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Sound field reproduction using directional loudspeakers and the equivalent acoustic scattering problem

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ABSTRACT

The problem is addressed of reproducing a desired sound field in the interior of a bounded region of space, using an array of loudspeakers that exhibit a first order acoustic radiation pattern. Previous work has shown that the computation of the required loudspeaker signals, in the case of omnidirectional transducers, can be determined by solving an equivalent scattering problem. This approach is extended here to the case of directional loudspeakers. It is shown that the loudspeaker complex coefficients can be computed by solving an equivalent scattering problem. These coefficients are given by the normal derivative of the total pressure field (incident field plus scattered field) arising from the scattering of the target field by an object with the shape of the reproduction region (the region bounded by the loudspeaker array) and with impedance boundary conditions. The expression for this impedance, or Robin, boundary condition is calculated from the radiation pattern of the loudspeakers, assuming that the latter can be expressed by a linear combination of a free field Green function and its gradient. The solution of the problem can be obtained in closed form for simple geometries of the loudspeaker array, such as a sphere, a circle or a plane, thus providing a meaningful improvement to sound field reproduction techniques such as Wave Field Synthesis or High Order Ambisonics. The method proposed is also valid for more general geometries, for which the computation of the solution should be performed by applying the Kirchhoff approximation or by means of numerical methods.

INTRODUCTION

The problem of reproducing a desired sound field with a two dimensional or three dimensional array of loudspeakers has recently obtained much interest from the scientific community. This problem involves the determination of the loudspeaker signals required for the reproduction of a given field, for a given arrangement of the loudspeakers. A variety of sound field reproduction methods have been proposed such as Ambisonics [1],[2], Wave Field Synthesis [3],[4], methods based on the solution of an inverse problem [5],[6] and other techniques presented in references [7]-[12], among others.

Several of the techniques above include the modelling of the loudspeaker array as a continuous distribution of secondary sources. The solution is given as a continuous source strength function, which is then discretized in order to obtain the driving function of the loudspeakers. In this paper, the same approach is adopted. Clearly, the use of a finite number of transducers rather than a continuous distribution of secondary sources generates errors in the reproduced field (spatial aliasing), but this problem will not be addresses in this paper. It is sufficient to mention that spatial aliasing artefacts are negligible when the spacing between loudspeakers is much smaller than the wavelength of the sound to be reproduced.

Most of the approaches mentioned above make the assumption that the loudspeakers radiate sound as omnidirectional ideal point sources (acoustic monopoles). This is a good approximation to a real transducer when the dimension of the latter is small in comparison to the operating wavelength of the sound to be reproduced. It is known however that this the loudspeaker radiation pattern becomes directional at higher frequencies. In this paper we will assume that the loudspeakers of the array exhibit a first order radiation pattern and that the sound field generated by a single loudspeaker can be described by the linear superposition of a monopole field and a dipole field.

In reference [13] it is shown that the sound field reproduction problem with a continuous layer of omnidirectional point sources is equivalent to an acoustical scattering problem. More specifically, it is shown that the source strength function required for the reproduction of a given sound field is equal to the normal derivative of the total field that is generated if the desired sound field is scattered by a sound soft object (pressure release boundary) with the shape of the continuous distribution of secondary sources. This concept is extended in this paper to the case of secondary sources with first order directivity. It is shown that, also in this case, the sound field reproduction problem is analogous to a scattering problem, with the difference that the boundary object will have an impedance condition, which is related to the directivity of the secondary sources adopted.

DEFINITION OF THE PROBLEM

A loudspeaker array is given, including an ideally continuous distribution of secondary sources arranged on the boundary $\partial\Lambda$ of a bounded set Λ (subset of either $\mathbb{R}^2$ or $\mathbb{R}^3$). As mentioned above, the assumption of a continuous distribution is a reasonably accurate model for a discrete distribution of secondary sources, provided that the distance between neighbouring sources is smaller than half of the wave length of the sound to be reproduced (in order to avoid spatial aliasing).

It is assumed that the sound field generated by each loudspeaker arranged at the location $\mathbf{y}_\ell \in \partial\Lambda$ can be expressed by

$$p_\ell(\mathbf{x}) = \alpha G(\mathbf{x},\mathbf{y}_\ell) + \frac{1-\alpha}{ik}\nabla_n G(\mathbf{x},\mathbf{y}_\ell) \quad (1)$$

with $0 \le \alpha \le 1$. $G(\mathbf{x},\mathbf{y}_\ell)$ is the free field Green function of the Helmholtz equation, given by

$$G(\mathbf{x},\mathbf{y}_\ell) = \frac{e^{ik|\mathbf{x}-\mathbf{y}_\ell|}}{4\pi|\mathbf{x}-\mathbf{y}_\ell|}, \quad \mathbf{x} \ne \mathbf{y}_\ell \quad (2)$$

whilst $\nabla_n G(\mathbf{x},\mathbf{y}_\ell)$ is its normal gradient with respect to the unitary vector $\hat{\mathbf{n}}$, normal to $\partial\Lambda$ at $\mathbf{y}_\ell$ and directed towards its exterior, as shown in Figure 1. This is given by

$$\nabla_n G(\mathbf{x},\mathbf{y}_\ell) = -ik\frac{e^{ik|\mathbf{x}-\mathbf{y}_\ell|}}{4\pi|\mathbf{x}-\mathbf{y}_\ell|}\cos\theta\left(1+\frac{i}{k|\mathbf{x}-\mathbf{y}_\ell|}\right), \quad (3)$$
$$\mathbf{x} \ne \mathbf{y}_\ell$$

where θ is the angle between the two vectors $\hat{\mathbf{n}}$ and $(\mathbf{x}-\mathbf{y}_\ell)$.

Formula (1) is a mathematical model of the acoustic radiation of secondary sources with first order directivity. The first term in (1) represent the field generated by a monopole, while the second term represents a dipole field.

Since a continuous distribution of secondary sources has been assumed, the sound field generated by the array is given by an integral of the form

$$\hat{p}(\mathbf{x}) = \int_{\partial\Lambda}\left(\alpha G(\mathbf{x},\mathbf{y}) + \frac{1-\alpha}{ik}\nabla_n G(\mathbf{x},\mathbf{y})\right) w(\mathbf{y})\, dS(\mathbf{y}), \quad (4)$$
$$\mathbf{x} \in \mathbb{R}^3$$

where the function $w(\mathbf{y})$ represents the strength of the secondary sources. In the mathematical literature (see for example [14]), the integral above is referred to as the *combined layer potential*, since it is the combination of a single layer potential (only monopole-like secondary sources) and a double layer potential (only dipole-like secondary sources)

The aim is to reproduce in Λ a given target sound field $p(\mathbf{x})$ with the loudspeaker array above. The target field is assumed to satisfy the homogeneous wave equation in Λ, that is

$$\nabla^2 p(\mathbf{x}) + k^2 p(\mathbf{x}) = 0, \quad \mathbf{x} \in \Lambda \quad (5)$$

The solution of this problem corresponds to the determination of the required source strength function $w(\mathbf{y})$ required for the reproduction of the desired field $p(\mathbf{x})$.

This problem involves the solution of an integral equation of the form

$$p(\mathbf{x}) = \int_{\partial\Lambda}\left(\alpha G(\mathbf{x},\mathbf{y}) + \frac{1-\alpha}{ik}\nabla_n G(\mathbf{x},\mathbf{y})\right) w(\mathbf{y}) dS(\mathbf{y}), \quad (6)$$
$$\mathbf{x} \in \partial\Lambda$$

(note that that the left hand side of this integral is the target sound field). We will not be concerned here with the existence and uniqueness of the solution. The interested reader can refer to [14] for an in depth discussion of this subject.

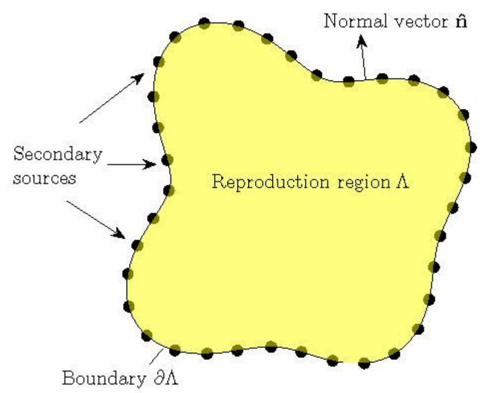

Figure 1: diagram of the distribution of secondary sources

EQUIVALENT SCATTERING PROBLEM

In order to analyse the equivalent scattering problem we create the fictitious scenario, in which the incident sound field $p(\mathbf{x})$ (the target sound field) is scattered by an impenetrable object with the shape of Λ and impedance (Robin) boundary conditions of the form

$$\alpha p_T(\mathbf{x}) + \frac{1-\alpha}{ik}\nabla_n p_T(\mathbf{x}) = 0 \quad (7)$$

The total field $p_T(\mathbf{x})$ is given by the sum of the incident field $p(\mathbf{x})$ and the scattered field $p_s(\mathbf{x})$. The case of $\alpha = 1$ corresponds to a sound soft object (pressure release boundaries) whilst the limiting case of $\alpha = 0$ determines a sound hard object (rigid boundaries). Note that this boundary condition has been chosen in view of the directivity pattern of the secondary sources, represented by equation (1).

The incident field can be expressed in the interior of Λ by the Kirchhoff Helmholtz integral [15], yielding

$$\int_{\partial\Lambda} G(\mathbf{x},\mathbf{y})\nabla_n p(\mathbf{y}) - \nabla_n G(\mathbf{x},\mathbf{y}) p(\mathbf{y}) dS(\mathbf{y})$$
$$= \begin{cases} p(\mathbf{x}) & \mathbf{x} \in \Lambda \\ p(\mathbf{x})/2 & \mathbf{x} \in \partial\Lambda \\ 0 & \mathbf{x} \notin \overline{\Lambda} \end{cases} \quad (8)$$

while the scattered field $p_s(\mathbf{x})$ is given, in the exterior of Λ, by

$$-\int_{\partial\Lambda} G(\mathbf{x},\mathbf{y})\nabla_n p_s(\mathbf{y}) - \nabla_n G(\mathbf{x},\mathbf{y}) p_s(\mathbf{y}) dS(\mathbf{y}) = \begin{cases} 0 & \mathbf{x}\in\Lambda \\ p_s(\mathbf{x})/2 & \mathbf{x}\in\partial\Lambda \\ p_s(\mathbf{x}) & \mathbf{x}\notin\overline{\Lambda} \end{cases} \tag{9}$$

Subtracting equation (9) from (8) and recalling the definition of the total field $p_T(\mathbf{x})$ yields

$$\int_{\partial\Lambda} G(\mathbf{x},\mathbf{y})\nabla_n p_T(\mathbf{y}) - \nabla_n G(\mathbf{x},\mathbf{y}) p_T(\mathbf{y}) dS(\mathbf{y}) = \begin{cases} p(\mathbf{x}) & \mathbf{x}\in\Lambda \\ [p(\mathbf{x}) - p_s(\mathbf{x})]/2 & \mathbf{x}\in\partial\Lambda \\ -p_s(\mathbf{x}) & \mathbf{x}\notin\overline{\Lambda} \end{cases} \tag{10}$$

Substituting the Robin boundary condition (7) in the equation above and after algebraic manipulation, we have that

$$\int_{\partial\Lambda} \left(\alpha G(\mathbf{x},\mathbf{y}) + \frac{1-\alpha}{ik}\nabla_n G(\mathbf{x},\mathbf{y}) \right) \frac{\nabla_n p_T(\mathbf{y})}{\alpha} dS(\mathbf{y}) = \begin{cases} p(\mathbf{x}) & \mathbf{x}\in\Lambda \\ (p(\mathbf{x}) - p_s(\mathbf{x}))/2 & \mathbf{x}\in\partial\Lambda \\ p_s(\mathbf{x}) & \mathbf{x}\in R^3\setminus\Lambda \end{cases} \tag{11}$$

Form a simple comparison of this result with equation (4), it can be observed that

$$w(\mathbf{y}) = \frac{\nabla_n p_T(\mathbf{y})}{\alpha} \tag{12}$$

namely the secondary source strength function $w(\mathbf{y})$ corresponds to the normal derivative of the total field, divided by α.

The equation above not only provides an expression for $w(\mathbf{y})$ that allows for an exact reproduction of the target field in Λ, but it also indicates that the reproduced sound field in the exterior of Λ is the scattered field $p_s(\mathbf{x})$ arising from the equivalent scattering problem.

It should be noted that a result analogous to equation (12) can be obtained by expressing $\nabla_n p_T(\mathbf{y})$ as a function of $p_T(\mathbf{y})$, namely

$$w(\mathbf{y}) = -ik\frac{p_T(\mathbf{y})}{1-\alpha} \tag{13}$$

This can expression can be used, for example, in the case of $\alpha = 0$ (dipole secondary sources).

Closed form expression for $w(\mathbf{y})$ can be derived for geometries such as spheres and circles or even lines or planes (as an extension of the method to unbounded regions), while approximate solutions can be obtained still in closed form by application of the physical optics approximation for direct scattering problems (Kirchhoff approximation). This assumes that the total field is zero in the region of the boundary that is not acoustically illuminated by the incident field and equals twice the incident field in the illuminated region (see [14] for a more detailed discussion).

EXAMPLE WITH SPHERICAL GEOMETRY

The example is provided of the case when the loudspeakers are arranged on a spherical surface. It is therefore assumed that the boundary $\partial\Lambda$ is a sphere of radius R.

The incident field can be expressed by [15]

$$p(\mathbf{x}) = \sum_{n=0}^{\infty} j_n(kx) \sum_{m=-n}^{n} A_{mn} Y_n^m(\hat{\mathbf{x}}) \tag{14}$$

where $j_n(kx)$ are spherical Bessel functions, $Y_n^m(\hat{\mathbf{x}})$ are spherical harmonics, $x = |\mathbf{x}|$, $\hat{\mathbf{x}} = \mathbf{x}/x$ and the coefficients A_{mn} depend on the given field. The scattered field can be expressed similarly by [15]

$$p_s(\mathbf{x}) = \sum_{n=0}^{\infty} h_n(kx) \sum_{m=-n}^{n} B_{mn} Y_n^m(\hat{\mathbf{x}}) \tag{15}$$

where $h_n(kx)$ is a spherical Hankel function of the first kind. The total field on the boundary $\partial\Lambda$ is given by the sum of the two equations above, with $x = R$.

The normal derivative $\nabla_n p_T(\mathbf{x})$ of the field on $\partial\Lambda$ is obtained by taking the normal derivative of the Bessel and Hankel functions with respect to x, namely

$$\nabla_n p_T(\mathbf{y}) = k\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \left[A_{mn} j_n'(kR) + B_{mn} h_n'(kR) \right] Y_n^m(\hat{\mathbf{y}}) \tag{16}$$

Applying the Robin boundary condition (7) yields

$$B_{mn} = -A_{mn}\frac{\alpha\, j_n(kR) - i(1-\alpha)\, j_n'(kR)}{\alpha\, h_n(kR) - i(1-\alpha) h_n'(kR)} \tag{17}$$

where the orthogonality relation of the spherical harmonics on the unit sphere has been used [14],[15]. The explicit expression for the source strength $w(\mathbf{y})$ can be calculated from equations (12), (16) and (17) and in view of the Wronskian relation

$$j_n'(z)h_n(z) - j_n(z)h_n'(z) = -\frac{i}{z^2} \tag{18}$$

After some algebraic manipulation we obtain

$$w(\mathbf{y}) = \frac{1}{kR^2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{A_{mn}Y_n^m(\hat{\mathbf{y}})}{i\alpha\, h_n(kR) + (1-\alpha) h_n'(kR)} \tag{18}$$

An similar result is derived also in [16]. The case of $\alpha = 1$, reported in the literature [10],[12],[17],[18], corresponds to a sound soft sphere and to the case of omnidirectional secondary sources. $\alpha = 0$ corresponds to a sound hard sphere and to secondary sources with dipole radiation pattern.

CONCLUSIONS

It has been shown that the problem of reproducing a target sound field with an array of loudspeakers with first order directivity is equivalent to an acoustical scattering problem. The source strength functions required for the reproduction of a given target sound field corresponds to the normal derivative of the total acoustic field generated when the target field is scattered by an object with the same shape of the loudspeaker array (the latter is assumed to be a continuous distribution of directional secondary sources). The scattering object has impedance boundaries, which are determined from the expression of the radiation pattern of the secondary sources. The case of spherical geometry has been presented as an example.

The results presented in this paper may help improve the design of signal processing strategies for sound field reproduction systems. Nevertheless, solution in closed form can be determined only for simple shapes of the loudspeaker array, or by using the physical optics approximation. Further work might be undertaken in order to derive the source strength functions for more complex geometries, either by explicit calculation or with numerical methods used also for the solution of scattering problems. It would be interesting to find out if this approach can be extended to secondary sources with higher order directivity.

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