phases, it is of second order in perturbation quantities and does not appear in the acoustic equations.

The two significant exchange terms that appear in the energy equations or thermodynamic relations are the heat exchange due to thermal conduction between gas and droplets and the enthalpy exchange, including the latent heat, associated with mass exchange. The dissipation resulting from relative motions of droplets and gas and the kinetic energy transfer due to mass exchange are second order perturbation quantities and are negligible in the acoustic approximation.

The detailed development of the general equations describing the system, as well as the approximations involved in the acoustic equations, are given by Marble.⁶

III. THE TWO-PHASE ACOUSTIC PROBLEM

The acoustic problem may be described by gas phase equations of (1) continuity, (2) motion, (3) energy, (4) vapor phase equation of continuity and liquid phase equations of (5) motion and (6) energy. The dependent variables are the velocities

and temperatures of the gas and droplet phase, the pressure of the gas phase, and the difference ($p_v^* - p_v$) between the vapor pressure and its

equilibrium value corresponding to the droplet temperature T_p . In matrix form these relations are

$$\begin{bmatrix} a \frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial t} & 0 \\ \frac{\partial}{\partial t} + \frac{\kappa_p}{\tau_v} & -\frac{\kappa_p}{\tau_v} & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial t} + \frac{\kappa_p}{\tau_T} & -\frac{\kappa_p}{\tau_T} \\ 0 & 0 & 0 & -\kappa_v \frac{\gamma}{\gamma - 1} \eta \frac{\partial}{\partial t} \\ -\frac{1}{\tau_v} & \frac{\partial}{\partial t} + \frac{1}{\tau_v} & 0 & 0 \\ 0 & 0 & -\frac{1}{\tau_T} & \frac{\partial}{\partial t} + \frac{1}{\tau_T} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial t} & -\frac{\kappa_p}{\tau_D} \\ \frac{a}{\gamma} \frac{\partial}{\partial x} & 0 \\ -\frac{\gamma - 1}{\gamma} \frac{\partial}{\partial t} & 0 \\ \kappa_v \frac{\partial}{\partial t} & \frac{\partial}{\partial t} + \frac{(1 - \kappa_v)}{\tau_D} \kappa_p \\ 0 & 0 \\ 0 & \frac{\eta}{\tau_D} \end{bmatrix} \begin{bmatrix} \frac{u}{a} \\ \frac{u_p}{a} \\ \frac{T}{T_0} \\ \frac{T_p}{T_0} \\ \frac{p}{p_0} \\ \frac{p_v^* - p_v}{p_0} \end{bmatrix} = [0] \quad (8)$$

and they are listed in the order of numbering given above. In these expressions κ_p denotes the ratio of liquid mass to gas mass while κ_v denotes the ratio of condensable vapor to total mass of gas. The ratio of specific heats, γ , is defined as c_p/c_v , and $a \equiv (\gamma RT)^{1/2}$ is the sound speed in the gas. The dependent variables are perturbation quantities; the variable T , for example, denotes the perturbation of the gas temperature from its equilibrium value. The same is true for T_p , p , p_v^* , and p_v . The matrix operator may be used to obtain a differential equation of the fifth order for any of the dependent variables, such as the pressure perturbation p/p_0 . For the acoustic problem, it is adequate to choose a solution

$$\exp \left[i \left(\frac{Kx}{a} - \omega t \right) \right] \quad (9)$$

for any of the variables where $K \equiv K_1 + iK_2$ is the complex characteristic value corresponding to a plane wave of frequency $\omega/2\pi$. The imaginary part K_2 of K is the attenuation coefficient. These characteristic values are obtained by setting equal to zero the determinant of the matrix equation (8) after substituting the appropriate exponential solution, Eq. (9). The result is a quadratic expression for the characteristic value K as

$$\left(\frac{K}{\omega} \right)^2 = \frac{(Z_1 - i\omega)(Z_2 - i\omega)(Z_3 - i\omega)}{(\Omega_1 - i\omega)(\Omega_2 - i\omega)(\Omega_3 - i\omega)}, \quad (10)$$

where the Ω 's are constants related to the three elementary relaxation times. The expressions for

Z_i, Ω_i are enumerated in the Appendix. The factored form is of interest here primarily because, when Z_i and Ω_i are not too different, the imaginary part of K has a relative maximum near each $\omega/\Omega_i = 1$.

General calculations for the imaginary part of K are easily performed numerically. It is convenient to take $\omega\tau_v$ as the dimensionless frequency parameter; this leaves the physical parameters $\tau_T/\tau_v, \tau_T/\tau_v, \kappa_p, \kappa_v$, and $\eta \equiv h_l/c_p T_0$ defining the system. The first two of these are usually near unity and we shall be interested in the influences of droplet and vapor mass fraction and the latent heat parameter.

Figure 1 shows the values of acoustic attenuation coefficient for a wide range of frequencies for several different values of droplet concentration κ_p . For

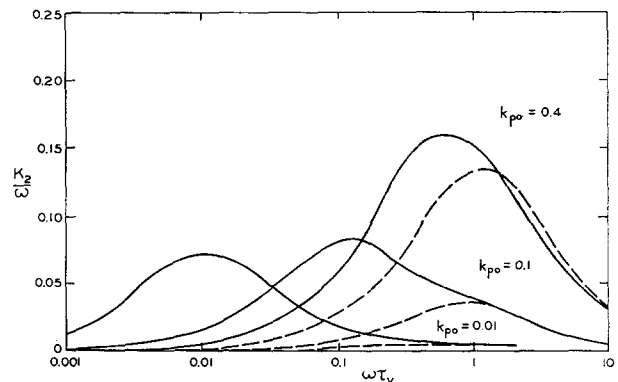


FIG. 1. Effect of phase exchange upon the acoustic damping caused by liquid droplets. Various liquid fractions. $\kappa_v = 0.5, h_l/RT_0 = 10, \gamma = 1.6, \tau_v = \tau_T = \tau_D$.

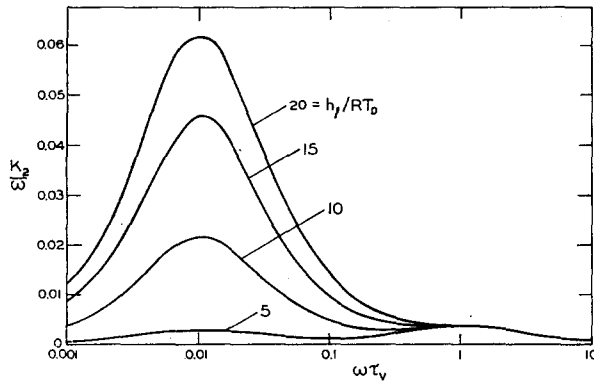


FIG. 2. Effect of latent heat parameter upon phase-exchange acoustic damping. $\kappa_v = \kappa_p = 0.01$, $\gamma = 1.6$, $\tau_v = \tau_T = \tau_D$.

each of these curves the vapor concentration and the latent heat parameter are held constant at $\kappa_v = 0.5$, $h_l/RT_0 = 10$; these are appropriate for some mixtures of potassium or sodium vapor in argon. For very wet mixtures, that is, when the liquid mass is 0.4 of the total gas mass ($\kappa_p = 0.4$), the attenuation curve is a typical one and differs largely in magnitude from the attenuation that would occur without phase transition. The neighboring broken line shows, in fact, the attenuation resulting from particle drag and heat transfer corresponding to the calculation of Epstein and Carhart. While there is some tendency to reduce the frequency of maximum attenuation, the principal effect of the phase exchange is to increase the attenuation by about 20%. This result is essentially what would be expected if the phase transition were simply another process similar to the drag and heat transfer damping.

When the mass of liquid is reduced until $\kappa_p = 0.1$, the unique character of the phase transition process becomes quite clear. The peak attenuation now occurs at a reduced frequency of about $\omega\tau_v = 0.1$, indicating that the physical process is no longer described by the time elementary constant τ_v . The attenuation that can be ascribed to phase exchange, the difference between the solid and broken curves, is now a major fraction of the total. When the liquid fraction is further reduced to $\kappa_p = 0.01$, the phase exchange damping dominates the process completely. The peak attenuation, which now appears near the low frequency of $\omega\tau_v = 0.01$, is now entirely associated with the vaporization-condensation mechanism. Furthermore, it is to be noted that the value of K_2/ω at the peak varies only slightly as κ_p is reduced from 0.1 to 0.01. Clearly, then, the peak attenuation

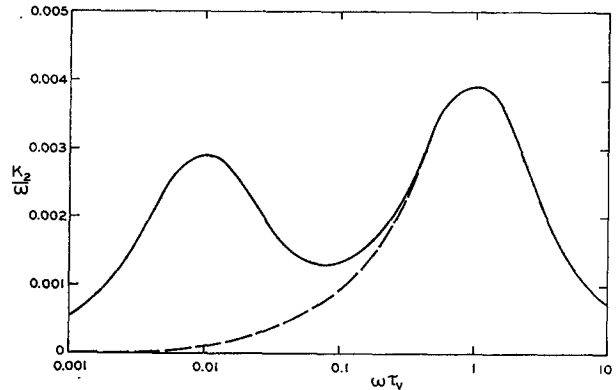


FIG. 3. Acoustic damping at very low liquid and vapor fractions. Broken line indicates damping without phase exchange. $\kappa_v = \kappa_p = 0.01$, $h_l/RT_0 = 5.0$, $\gamma = 1.6$, $\tau_v = \tau_T = \tau_D$.

does not depend upon the total liquid mass present.

The calculations of Fig. 1 suggest that the phase exchange mechanism dominates the attenuation process for low droplet mass fraction, that the peak attenuation moves toward low frequencies as κ_p is reduced, and that peak attenuation is relatively independent of droplet mass fraction.

Figure 2 illustrates the influence of the important latent heat parameter h_l/RT_0 . In addition to the droplet concentration $\kappa_p = 0.01$, other physical quantities are the same as in Fig. 1. The strong effect upon attenuation coefficient of increasing the latent heat has no influence upon the frequency at which the peak occurs. Moreover, the latent heat variation has no influence upon the primary attenuation at $\omega\tau_v \sim 1$.

Finally, the dependence of attenuation upon the vapor concentration κ_v is examined in Fig. 3 by considering an example where $\kappa_v = \kappa_p = 0.01$. These conditions are fairly typical of "fogs" and only the value of $\gamma = 1.6$ differentiates this case from an atmospheric fog. Here, the two attenuation peaks are observed. The one on the right with its peak at $\omega\tau_v \sim 1$ is due to viscous damping and conductive heat transfer only; this is clear from the broken curve which gives attenuation due only to these factors. The peak on the left-hand side results entirely from the phase exchange phenomena; the effect of reducing κ_v has been to decrease the value of attenuation at this peak. Comparison with the curve for $\kappa_p = 0.01$ in Fig. 1 gives an estimate of the resulting reduction. These values of K_2 , while small, still represent an appreciable spatial damping; a plane wave loses 50% of radiated power in passing 20 wavelengths through such a medium.

The unusual feature of the results is that the absorption band associated with the phase transition process is displaced so far from bands corresponding to any of the natural time constants; this suggests that a coupled relaxation process is dominating this phenomenon, and it is our aim to explore what the nature of this coupling may be.

IV. THE COUPLED RELAXATION PROCESS

It is not difficult to argue that the relatively long relaxation process, whose effect is observed in the calculated results, is associated with coupling

between phase change and heat transfer mechanisms and not with the viscous drag forces experienced by the droplets. To investigate the remaining thermodynamic processes, consider a container of gas-vapor-droplet mixture, at rest and in equilibrium at time $t = 0$. Then let us impose a small instantaneous change in gas temperature and pressure. Then the dependent variables T , T_p , p , and p_v represent the displacements in gas temperature, droplet temperature, gas pressure, vapor pressure from the equilibrium values T_0 , p_0 . The relaxation process is described by the system

$$\begin{bmatrix} -\frac{\partial}{\partial t} & 0 & \frac{\partial}{\partial t} & -\frac{\kappa_p}{\tau_D} \\ \frac{\partial}{\partial t} + \frac{\kappa_p}{\tau_T} & -\frac{\kappa_p}{\tau_T} & -\frac{\gamma-1}{\gamma} \frac{\partial}{\partial t} & 0 \\ -\frac{1}{\tau_T} & \frac{\partial}{\partial t} + \frac{1}{\tau_T} & 0 & \frac{\eta}{\tau_D} \\ 0 & -\kappa_v \frac{\gamma}{\gamma-1} \eta \frac{\partial}{\partial t} & \kappa_v \frac{\partial}{\partial t} & \frac{\partial}{\partial t} + \frac{(1-\kappa_v)\kappa_p}{\tau_D} \end{bmatrix} \begin{bmatrix} \frac{T}{T_0} \\ \frac{T_p}{T_0} \\ \frac{p}{p_0} \\ \frac{p_v^e - p_v}{p_0} \end{bmatrix} = [0] \tag{11}$$

and according to our choice of initial conditions, the solutions are of the form

$$\begin{aligned} \frac{T}{T_0} &= A_0 \exp\left(-\frac{t}{\tau}\right) + A_1 \left[1 - \exp\left(-\frac{t}{\tau}\right)\right], \\ \frac{T_p}{T_0} &= A_1 \left[1 - \exp\left(-\frac{t}{\tau}\right)\right], \\ \frac{p}{p_0} &= B_0 \exp\left(-\frac{t}{\tau}\right) + B_1 \left[1 - \exp\left(-\frac{t}{\tau}\right)\right], \\ \frac{p_v^e - p_v}{p_0} &= -\kappa_v B_0 \exp\left(-\frac{t}{\tau}\right). \end{aligned} \tag{12}$$

Here, A_0 and B_0 are the initial temperature and pressure displacements; A_1 and B_1 are the final perturbations in temperature and pressure after equilibrium is re-established. The decay time τ appears as the characteristic value of the quadratic relation obtained by setting to zero the determinant arising from substituting the solution (12) into (11). The roots are algebraic functions of the physical constants of the system; for small values of the droplet weight fraction, the situation in which we are particularly interested, the roots take the simple form

$$\tau_1 = \tau_T \left\{ 1 + \frac{\gamma}{\gamma-1} \eta^2 \kappa_v \frac{\tau_T}{\tau_D} \right\}^{-1}, \tag{13}$$

$$\tau_2 = \frac{\tau_D}{\kappa_p} \left\{ \frac{1 + [\gamma/(\gamma-1)]\eta^2 \kappa_v (\tau_T/\tau_0)}{1 + (\gamma-1)\kappa_v \{[\gamma/(\gamma-1)]\eta - 1\}^2} \right\}. \tag{14}$$

The character of these roots is brought out more clearly by further restricting our consideration to small concentrations of vapor; this corresponds to the usual vapor fog in an inert gas. When $\kappa_v \ll 1$, the bracketed terms in (13) and (14) become unity and

$$\begin{aligned} \tau_1 &= \tau_T, \\ \tau_2 &= \tau_D/\kappa_p. \end{aligned} \tag{15}$$

Thus, one of the composite decay processes is of the same value as the elementary relaxation time τ_T . The value of τ_2 , however, is much larger than the elementary relaxation times when κ_p is small. It is this root, $\tau_2 = \tau_D/\kappa_p$, that accounts for the attenuation at low frequencies; this is seen directly because the frequency of maximum damping in Fig. 1 decreases in proportion to κ_p .

The mechanisms by which these composite relaxation processes take place are found most easily by investigating the "natural modes" associated with each of the roots. In the conventional manner, this leads to an initial value for

$$\frac{A_0}{B_0} \sim \frac{\Delta T/T_0}{\Delta p/p_0}$$

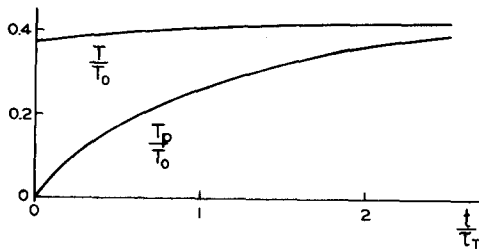


FIG. 4. Relaxation of droplet and gas temperatures toward thermal equilibrium after isentropic compression; fast mode.

that excites a decay process having one distinct time constant. For algebraic simplicity, consider the case of small concentration of condensable vapor, $\kappa_v \ll 1$. Solution for the rapid decay time, $\tau = \tau_T$, requires that

$$\begin{aligned} \frac{A_0}{B_0} &= \frac{\gamma - 1}{\gamma} \frac{1}{\eta}, \\ \frac{A_1}{B_0} &= \frac{\gamma - 1}{\gamma} \frac{1}{\eta} \left(1 - \frac{\gamma - 1}{\gamma} \frac{\kappa_p \tau_T}{\kappa_v \tau_D} \frac{1}{\eta^2} \right)^{-1}, \\ \frac{B_1}{B_0} &= 1 + \frac{\gamma - 1}{\gamma} \frac{1}{\eta} \left(\frac{\gamma - 1}{\gamma} \frac{\kappa_p \tau_D}{\kappa_v \tau_T} \eta^2 - 1 \right)^{-1}, \end{aligned} \tag{16}$$

while the solution for the slow decay time, $\tau = \tau_D/\kappa_p$, requires that

$$\begin{aligned} \frac{A_0}{B_0} &= -\kappa_v \frac{\tau_T}{\tau_D} \eta, \\ \frac{A_1}{B_0} &= (\gamma - 1)\kappa_v(\eta - 1), \\ \frac{B_1}{B_0} &= 1 + \gamma\kappa_v(\eta - 1). \end{aligned} \tag{17}$$

The significant contrast to be observed is that, when the pressure of the system is raised, the rapid decay requires an accompanying gas temperature

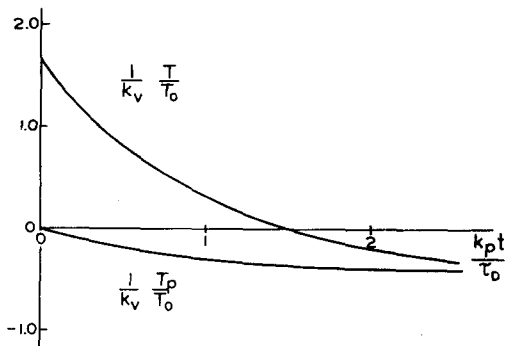


FIG. 5. Relaxation of droplet and gas temperatures toward thermal equilibrium after isentropic compression; slow mode.

rise of the same order of magnitude. The slow decay, under the same pressure disturbance, requires an accompanying gas temperature disturbance of the opposite sign and proportional to the concentration of condensable vapor.

The response of this system to a sound wave corresponds to an initial disturbance such that temperature and pressure displacements of the gas phase are related by the isentropic law, $\Delta T/T_0 = [(\gamma - 1)/\gamma](\Delta p/p_0)$, and the displacements of the liquid state are zero. This initial condition excites modes of decay corresponding to both (16) and (17). The rapid and slow components of gas and droplet temperature relaxation are shown in Figs. 4 and 5, corresponding to a unit isentropic temperature disturbance. In each case, the pressure change during relaxation is small enough that it will not influence our discussion.

The fast decay process, Fig. 4, is normalized with respect to the thermal equilibration time and shows that the significant change during relaxation is the increase in droplet temperature toward the gas temperature. This rapid increase in droplet temperature is entirely a heat transfer phenomenon. The initial increase in gas pressure raises the temperature of the gas above the droplet temperature and, because the droplet mass is small, the droplets heat up with relatively little change in gas temperature. Regardless of how small the mass fraction of liquid or vapor phase may be, the process takes place in the same manner. For the droplet surface area, through which the thermal transport process takes place, and the mass of liquid whose temperature is undergoing change, both vary as the quantity of liquid is changed. This decay time is therefore independent of liquid mass fraction.

The slow decay process, Fig. 5, is normalized with respect to the long composite relaxation time, τ_D/κ_p , and the ordinates are multiplied by the factor $1/\kappa_v$. For small values of condensable vapor concentration, the actual values of temperature change are smaller than those shown in Fig. 4. In contrast to the rapid decay mode, the slow decay mode shows the dominant trend to be the change in gas temperature toward the droplet temperature. Here, the droplet temperature and the pressure of the vapor component of the gas correspond very nearly to saturation equilibrium; vaporization and heat transfer continually cool the gas. When the liquid mass is reduced to a very small fraction, the rate at which the gas is cooled is reduced in proportion to their masses. Thus,

instead of the rate being $\sim 1/\tau_D$ as it would be if the droplet temperature were changing, the rate is $\sim \kappa_p/\tau_D$ which leads to the observed relaxation time.

For a given increase of pressure in the total system, then, the equilibrium is achieved in two steps: (i) droplets are warmed until droplet temperature and vapor pressure nearly reach saturation equilibrium; (ii) the gas is cooled, very slowly, by droplet vaporization until the temperatures are nearly equal, always keeping the droplet temperature and vapor pressure near saturation equilibrium.

V. APPROXIMATE ATTENUATION FOR LOW FREQUENCIES

From the mechanism we have described, it is clear that the low-frequency attenuation observed in the numerical calculations results from a coupled relaxation process having a time constant approximately equal to τ_D/κ_p . This is long in comparison with the physical time constants of the system so that, if we investigate the system response on a time scale τ_D/κ_p , we may assume that processes having relaxation times of the order τ_v, τ_T, τ_D are in equilibrium. Formally we accomplish this analytically by permitting $\tau_v \rightarrow 0, \tau_T \rightarrow 0, \tau_D \rightarrow 0$ while τ_D/κ_p remains finite. The operator in Eq. (8), applied to the pressure fluctuation p/p_0 , then becomes

$$\frac{\partial}{\partial t} \left(\frac{\partial^2}{a^2 \partial t^2} - \frac{\partial^2}{\partial x^2} \right) \frac{p}{p_0} + \frac{\kappa_p}{\tau_D} \left[1 - \kappa_v \left(1 - \frac{\gamma}{\gamma - 1} \eta^2 \right) \right] \left(\frac{\partial^2}{a_*^2 \partial t^2} - \frac{\partial^2}{\partial x^2} \right) \frac{p}{p_0} = 0, \quad (18)$$

where

$$\frac{a_*^2}{a^2} = \frac{1 + \kappa_v \{ [\gamma/(\gamma - 1)] \eta - 1 \}}{1 + (\gamma - 1) \kappa_v \{ [\gamma/(\gamma - 1)] \eta - 1 \}^2}. \quad (19)$$

The quantity a_* has the physical significance of the sound propagation speed with all elementary relaxation processes in equilibrium. Thus, as $\tau_v, \tau_T, \tau_D \rightarrow 0$, the velocity of acoustic disturbances becomes a_* .

Returning to the case of small vapor fraction, the expression for the attenuation coefficient is found to be

$$\frac{K_2}{\omega} \cong \frac{\gamma}{2} \kappa_v (\eta - 1)^2 \frac{\omega \tau_D / \kappa_p}{1 + (\omega \tau_D / \kappa_p)^2}. \quad (20)$$

This result should be a good approximation for low-frequency attenuation of the system where

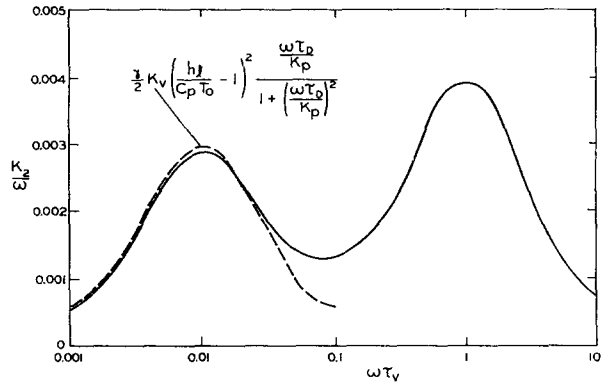


FIG. 6. Comparison of approximate solutions for low frequency attenuation with detailed numerical calculation. $\kappa_v = \kappa_p = 0.01, h_l/RT_0 = 5.0, \gamma = 1.6, \tau_v = \tau_T = \tau_D$.

attenuation associated with the elementary relaxation processes is negligible. This attenuation coefficient has a maximum value at

$$\omega = \kappa_p / \tau_D = \omega^* \quad (21)$$

at which frequency the value of K_2 is

$$\left(\frac{K_2}{\omega} \right)_{\max} = \frac{1}{4} \gamma \kappa_v (\eta - 1)^2. \quad (22)$$

That it is indeed a very good approximation is shown in Fig. 6, where the attenuation coefficient calculated from Eq. (20) is shown as the broken line to be compared with the detailed numerical calculations.

The results of our investigation of low-frequency sound attenuation in gas mixtures containing wet vapor may be conveniently summarized as follows.

(i) The maximum attenuation coefficient occurs at a frequency

$$\omega^* = 3 \kappa_p \left(\frac{\rho}{\rho_l} \right) \frac{D}{\sigma^2} \quad (23)$$

and thus varies directly as the droplet mass fraction and inversely as the square of droplet radius.

(ii) The attenuation coefficient is, for low concentrations of vapor and liquid, very nearly

$$\frac{K_2}{\omega} = \frac{1}{2} \gamma \kappa_v (\eta - 1)^2 \frac{\omega / \omega^*}{1 + (\omega / \omega^*)^2}. \quad (24)$$

(iii) For a given system and droplet size, the maximum attenuation varies directly as the vapor fraction κ_v and directly as $(\eta - 1)^2$, the square of the latent heat parameter.

The variation of attenuation coefficient with latent heat, shown in Fig. 2, is well reproduced by the factor $(\eta - 1)^2$ in Eq. (24).

ACKNOWLEDGMENT

This work was performed with the financial support of the Aerospace Research Laboratories, Wright-Patterson Air Force Base, Contract F 33615-69-C-1069.

APPENDIX

In the dispersion relation

$$\left(\frac{K}{\omega}\right)^2 = \frac{(Z_1 - i\omega)(Z_2 - i\omega)(Z_3 - i\omega)}{(\Omega_1 - i\omega)(\Omega_2 - i\omega)(\Omega_3 - i\omega)}$$

the various terms are defined as follows:

$$\begin{aligned} Z_1 &= \frac{a^2}{a_v^2 \tau_v}, \\ Z_{2,3} &= -\frac{1}{2} \left(\frac{a^2}{a_T^2 \tau_T'} + \frac{a^2}{a_D^2 \tau_D'} \right) \pm \frac{1}{2} \left[\left(\frac{a^2}{a_T^2 \tau_T'} + \frac{a^2}{a_D^2 \tau_D'} \right)^2 - \frac{4\alpha}{a_{TD}^2} \frac{a^2}{\tau_T' \tau_D'} \right]^{1/2}, \\ \Omega_1 &= \frac{1}{\tau_v}, \\ \Omega_{2,3} &= -\frac{1}{2} \left(\frac{1}{\tau_T'} + \frac{1}{\tau_D'} \right) \pm \left[\left(\frac{1}{\tau_T'} + \frac{1}{\tau_D'} \right)^2 - \frac{4\alpha}{\tau_T' \tau_D'} \right]^{1/2}, \\ \alpha &= \frac{1}{1 + \kappa_p} \left(1 + \frac{1}{\kappa_p + [\kappa_v/(1 - \kappa_v)][\gamma/(\gamma - 1)]\eta^2} \right), \\ \frac{a_v^2}{a^2} &= \frac{1}{1 + \kappa_p}, \\ \frac{a_T^2}{a^2} &= \frac{1 + \kappa_p}{1 + \gamma\kappa_p}, \\ \frac{a_D^2}{a^2} &= \frac{(1 - \kappa_v)\kappa_p + \kappa_v[\gamma/(\gamma - 1)]\eta^2}{\gamma\kappa_p\kappa_v + (1 - \kappa_v)\kappa_p + \kappa_v[\gamma/(\gamma - 1)]\eta^2}, \\ \frac{a_{TD}^2}{a^2} &= \frac{(1 - \kappa_v)(1 + \kappa_p) + \kappa_v[\gamma/(\gamma - 1)]\eta^2}{(1 - \kappa_v)(1 + \kappa_p) + \gamma\kappa_v(1 + \kappa_p) - 2\gamma\kappa_v\eta + \gamma\kappa_v[\gamma/(\gamma - 1)]\eta^2}, \\ \tau_T' &= \tau_T(1 + \kappa_p)^{-1} \\ \tau_D' &= \tau_D\{(1 - \kappa_v)\kappa_p + \kappa_v[\gamma/(\gamma - 1)]\eta^2\}^{-1}. \end{aligned}$$

* Present address: Stanford Research Institute, Palo Alto, California.

¹ C. J. T. Sewell, *Phil. Trans. Roy. Soc. (London)* **A210**, 239 (1910).

² P. S. Epstein, *Contributions to Applied Mechanics* (California Institute of Technology, Pasadena, California, 1941), Theodore von Kármán Anniversary Volume, p. 162.

³ P. S. Epstein and R. R. Carhart, *J. Acoust. Soc. Am.* **25**, 553 (1953).

⁴ S. Temkin and R. A. Dobbins, *J. Acoust. Soc. Am.* **40**, 317 (1966).

⁵ K. Oswatitsch, *Physik Z.* **42**, 365 (1941).

⁶ D. C. Wooten, Ph.D. thesis, California Institute of Technology (1966), pp. 1-72.

⁷ F. E. Marble, *Astronautica Acta* **14**, 585 (1969).