Source Localization Using Recursively Applied and Projected (RAP) MUSIC

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Abstract†

A new method for source localization is described that is based on a modification of the well-known MUSIC algorithm. In classical MUSIC, the array manifold vector is projected onto an estimate of the signal subspace, but errors in the estimate can make location of multiple sources difficult. Recursively applied and projected (RAP) MUSIC uses each successively located source to form an intermediate array gain matrix, and projects both the array manifold and the signal subspace estimate into its orthogonal complement. The MUSIC projection is then performed in this reduced subspace. Using the metric of principal angles, we describe a general form of the RAP-MUSIC algorithm for the case of diversely polarized sources. Through a uniform linear array simulation, we demonstrate the improved Monte Carlo performance of RAP-MUSIC relative to MUSIC and two other sequential subspace methods, S and IES-MUSIC.

1. Introduction

The multiple signal classification (MUSIC) [1, 2] method for source localization is a computationally attractive alternative to least-squares, since the locations of each source can be found separately, thus avoiding high dimensional searches of non-convex cost functions. Two problems, however, often arise in practice. First, errors in estimating the signal subspace can make it difficult to differentiate "true" from "false" peaks in the MUSIC metric. Second, automatically finding several local maxima in the MUSIC metric becomes difficult as the dimension of the source space increases.

We develop here an algorithm, RAP-MUSIC, that overcomes these problems by using a recursive procedure in which each source is found as the global maximizer of a different cost function. In essence, the method works by applying a MUSIC search to a modified problem in which we first project both the estimated signal subspace and the array manifold vector away from the subspace spanned by the sources that have already been found. We describe the RAP-MUSIC method using *principal angles* [3] which provide a framework for comparing signal subspaces and have previously been used in other related subspace signal processing problems [4, 5, 6].

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In Section 2 and 3 we describe the problem formulation, and for comparative purposes, we describe classical MUSIC in terms of principal angles. We then develop RAP-MUSIC in Section 4 for the general case of "diversely polarized" sources [1, 2]. A comparison of this method, both in terms of formulation using principal angles and Monte-Carlo performance evaluation, is presented for two alternative sequential algorithms, S- and IES-MUSIC [7, 8].

2. Background

We consider the problem of estimating the parameters of r sources impinging on an m sensor element array. Each source is represented by an $m > r$ (possibly complex) array manifold vector $\boldsymbol{a}(\theta)$. Each source parameter θ may be multidimensional, and the collection of the r manifold parameters is designated $\Theta = \{\theta_1, \ldots, \theta_r\}$. The manifold vectors collectively form an $m \times r$ array transfer matrix

$$\boldsymbol{A}(\Theta) = [\boldsymbol{a}(\theta_1), \ldots, \boldsymbol{a}(\theta_r)] \tag{1}$$

which we assume to be of full column rank r for any set of r distinct source parameters Θ, i.e., no array ambiguities exist. Associated with each array vector is a time series $s(t)$, and the data are acquired as $\boldsymbol{x}(t) = \boldsymbol{A}(\theta)\boldsymbol{s}(t) + \boldsymbol{n}(t)$, where $\boldsymbol{s}(t)$ is the vector of r time series at time t. The data are assumed to be white, i.e., the additive noise vector $\boldsymbol{n}(t)$ has zero mean and covariance $E\{\boldsymbol{n}(t)\boldsymbol{n}^H(t)\} = \sigma_n^2\boldsymbol{I}$, where superscript "^{H}" denotes the Hermitian transpose.

The autocorrelation of $\boldsymbol{x}(t)$ can be partitioned as

$$\begin{aligned}\boldsymbol{R} &= E\{\boldsymbol{x}(t)\boldsymbol{x}^H(t)\} \\ &= \boldsymbol{A}(\Theta)(E\{\boldsymbol{s}(t)\boldsymbol{s}^H(t)\})\boldsymbol{A}(\Theta)^H + \sigma_n^2\boldsymbol{I} \\ &= \Phi[\Lambda + \sigma_n^2\boldsymbol{I}]\Phi^H = \Phi_s\Lambda_s\Phi_s^H + \Phi_n\Lambda_n\Phi_n^H\end{aligned} \tag{2}$$

where we have assumed that the time series $\boldsymbol{s}(t)$ are uncorrelated with the noise. We assume that the correlation of the signal time series yields a full rank matrix $\boldsymbol{P} = E\{\boldsymbol{s}(t)\boldsymbol{s}^H(t)\}$, and $\boldsymbol{A}(\Theta)\boldsymbol{P}\boldsymbol{A}(\Theta)^H$ can be eigendecomposed as $\Phi_s\Lambda\Phi_s^H$, such that $\mathrm{span}(\boldsymbol{A}(\Theta)) = \mathrm{span}(\Phi_s)$. The r eigenvalues of this decomposition combine with the noise covariance to form $\Lambda_s = \Lambda + \sigma_n^2\boldsymbol{I}$, with the eigenvalues in the diagonal Λ_s arranged in decreasing order. The diagonal Λ_n comprises the $m - r$ repeated eigenvalues σ_n^2. Thus *(2)* represents the well-known partitioning of the covariance matrix into *signal subspace* ($\text{span}(\Phi_s)$) and *noise-only subspace* ($\text{span}(\Phi_n)$) terms.

Let $\hat{\boldsymbol{R}}$ denote the sample covariance matrix estimate of $\boldsymbol{R}$ obtained by averaging the outer products of the data vectors. Accordingly, we designate the first r eigenvectors of $\hat{\boldsymbol{R}}$ as $\hat{\Phi}_s$, i.e., a set of vectors which span our estimate of the signal subspace; similarly we designate the estimated noise-only subspace $\hat{\Phi}_n$ using the remaining eigenvectors.

Finally, we generalize the array manifold vector for the case of vector sources representing, for instance, diverse polarization [1, 2] in conventional array processing or current dipoles in EEG and MEG source localization [9, 10, 11]. In this case, the array manifold vector is the product of an array matrix and a polarization or orientation vector,

$$\boldsymbol{a}(\theta) = \boldsymbol{G}(\rho)\phi \tag{3}$$

and we may view the parameter set for each source as $\theta = \{\rho, \phi\}$, comprising quasi-linear orientation parameters ϕ and nonlinear location parameters ρ.

3. MUSIC and principal angles

The MUSIC algorithm [1, 2] finds the source locations as those for which the corresponding array vector is nearly orthogonal to the noise-only subspace, or equivalently, projects almost entirely into the estimated signal subspace. For the diversely polarized case, the problem becomes more complex since the signal or noise-only subspaces must be compared with the entire span of the gain matrix $\boldsymbol{G}(\rho)$. A natural way to compare these two subspaces is through use of *principal angles* [3] or *canonical correlations* (i.e. the cosines of the principal angles) (cf. [6]).

Let q denote the minimum of the ranks of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. The "subspace correlation" is a vector containing the cosines of the q principal angles that reflect the similarity between the subspaces spanned by the columns of the two matrices. The elements of the subspace correlation vector are ranked in decreasing order, and we denote the cosine of the smallest principal angle (i.e., the largest canonical correlation) as

$$subcorr_1(\boldsymbol{A}, \boldsymbol{B}) \tag{4}$$

If $subcorr_1(\boldsymbol{A}, \boldsymbol{B}) = 1$, then the two subspaces have at least a one dimensional subspace in common. Conversely, if $subcorr_1(\boldsymbol{A}, \boldsymbol{B}) = 0$, then the two subspaces are orthogonal. These subspace correlations are readily computed using SVDs as described in [3] and reviewed in [9].

The MUSIC algorithm finds the source locations as those for which the principal angle between the array vector and the noise-only subspace is maximum. Equivalently the sources are chosen as those that minimize the noise-only subspace correlation $subcorr_1(\boldsymbol{a}(\theta), \hat{\Phi}_n)$ or equivalently maximize the signal subspace correlation $subcorr_1(\boldsymbol{a}(\theta), \hat{\Phi}_s)$. Since the first argument is a vector and the second is already orthogonalized, the square of this signal subspace correlation is easily shown to be

$$subcorr_1(\boldsymbol{a}(\theta), \hat{\Phi}_s)^2 = \frac{(\boldsymbol{a}(\theta)^H \hat{\Phi}_s \hat{\Phi}_s^H \boldsymbol{a}(\theta))}{(\boldsymbol{a}(\theta)^H \boldsymbol{a}(\theta))} \tag{5}$$

where the right hand side is the standard metric used in MUSIC [1, 2]. Practical considerations in low-rank E/MEG source localization lead us to prefer the use of the signal rather than the noise-only subspace [9, 11, 12]. The development below in terms of the signal subspace is readily modified to computations in terms of the noise-only subspace.

Principal angles can also be used to represent the MUSIC metric for diversely polarized sources [1, 2] and E/MEG dipole localization [10]. In this case, the algorithm must compare the entire space spanned by the gain matrix $\boldsymbol{G}(\rho)$ with the signal subspace. It is again straightforward to equate the subspace correlation with Schmidt's diversely polarized MUSIC solution,

$$subcorr_1(\boldsymbol{G}(\rho), \hat{\Phi}_s)^2 = \lambda_{\max}(\boldsymbol{U}_{\boldsymbol{G}}^H \hat{\Phi}_s \hat{\Phi}_s^H \boldsymbol{U}_G), \tag{6}$$

where $\boldsymbol{U}_{\boldsymbol{G}}$ is the orthogonalization of $\boldsymbol{G}(\rho)$ and $\lambda_{\max}(\ )$ is the maximum eigenvalue of the enclosed expression.

The source locations ρ can be found as those for which *(6)* is approximately unity. The quasi-linear parameters ϕ can then be found as the eigenvector corresponding to the maximum eigenvalue in *(6)*. Equivalently, the singular vectors from the SVDs performed to compute $subcorr_1(\ )$ can be used to form ϕ (see the appendix in [9] for further details).

4. RAP-MUSIC

If the r-dimensional signal subspace is estimated perfectly, then the sources are simply found as the r global maximizers of *(6)*. Errors in our estimate $\hat{\Phi}_s$ reduce *(6)* to a function with a single global maximum and at least $(r-1)$ local maxima. Finding the first source is simple: over a sufficiently densely sampled grid of the nonlinear parameter space ρ, find the global maximum of *(6)*,

$$\hat{\rho}_1 = \underset{\rho}{arg\,max}(subcorr_1(\boldsymbol{G}(\rho), \hat{\Phi}_s)) \tag{7}$$

We then extract the corresponding eigenvector to form the quasi-linear parameter estimate $\hat{\phi}$. The estimate of the parameters of the first source is denoted $\hat{\theta}_1 = \{\hat{\rho}_1, \hat{\phi}_1\}$ and the first estimated array manifold vector is formed as

$$\boldsymbol{a}(\hat{\theta}_1) = \boldsymbol{G}(\hat{\rho}_1)\hat{\phi}_1 \tag{8}$$

Identifying the remaining local maxima becomes more difficult since nonlinear search techniques may miss shallow or adjacent peaks and return to a previous peak. We also need to locate the r best peaks, rather than any r local maxima. Numerous techniques have been proposed in the past to enhance the "peak-like" nature of the spectrum (cf.

[4]), so that identifying these peaks becomes simpler. Nonetheless, "peak-picking" algorithms rapidly become complex and subjective as the number of sources increases and the dimensionality of ρ increases.

The novelty of RAP-MUSIC is to avoid this peak-picking problem entirely. We instead remove the component of the signal subspace that is spanned by the first source, then perform a search to find the second source as the global maximizer over this modified subspace. In this way we replace the problem of finding r local maxima with one in which we recursively find the sources by finding r global maxima.

The method can be viewed simply in terms of the subspace correlation functions described above. Define the orthogonal projection operation for $\boldsymbol{a}(\hat{\theta}_1)$ as

$$\Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)} = (\boldsymbol{I} - (\boldsymbol{a}(\hat{\theta}_1)\boldsymbol{a}^H(\hat{\theta}_1))/(\boldsymbol{a}^H(\hat{\theta}_1)\boldsymbol{a}(\hat{\theta}_1))) \qquad (9)$$

and apply this operator to both arguments of the $subcorr_1(\ )$ function. The second source is then found as the global maximizer

$$\hat{\rho}_2 = \underset{\rho}{arg\ max}\left(subcorr_1\left(\Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)}\boldsymbol{G}(\rho), \Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)}\hat{\Phi}_s\right)\right) \qquad (10)$$

Here we have projected both our signal subspace estimate and the multidimensional array manifold away from the first solution, then found the maximum subspace correlation (minimum principal angle) between these two projected spaces. After the maximization, the quasi-linear parameters are again easily extracted and the second array manifold vector estimated as $\boldsymbol{a}(\hat{\theta}_2) = \boldsymbol{G}(\hat{\rho}_2)\hat{\phi}_2$. We then form the orthogonal projection operator for the combination of the first two sources, and proceed recursively.

By extension, the kth recursion, $k = 1, \ldots, r$, of RAP-MUSIC is

$$\hat{\rho}_k = \underset{\rho}{arg\ max}\left(subcorr_1\left(\Pi^{\perp}_{\hat{\boldsymbol{A}}_{k-1}}\boldsymbol{G}(\rho), \Pi^{\perp}_{\hat{\boldsymbol{A}}_{k-1}}\hat{\Phi}_s\right)\right) \qquad (11)$$

where we define

$$\hat{\boldsymbol{A}}_{k-1} \equiv \left[\boldsymbol{a}(\hat{\theta}_1)\ \ldots\ \boldsymbol{a}(\hat{\theta}_{k-1})\right] \qquad (12)$$

as formed from the array manifold estimates of the previous $k-1$ recursions, and

$$\Pi^{\perp}_{\hat{\boldsymbol{A}}_{k-1}} \equiv (\boldsymbol{I} - \hat{\boldsymbol{A}}_{k-1}(\hat{\boldsymbol{A}}^H_{k-1}\hat{\boldsymbol{A}}_{k-1})^{-1}\hat{\boldsymbol{A}}^H_{k-1}) \qquad (13)$$

is the orthogonal projection operator for this matrix.

5. Other sequential forms of MUSIC

In [9], we introduced a preliminary version of RAP-MUSIC that we now refer to as R-MUSIC†, developing the algorithm from a subspace "distance" perspective [3], with specific application to the E/MEG source localization problem. Successful demonstration of R-MUSIC in a blind test for E/MEG source localization can be found in [13], and a more complete description of the application of R-MUSIC to the E/MEG problem can be found in [11].

†In [9, 13], we referred to R-MUSIC as RAP-MUSIC, and while related, these two methods are distinct.

To demonstrate the promise of RAP-MUSIC, we present here a comparison with classical MUSIC and two other "sequential" methods: S-MUSIC [7] and IES-MUSIC [8]. While the RAP-MUSIC method is applicable to any problem for which MUSIC can be used, we restrict our discussion here to a scalar source parameter θ and to two sources. All of the methods used in our comparison find the first source in the same way, i.e. as the global maximizer of $subcorr_1(\boldsymbol{a}(\theta), \Phi_s)$. The manner in which the second source is found differs for each method.

In S-MUSIC [7], we apply the projection operator *(9)* only to the array manifold and find the second source as $arg\ max_{\theta}\ g(\theta)$, where

$$g(\theta) = \frac{\left(\boldsymbol{a}^H(\theta)\Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)}\hat{\Phi}_s\hat{\Phi}_s^H\Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)}\boldsymbol{a}(\theta)\right)}{\left\|\Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)}\boldsymbol{a}(\theta)\right\|^2} \qquad (14)$$

In IES-MUSIC [8], the denominator of *(14)* is dropped and the following modification used

$$g(\theta, \hat{\rho}) = \boldsymbol{a}^H(\theta)\left(\boldsymbol{I} - \hat{\rho}^*\Pi_{\boldsymbol{a}(\hat{\theta}_1)}\right)\hat{\Phi}_s\hat{\Phi}_s^H(\boldsymbol{I} - \hat{\rho}\Pi_{\boldsymbol{a}(\hat{\theta}_1)})\boldsymbol{a}(\theta) \qquad (15)$$

This measure is equivalent to S-MUSIC for $\hat{\rho} = 1$ and MUSIC for $\hat{\rho} = 0$. In [8], an optimal scalar ρ is derived for the case of two sources, but this scalar requires knowledge of the two sources θ_1 and θ_2. Since these parameters are unknown, IES-MUSIC first obtains the estimated locations θ_1 and θ_2 from another approach, such as MUSIC, from which it forms the estimate $\hat{\rho}$. After this step $g(\theta, \hat{\rho})$ is maximized to find the second source.

These algorithms may be summarized and compared using the subspace correlation function as follows.

MUSIC:

$$\hat{\theta}_2 = arg\ max\ subcorr_1\{\boldsymbol{a}(\theta), \hat{\Phi}_s\} \qquad \hat{\theta}_2 \neq \hat{\theta}_1 \qquad (16)$$

S-MUSIC:

$$\hat{\theta}_2 = arg\ max\ subcorr_1\left\{\Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)}\boldsymbol{a}(\theta), \hat{\Phi}_s\right\} \qquad (17)$$

IES-MUSIC ($\hat{\rho}$ defined in [8]):

$$\hat{\theta}_2 = arg\ max\left\|\boldsymbol{I} - \hat{\rho}^*\Pi_{\boldsymbol{a}(\hat{\theta}_1)}\boldsymbol{a}(\theta)\right\|^2 \cdot subcorr_1\left\{(\boldsymbol{I} - \hat{\rho}^*\Pi_{\boldsymbol{a}(\hat{\theta}_1)})\boldsymbol{a}(\theta), \hat{\Phi}_s\right\}^2 \qquad (18)$$

RAP-MUSIC:

$$\hat{\theta}_2 = \mathit{arg\ max\ subcorr}_1\left\{\Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)}\boldsymbol{a}(\theta), \Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)}\hat{\Phi}_s\right\} \tag{19}$$

In *(17)* and *(18)*, the first argument is a vector and the second argument is already orthogonal. Thus *(17)* and *(18)* are readily seen to be equivalent to *(14)* and *(15)*, respectively. Conversely, by explicitly forming the orthogonalizations required in *(19)*, we may express RAP-MUSIC in a form comparable to *(14)* and *(15)* as

$$\left(\mathit{subcorr}_1\left\{\Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)}\boldsymbol{a}(\theta), \Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)}\hat{\Phi}_s\right\}\right)^2 = \frac{\left(\boldsymbol{a}^H(\theta)\Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)}\hat{\Phi}_s\left(\hat{\boldsymbol{V}}_{\boldsymbol{a}(\hat{\theta}_1)}\hat{\Sigma}^{-2}_{\boldsymbol{a}(\hat{\theta}_1)}\hat{\boldsymbol{V}}^H_{\boldsymbol{a}(\hat{\theta}_1)}\right)\hat{\Phi}^H_s\Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)}\boldsymbol{a}(\theta)\right)}{\left\|\Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)}\boldsymbol{a}(\theta)\right\|^2} \tag{20}$$

To form this equivalence we have orthogonalized the projected signal subspace using an SVD as

$$\Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)}\hat{\Phi}_s = \hat{\boldsymbol{U}}_{\boldsymbol{a}(\hat{\theta}_1)}\hat{\Sigma}_{\boldsymbol{a}(\hat{\theta}_1)}\hat{\boldsymbol{V}}^H_{\boldsymbol{a}(\hat{\theta}_1)} \tag{21}$$

and retained only the non-zero singular value components of *(21)*. We note that

$$\hat{\boldsymbol{U}}_{\boldsymbol{a}(\hat{\theta}_1)} = \Pi^{\perp}_{\boldsymbol{a}(\hat{\theta}_1)}\hat{\Phi}_s\hat{\boldsymbol{V}}_{\boldsymbol{a}(\hat{\theta}_1)}\hat{\Sigma}^{-1}_{\boldsymbol{a}(\hat{\theta}_1)}, \tag{22}$$

and substitution using *(22)* and the idempotent property of projection operators yields *(20)*.

When viewed in terms of the subspace correlations, we see that the clear difference between RAP-MUSIC and the other sequential forms is that the projection operator is applied to both arguments before computing the subspace correlation, rather than just to the array manifold as in the case of S and IES MUSIC.

6. Simulation

We have followed the simulations in [8] in order to draw performance comparisons between these various forms of recursive MUSIC. The sensor array is the conventional uniform linear array of sensors spaced a half-wavelength apart. The sources are far-field narrow-band and impinging on the array from scalar direction θ. The array manifold vector may therefore be specified as

$$\boldsymbol{a}(\theta) = [1, e^{i\pi\sin\theta}, \ldots, e^{i\pi(m-1)\sin\theta}]^T, \tag{23}$$

where $\theta = 0$ is broadside to the array, and $\|\boldsymbol{a}(\theta)\|^2 = m$. The source time series are assumed to be complex zero-mean Gaussian sequences with covariance matrix $\boldsymbol{P}$. We assume fifteen sensor elements and two sources at 25 and 30 degrees. The source covariance matrix is specified as

$$\boldsymbol{P} = \begin{bmatrix} 1 & \gamma \\ \gamma^* & 1 \end{bmatrix} \tag{24}$$

where $|\gamma| \le 1$ determines the degree of correlation between these two sources of equal power. The variance of the noise is set to unity, such that the signal to noise power ratio is also unity.

We simulate n samples of both the signal and noise, form the estimated data covariance matrix, then extract the matrix $\hat{\Phi}_s$ comprising the two estimated signal subspace vectors. The noise variance is estimated as the mean of the noise-only subspace eigenvalues. For each realization, we find the maxima of the MUSIC measure in a region about each of the true solutions. The source with the better correlation was considered source θ_1. The second source θ_2 was then found by maximizing the appropriate measure, *(16)*-*(19)*. Since IES-MUSIC is a "two-pass" algorithm, i.e., it requires an initial estimate of both source parameters, we used the RAP-MUSIC source estimates for the initial estimate, as the RAP-MUSIC solution was on average superior to the MUSIC and S-MUSIC estimates. We also ran as a comparison IES-MUSIC with ρ set to the true optimal value using the true source angles.

In [8], closed-form formulae are presented for calculating the theoretical error variance of MUSIC, S-MUSIC, and IES-MUSIC. For each estimator, we also calculated a numerical root mean squared (RMS) error,

$$\mathrm{RMS} = \left(\frac{1}{\mathrm{Runs}}\sum_{i=1}^{\mathrm{Runs}}(\hat{\theta}_2(i) - \theta_2)^2\right)^{1/2} \tag{25}$$

where $\hat{\theta}_2(i)$ represents the estimate from the i th Monte Carlo run. In each of these runs, we determined which of the two MUSIC peaks in the regions about the true answer was greater and declared this source as $\hat{\theta}_1$. We then estimated the second source, then tabulated the actual number of runs used for both $\theta_2 = 25$ or 30 degrees, which is approximately evenly split at about 250 Monte Carlo runs each.

In Table 1, we held the number of time samples constant at $n = 1000$ and varied the degree of correlation between the two sources. For uncorrelated sources, $\gamma = 0$, all measures performed similarly, as also demonstrated in [8]. The differences in performance begin to arise at $\gamma = 0.7$, as tabulated in our table, where we observe that IES-MUSIC and RAP-MUSIC have RMS error about 25% better than MUSIC and S-MUSIC. At $\gamma = 0.925$, we see that RAP-MUSIC continues to have performance comparable to that of perfect IES-MUSIC, but that estimated IES-MUSIC is beginning to degrade in comparison; MUSIC and S-MUSIC have RMS error almost twice that of IES-MUSIC and RAP-MUSIC at this point. By $\gamma = 0.975$, all methods are experiencing comparable dif-

Table 1: Comparison of Analytic Std. Devs. and RMS error. The number of time samples remains constant at 1000, and the correlation γ between the two sources is varied. For each of the 500 Monte Carlo realizations, source 1 (either 25 or 30 degrees) was selected as the source with the highest MUSIC peak. The theoretical and root mean squared (RMS) error of the second source is tabulated. IES-MUSIC is shown both with its scalar ρ set using the true parameters and with its scalar estimated. MUSIC was unreliable in locating the second peak for $\gamma = 0.975$,

	n	1000		1000		1000		1000		1000		1000	
	γ	0.975		0.950		0.925		0.900		0.800		0.700	
	θ_2 (deg)	25	30	25	30	25	30	25	30	25	30	25	30
	Runs	254	246	236	264	270	230	242	258	241	259	231	269
MUSIC (deg)	Theoretical	0.531	0.555	0.294	0.308	0.215	0.224	0.174	0.182	0.110	0.116	0.088	0.092
	RMS err	- -	- -	0.494	0.389	0.214	0.232	0.165	0.171	0.105	0.115	0.087	0.088
S-MUSIC	Theoretical	0.534	0.559	0.297	0.310	0.216	0.226	0.175	0.183	0.111	0.116	0.089	0.093
	RMS err	0.834	0.818	0.278	0.283	0.184	0.202	0.146	0.160	0.101	0.112	0.082	0.086
IES-MUSIC	Theoretical	0.083	0.087	0.065	0.069	0.062	0.064	0.060	0.063	0.059	0.062	0.059	0.061
	RMS err, ρ	0.461	0.484	0.074	0.077	0.070	0.069	0.066	0.065	0.066	0.071	0.062	0.067
	RMS err, $\hat{\rho}$	0.946	0.919	0.189	0.184	0.110	0.116	0.084	0.084	0.071	0.075	0.063	0.068
RAP-MUSIC	RMS err	0.879	0.863	0.150	0.153	0.083	0.093	0.070	0.070	0.067	0.069	0.061	0.065

ficulty in estimating the sources. MUSIC is particularly poor at this correlation, since in many trials an adequately detectable peak did not occur in the region around the true answer.

In general, the RMS error of MUSIC and S-MUSIC match the theoretical bounds established in [8] quite well, and our RMS errors agree well with those presented in [8] for their comparable cases. RAP-MUSIC consistently maintains an improved RMS error over that of IES-MUSIC, and we note again that IES-MUSIC depends on some other technique in order to arrive at an initial set of source estimates. IES-MUSIC performance using the optimally designed ρ agrees quite well with the theoretical bounds, but this performance obviously requires prior knowledge of the true solution.

These RMS errors were calculated at a relatively large number of time samples. We also tested small sample performance, in which we held the correlation constant at $\gamma = 0.9$ and varied the number of time samples; for brevity we do not include the table of results. At lower numbers of time samples, we generally had a difficult task determining a second MUSIC peak, and the MUSIC results were unreliable. As in Table 1, RAP-MUSIC consistently maintained improved performance over the other methods, and the performances were generally in good agreement with the theoretical bounds established by [8].

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