Source-Induced Optical Noise in Polarization Measurements

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***Abstract*—Formalism is derived for the autocovariance of the noise induced by a source of general statistics, in measurements of the Stokes vector at the output of an optical medium. The autocovariance is given in terms of the source statistics and the medium transfer properties. Specific results and an experimental demonstration are provided for a polarized laser source. The results establish phase-noise-related bounds to the errors in polarization measurements with a continuous-wave laser source.**

***Index Terms*—Optical noise, phase noise, polarization measurements, polarization-mode dispersion (PMD), statistical optics.**

I. Introduction

SOURCE-INDUCED optical noise may become a performance-limiting factor in numerous applications, such as optical communications [1], optical sensors [2]–[4], optical code-division multiple-access schemes [5], reflectometry, and more. Fluctuations of the optical source amplitude appear as intensity noise at the output of a square law detector. When an optical medium contains multiple paths, the source phase noise may also give rise to intensity noise upon detection. The transfer properties of an optical medium filter the optical noise and alter its statistics. While optical noise in intensity measurements was studied by many [2]–[4], [6], [7], its effect on Polarimetric measurements was rarely treated in the literature.

Recently, the effect of source-induced noise in measurements of the Stokes parameters was studied in both theory [8] and experiment [9]. It has been shown that when measured at the output of an optical medium, each Stokes parameter may be accompanied by noise of different magnitude and statistical properties [8]. This noise can limit the accuracy of polarization measurements in various applications, such as polarization-mode dispersion (PMD) monitoring. Our previous studies were limited to thermal sources which are characterized by Gaussian statistics.

In this letter, we generalize the work of Weissman [6], and obtain an expression for the autocovariance functions of the Stokes parameters at the output of a linear optical medium. These functions are given in terms of fourth-order moments of the optical field of a general source, and the transfer properties of the medium. Emphasis is given to the specific case of a polarized laser source with predominant phase noise. The obtained formalism is demonstrated experimentally for the noise spectrum at the output of a first-order PMD medium. Finally, bounds to the errors in measuring the state of polarization (SOP) in the presence of first-order PMD are discussed.

Manuscript received January 17, 2006; revised April 9, 2006. This work was supported in part by the Israel Science Foundation. The work of A. Zadok was supported by the Israel Clore Foundation under a Ph.D. study scholarship.

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Digital Object Identifier 10.1109/LPT.2006.877012

II. Autocovariance of Stokes Parameters

A. Source of General Statistics

Let $\vec{E}_{\rm in}(t)$, $\vec{E}_{\rm out}(t)$ denote Jones column vectors describing the instantaneous field at the input and output of a linear, time invariant optical medium, respectively. The components of $\vec{E}_{\rm in}(t)$, $\vec{E}_{\rm out}(t)$ are assumed to be stationary random processes. We define the instantaneous coherency matrices of $\vec{E}_{\rm in}(t)$, $\vec{E}_{\rm out}(t)$ as [6], [8]

$$\mathbf{J}_{\rm in,out}(t_1,t_2)=\vec{E}_{\rm in,out}(t_2)\,\vec{E}^{\dagger}_{\rm in,out}(t_1). \tag{1}$$

The output optical field $\vec{E}_{\rm out}(t)$ is related to $\vec{E}_{\rm in}(t)$ though the impulse response matrix of the medium $\mathbf{h}(t)$

$$\vec{E}_{\rm out}(t)=\mathbf{h}(t)*\vec{E}_{\rm in}(t)=\int_{-\infty}^{\infty}\mathbf{h}(t-\xi)\vec{E}_{\rm in}(\xi)\,d\xi. \tag{2}$$

Using (2), we can relate $\mathbf{J}_{\rm out}(t_1,t_2)$ to $\mathbf{J}_{\rm in}(t_1,t_2)$ [6]

$$\begin{aligned}\mathbf{J}_{\rm out}(t_1,t_2)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{h}(t_2-\xi)\vec{E}_{\rm in}(\xi)\vec{E}^{\dagger}_{\rm in}(\eta)\,\mathbf{h}^{\dagger}(t_1-\eta)d\xi d\eta\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{h}(t_2-\xi)\,\mathbf{J}_{\rm in}(\eta,\xi)\mathbf{h}^{\dagger}(t_1-\eta)\,d\xi d\eta.\end{aligned} \tag{3}$$

The instantaneous Stokes parameters $S_k(t)$ are given by

$$S_k(t)={\rm tr}\left[\mathbf{J}_{\rm out}(t,t)\,\boldsymbol{\sigma}_k\right] \tag{4}$$

where $k=0,\ldots,3$ and $\boldsymbol{\sigma}_k$ are the Pauli spin matrices

$$\boldsymbol{\sigma}_0=\mathbf{I},\ \boldsymbol{\sigma}_1=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\ \boldsymbol{\sigma}_2=\begin{bmatrix}0&1\\1&0\end{bmatrix},\ \boldsymbol{\sigma}_3=\begin{bmatrix}0&-j\\j&0\end{bmatrix}. \tag{5}$$

Combining (3) and (4), we obtain

$$\begin{aligned}S_k(t)&={\rm tr}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{h}(t-\xi)\,\mathbf{J}_{\rm in}(\eta,\xi)\mathbf{h}^{\dagger}(t-\eta)\,d\xi d\eta\boldsymbol{\sigma}_k\right]\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}{\rm tr}\left[\mathbf{h}^{\dagger}(t-\eta)\,\boldsymbol{\sigma}_k\mathbf{h}(t-\xi)\,\mathbf{J}_{\rm in}(\eta,\xi)\right]d\xi d\eta.\end{aligned} \tag{6}$$

We define $\mathbf{M}_k(t_1,t_2)\equiv\mathbf{h}^{\dagger}(t_1)\,\boldsymbol{\sigma}_k\mathbf{h}(t_2)$. The autocovariance functions $C_k(\tau)$ of the Stokes parameters are expressed as [6]

$$\begin{aligned}C_k(\tau)&=\langle S_k(\tau)\,S_k(0)\rangle-\langle S_k(\tau)\rangle\,\langle S_k(0)\rangle\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle{\rm tr}\left[\mathbf{M}_k(\tau-\eta,\tau-\xi)\,\mathbf{J}_{\rm in}(\eta,\xi)\right]\\&\qquad\cdot\,{\rm tr}\left[\mathbf{M}_k(-\mu,-\nu)\,\mathbf{J}_{\rm in}(\mu,\nu)\right]\rangle\\&\qquad\cdot\,d\eta d\xi d\mu d\nu\\&\quad-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}{\rm tr}\left[\mathbf{M}_k(\tau-\eta,\tau-\xi)\,\langle\mathbf{J}_{\rm in}(\eta,\xi)\rangle\right]\\&\quad\cdot\,{\rm tr}\left[\mathbf{M}_k(-\mu,-\nu)\,\langle\mathbf{J}_{\rm in}(\mu,\nu)\rangle\right]d\eta d\xi d\mu d\nu.\end{aligned} \tag{7}$$

In (7), $\langle\ \rangle$ denote the ensemble average. Using the identity $\mathrm{tr}\,(\mathbf{AB})\cdot\mathrm{tr}\,(\mathbf{CD}) = \mathrm{tr}\,[(\mathbf{A}\otimes\mathbf{C})\,(\mathbf{B}\otimes\mathbf{D})]$, where $\otimes$ denotes Kronecker product and $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are square matrices of equal dimensions, we obtain

$$\begin{aligned} C_k(\tau) = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \\ &\cdot \mathrm{tr}\,\{[\mathbf{M}_k(\tau-\eta,\tau-\xi)\otimes\mathbf{M}_k(-\mu,-\nu)] \\ &\quad\cdot[\langle\mathbf{J}_{\mathrm{in}}(\eta,\xi)\otimes\mathbf{J}_{\mathrm{in}}(\mu,\nu)\rangle \\ &\qquad-\langle\mathbf{J}_{\mathrm{in}}(\eta,\xi)\rangle\otimes\langle\mathbf{J}_{\mathrm{in}}(\mu,\nu)\rangle]\}\,d\eta d\xi d\mu d\nu. \end{aligned} \tag{8}$$

The first term in the integrand represents transfer properties of the optical medium, and the second term describes stochastic properties of the input signal. The radio-frequency (RF) spectrum is given by the Fourier transform of (8).

B. Polarized Continuous-Wave (CW) Laser Source

Let the field at the optical medium input be that of a CW polarized laser source, of optical frequency ω_0, intensity I_{in}, and fixed unit Jones vector $\hat{\varepsilon}_0$. The instantaneous optical phase of the laser source is represented by a stationary stochastic process, denoted by $\varphi\,(t)$. The optical field can be expressed as [3], [4]

$$\vec{E}_{\mathrm{in}}\,(t) = \sqrt{I_{\mathrm{in}}}\exp\left[j\omega_0 t + j\varphi\,(t)\right]\hat{\varepsilon}_0. \tag{9}$$

The input instantaneous coherency matrix can be separated into a fixed matrix, multiplied by a scalar phase term

$$\mathbf{J}_{\mathrm{in}}\,(t_1,t_2) = I_{\mathrm{in}}\mathbf{J_0}\exp\left[j\varphi\,(t_2) - j\varphi\,(t_1)\right] \tag{10}$$

where $\mathbf{J_0} \equiv \hat{\varepsilon}_0\hat{\varepsilon}_0^{\dagger}$. The second-order moment of the random phase term is given by [3], [4]

$$\langle\exp\left[j\varphi\,(t_2) - j\varphi\,(t_1)\right]\rangle = \exp\left(\frac{-|t_2-t_1|}{\tau_c}\right) \equiv \Gamma_2\,(t_1,t_2) \tag{11}$$

with τ_c denoting the source coherence time [7]. The fourth-order moments can be expressed as (12), shown at the bottom of the page [3], [4]. Using (6), (10)–(12), the mean output Stokes parameters are

$$\langle S_k\rangle = I_{\mathrm{in}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{tr}\left[\mathbf{M}_k\,(-\mu,-\nu)\,\mathbf{J}_0\right]\Gamma_2\,(\mu,\nu)\,d\mu d\nu. \tag{13}$$

Similarly, we may express the autocovariance functions

$$\begin{aligned} C_k(\tau) = &I_{\mathrm{in}}^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \\ &\cdot\mathrm{tr}\,\{[\mathbf{M}_k(\tau-\eta,\tau-\xi)\otimes\mathbf{M}_k(-\mu,-\nu)] \\ &\quad\cdot(\mathbf{J}_0\otimes\mathbf{J}_0)\} \\ &\cdot[\Gamma_4(\eta,\xi,\mu,\nu) - \Gamma_2(\eta,\xi)\Gamma_2(\mu,\nu)]\,d\eta d\xi d\mu d\nu. \end{aligned} \tag{14}$$

C. First-Order PMD Medium

The expressions in (13) and (14) can be applied to the output of an optical medium with first-order PMD. The impulse response matrix of the medium is described by its differential group delay (DGD) τ_0 and differential phase $\Delta\phi_m$. Without loss of generality, we express $\mathbf{h}\,(t)$ in the basis of the principal axes of the medium, and use an arbitrary input SOP $\hat{\varepsilon}_0$

$$\mathbf{h}(t) = \begin{bmatrix} \exp\left(-\frac{j}{2}\Delta\phi_m\right)\delta\left(t-\frac{1}{2}\tau_0\right) & 0 \\ 0 & \exp\left(\frac{j}{2}\Delta\phi_m\right)\delta\left(t+\frac{1}{2}\tau_0\right) \end{bmatrix} \tag{15}$$

$$\hat{\varepsilon}_0 = \left[\cos(\theta)\exp\left(-\frac{j}{2}\Delta\phi_{\mathrm{in}}\right) \quad \sin(\theta)\exp\left(\frac{j}{2}\Delta\phi_{\mathrm{in}}\right)\right]^T. \tag{16}$$

Using (10)–(16), the average Stokes vector at the output of the optical medium is given by

$$\begin{aligned} &\langle S_0\rangle = I_{\mathrm{in}}, \quad \langle S_2\rangle = I_{\mathrm{in}}\exp\left(\frac{-\tau_0}{\tau_c}\right)\sin(2\theta)\cos\left(\Delta\phi\right) \\ &\langle S_1\rangle = I_{\mathrm{in}}\cos(2\theta), \quad \langle S_3\rangle = I_{\mathrm{in}}\exp\left(\frac{-\tau_0}{\tau_c}\right)\sin\left(2\theta\right)\sin\left(\Delta\phi\right) \end{aligned} \tag{17}$$

where $\Delta\phi \equiv \Delta\phi_{\mathrm{in}} + \Delta\phi_m$. It is seen that the phase noise introduces a decrease in the degree of polarization (DOP). For the autocovariance functions $C_k\,(\tau)$, $k = 0,\ldots,3$, we define $\alpha \equiv (-2\,|\tau| + |\tau+\tau_0| + |\tau-\tau_0|)/\tau_c$, and obtain

$$\begin{aligned} &C_{2,3}(\tau) = \frac{1}{2}I_{\mathrm{in}}^2\sin^2(2\theta)\exp\left(-\frac{2\tau_0}{\tau_c}\right)[e^{\alpha}-1\pm\cos(2\Delta\phi)(e^{-\alpha}-1)] \\ &C_{0,1}(\tau) = 0. \end{aligned} \tag{18}$$

Note that $C_{2,3}\,(\tau) = 0$, for $|\tau| > \tau_0$.

III. Experiment

Application of the theoretical results for an optical medium of first-order PMD was demonstrated experimentally. The medium consisted of a polarization beam splitter (PBS) and a polarization beam combiner (PBC), connected by two sections of polarization-maintaining fibers with different lengths (see Fig. 1). This configuration allowed for long DGD of ~30 nS for which the source induced optical noise was dominant. The input light from a CW distributed feedback laser source was linearly polarized at $\theta = 45^\circ$ with respect to the principal axes of the PBS. A measurement of $S_2\,(t)$ was performed at the PBC output. To that

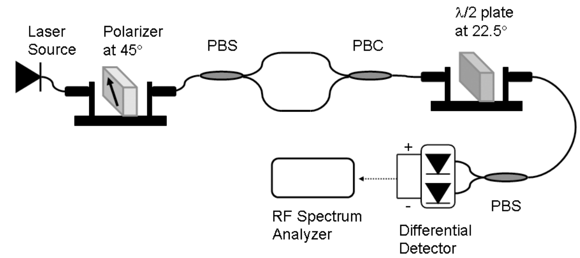

Fig. 1. Experimental setup for measurement of the spectrum of S_2, at the output of a first-order PMD medium. All fibers are polarization maintaining.

$$\begin{aligned} &\langle\exp[j\varphi\,(t_4) - j\varphi\,(t_3) + j\varphi\,(t_2) - j\varphi\,(t_1)]\rangle \\ &\quad = \exp\left[\frac{(-\,|t_2-t_1| - |t_4-t_3| - |t_2-t_3| - |t_4-t_1| + |t_4-t_2| + |t_3-t_1|)}{\tau_c}\right] \equiv \Gamma_4\,(t_1,t_2,t_3,t_4) \end{aligned} \tag{12}$$

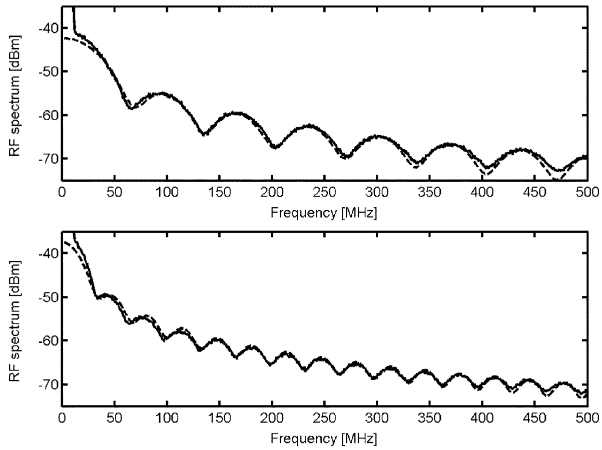

Fig. 2. Experimental (solid) and theoretical (dashed) power spectra of S_2 at the output of a first-order PMD medium. DGD is 15, 30 nS (top, bottom).

end, a broadband polarimeter, consisting of $\lambda/2$ plate, a second PBS, and a broadband differential detector, was used [9]. The polarimeter output was connected to an RF spectrum analyzer yielding the Fourier transform of $C_2(\tau)$.

In the experimental setup, $\Delta\phi_m$ varied randomly within the measurement duration. The autocovariance functions of (18) are, therefore, averaged with respect to $\Delta\phi$

$$\langle C_2(\tau)\rangle_{\Delta\phi} = \langle C_3(\tau)\rangle_{\Delta\phi} = \frac{1}{2} I_{\text{in}}^2 \exp\left(\frac{-2\tau_0}{\tau_c}\right)[\exp(\alpha) - 1]. \tag{19}$$

Fig. 2 shows the measured power spectrum of S_2, as well as the Fourier transform of (19), for two values of DGD: 15 (top) and 30 nS (bottom). Close agreement of theory and experiment is evident. The fitted coherence time of the laser source was found to be 61 nS. The measured DOP was 0.78 and 0.61 in agreement with (17).

IV. Discussion

The expressions obtained above can be utilized in estimating the phase-noise-induced errors in SOP measurements. Consider a measurement of $\vec{S}$ at the output of first-order PMD medium, averaged over a duration T. The mean and variance of S_k are [10]

$$W_k = \langle S_k\rangle T \tag{20}$$

$$\sigma_k^2 = \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} C_k(\xi-\eta)\,d\eta\,d\xi = \int_{-T/2}^{T/2}\int_{\xi-T/2}^{\xi+T/2} C_k(\tau)\,d\tau\,d\xi. \tag{21}$$

In many practical situations τ_c is much longer than τ_0. For this condition, the autocovariance functions can be approximated for $|\tau| \le \tau_0$ by

$$C_{2,3}(\tau) \approx I_{\text{in}}^2 \sin^2(2\theta)[1 \mp \cos(2\Delta\phi)]\frac{\tau_0 - |\tau|}{\tau_c}. \tag{22}$$

If in addition $T \gg \tau_0$, we obtain

$$\begin{aligned}\sigma_{2,3}^2 &\approx I_{\text{in}}^2 \sin^2(2\theta)\int_{-T/2}^{T/2}\int_{-\infty}^{\infty} C_{2,3}(\tau)\,d\tau\,d\xi \\ &= I_{\text{in}}^2 \sin^2(2\theta)[1 \mp \cos(2\Delta\phi)]\frac{\tau_0^2}{\tau_c}T.\end{aligned} \tag{23}$$

The azimuth and elevation angles on the Poincaré sphere are given by $\tan(\beta) = W_2/W_1$ and $\tan(\delta) = W_3/\sqrt{(W_1^2 + W_2^2)}$, respectively. The variances of these angles may be represented as an uncertainty area on the unit sphere, surrounding the mean SOP: $dA = \sigma_\beta\sigma_\delta\cos(\delta)$. This area can be used as a quantitative estimate of the phase-noise-induced errors in SOP measurements. Assuming sequential measurements of Stokes parameters by a single broadband polarimeter, the uncertainty area can be related to $\sigma_{2,3}^2$, leading to

$$dA \le \frac{\tau_0^2}{(\tau_c T)}. \tag{24}$$

For the experimental parameters of Section III, $dA \approx 10^{-6}$. In wavelength swept Polarimetric measurements, the coherence time of the laser source is typically two orders of magnitude longer leading to proportionally smaller uncertainty.

In conclusion, the autocovariance of source-induced optical noise in Stokes parameters measurements was formulated at the output of a linear time-invariant optical medium. The formalism is applicable for general source statistics, with specific results presented for a polarized CW laser source. Experimental results were obtained for the output of a first-order PMD medium. Phase-noise-induced bounds to the errors in polarization measurements were established.

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