Published in final edited form as:

J Educ Psychol. 2010 November ; 102(4): 843–859. doi:10.1037/a0019824.

Sources of Group and Individual Differences in Emerging Fraction Skills

Steven A. Hecht and
Children's Learning Institute at University of Texas Health Science Center at Houston

Kevin J. Vagi
Department of Psychology, Florida Atlantic University

Abstract

Results from a two year longitudinal study of 181 children from fourth through fifth grade are reported. Levels of growth in children's computation, word problem, and estimation skills using common fractions were predicted by working memory, attentive classroom behavior, conceptual knowledge about fractions, and simple arithmetic fluency. Comparisons of 55 participants identified as having mathematical difficulties to those without mathematical difficulties revealed that group differences in emerging fraction skills were consistently mediated by attentive classroom behavior and conceptual knowledge about fractions. Neither working memory nor arithmetic fluency mediated group differences in growth in fraction skills. It was also found that the development of basic fraction skills and conceptual knowledge are bidirectional in that conceptual knowledge exerted strong influences on all three types of basic fraction skills, and basic fraction skills exerted a more modest influence on subsequent conceptual knowledge. Results are discussed with reference to how the identification of potentially malleable student characteristics that contribute to the difficulties that some students have with fractions informs interventions and also will contribute to a future theoretical account concerning how domain general and domain specific factors influence the development of basic fraction skills.

Keywords

Fraction skills; Mathematical learning disabilities; Mathematical learning difficulties; Mathematical development

One of the most persistent problems for children with mathematical difficulties is solving problems involving fractions (Algozzine, O'Shea, Crews, & Stoddard, 1987; Hecht, Vagi, & Torgesen, 2007; National Mathematics Advisory Panel, NMAP, 2008). For many of these children, this constitutes a major obstacle to their movement beyond basic math to more advanced topics in later elementary school and beyond (Hecht, Close, & Santisi, 2003; Heller, Post, Behr, & Lesh, 1990; Loveless, 2003; NMAP, 2008). Nationally representative

Correspondence concerning this article should be addressed to Steven A. Hecht, Children's Learning Institute, Department of Pediatrics, University of Texas Health Science Center at Houston, 7000 Fannin, Suite 2441, Houston, TX 77030. Steven.A.Hecht@uth.tmc.edu.
Kevin J. Vagi is now at Epidemic Intelligence Service program, the Centers for Disease Control and Prevention.

studies also show that difficulties with fractions are commonplace (Hope & Owens, 1987; NMAP, 2008; Smith, 1995). For example, only approximately 25% of fourth graders could correctly identify among four common fraction numerals the one that was closest to ½ (NAEP, 2009). Basic fraction skills are especially difficult for children with mathematical difficulties (Cawley, Parmer, Yan, Miller, 1998; Cawley, 1985). Unfortunately, relatively little research has been conducted to identify the variables that contribute to the difficulties that children with math difficulties have in the domain of fractions (NMAP, 2008; Mazzocco & Devlin, 2008). Such understanding would help guide theoretical accounts of fraction skills development, and provide practical guidance for early identification and treatment of difficulties in acquiring basic fraction skills.

The purpose of the present study was to examine the factors that may mediate performance in three types of basic fraction skills: fraction computation (e.g., 1/2 + 1/3 = ?), fraction estimation (e.g., which number is closest to the answer for 9/10 + 11/12 = ?, 1, 15, 20 or 120?), and word problem solving with fraction quantities (e.g., John ate 1/2 of a cake, and Cindy at 1/4 of the same cake. How much cake is left over?). We focused on these three aspects of basic fraction skills because these problem types are prominent in elementary school curricula (National Council for Teachers of Mathematics, 2000; Kilpatrick, Swafford, Findell, & Bradford, 2001), each creating a range of skill development by the beginning of fourth grade (Hecht, Close, & Santisi, 2003; Mazzocco & Devlin, 2008; Siegler, 2009).

In the literature concerning correlates of mathematical learning difficulties, prior work has recurrently identified a limited set of attributes that seem to characterize children with poor mathematical achievement (Gersten, Jordan, & Flojo, 2005; Mabbott & Bisanz, 2008; Geary, 1993; Jordan & Hanich, 2003). Some of these attributes are *domain-specific* because they are used to solve a specific type of math problem (c.f., Kail, 2004). In particular, a hallmark characteristic of children with math difficulties is reliance on more error prone and slower counting based strategies than retrieval from long-term memory of factual knowledge to find the answers to simple arithmetic problems (aka number combinations; e.g., Cawley, et al., 1998; Geary, 1993; Jordan & Hanich, 2003; Mabbot & Bisanz, 2008). Children with math difficulties also seem to exhibit poorly developed conceptual knowledge concerning principles of counting and calculation (e.g., order irrelevance when counting objects, commutativity, inversion; Jordan, Hanich, & Kaplan, 2003; Mabbot & Bisanz, 2008; Geary, 1994). The contribution of domain specific types of knowledge does not, however, provide a complete description of mathematical learning difficulties. There is also evidence for the role of more domain general cognitive processes in mathematical learning difficulties; that is, processes that are not specific to a particular area of mathematical skill. In particular, mathematical difficulties are associated with poorer working memory skills (e.g., LeFevre, DeStefano, Coleman, & Shanahan, 2005; Kail & Hall, 1999; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Hitch & McAuley, 1991) and behavioral inattention (Cirino, Ewing-Cobbs, Barnes, Fuchs, & Fletcher, 2007; Zentall, 1990; Passolunghi & Pazzagliab, 2004). Consistent with this literature, we adopted the perspective that difficulties in basic fraction skills are secondary to factors that are both domain-specific and domain-general (c.f., Kail, 2004). We summarize this prior work as it provided the basis for hypothesizing that certain domain-specific and domain-general variables would be important for acquiring basic fraction skills.

Prior Work

Domain Specific Knowledge and Basic Fraction Skills

Conceptual Knowledge—In dealing with fractions, children are confronted with learning how to both make sense of the fraction symbols and also operations with rational quantities (Cramer, Wyberg, & Leavitt, 2008; Hecht, et al., 2007; Kieran, 1993; Stafylidou

NIH-PA Author Manuscript

NIH-PA Author Manuscript

& Vosniadou, 2004). These conceptual understandings seem to be particularly difficult for children with mathematical difficulties (Butler, Miller, Crehan, Babitt, & Pierce, 2003; Mazzocco & Devlin, 2008). It has been widely accepted that elementary school children should be provided instruction for understanding part–whole and measurement interpretations of common fractions (Cramer, et al; 2008; English & Halford, 1995; Kilpatrick, et al., 2001). *Part–whole* means that fractions represent the parts of an entire object or set of objects indicated by the fraction symbols. For example, *1–2* can refer to a pie with half of it eaten, two pies with one of them eaten, and so on. *Measurement* refers to the fact that fractions are numbers that reflect cardinal size. For example, fractions can be ordered from lowest to highest (e.g., 1/4, 1/3, 1/2, 2/2; Smith, Solomon, & Carey, 2005; Mazzocco & Devlin, 2008).

Conceptual knowledge about rational number units might aid some students in selecting appropriate procedures for solving fraction computation, estimation, and word problems (Byrnes & Wasik, 1991; Hecht, 1998; Hecht, Close,& Santisi, 2003; Hiebert, 1986; Hiebert & LeFevre, 1986). Conceptual understandings about fraction symbols can also be used to detect or avoid procedural mistakes while solving fraction computation problems. Qualitative interview studies have documented children's use of part–whole and measurement interpretations of fractions to select and to monitor the success of computation procedures (Mack, 1990; Streefland, 1993; see also Pitkethly & Hunting, 1996, for a review). Conceptual understandings concerning how and why operations with fractions work can also be used to solve some math problems involving fractions (see e.g., Ball, 1993; Cramer, Post, del Mas, 2002; Sherman & Bisanz, 2009). For example, some students might add fractions by shading a pictorial representation of part-whole relations (e.g., child adds ½ + ¼ by shading ½ and ¼ of a pie and noting that ¾ of the pie is then shaded to determine the answer). Translation of word problems into appropriate computations usually requires the construction of accurate mental models for the situations conveyed by the word problems (see e.g., Hegarty, Mayer, & Monk, 1995; Stern, 1993). Students who construct an inaccurate mental model for the situation conveyed by a word problem may be more likely to set up the wrong fraction computation problem to solve the word problem. Finally, conceptual understandings are likely needed for estimation of fractions. Conceptual knowledge can aid students with respect to converting fractional quantities into approximate whole number quantities during the process of estimating an answer (Behr & Post, 1986; Case & Sowder, 1990; Hecht, 1998).

There is some suggestive evidence that poorly developed conceptual knowledge about fractions contributes to group differences in fraction computation, estimation, and word problem performance. One study reported that sixth-, seventh-, and eighth-grade children with math difficulties scored lower than typically achieving children on conceptual knowledge tasks that involved ordering from lowest to highest both part-whole representations of fractions (i.e., pictorial representations of fractions, such as circles with parts shaded) and common fraction numerals (Mazzocco & Devlin, 2008). Conceptual knowledge is also correlated with individual differences in fraction computation, estimation, and word problem performance. Byrnes and Wasik (1991) reported that pretested conceptual knowledge correlated with how well children benefited from instruction in fraction computation, accounting for approximately 30% of the variance in fraction addition skills. Siegler (2009) reported that 6^{th} and 8^{th} grader's performance on a task involving placing common fractions on a number line (which assesses a student's measurement knowledge of cardinal size of fractions) correlated substantially with concurrent fraction computation skills (rxy's ranged from .55 to .70). Hecht (1998) provided quantitative evidence that conceptual knowledge *uniquely* contributes to variability in middle school student's fraction concurrent fraction computation, estimation, and word problem performance, with procedural knowledge recognition, simple arithmetic fluency, word level reading, and vocabulary knowledge accounted for. Hecht and his colleagues (2003) extended the findings from Hecht (1998) on the unique role of conceptual knowledge in a concurrent sample of 105 fifth graders.

Simple Arithmetic Knowledge—Simple arithmetic knowledge for whole numbers has been largely assumed to be a precursory skill that must be mastered with proficiency in order to effectively progress to more complex mathematical operations (see e.g., Gagne, 1983; Goldman, Mertz, & Pelligrino, 1989; Kilpatrick, et al., 2001; NMAP, 2008; Pressley, 1986). Indeed, the predominant view that mathematical knowledge is hierarchical (Aunola, Leskinen, Lerkkanen, & Nurmi, 2004; Fuchs, Fuchs, Compton, Powell, Seethaler, Capizzi, Schatschneider, & Fletcher, 2006) implies that mastery of more complex mathematics, such as fractions, are dependent on efficiency in more basic foundational skills, such as simple arithmetic. Typically achieving children tend to rely on the automatic and less error prone retrieval strategy to solve simple arithmetic, which is thought to enable them to devote limited attentional resources toward selecting and implementing fraction procedures. Meanwhile, children with math difficulties are at a supposed disadvantage because limited memory resources must be devoted to calculating simple arithmetic at the expense of other processes needed to solve the problem (e.g., Geary, 1999; Hecht, 1998; Mayer, 1985; Zentall, 1990). However, there is no direct empirical evidence that arithmetic efficiency contributes to performance differences between children with mathematical difficulties and typically achieving children with respect to basic fraction skills. This represents an important gap in our knowledge. On the one hand, efficient simple arithmetic performance seems important during the process of solving, say, fraction computation problems (e.g., one must add whole numbers in the numerator to solve ¼ + ¼). Indeed, there is evidence from studies collected at one time point suggesting that simple arithmetic fluency is an important source of emerging individual differences in basic fraction skills. In particular, simple arithmetic efficiency seems to consistently uniquely correlate with concurrent fraction computation accuracy (Hecht, 1998; Hecht, et al., 2003). On the other hand, simple arithmetic efficiency may not uniquely influence growth over time in basic fraction skills or explain ability group differences in fraction skills. Consistent with the latter view, fraction computation mistakes made by late elementary school students rarely involve miscalculations (i.e., less than 5% of errors coded; Hecht, et al., 2007).

Domain General Cognitive Abilities and Basic Fraction Skills

Working Memory—Working memory tends to be impaired in children with mathematical difficulties relative to typically achieving students (see e.g., Geary, 1993; Hitch & McAuley, 1991; Passolunghi & Pazzagliab, 2004; Pickering & Gathercole, 2004; Raghubar, Barnes, & Hecht, 2010; Siegel & Ryan, 1989; Swanson & Sachse-Lee, 2001). In the current context, working memory refers to concurrent storage and manipulation of the information necessary to perform a mental task. During on-line performance, representations of the terms and operators as well as the generated answer should at some point be maintained in working memory (c.f., Geary, 1993; Lemaire, Abdi, & Fayol, 1996; Logie & Baddeley, 1987). For example, the child might verbally encode “two minus one-third equals” while either solving or estimating the answer to the fraction computation problem “2-1/3 = ”. With a poor working memory system, the representation of the encoded quantities are more likely to decay before completing a procedural strategy, such as converting the 2 into a rational number (i.e., 6/3; see Geary, 1993 for a related proposal in the case of simple arithmetic). With respect to fraction word problem solving, it is likely that an efficient working memory system is required to construct an accurate mental model of the situations conveyed by the word problem (c.f., Kintsch & Greeno, 1985; Swanson & Sachse-Lee, 2001). Hecht and his colleagues (Hecht, et al., 2003) reported a unique association between working memory and concurrent fraction word problem performance.

Attention—It is known that children with mathematics difficulties tend to engage in less attending behavior during math instruction (Bryan, 1974, McKinney & Speece, 1986). Children's attentive behavior during math instruction may be an important contributor to how well children benefit from formal fractions instruction. A child is certainly not in a good position to acquire fraction-related knowledge from a classroom lesson if she or he does not pay attention to instruction and carry out tasks prescribed by the teacher (Green, Forehand, Beck, & Vosk, 1980; Hecht & Greenfield, 2001; Wentzel, 1991). Children's ability to engage in on-task behaviors, including returning to an on-task behavior after an attentional shift (e.g., distraction from a peer), enables the child to form representations of mathematical information within the working memory system. Accordingly, the frequency of attentive behaviors in the classroom appears to correlate with individual differences in the acquisition of academic skills (Bennett, Gottesman, Rock, & Cerullo, 1993; Fuchs et al., 2006; McKinney & Speece, 1986; Wentzel, 1991). For example, Fuchs, et al., (2006) reported a correlation between teacher ratings of attentive behavior in the classroom of .62 and .51 for mathematical computation and word problem solving, respectively in third graders. Hecht et al., (2003) found that classroom attentive behavior was a relatively strong unique predictor of concurrent individual differences in fraction computation, estimation, and word problem performance in fifth graders, while controlling for mathematical knowledge, working memory, and word level reading.

Considering Potential Bi-Directional Relations Between Fraction Problem Solving and Conceptual Knowledge

Perhaps the most prevalent position is that conceptual understandings lead to further acquisition and use of procedures for solving math problems involving fractions (Byrnes & Wasik, 1991; Hecht, 1998; Hiebert & LeFevre, 1986; Kilpatrick, et al., 2001). An alternative view is that learning procedures for how to solve fraction computation, word problem solving, or estimation provides a child with additional practice and feedback concerning the accuracy of the mental representations of conceptual knowledge used during problem solution (Gagne, 1983; VanLehn, 1990). Some have proposed a third view, which is that concepts and procedures influence each other rather than being independent in development (Baroody & Ginsburg, 1986; Siegler & Stern, 1998; Sophian, 1997), such as via an iterative or hand-over-hand fashion (Rittle-Johnson & Siegler, 1998; Rittle-Johnson, Siegler, & Alibali, 2001). It has also been proposed that once math procedures are accessed and used with some degree of automaticity, the child is able to devote limited attentional resources to mental processes that make use of conceptual understandings (e.g., Fuson, 1988; Gagne, 1983; Karmiloff-Smith, 1992). Procedural knowledge might be needed before a child can reflect on "why (the procedure) has the steps it has, why it has certain conditions on its applicability, and why it succeeds when those conditions are met and the steps are followed accurately" (VanLehn, 1990, p. 38). Thus, the process of carrying out procedures and then using conceptual knowledge to make sense of the results obtained from the algorithmic steps has been hypothesized as a mechanism for learning conceptual knowledge (Gagne, 1983; Hiebert, 1986; VanLehn & Brown, 1980). The view that learning how to solve math problems involving fractions leads to enhanced conceptual knowledge would be supported by unique associations between initial fraction computation, estimation, and word problem skills and emerging individual differences in conceptual knowledge. For example, Rittle-Johnson, et al., (2001) showed that fifth- and sixth-grader's initial procedural knowledge concerning how to use a number line uniquely contributed to improvements in conceptual understandings about placement of decimal fractions on the number line task, while controlling for pretested conceptual knowledge.

Rationale of Current Study

Two strategies were employed in this study for understanding the domain specific and domain general contributors to fraction-related performance in typically developing children and in children with math difficulties. The first was to conduct a two-year longitudinal study that involved prediction of emerging individual differences among the entire range of basic fraction skills achievement in our sample. The second strategy was to combine this approach with analyses that compare children with math difficulties and children who are typically achieving over time – to obtain a clearer view of the factors that contribute to emerging difficulties in fraction skills (Hecht, et al., 2007). There are currently no studies that have used either approach for understanding both the unique correlates of emerging individual differences in basic fraction skills or group differences in fraction computation, estimation, or word problem performance. We therefore sought to extend the current body of literature by combining these two approaches with mediation analysis, a technique that can be used for determining which variables underlie group differences in mathematical performance (c.f., Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Hecht, et al,. 2003). Mediational analysis requires examination of three logically related research questions (Baron & Kenny, 1986; MacKinnon, 2008). In keeping with Baron and Kenny (1986), we first asked whether or not group differences would emerge with respect to performance on the fraction-related variables. It is also necessary to establish which potential mediator variables are uniquely associated with emerging individual differences in the fraction outcomes. Thus, our second research question asked whether or not working memory or classroom attentive behavior (both domain general abilities), and simple arithmetic efficiency or conceptual knowledge (both domain specific factors) would uniquely predict emerging individual differences in fraction computation, estimation, and word problem performance across the entire range of skill in our sample. Our third research question asked whether or not any of these domain general and domain specific variables could explain emerging group differences in the three fraction outcomes. The longitudinal data also allowed us to examine a fourth research question, which asked whether or not there is evidence for bidirectional relations between conceptual knowledge and fraction computation, estimation, and word problem performance.

Method

Participants

Fourth graders were 260 children served by one of nine public schools in South East Florida and tested in the winter (time 1). During the spring of their fifth grade year (time 2), 181 of the original sample participated again. Because teachers distributed consent forms to parents, the exact number of parents who received the consent forms can not be determined. Incentives such as small gifts to children who returned parent permission forms were provided. At time 1, we obtained the sample by using the following procedure. Initial screening was done by asking teachers to identify any child that they thought was “math disabled”. Teachers were also asked to identify all other students that were not experiencing math difficulties. Parent permission forms were distributed to all children so identified by their teachers. The regular school teacher obtained parent permission forms for children enrolled in special education math classrooms as well to increase the sample size of participants with mathematical difficulties. All children who returned an affirmative parent permission form regardless of math difficulty status were tested. No test-giver or child was aware of the participant’s designation concerning math disability status as given by the teacher. At time 2, all children who participated at time 1 were given a parental permission form. Exclusionary criteria for initial screening included: children deemed well below average in intelligence by his/her teacher; 2) children with severe behavior disorders, and 3) students who did not speak fluent English. All other children who returned an affirmative parent permission form were tested.

When students were in fifth grade, 181 of the original sample agreed to participate again and were tested at time 2. The final sample involved 31 and 39 classrooms (during wave 1 and wave 2, respectively) from the same nine schools. Attrition analyses (repeated measures) revealed that the 181 children with complete data did not differ from the 79 children who did not participate at 5^{th} grade on the study variables at baseline (i.e., all p's > .05). This supports the assumption that data were missing at random. Of the 55 participants with math difficulties, 27 (49%) were female, the racial/ethnic composition was 12 (21.8%) White, 10 (18.2%) Hispanic, 32 (58.2%) African American, and 1 (1.8%) Other, and on the basis of family income, 11 participants (i.e., 20.0% of the sample) were eligible for free or reduced-price lunch and 9 (16.4%) had a school identified disability. Of the 126 typically achieving children, 74 (58.7%) were female, the racial/ethnic composition was 57 (45.2%) White, 26 (20.6%) Hispanic, 40 (31.7%) African American, and 3 (2.4%) other, and 13 participants (i.e., 10.3% of the sample) were eligible for free or reduced-price lunch and 3 (2.4%) had a school identified disability.

Determination of Math Difficulty Status

Math difficulty status was based on the cutoff criteria used by Siegel and Ryan (1989; see also Geary, 2004). The sample included 55 students with math difficulties (MD), who scored at or below the 25^{th} percentile on the Woodcock-Johnson III Calculation composite (WJIII; Woodcock, McGrew, & Mather, 2001) in the fall of fourth grade. The Calculation composite is based on both the Calculation and Math Fluency measures. The typically achieving sample included 126 4^{th} grade children who scored above the 40^{th} percentile on the calculation composite (percentile range = 41 to 100). The 40^{th} percentile is the cut point that is used in the research literature frequently for designating lack of difficulty (Fuchs, Fuchs, Stuebing, Fletcher, Hamlett, & Lambert, 2008). Of the 55 students initially classified as MD, 38 children exhibited persistent MD in fifth grade, while 17 students tested out of our a priori definition for MD status on the basis of exceeding our initial percentile cutoff score of 25 (range = 29–39) on the WJIII calculation composite. No children who were originally classified as typically achieving in fourth grade scored within the bounds of the MD classification in fifth grade. Age based standard scores for full scale intelligence on the Wechsler Abbreviated Scales of Intelligence (WASI; Wechsler; 1999; 2-form version; which is normed to have a mean of 100 and SD of 15) placed all children within both groups at the normal range in general intelligence (i.e., age-based standard score of at least 80 – the 9^{th} percentile - or above) in fourth grade. Children's mean estimated Full Scale IQ score at the fall of fourth grade was 96.1 (SD = 10.4) and 103.7 (SD = 11.20) for children in the math difficulties and typically achieving groups, respectively.

Measures

Math textbooks used in a participating school district and two elementary school teachers were consulted to ensure that the measures of basic fraction skills and conceptual knowledge contained adequate coverage of subskills for the age-range studied. The basic fraction skills and conceptual knowledge measures were similar or identical to those employed in previous concurrent studies of elementary and middle school children's fraction performance (Hecht, 1998; Hecht, et al., 2003).

Three tests measured each of the considered types of basic fraction skills

1. Fraction computation. There were 12 addition and 12 multiplication computation problems. For each operation, there were two items for each of six item types: (a) proper fractions, same denominators; (b) proper fractions, different denominators; (c) one term mixed fraction and the other term a proper fraction with same denominators (e.g., 2 3/5 + 1/5); (d) one term a proper fraction and the other term a whole number (e.g., 2 + 1/4), one term a proper fraction and the second term a proper fraction with a zero numerator (e.g., 1/4 + 0/4); and (e) one term mixed fraction and the other term a proper fraction with different denominators (e.g., 2 ½ + 1/4). There were also two fraction division problems, involving proper fractions with same denominators. Coefficient alpha on this sample was .80 and .87 in fourth and fifth grade, respectively. The total correct answers were recorded.

2. Word problems. This test was composed of 12 word problems that involved fractional quantities. Four problems were correctly solved by addition of the quantities, four by multiplying, and four by dividing. Within each set of four problems, two involved one fraction and one whole number, and two involved fractions only. Students were instructed to provide the correct mathematical equation to determine the answer to the word problem. Coefficient alpha on this sample was .81 and .85 in fourth and fifth grade, respectively. The total correct mathematical equation set-ups was recorded.

3. Fraction estimation. This test was composed of 12 items. Students were presented a fraction computation problem (e.g., 99/100 + 99/100 =) with four alternatives (1, 10, 100, 1000), and circled the closest whole number to the correct sum. Coefficient alpha on this sample was .84 and .91 in fourth and fifth grade, respectively. The total correct answers were recorded.

Two tasks assessed part-whole conceptual knowledge about fraction symbols

4. Picture-symbol. For each item, students indicated the fraction of a polygon, or set of polygons, that was shaded. Students wrote the indicated fraction in symbolic form (e.g., ½). There were 13 items. Coefficient alpha on this sample was .91 and .90 in fourth and fifth grade, respectively. The total correct answers were recorded.

5. Symbol-picture. For each item, participants were presented a fraction symbol (e.g., ¾). Beside each fraction symbol there was a polygon figure or set of figures. Students shaded the polygon(s) in the amount indicated by the fraction symbol. There were 18 items. Coefficient alpha on this sample was .85 and .86 in fourth and fifth grade, respectively. Total correct answers were recorded.

One task assessed measurement conceptual knowledge about fraction symbols

6. Size of fraction. For each item, students were presented a pair of numbers (e.g., ½ and 1). Students indicated which of the two numbers was the largest. There were 24 items. Coefficient alpha on this sample was .89 and .91 in fourth and fifth grade, respectively. Total correct answers were recorded.

One task assessed conceptual knowledge concerning adding rational numbers using pictures

7. Picture computation. For each item, participants were presented two circles, or two squares or two rectangles (all equal in size). Part of each figure was shaded. Next to each pair of pictures was a blank picture that was the same sized shape (or shapes) as the preceding two. Students were instructed to add the amount shaded in the two pictures and draw the answer using the provided blank pictures. For example, if one picture depicted a square with ¼ shaded, and another square with 2/4 shaded, students would need to shade ¾ of the provided blank square to get this item correct. Thus, this task measures the understanding of how rational quantities depicted in pictorial form (as

NIH-PA Author Manuscript

NIH-PA Author Manuscript

opposed to mathematical symbol form) can be combined. There were 13 items, with six items adding up to one or less and the remaining items with sums greater than one whole. Coefficient alpha on this sample was .88 and .84 in fourth and fifth grade, respectively. The total correct answers were recorded.

Two tasks measured simple arithmetic knowledge

8. Simple arithmetic – addition. For each of the 22 items, participants were presented in the middle of a computer screen a simple arithmetic problem, written in standard horizontal form (e.g., 2 + 5 =). Students were instructed to say the answer to each simple arithmetic problem as fast as possible. Immediately after an answer was stated, students were asked to give an immediate retrospective verbal report by verbalizing the strategy that was used to solve the given problem. For each trial, the time taken between when the problem appeared on the screen and when the child said the answer in a microphone was recorded. Total number of arithmetic problems that were solved via retrieval was recorded (which we refer to as "retrieval use"). A trial was coded as retrieval if the child provided a verbal indication (e.g., child said "just knew it" or "popped into my head") immediately after the child gave an answer. Mean accuracy and mean latency was also recorded.

9. Simple arithmetic – multiplication. The task was identical to the simple arithmetic – addition task, except the items were multiplication.

One task measured working memory skills

10. Counting span. For each trial, participants were first shown a screen with circles and squares, and asked to utter the number of circles. Next, participants were shown additional screens; for each screen participants indicated aloud the number of circles. The experimenter pushed the spacebar immediately after the child said the number of circles, which caused the next screen to appear. After students were presented a certain number of screens, they were asked to indicate the number of circles counted from each screen in exact order of presentation. Participants were given two practice trials involving two screens and were provided corrective feedback when necessary. Each of the test trials involved 2, 3, 4, 5, or 6 screens (10 trials altogether). Coefficient alpha on this sample was .80 in fourth grade. The total correct trials were recorded.

Word level reading ability was used as a control variable

11. Word level reading. The Real Words test, forms A & B from the Test of Word Reading Efficiency (TOWRE; Torgesen, Wagner, & Rashotte, 1999) was used. Students were instructed to say the words printed on the card as fast as possible. The average total number of correctly read words across forms A and B was recorded. As reported by the test developer, coefficient alpha for this task is above .90 for the age range studied here (Torgesen et al., 1999).

One measure was used to assess attentive behavior in the classroom

12. Attentive behavior. Participant's teacher rated each student's attentive behaviors in the classroom during math instruction. Items were from the cooperation subscale of the Social Skills Rating System (Gresham & Elliott, 1990). The SSRS is a norm-referenced rating scale, with cooperation items derived from classroom behaviors that teachers consider to be indicative of attentive on-task behavior (Demaray, Ruffalo, Carlson, Busse, Olson, McManus, & Leventhal, 1995), and that correlate significantly with direct observations of on-task behavior (Gresham & Elliott, 1990). As reported by the test developer, coefficient alpha is .92 for elementary school children (Gresham & Elliott, 1990).

Procedure

Children were individually administered the WJIII Calculation and Math Fluency measures and the WASI Vocabulary and Matrix Reasoning measures which were used to create the overall full scale IQ estimate. Children were also individually administered the counting span measure of working memory, and the simple arithmetic tasks. The remaining measures were stapled together in a test booklet for group administration: fraction computation, fraction word problems, fraction estimation, picture symbol, picture computation, symbol picture, size of fraction, and the Woodcock Johnson III calculation measure. Directions for each measure in the test booklet were printed at the beginning of each measure. The test booklet measures were arranged in one of six different random orders.

All children were administered the measures in two sessions. For the group administered tasks, participants were told to complete the test booklet at their own pace and were encouraged to ask questions when necessary. Group administration was done in a vacant classroom, library, or the cafeteria. Students completed the group session test booklet in approximately 45 to 60 min. Size of groups varied, and trained research assistants were present to monitor students as they performed the tasks. The second session was conducted in a quiet area on school grounds and entailed individual administration of the remaining tasks. Students completed these individually administered tasks in approximately 30–40 minutes. During the spring of fourth grade, children's teacher rated each child's attentive on-task behavior in the classroom during math instruction.

Results

Data screening and construction of variables

Procedures outlined by Tabachnick and Fidell (2007) were used to screen the data. Square root transformation of skewed variables did not result in meaningful changes in correlations among these variables with the other variables. Thus, non-transformed scores were used in all of the analyses. To account for the dependency among observations (students) within clusters (classrooms), we conducted all analyses using the "complex analysis" feature in Mplus, Version 4.21 (Muthen & Muthen, 2007), in which the models were estimated via the maximum likelihood estimation method with robust standard errors (Williams, 2000). Bivariate correlations among the individual measures are reported in Table 1. For constructs for which we had more than one measure, we created weighted composite variables using principle components factor analysis across the variables that measured that particular construct. Each principal components factor analysis yielded only one factor, and therefore no rotation was necessary. Composite variables were created for part-whole conceptual knowledge about fraction symbols (in which we created a composite score for Picture symbol and Symbol picture). Two simple arithmetic composite variables were created. One composite variable, called arithmetic solution processes, was composed of the average proportion of addition and multiplication trials that were both solved correctly (these were the Arithmetic – addition accuracy and Arithmetic – multiplication accuracy variables) and solved via the retrieval strategy (these were the Arithmetic – addition retrieval use and Arithmetic – multiplication retrieval use variables). Consistent with Siegler and his colleague's most recent Strategy Choice and Discovery Simulation (SCADS*; Siegler & Araya, 2005), both accuracy and retrieval frequency in simple arithmetic should substantially reflect the same underlying cognitive processes. The accuracy variable represents the efficiency that students can select strategies that lead to a correct answer. The frequency of retrieval use variable represents student's ability to select the most efficient retrieval strategy instead of using more time consuming and error prone non-retrieval strategies (Geary, 1993; Hecht, 1999; Siegler, 1988). The other arithmetic composite variable was arithmetic latencies. Arithmetic latencies are substantially correlated with arithmetic solution processes (Hecht, et al, 2003), yet other variables also contribute to solution latencies (such as general attention mechanisms and speed of retrieval (Hecht, 1999; Kail, 2004; Widaman, Little, Geary, & Cormier, 1992).

Group Differences in Performance

Our first research question asked whether or not group differences would emerge with respect to performance on the math-related variables. Table 2 depicts the means and standard deviations for each of the observed variables segregated by group (typically achieving and math difficulties). The table also includes omnibus F values and effect sizes (ESs) comparing the groups. All effect sizes were expressed in terms of Cohen’s d, which is the difference between means divided by the pooled SD (Hedges & Olkin, 1985). One-way analyses of variance (ANOVAs) were used to determine the statistical significance of group differences on both fourth grade and fifth grade performance, and improvement scores on all dependent measures. The one-way ANOVA on improvement scores (computed as fifth grade minus fourth grade performance) is equivalent to the statistical analysis results obtained from a 2-way ANOVA for the time by group interaction.

Fourth and fifth grader’s performance on each measure differed by group, with the math difficulties group showing consistently poorer performance than the typically achieving group. Thus, the present group of students with math difficulties showed general impairments in math and other abilities (i.e., working memory, word level reading, estimated full scale IQ, and frequency of attentive behavior in the classroom), rather than a more specific impairment in math alone. In general, improvement scores were significantly different from zero, unless otherwise noted in Table 2. Group differences in the amount of improvement in scores between fourth and fifth grade were, in general, non-significant on most of our variables. Notable exceptions were for fraction computation, estimation, and word problem skills, which showed significantly larger gains in performance across grades for the typically achieving group than the mathematical difficulties group.

Prediction of Individual Differences in Fraction Performance

Our second research question asked whether or not domain general abilities such as working memory and classroom attentive behavior, and domain specific skills such as simple arithmetic efficiency and conceptual knowledge uniquely predicted emerging individual differences across the entire range of abilities in our sample with respect to fraction computation, estimation, and word problem performance. Multiple regression analyses were carried out using hierarchical regression procedures. The unique proportion of variance in fifth grade performance accounted for by the predictors measured in fourth grade is reported in Table 3. In these analyses, we controlled for the autoregressive effects of fourth grade math ability, estimated full scale IQ, and word level reading. Autoregressive refers to models where a variable is regressed on itself at a prior time period. By including fourth grade math ability in the first step of these analyses, we were able to predict changes in relative ordering of participant’s fraction skills (i.e., growth) during the time points under consideration (Gollob & Reichardt, 1987; Hecht, Torgesen, Wagner, & Rashotte, 2001;Kessler & Greenberg, 1981). In the second step (see panel 2), both estimated full scale IQ and word level reading were entered to control for general (global) learning mechanisms that could also underlie the observed associations between the predictors and math-related outcomes (cf., Bull & Johnston, 1997;Kail, 2004). Panel three in Table 3 depicts the proportion of unique variance explained by each of the mathematical knowledge and cognitive abilities, while only holding constant the effects of the control variables entered in the prior steps. These results provide the first empirical evidence for a unique contribution of each predictor to growth in fraction computation, word problem, and estimation performance during the fourth to fifth grade years. The findings suggest that arithmetic fluency, conceptual knowledge, working memory, and classroom attentive behavior uniquely contribute to growth in each type of basic fraction skills. Also noteworthy is that the percentage of variance attributable to the autoregressive effect of fourth grade math ability was quite substantial relative to any additional variance in fifth grade levels of fraction skill captured by the other predictors. Thus, the relative ordering of children's fraction problem performance was highly consistent from year-to-year.

Having established unique influences of each of the predictors on subsequent fraction computation, word problem, and estimation skills, an obvious next question is whether the contribution of these variables are independent of or redundant with one another (see panel 4, in Table 3). Correlations among the predictor variables indicate that they share common variance. The origin of the contribution of a given predictor variable on a fraction problem could be variance that is common to the other mathematical-related abilities in the model, in which case the influence would be redundant with that of the other abilities. Alternatively, the origin could be variance that is unique to the given mathematical-related ability, in which case the contributions would be independent of the other abilities. A third possibility is some mix of independence and redundancy. To find out whether the contributions of the mathematical predictors were redundant or independent, we used a multiple regression model wherein all variables were entered as simultaneous predictors of variability in basic fraction skills. Results indicate substantial redundancy in terms of predicting later variability in fraction problem solving. Yet, both classroom attentive behavior and at least one conceptual knowledge variable (which differed depending on the specific fraction outcome) were consistent unique predictors of growth in all three types of fraction problem solving, while controlling for all other variables and each other. Regarding the effects of conceptual knowledge, picture computation emerged as a consistent unique predictor of growth in basic fraction skills. Also, the measurement interpretation of fractions uniquely predicted growth in fraction estimation skills, while controlling for the other variables, including picture computation.

Testing of Mediation Effects

Our third research question asked whether or not domain-general and domain-specific predictor variables emerged as significant mediators of group differences in the three fraction outcomes. If children with math difficulties show deficits in fraction problem solving due to levels of performance on any of our predictors of individual differences in fraction outcomes, then one or more of these variables will emerge as significant mediators of the group contrast (c.f. Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007). Potential mediation is indicated when the product of the effect of the group contrast variable to the fourth-grade mediator (a) and the mediator to the fifth grade fraction outcome effect (b), is significantly different from zero (Baron & Kenny, 1986; MacKinnon, 2008; Preacher & Hayes, 2004). Our potential mediators were fourth grade levels of performance on the following domain specific variables: part-whole, size of fraction, picture computation, arithmetic solution processes, and arithmetic latency, and the following domain general variables: working memory and classroom attention behavior. To test for mediation effects, both the (a) and (b) parameters must be significantly different from zero (Baron & Kenny, 1986; MacKinnon, 2008). Both the (a) and (b) parameters are the unstandardized regression coefficients. For all mediation analyses, the (a) parameter was estimated while controlling for fourth grade levels of full scale IQ, and word level reading, and the (b) parameter was estimated while controlling for fourth grade levels of IQ, word level reading, and also the autoregressive effect of prior fourth grade math ability. Also, in these analyses, the (b) parameter must be estimated while controlling for the effects of the group contrast (Baron & Kenny, 1986; Mackinnon, 2008). The product of (a) and (b) is usually tested by the Sobel test (Geary, et al., 2007), which provides a pooled standard error based on the delta method

(Sobel, 1982). If the mediational effect is significant and reduces the group contrast (c) to nonsignificance (p's > .05), full mediation is implied; partial mediation is implied if the group contrast remains significant.

All mediation models conducted are shown in Table 4. The (a) and (b) parameters and the product (a × b) is depicted in this table, along with associated standard errors. Panel 1 shows the results when each of the mediator measures was entered alone. Panel 2 shows the same results when all predictors were entered simultaneously. The advantage of entering the mediating variables simultaneously is that one learns if the mediation is independent of the effects of the other mediators (Cole & Maxwell, 2003). The results of these mediational models indicated that both conceptual knowledge and attentive classroom behavior emerged as unique mediators of group differences in fraction computation, word problem, and estimation skills (see column (a × b) in Table 4). For conceptual knowledge, it is noted that picture computation, which requires application of part whole knowledge to compute fraction sums, was the consistent unique predictor of each type of fraction outcome. In all, these results provide the first evidence that the reason for group differences in emerging fraction skills is that difficulties in both conceptual knowledge and classroom attentive behavior lead to difficulties in fraction problem solving.

It was determined whether or not complete or partial mediation emerged when each of the mediators was entered simultaneously into the regression equation (not shown in Table 4). For fraction computation skills, there was complete mediation. That is, the unstandardized regression coefficient for group was not significant ($b = .71$, $SE =.63$, $p > .10$). For fraction word problems, there was marginally significant evidence for persistent differences between the typically achieving and math difficulty groups, ($b = 1.10$, $SE = .57$, $p < .056$). For estimation skills, there was a significant persistent difference between the math difficulty versus typical achievers ($b = 1.74$, $SE = .72$, $p < .05$). In all, these results indicate that ability group differences in both fraction computation and word problem solving were substantially explained by our predictors, while there was still a robust ability group difference with respect to fraction estimation skills.

Bidirectional relations between fraction problem solving and conceptual knowledge

Our fourth research question asked whether or not evidence for bidirectional relations between conceptual knowledge and performance on fraction computation, estimation, and word problem solving would emerge. We determined whether or not fourth grade levels of fraction computation, estimation, and word problem performance uniquely predicted fifth grade conceptual knowledge. The results depicted in Table 5 reveal that variability in each of the fraction outcomes uniquely predicted emerging individual differences in measurement and picture computation conceptual knowledge. Only word problem performance predicted later part whole knowledge. When all three types of basic fraction skills were included in the regression equation, along with the control variables, we found that word problem skills emerged as the only unique predictor of part whole and measurement conceptual knowledge. Fraction computation emerged as the sole unique predictor of growth in picture computation. In all, these analyses provide the first evidence that learning how to solve math problems involving common fractions contributes to emerging individual differences in conceptual fraction knowledge. These findings are consistent with Rittle-Johnson, Siegler, and Alibali (2001), who reported similar relations in 5^{th} and 6^{th} graders learning conceptual and procedural knowledge in the domain of decimal fractions. These results are consistent with theories that assume that procedural and conceptual knowledge influence each other's development in some way (Rittle-Johnson, et al., 2001;Siegler & Stern, 1998;Baroody & Ginsburg, 1986).

Discussion

We asked four related questions to further our understanding of emerging fraction computation, estimation, and word problem performance. Our first research question asked whether or not group differences between typically achieving students and children with math difficulties would emerge with respect to performance on the math-related variables. What we found was that the group differences on all of the measures were pervasive, favoring the typically achieving group. In fact, most of the effect sizes were moderate to very large during both fourth- and fifth-grades, suggesting that a consistently wide disparity exists in the realm of fraction related factors between typical achievers and children who are classified as having a math difficulty. We also found that improvements in fraction computation, estimation, and word problem performance among the math difficulties students were significantly less than change scores made by the typically achieving group. This is consistent with a pattern showing increasingly diminishing rates of improvement for children with math difficulties when compared to typically achieving children (i.e., Matthew Effects; Stanovich, 1986).

Our second research question asked whether or not working memory, classroom attentive behavior, simple arithmetic efficiency, or conceptual fraction knowledge uniquely predicted emerging individual differences across the entire range of abilities in our sample with respect to fraction computation, estimation, and word problem performance. Consistent with previous cross-sectional studies (Hecht, 1998; Hecht et al., 2003; 2007; Mazzocco & Devlin, 2008; Siegler, 2009), results from the present two-year longitudinal study suggest that children’s fraction computation, word problem, and estimation skills are uniquely predicted by conceptual knowledge, arithmetic fluency, working memory, and attentive behaviors in the classrooms. One new finding from the present longitudinal study was that these child characteristics predicted growth in fraction problem solving skills over time, while controlling for the autoregressive effect of prior fourth grade fraction ability, estimated full scale IQ, and word level reading. These findings provide evidence for identification of malleable factors that might lead to difficulties that many children have with fractions. Thus, the findings can provide practical guidance about both detection and treatment of difficulties in the realm of common fractions.

In addition to its practical importance, the findings provide theoretical insight into the nature of mathematical development by supporting the view that conceptual knowledge is important for growth in math skills. Four reports by the National Academies Press (Bransford, Brown, & Cocking, 2000; Donovan & Bransford, 2005; Kilpatrick, et al., 2001; Pellegrino, Chudowsky, & Glaser, 2001) reviewed the major cognitive science perspectives on which an information processing theory of math skills should be constrained. These reports stressed the necessity of working to obtain a body of coordinated, meaningful conceptual knowledge that reveals the logical structure of the discipline (e.g., common fractions). This includes multiple representations of the same fractional unit (e.g., fraction circle and number line pictorial models, and standard math symbols such as common fraction numerals; Cramer, et al., 2002). The ability to conceptually understand numerals, operations, and applications has been cited in the mathematical cognition literature as a defining characteristic of the emerging construct referred to as *number sense* (Berch, 2005; Gersten & Chard, 1999; Greeno, 1991; Robinson, Menchetti, & Torgesen, 2002). It has been asserted that number sense may be as important to mathematical learning as oral language skills, such as phonemic awareness, are to reading (Gersten & Chard, 1999). Part whole and measurement conceptual fraction knowledge appear to be two types of fraction number sense that are important for the development of fraction computation, estimation, and word problem skills.

Another contribution of this study is that we found new evidence that attention in the math classroom was a consistent unique predictor of *growth* in all three types of fraction problem types studied here. Others have shown concurrent relations between teacher ratings of on-task attentive classroom behaviors and concurrent levels of performance in fractions (Hecht, et al., 2003) and other types of math achievement including both whole number arithmetic and word problem solving (Fuchs, et al., 2005, 2006; Green, et al., 1980; McKinney & Speece, 1986; Raghubar, Cirino, Barnes, Ewing-Cobbs, Fletcher, & Fuchs, 2009). It is noted that further work is needed to determine which specific aspects of attention are captured by more global estimates, obtained via teacher ratings, of student's attention-related behaviors in the classroom. For example, behavioral inhibition and sustained attention are at least theoretically related to variability in mathematical performance (see e.g., Passolunghi & Pazzagliab, 2004; Passolunghi & Siegel, 2000; Swanson and Beebe-Frankenberger, 2004; Zentall, 1993).

Our third research question asked whether or not domain-general and domain-specific predictor variables emerged as significant mediators of group differences in the three fraction outcomes. This study provides the first direct empirical test concerning which child characteristics underlie the difficulties that children with math difficulties have with fractions written in common fraction notation. We found that both conceptual knowledge and attentive behavior were consistent mediators of ability group differences in emerging fraction computation, word problem, and estimation skills. In fact, ability group differences were completely explained for both fraction computation and word problem performance. We note that both working memory and simple arithmetic fluency were not consistently significant mediators of group differences in fraction computation, word problem, or estimation skills. With the control variables included in the model, both working memory and simple arithmetic fluency uniquely predicted growth in all three types of basic fraction skills studied here. However, we did not find statistically significant evidence that either of these measures mediated ability group differences in basic fraction skills. Thus, a foundational difficulty with arithmetic fluency as an explanation for learning difficulties in the domain of fractions receives no support in this study. It is widely believed that children with math difficulties have overly limited attentional resources available to solve math problems, as a result of either poor working memory resources in general or due to the preponderance of attention demanding and error prone counting-based simple arithmetic strategies that compete with available attentional resources (Gagne, 1983; NMAP, 2008; Zentall, 1990). However, the current results do not provide support for this mechanism as an explanation for why children with math difficulties showed poorer performance on the fraction computation, fraction estimation, and fraction word problems tests than typically achieving students. Thus, interventions designed to enhance simple arithmetic fluency may have limited effects on the acquisition of problem solving in the domain of fractions. A recent study by Fuchs, et al., (2005) suggested that there may be some degree of independence between improvements in simple arithmetic fluency and improvements in other types of math problem solving skills in response to intensive remedial instruction. Fuchs et al (2005) reported that first grader's computation, concepts/applications, and story problem solving skills could be substantially enhanced, yet simple arithmetic fluency was not affected by similar intensity of remedial instruction. Fuchs and colleagues (2005) aptly pointed out that this finding is similar to what is found in reading research; namely, that word level reading accuracy can be substantially enhanced without much effect on other reading outcomes, such as reading fluency (see e.g., Torgesen, Wagner, Rashotte, Rose, Lindamood, Conway, & Garvan, 1999).

Our fourth research question asked whether or not there is evidence for bidirectional relations between conceptual knowledge and performance on fraction computation, estimation, and word problem solving. Of the three alternative views of relations between conceptual knowledge and the acquisition of basic fraction skills we reviewed in the introduction, our results generally support the view that the relations are bidirectional. We found that each of our conceptual knowledge measures was uniquely predicted by at least one of the fraction problem types, while controlling for the autoregressive effects of prior conceptual knowledge, estimated full scale IQ, and word reading. These findings are consistent with theories that propose that conceptual and procedural knowledge come to mutually influence each other's development (Byrnes & Wasik, 1991; Rittle-Johnson, et al., 2001), rather than with views on the independence of learning of procedural knowledge and conceptual knowledge (c.f., Nesher, 1986; Resnick & Omanson, 1987). This research suggests that, in the domain of common fractions, teaching children conceptual knowledge is likely to lead to improved procedural skills and teaching children procedural knowledge is likely to have some positive effects on children's acquisition of conceptual understandings (see Rittle Johnson, et al., 2001 for similar relations in the case of decimal fractions). These findings suggest that a "reform oriented" common fractions curriculum might be especially beneficial for MD children, especially if links between the types of conceptual and procedural knowledge studied here are explicitly taught. Fraction circles and number lines are examples of promising types of models for fostering student understanding of both part-whole and measurement conceptual knowledge about common fraction numerals and operations with rational quantities (see e.g., Cramer & del Mas, 2002; Cramer, Wyberg, Leavitt, 2008). In this context, reform oriented instruction refers to infusing conceptual understanding and collaborative learning into instruction. Reform oriented instruction is typically associated with instructional principles represented by the National Council of Teachers of Mathematics (NCTM) standards (Saxe, et al., 1999). Researchers have documented that reform oriented instruction is associated with gains in student math achievement (Gearhart, et al., 1999; Stein, Grover, & Henningsen, 1996), such as in the domain of fractions (Saxe, et al., 1999).

Regarding other directions for future research, we mention four limitations of this study. First, it seems important to determine how the current findings generalize to other types of basic fractions skills, such as decimal fractions and percents. Second, there may be other kinds of domain-specific knowledge types and domain general abilities not investigated here that contribute to individual and group differences in fraction computation, estimation, and word problem performance. For example, although both the part-whole and measurement concepts for fractions might be more relevant in terms of the types of problems that most elementary students are typically provided, there are other kinds of conceptual understandings about fractions that should be related to certain types of basic fraction skills that students are exposed to in middle school and beyond. Other meaningful conceptual interpretations for fractions include an operator (or scalar) that can shrink or stretch another quantity (e.g., Behr, Harel, Post, & Lesh, 1993) as a quotient of two quantities (Kieren, 1988), probability (Ball, 1993), and rate (Ball, 1993). For example, it is likely that one or more of these types of conceptual knowledge (e.g., rate) might be uniquely predictive of individual differences in certain types of word problems, such as "For every 2 chocolate éclairs, we serve 1 glass of milk. How many glasses of milk do we need if we are serving 8 chocolate éclairs?". Other domain-specific factors include affect (especially math anxiety) and motivational issues that are specific to math, which may gain importance as children move from beginning to more complex (i.e., difficult) mathematical skills during the grade school years (Ashcraft, 2002; Cacioppo, Petty, Feinstein, & Jarvis, 1996; McLeod, 1990).

Concerning domain-general abilities, prior work suggests that mathematical difficulties are associated with visual-spatial skills, at least for some students (Geary, 1993, 1994). Visual-spatial working memory is associated with individual differences in mathematical performance, as measured by standardized and experimental math tests, at different ages and for different mathematical domains (Bull, Espy, & Wiebe, 2008; Holmes, Adams, &

Hamilton, 2008; Raghubar, et al., 2010). It is possible that certain visual-spatial abilities and visual-spatial working memory are related to fraction-related variables, such as using explicitly presented or child constructed mental models of both part-whole and measurement interpretations of fractions. Further work is also needed to determine whether or not other measures of working memory are related to growth in basic fraction skills. The currently used counting span working memory task involves simultaneous counting of circles while maintaining previously counted amounts. Counting involves substantial verbal working memory resources (see e.g., Hecht, 2002; Logie & Baddeley, 1987). There is a substantial domain general component to working memory tasks that involve verbal processing of any information (see e.g., Wilson & Swanson, 2001). However, it is noteworthy that the literature shows consistent effects for the relations of working memory and math when the task involves numerical information, but not when the measure includes word and sentence processing (Hitch & McAuley, 1991; Passolunghi & Siegel, 2004; Siegel & Ryan, 1989; reviewed in Raghubar et al., 2010).

A third limitation of the current study to be addressed in future research concerns how the nature of classroom instruction moderates the relations between domain-specific and domain-general variables and fraction skills. For example, on the one hand, it is possible that existing variability in classroom attentive behavior may not be such an important variable when students are consistently engaged in activities that are both instructive and intrinsically interesting (Bottge, 2001). On the other hand, classroom attentive behavior may become increasingly important in the face of the greater complexity of mathematical topics that children study during the grade school years. Blair, Gamson, Thorne, & Baker (2004) have argued that as children move from the earlier to later grades, mathematics curricula become more complex and increasingly require domain general cognitive abilities, such as attention skills, during instruction. Instructional effects may also moderate the relations between conceptual knowledge and basic fraction skills. Relations between conceptual knowledge and basic fraction skills are likely to depend considerably on the kinds of understandings that children are actually taught during formal math instruction (Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier, & Human, 1997).

A fourth limitation of the current study that should be addressed by future research concerns identification of factors that best predict growth in fraction-related variables during both earlier (i.e., before fourth grade) and later time points than studied here. In particular, given the importance of part-whole and measurement conceptual knowledge, it would seem important to examine the factors that contribute to growth in these understandings during the early elementary school years. Then, if children with conceptual knowledge deficits can be identified early, intervention efforts may be more effective and some concomitants of mathematical failure (math anxiety, poor academic self concept) might be avoided or at least mitigated.

In conclusion, the findings suggest that the domain specific and domain general variables that we studied are important unique correlates of growth in fraction skills during the fourth- to fifth-grade time period. Both conceptual knowledge and attentive behavior in the classroom appear to consistently explain why children with mathematical difficulties struggle with fractions. The assumption that child attributes are either domain general or domain specific in nature is consistent with an established tradition in the cognitive development literature (Kail, 2004), and is a useful distinction for ordering the variables that contribute to mathematical development (see e.g., Alexander & Judy, 1988; Fuchs, et al., 2006). The current and prior work concerning identification of both student and contextual factors (e.g., classroom environment) will enable a future integrated information processing theory concerning how both intrinsic and extrinsic factors simultaneously influence the development of specific types of mathematical skill, including fraction computation, estimation, and word problem performance.

Acknowledgments

This research was supported by Eunice Kennedy Shriver National Institute of Child Health and Human Development grant 5R03HD044599–02 awarded to Steven A. Hecht. We are very grateful to the research assistants and the children and teachers who participated in this project for their generous cooperation. The contents of the article are solely the responsibility of the authors and do not necessarily represent the official views of the National Institutes of Health. All involvement by Kevin J. Vagi in the conduct of this study occurred while he was a graduate student at Florida Atlantic University.

References

Alexander PA, Judy JE. The interaction of domain-specific and strategic knowledge in academic performance. Review of Educational Research 1988;58:375–404.

Algozzine B, O’Shea D, Crews W, Stoddard K. Analysis of mathematics competence of LD adolescents. The Journal of Special Education 1987;21:97–107.

Ashcraft MH. Math anxiety: Personal, Educational, and Cognitive Consequences. Current Directions in Psychological Science 2002;11:181–185.

Aunola K, Leskinen E, Lerkkanen M, Nurmi J. Developmental dynamics of math performance from preschool to grade 2. Journal of Educational Psychology 2004;96:699–713.

Ball, DL. Halves, pieces, and twoths: Constructing representational contexts in teaching fractions. In: Carpenter, T.; Fennema, E.; Romberg, T., editors. Rational numbers: An integration of research. Hillsdale, NJ: Erlbaum; 1993. p. 157-196.

Ball, DL.; Peoples, B. Assessing a student’s mathematical knowledge by way of interview. In: Schoenfeld, A., editor. Assessing mathematical Proficiency. Vol. 53. MSRI Publications; 2007. p. 213-267.

Baroody, AJ.; Ginsburg, HP. The relationship between initial meaning and mechanical knowledge of arithmetic. In: Hiebert, J., editor. Conceptual and procedural knowledge: The case of mathematics. Hillsdale, NJ: Erlbaum; 1986.

Baron RM, Kenny DA. The mediator-moderator distinction in social psychology: Conceptual, strategic, and statistical considerations. Journal of Personality and Social Psychology 1986;51:1173–1182. [PubMed: 3806354]

Behr, M.; Harel, G.; Post, T.; Lesh, R. Rational Numbers: Toward a Semantic Analysis - Emphasis on the Operator Construct. In: Carpenter, T.; Fennema, E.; Romberg, T., editors. Rational Numbers: An Integration of Research. Hillsdale, NJ: Lawrence Erlbaum Associates; 1993. p. 13-47.

Behr, M.; Post, TR. Estimation and children’s concept of rational number size. In: Schoen, H.; Zweng, M., editors. Estimation and Mental Computation: 1986 NCTM Yearbook. Reston, VA: National Council of Teachers of Mathematics; 1986. p. 103-111.

Bennett RE, Gottesman RL, Rock DA, Cerullo F. Influence of behavior perceptions and gender on teachers’ judgments of students’ academic skill. Journal of Educational Psychology 1993;85:347–356.

Berch DB. Making sense of number sense: Implications for children with mathematical disabilities. Journal of Learning Disabilities 2005;38:333–339. [PubMed: 16122065]

Blair C, Gamson D, Thorne S, Baker D. Rising mean IQ: Changing cognitive demand of mathematics education for young children, population exposure to formal schooling, and the neurobiology of the prefrontal cortex. Intelligence 2005;33:93–106.

Bottge BA. Using intriguing problems to improve math skills. Educational Leadership 2001;58:68–72.

Bransford, JD.; Brown, AL.; Cocking, RR., editors. How people learn. Washington, DC: National Academies Press; 2000.

Bryan TS. An Observational Analysis of Classroom Behaviors of Children with Learning Disabilities. Journal of Learning Disabilities 1974;7:26–34.

Bull RB, Espy KA, Wiebe SW. Short-term memory, working memory and executive functioning in preschoolers: Longitudinal predictors of mathematical achievement. Developmental Neuropsychology 2008;33:205–228. [PubMed: 18473197]

Bull R, Johnston RS. Children's arithmetical difficulties: Contributions from processing speed, item identification, and short-term memory. Journal of Experimental Child Psychology 1997;65:1–24. [PubMed: 9126630]

Butler FM, Miller SP, Crehan K, Babbitt B, Pierce T. Fraction instruction for students with mathematics disabilities: Comparing two teaching sequences. Learning Disabilities Research & Practice 2003;18(2):99–111.

Byrnes JP, Wasik BA. Role of conceptual knowledge in mathematical and procedural Learning. Developmental Psychology 1991;27:777–786.

Cacioppo J, Petty R, Feinstein J, Jarvis W. Dispositional differences in cognitive motivation: The life and times of individuals varying in need for cognition. Psychological Bulletin 1996;119:197–253.

Case R, Sowder J. The development of computational estimation: A neo-Piagetian analysis. Cognition and Instruction 1990;7:79–104.

Cawley, JF. Learning disabilities and the secondary school. In: Cawley, JF., editor. Secondary school mathematics for the learning disabled. Rockville, MA: Aspen Systems Corporation; 1985. p. 1-27.

Cawley JF, Parmer RS, Yan WE, Miller JH. Arithmetic computation abilities of students with learning disabilities: Implications for instruction. Learning Disabilities Research and Practice 1996;11(4): 230–237.

Cirino PT, Ewing-Cobbs L, Barnes M, Fuchs LS, Fletcher JM. Cognitive arithmetic differences in learning disability groups and the role of behavioral inattention. Learning Disabilities Research and Practice 2007;22:25–35.

Cole DA, Maxwell SE. Testing mediational models with longitudinal data: questions and tips in the use of structural equation modeling. Journal of Abnormal Psychology 2003;112:558–577. [PubMed: 14674869]

Cramer KA, Post T, del Mas R. Initial fraction learning by fourth- and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the Rational Number Project Curriculum. Journal for Research in Mathematics Education 2002;33:111–144.

Cramer K, Wyberg T, Leavitt S. The Role of Representations in Fraction Addition and Subtraction. Mathematics Teaching in the Middle School 2008;13:490–496.

Davis RB, McKnight C. The influence of semantic content on algorithmic behavior. The Journal of Mathematical Behavior 1980;3:39–87.

Demaray MK, Ruffalo SL, Carlson J, Buss RT, Olson AE, McManus SM, Leventhal A. Social skills assessment: A comparative evaluation of six published rating scales. School Psychology Review 1995;24:648–671.

Donovan, MS.; Bransford, JD. How students learn: Mathematics in the classroom. Washington, DC: National Academies Press; 2005.

English, LD.; Halford, GS. Mathematics education: Models and processes. Mahwah, NJ: Erlbaum; 1995.

Fuchs LS, Compton DL, Fuchs D, Paulsen K, Bryant JD, Hamlett CL. The prevention, identification, and cognitive determinants of math difficulty. Journal of Educational Psychology 2005;97:493–513.

Fuchs LS, Fuchs D, Compton DL, Powell SR, Seethaler PM, Capizzi AM, Schatschneider C, Fletcher JM. The cognitive correlates of third-grade skill arithmetic, algorithm computation, and arithmetic word problems. Journal of Educational Psychology 2006;98:394–409.

Fuchs LS, Fuchs D, Stuebing K, Fletcher JM, Hamlett CL, Lambert W. Problem solving and computational skill: Are they shared or distinct aspects of mathematical cognition. Journal of Educational Psychology 2008;100:30–47. [PubMed: 20057912]

Fuson, K. Children's counting and concepts of number. New York, NY: Springer-Verlag; 1988.

Gagne RM. Some issues in the psychology of mathematics instruction. Journal for Research in Mathematics Education 1983;14:275–282.

Geary DC. A componential analysis of an early learning deficit in mathematics. Journal of Experimental Child Psychology 1990;49:363–383. [PubMed: 2348157]

Geary DC. Mathematical disabilities: Cognitive, neuropsychological, and genetic components. Psychological Bulletin 1993;114:345–362. [PubMed: 8416036]

Geary, DC. Children's mathematical development: Research and practical applications. Washington, DC: American Psychological Association; 1994.

Geary DC. Sex differences in mathematical abilities: Commentary on the math-fact retrieval hypothesis. Contemporary Educational Psychology 1999;24:267–274. [PubMed: 10373316]

Geary DC. Mathematics and learning disabilities. Journal of Learning Disabilities 2004;37:4–15. [PubMed: 15493463]

Geary DC, Brown SC, Samaranayake VA. Cognitive addition: A short longitudinal study of strategy choice and speed-of-processing differences in normal and mathematically disabled children. Developmental Psychology 1991;27:787–797.

Geary DC, Hoard MK, Byrd-Craven J, Nugent L, Numtee C. Cognitive mechanisms underlying achievement deficits in children with mathematical learning disability. Child Development 2007;78:1343–1359. [PubMed: 17650142]

Geary DC, Hoard MK, Hamson CO. Numerical and arithmetical cognition: Patterns of functions and deficits in children at risk for mathematical disability. Journal of Experimental Child Psychology 1999;74:213–239. [PubMed: 10527555]

Gersten R, Chard D. Number sense: Rethinking arithmetic instruction for students with mathematics difficulties. Journal of Learning Disabilities 1999;33:18–29.

Gersten R, Jordan NC, Flojo JR. Early identification and interventions for students with mathematics difficulties. Journal of Learning Disabilities 2005;38:293–304. [PubMed: 16122059]

Goldman SR, Mertz DL, Pellegrino JW. Individual differences in extended practice functions and solution strategies for basic addition facts. Journal of Educational Psychology 1989;81:481–496.

Gollob, HF.; Reichardt, CS. Interpreting and estimating indirect effects assuming time lags really matter. In: Collins, LM.; Horn, JL., editors. Best Methods for the Analysis of Change: Recent Advances, Unanswered Questions, Future Directions. Washington, DC: American Psychological Association; 1991. p. 243-259.

Green KD, Forehand RL, Beck SJ, Vosk B. An assessment of the relationship among measures of children's social competence and children's academic achievement. Child Development 1980;51:1149–1156.

Greeno JG. Understanding and procedural knowledge in mathematics instruction. Educational Psychologist 1978;12:262–283.

Greeno JG. Number Sense as Situated Knowing in a Conceptual Domain. Journal for Research in Mathematics Education 1991;22:170–218.

Gresham, FM.; Elliott, SN. Social Skills Rating System. Circle Pines, MN: American Guidance Service; 1990.

Hecht SA. Toward an information processing account of individual differences in fraction skills. Journal of Educational Psychology 1998;90:545–559.

Hecht SA. Individual solution processes while solving addition and multiplication math facts in adults. Memory and Cognition 1999;27:1097–1107.

Hecht SA. Counting on Working Memory in Simple Arithmetic When Counting is Used for Problem Solving. Memory and Cognition 2002;30:447–455.

Hecht S, Close L, Santisi M. Sources of individual differences in fraction skills. Journal of Experimental Child Psychology 2003;86:277–302. [PubMed: 14623213]

Hecht SA, Greenfield DB. Comparing the predictive validity of first grade teacher ratings and reading related tests on third grade levels of reading skills in young children exposed to poverty. School Psychology Review 2001;30:50–69.

Hecht SA, Torgesen JK, Wagner RK, Rashotte CA. The relations between phonological processing abilities and emerging individual differences in mathematical computation skills: A longitudinal study from second- to fifth-grade. Journal of Experimental Child Psychology 2001;79:192–227. [PubMed: 11343408]

Hecht, SA.; Vagi, KJ.; Torgesen, JK. Fraction skills and proportional reasoning. In: Berch, DB.; Mazzocco, MMM., editors. Why is Math So Hard for Some Children? The Nature and Origins of

Mathematical Learning Difficulties and Disabilities. New York: Brookes Publishing; 2007. p. 121-132.

Hedges, LV.; Olkin, I. Statistical methods for meta-analysis. Orlando, FL: Academic Press; 1985.

Hegarty M, Mayer RE, Monk CA. Comprehension of arithmetic word problems: A comparison of successful and unsuccessful problem solvers. Journal of Educational Psychology 1995;87:18–32.

Hiebert, J. Conceptual and procedural knowledge: The case of mathematics. Mahwah, NJ: Lawrence Erlbaum Associates; 1986.

Hiebert, J.; Carpenter, TP.; Fennema, E.; Fuson, KC.; Wearne, D.; Murray, H.; Olivier, A.; Human, P. Making sense: Teaching and learning mathematics with understanding. Portsmouth, NH: Heinemann; 1997.

Hiebert, J.; LeFevre, P. Conceptual and procedural knowledge in mathematics: An introductory analysis. In: Hiebert, J., editor. Conceptual and procedural knowledge. Mahwah, NJ: Lawrence Erlbaum Associates; 1986. p. 1-27.

Hitch GJ, McAuley E. Working memory in children with specific arithmetical learning disabilities. British Journal of Psychology 1991;82:375–386. [PubMed: 1954527]

Holmes J, Adams JW, Hamilton CJ. The relationship between visuospatial sketchpad capacity and children's mathematical skills. European Journal of Cognitive Psychology 2008;20:272–289.

Hope J, Owens O. An analysis of the difficulty of learning fractions. Focus on Learning Problems in Mathematics 1987;9:25–39.

Jordan NC, Hanich LB. Characteristics of children with moderate mathematics deficiencies: A longitudinal perspective. Learning Disabilities Research and Practice 2003;18:213–221.

Jordan NC, Hanich LB, Kaplan D. A longitudinal study of mathematical competencies in children with specific mathematics difficulties versus children with co-morbid mathematics and reading difficulties. Child Development 2003;74:834–850. [PubMed: 12795393]

Kail RV. Cognitive Development Includes Global and Domain-Specific Processes. Merrill-Palmer Quarterly 2004;50:445–455.

Kail R, Hall LK. Sources of developmental change in children's word problem performance. Journal of Educational Psychology 1999;91:660–668.

Karmiloff-Smith, A. Beyond Modularity. Cambridge, MA: MIT Press; 1992.

Kessler, RC.; Greenberg, DF. Linear Panel Analysis. NY: Academic Press; 1981.

Kieren, TE. Personal knowledge of rational numbers: Its intuitive and formal development. In: Hiebert, J.; Behr, M., editors. Number concepts and operations in the middle grades. Reston, VA: National Council of Teachers of Mathematics; 1988. p. 162-181.

Kilpatrick, J.; Swafford, J.; Findell, Bradford. Adding it up: helping children learn mathematics. Washington, DC: National Academy Press; 2001.

Kintsch W, Greeno JG. Understanding and solving word arithmetic problems. Psychological Review 1985;92:109–129. [PubMed: 3983303]

LeFevre, J.; DeStefano, D.; Coleman, B.; Shanahan, T. Mathematical cognition and working memory. In: Campbell, JID., editor. Handbook of Mathematical Cognition. New York: Psychology Press; 2005. p. 361-377.

Lemaire P, Abdi H, Fayol M. The role of working memory resources in simple cognitive arithmetic. European Journal of Cognitive Psychology 1996;8:73–103.

Logie RH, Baddeley AD. Cognitive processes in counting. Journal of Experimental Psychology: Learning, Memory, & Cognition 1987;13:310–326.

Loveless T. Trends in math: The importance of basic skills. Brookings Review 2003;21:40–43.

Mabbott DJ, Bisanz J. Computational skills, working memory, and conceptual knowledge in older children with mathematics learning disabilities. Journal of Learning Disabilities 2008;41:15–28. [PubMed: 18274501]

Mack NK. Learning fractions with understanding: Building on informal knowledge. Journal for Research in Mathematics Education 1990;21:16–32.

Mack, NK. Learning rational numbers with understanding: The case of informal knowledge. In: Carpenter, TP.; Fennema, E.; Romberg, TA., editors. Rational numbers: An integration of research. Mahwah, NJ: Lawrence Erlbaum Associates; 1993. p. 85-105).

Mack NK. Confounding whole-number and fraction concepts when building on informal knowledge. Journal for Research in Mathematics Education 1995;26:422–441.

MacKinnon, DP. Introduction to statistical mediation analysis. New York: Lawrence Erlbaum Associates; 2008.

Mayer, RE. Implications of cognitive psychology for instruction in mathematical problem solving. In: Silver, EA., editor. Teaching and learning mathematical problem solving:multiple research perspectives. NJ: Lawrence Erlbaum; 1985.

Mazzocco MMM, Devlin KT. Parts and holes: Gaps in rational number sense in children with vs. without mathematical learning disability. Developmental Science 2008;11:681–691. [PubMed: 18801123]

Mazzocco MMM, Myers GF. Complexities in identifying and defining mathematics learning disability in the primary school age years. Annals of Dyslexia 2003;53:218–253. [PubMed: 19750132]

McKinney JD, Speece DL. Academic consequences and longitudinal stability of behavioral subtypes of learning disabled children. Journal of Educational Psychology 1986;78:365–372.

McLeod DB. Information processing theories and mathematics learning: The role of affect. International Journal of Educational Research 1990;14:13–29.

Muthen, LK.; Muthen, BO. Mplus User’s Guide. 4. Los Angeles, CA: Muthen & Muthen; 2007.

National Council for Teachers of Mathematics. Principles and standards for school mathematics. Reston, VA: Author; 2000.

National Mathematics Advisory Panel. Foundations for success: The final report of the national mathematics advisory panel. Washington, DC: U.S. Department of Education; 2008.

National Mathematics Advisory Panel. Chapter 4: Report of the Task Group on Learning Processes. Washington, DC: U.S. Department of Education; 2008.

Nesher P. Are mathematical understanding and algorithmic performance related. For the Learning of Mathematics 1986;6:2–9.

Ni Y. How valid is it to use number lines to measure children’s conceptual knowledge about rational number? Educational Psychology 2000;20:139–152.

Ohlsson S, Rees E. The Function Conceptual Understanding in the Learning of Arithmetic Procedures. Cognition & Instruction 1991;8:103–179.

Passolunghi MC, Pazzagliab F. Individual differences in memory updating in relation to arithmetic problem solving. Learning and Individual Differences 2004;14:219–230.

Passolunghi MC, Siegel LS. Short-Term Memory, Working Memory, and Inhibitory Control in Children with Difficulties in Arithmetic Problem Solving. Journal of Experimental Child Psychology 2000;80:44–57. [PubMed: 11511134]

Passolunghi MC, Siegel LS. Working memory and access to numerical information in children with disability in mathematics. Journal of Experimental Child Psychology 2004;88:348–367. [PubMed: 15265681]

Pellegrino, JW.; Chudowsky, N.; Glaser, R., editors. Knowing what students know. Washington, DC: National Academies Press; 2001.

Pickering SJ, Gathercole SE. Distinctive working memory profiles in children with special needs. Educational Psychology 2004;24:393–408.

Pitkethly A, Hunting RP. A review of recent research in the area of initial fraction concepts. Educational Studies in Mathematics 1996;30:5–38.

Preacher KJ, Hayes AF. SPSS and SAS procedures for estimating indirect effects in simple mediation models. Behavior Research Methods, Instruments, & Computers 2004;36:717–731.

Pressley M. The relevance of the good strategy user model to the teaching of mathematics. Educational Psychologist 1986;21:139–161.

Raghubar KP, Barnes M, Hecht SA. Working Memory and Mathematics: A Review of Developmental, Individual Difference, and Cognitive Approaches. Learning and Individual Differences 2010;20:110–122.

Raghubar KP, Cirino PT, Barnes MA, Ewing-Cobbs L, Fletcher JM, Fuchs LS. Errors in multi-digit arithmetic and behavioral inattention in children with math difficulties. Journal of Learning Disabilities 2009;42:356–371. [PubMed: 19380494]

Resnick, LB.; Ford, WW. The psychology of mathematics for instruction. Hillsdale, NJ: Lawrence Erlbaum; 1981.

Resnick, L.; Omanson, S. Learning to understand arithmetic. In: Glaser, R., editor. Advances in Instructional Psychology. Vol. 3. Hillsdale, NJ: Erlbaum; 1987. p. 41-95.

Rittle-Johnson, B.; Siegler, RS. The relations between conceptual and procedural knowledge in learning mathematics: A review. In: Donlan, C., editor. The development of mathematical skill. Hove, England: Psychology Press; 1998. p. 75-110.

Rittle-Johnson B, Siegler RS, Alibali MW. Developing conceptual understanding and procedural skill in mathematics: An iterative process. Journal of Educational Psychology 2001;93:346–362.

Robinson CS, Menchetti BM, Torgesen JK. Toward a two-factor theory of one type of mathematics disabilities. Learning Disabilities Research & Practice 2002;17:81–89.

Saxe GB, Gearhart M, Seltzer M. Relations between classroom practices and student learning in the domain of fractions. Cognition and Instruction 1999;17:1–24.

Sherman J, Bisanz J. Equivalence in symbolic and non-symbolic contexts: Benefits of solving problems with manipulatives. Journal of Educational Psychology 2009;101:88–100.

Siegal M, Smith JA. Toward making representation count in children's conceptions of fractions. Contemporary Educational Psychology 1997;22:1–22.

Siegel LS, Ryan EB. The development of working memory in normally achieving and subtypes of learning disabled children. Child Development 1989;60:973–980. [PubMed: 2758890]

Siegler RS. Strategy choice procedures and the development of multiplication skill. Journal of Experimental Psychology: General 1988;117:258–275. [PubMed: 2971762]

Siegler, RS. There's Nothing As Useful As a Good Theory. Plenary Address to the Japanese Psychological Association; August 27; Kyoto, Japan. 2009.

Siegler, RS.; Araya, R. A computational model of conscious and unconscious strategy discovery. In: Kail, RV., editor. Advances in child development and behavior. Vol. 33. Oxford, UK: Elsevier; 2005. p. 1-42.

Siegler RS, Stern E. Conscious and unconscious strategy discoveries: A microgenetic analysis. Journal of Experimental Psychology: General 1998;127:377–397. [PubMed: 9857493]

Smith JP. Competent reasoning with rational numbers. Cognition and Instruction 1995;13:3–50.

Smith C, Solomon G, Carey S. Never getting to zero: Elementary school students' understanding of the infinite divisibility of number and matter. Cognitive Psychology 2005;51:101–140. [PubMed: 16081058]

Sobel, ME. Asymptotic confidence intervals for indirect effects in structural equation models. In: Leinhardt, S., editor. Sociological Methodology. Washington, DC: American Sociological Association; 1982. p. 290-312.

Sophian C. Beyond competence: The significance of performance for conceptual development. Cognitive Development 1997;12:281–303.

Sophian C, Garyantes D, Chang C. When three is less than two: early developments in children's understanding of fractional quantities. Developmental Psychology 1997;33:731–744. [PubMed: 9300207]

Stafylidou S, Vosniadou S. The development of students' understanding of the numerical value of fractions. Learning and Instruction 2004;14:503–518.

Stanovich KE. Matthew effects in reading: Some consequences of individual differences in the acquisition of literacy. Reading Research Quarterly 1986;21:360–406.

Stein MK, Grover BW, Henningsen M. Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. American Educational Research Journal 1996;2:50–80.

Stern E. What makes certain arithmetic word problems involving the comparison of sets so difficult for children? Journal of Educational Psychology 1993;85:7–23.

Streefland, L. Fractions: A realistic approach. In: Carpenter, TP.; Fennema, E.; Romberg, TA., editors. Rational numbers: An integration of research. Mahwah, NJ: Lawrence Erlbaum Associates; 1993. p. 289-304.

Swanson HL, Beebe-Frankenberger M. The relationship between working memory and mathematical problem solving in children at risk and not at risk for math disabilities. Journal of Education Psychology 2004;96:471–491.

Swanson HL, Sachse-Lee C. Mathematical problem solving and working memory in children with learning disabilities: Both executive and phonological processes are important. Journal of Experimental Child Psychology 2001;79:294–321. [PubMed: 11394931]

Tabachnick, BG.; Fidell, LS. Using Multivariate Statistics. 5. New York, NY: Allyn & Bacon; 2007.

Torgesen JK, Wagner RK, Rashotte CA, Rose E, Lindamood P, Conway T, Garvan C. Preventing reading failure in young children with phonological processing disabilities: Group and individual responses to instruction. Journal of Educational Psychology 1999;91:579–593.

VanLehn, K. Mind bugs: The origins of procedural misconceptions. Cambridge, MA: MIT Press; 1990.

VanLehn, K.; Brown, J. Planning nets: A representation for formalizing analogies and semantic models of procedural skills. In: Snow, RE.; Federico, PA.; Montague, WE., editors. Aptitude, learning, and instruction. Hillsdale, NJ: Lawrence Erlbaum; 1980. p. 95-137.

Wagner RK, Torgesen JK, Rashotte CA. Development of reading-related phonological processing abilities: New evidence of bidirectional causality from a latent variable longitudinal study. Developmental Psychology 1994;30:73–87.

Wechsler, D. Wechsler Abbreviated Scale of Intelligence. NY: Psychological Corporation; 1999.

Wentzel KR. Social competence at school: Relation between social responsibility and academic achievement. Review of Educational Research 1991;61:1–24.

Widaman KF, Little TD, Geary DC, Cormier P. Individual differences in the development of skill in mental addition: Internal and external validation of chronometric models. Learning and Individual Differences 1992;4:167–213.

Williams RL. A note on robust variance estimation for cluster-correlated data. Biometrics 2000;56:645–646. [PubMed: 10877330]

Wilson KM, Swanson HL. Are mathematics disabilities due to a domain-general or a domain-specific working memory deficit? Journal of Learning Disabilities 2001;34:237–248. [PubMed: 15499878]

Woodcock, RW.; McGrew, KS.; Mather, N. Woodcock-Johnson III: Complete Battery Tests of Achievement. Itasca, IL: Riverside Publishing; 2001.

Zentall SS. Fact-retrieval and math problem solving by learning disabled, attention-disordered, and normal adolescents. Journal of Educational Psychology 1990;82:856–865.

Zentall SS. Research on the Educational Implications of Attention Deficit Hyperactivity Disorder. Exceptional Children 1993;60:143–153.

NIH-PA Author Manuscript

NIH-PA Author Manuscript

Table 1

Bivariate correlations among the predictors and basic fraction skills.

Variable	1	2	3	4	5	6
Fourth Grade						
1. Part-whole conceptual knowledge composite	---					
2. Size of fraction	.48	---				
3. Picture computation	.77	.49	---			
4. Arithmetic solution processes composite	.37	.35	.31	---		
5. Arithmetic latencies composite	.33	.31	.28	.53	---	
6. Word level reading	.32	.30	.27	.13	.41	---
7. Working memory	.36	.24	.31	.33	.29	.30
8. Attentive behavior	.63	.54	.66	.47	.36	.27
9. WASI Full Scale IQ	.42	.30	.43	.27	.25	.28
10. Fraction computation	.59	.51	.56	.44	.47	.35
11. Fraction word problems	.60	.44	.62	.38	.45	.45
12. Fraction estimation	.42	.33	.47	.18	.19	.18
Fifth Grade						
13. Part-whole conceptual knowledge composite	.72	.49	.59	.36	.40	.34
14. Size of fraction	.47	.58	.59	.35	.32	.27
15. Picture computation	.66	.47	.77	.37	.34	.37
16. Arithmetic solution processes composite	.43	.36	.39	.75	.52	.17
17. Arithmetic latencies composite	.30	.29	.32	.46	.73	.32
18. Word level reading	.31	.23	.29	.16	.34	.71
19. Working memory	.43	.27	.30	.35	.37	.36
20. Fraction computation	.59	.49	.62	.42	.41	.25
21. Fraction word problems	.63	.50	.65	.46	.49	.47
22. Fraction estimation	.52	.47	.58	.29	.34	.29

	7	8	9	10	11	12
7. Working memory	---					
8. Attentive behavior	.36	---				
9. WASI Full Scale IQ	.24	.33	---			

	7	8	9	10	11	12
10. Fraction computation	.35	.51	.35	---		
11. Fraction word problems	.39	.54	.56	.55	---	
12. Fraction estimation	.25	.34	.40	.43	.44	---
Fifth Grade						
13. Part-whole conceptual knowledge composite	.32	.55	.39	.48	.57	.36
14. Size of fraction	.26	.46	.42	.45	.53	.38
15. Picture computation	.31	.61	.43	.58	.63	.46
16. Arithmetic solution processes composite	.35	.50	.30	.50	.41	.23
17. Arithmetic latencies composite	.28	.34	.29	.44	.41	.24
18. Word level reading	.18	.30	.41	.30	.46	.21
19. Working memory	.72	.36	.24	.37	.36	.20
20. Fraction computation	.43	.61	.34	.66	.50	.39
21. Fraction word problems	.42	.64	.49	.60	.76	.41
22. Fraction estimation	.35	.55	.35	.54	.53	.52

	13	14	15	16	17	18
13. Part-whole conceptual knowledge composite	---					
14. Size of fraction	.50	---				
15. Picture computation	.68	.58	---			
16. Arithmetic solution processes	.44	.38	.47	---		
17. Arithmetic latencies composite	.39	.37	.42	.51	---	
18. Word level reading	.35	.26	.32	.15	.31	---
19. Working memory	.40	.32	.38	.36	.38	.24
20. Fraction computation	.58	.49	.61	.51	.47	.23
21. Fraction word problems	.60	.52	.70	.52	.49	.43
22. Fraction estimation	.46	.49	.58	.34	.38	.33

	19	20	21	22
19. Working memory	---			
20. Fraction computation	.38	---		
21. Fraction word problems	.42	.59	---	
22. Fraction estimation	.28	.57	.56	---

All correlations significant ($p <$ at most .05).

Table 2

Mean Performance by Group (SD's in parentheses).

	Group			
Measure (Max Score)	**Typically Achieving (n = 126)**	**Math Difficulties (n = 55)**	**F (1,179)**	**Effect Size (d)**
1. Fraction computation (26)				
4th grade	9.85(3.85)	4.89(2.90)	73.72	1.39
5th grade	14.46(4.66)	8.22(4.04)	72.97	1.38
Improve	4.61(4.22)	3.33(3.63)	3.84	.32
2. Fraction estimation (12)				
4th grade	3.84(3.19)	1.64(1.61)	23.72	2.22
5th grade	6.88(3.73)	2.31(1.94)	73.83	1.39
Improve	3.04(3.72)	.67(2.01)	19.67	.72
3. Fraction word problems (16)				
4th grade	9.20(3.20)	4.55(2.85)	86.58	1.50
5th grade	13.12(3.32)	7.22(3.24)	122.35	1.81
Improve	3.92(2.86)	2.67(2.51)	7.85	.45
4. Arithmetic -addition retrieval use (22)				
4th grade	11.60(6.68)	7.20(6.41)	17.06	.67
5th grade	16.73(5.49)	11.22(6.51)	34.35	.95
Improve	5.14(6.73)	4.02(4.25)	1.26ns	.18
5. Arithmetic -multiplication retrieval use (22)				
4th grade	15.55(6.13)	9.96(7.77)	26.75	.84
5th grade	19.13(3.62)	12.45(7.68)	63.22	1.29
Improve	3.58(6.54)	2.49(3.68)	1.26ns	.19
6. Arithmetic - addition accuracy (22)				
4th grade	21.07(3.40)	15.13(7.69)	52.27	1.17
5th grade	21.30(2.69)	15.78(7.41)	54.07	.87
Improve	.23(2.19)ns	.66(4.67)ns	.72 ns	.14
7. Arithmetic –multiplication accuracy (22)				
4th grade	21.86(.99)	16.91(7.04)	59.99	1.25
5th grade	21.88(.96)	18.95(5.78)	30.85	.90
Improve	.02(1.32)ns	2.04(4.37)	22.35	.77
8. Arithmetic – addition latency				
4th grade	2.61(.83)	3.85(1.03)	73.83	.47
5th grade	2.17(.77)	3.32(.96)	73.11	1.38
Improve	−.44(.75)	−.53(.89)	.54ns	.11
9. Arithmetic – multiplication latency				
4th grade	2.89(1.28)	5.02(2.03)	72.41	1.38
5th grade	2.35(1.01)	4.13(1.95)	65.87	1.31
Improve	−.54(1.18)	−.88(1.37)	1.26ns	.27
10. Symbol Picture (16)				

	Group			
Measure (Max Score)	Typically Achieving (n = 126)	Math Difficulties (n = 55)	F (1,179)	Effect Size (d)
4th grade	12.33(2.78)	8.24(4.06)	61.92	1.27
5th grade	14.05(2.02)	10.55(3.93)	62.57	1.28
Improve	1.71(2.59)	2.31(3.23)	1.81ns	.21
11. Picture Symbol (13)				
4th grade	9.84(3.17)	4.33(3.36)	112.06	1.71
5th grade	11.81(3.01)	6.67(3.44)	105.58	1.66
Improve	1.97(2.84)	2.35(2.82)	.72ns	.13
12. Size of fraction (22)				
4th grade	17.98(3.82)	13.02(5.54)	48.45	1.13
5th grade	19.82(2.44)	15.91(3.75)	69.61	1.15
Improve	1.84(3.48)	2.89(5.15)	2.54ns	.26
13. Picture Computation (13)				
4th grade	9.98(3.02)	5.78(2.91)	75.26	1.41
5th grade	11.30(2.02)	7.11(2.83)	128.03	1.83
Improve	1.33(2.21)	1.33(2.44)	.01ns	.00
14. Working memory (12)				
4th grade	5.08(2.26)	3.13(1.94)	31.09	.90
5th grade	6.33(1.91)	4.15(1.90)	49.03	1.13
Improve	1.25(1.82)	1.02(1.35)	.72ns	.14
15. Attentive behavior (30)				
4th grade	25.04(5.13)	18.00(5.21)	71.35	1.37
5th grade	21.74(5.57)	13.42(4.69)	93.45	1.56
Improve	−3.30(4.62)	−4.58(7.02)	2.17ns	.23
16. Word level reading (104)				
4th grade	71.56(9.26)	62.85(13.00)	26.51	.83
5th grade	78.19(9.47)	70.00(13.29)	22.12	.76
Improve	6.63(6.59)	7.16(12.17)	.18ns	.06
17. WASI Full Scale IQ (157)				
4th grade	103.68(11.24)	96.06(10.41)	18.35	.88

Mean total correct, except for arithmetic latency and retrieval use. Arithmetic latency expressed in terms of seconds (i.e., millisecond raw data was divided by 1000). Arithmetic retrieval use expressed as mean total number of problems solved via the retrieval strategy. All values p < at most .05 unless otherwise noted.

NIH-PA Author Manuscript

NIH-PA Author Manuscript

Table 3

Proportion of variance in computation, estimation, and word problem performance explained by predictors (n = 181).

	Computation		Word Problem		Estimation	
Step and measure	Unique R^2	Total R^2	Unique R^2	Total R^2	Unique R^2	Total R^2
Panel 1. Prior fourth grade math entered						
1a. Prior ability	.43***	.43***	.58***	.58***	.27***	.27***
Panel 2. Full scale IQ and reading entered after prior fourth grade math						
2a. FS IQ	.01*		.00		.02*	
2b. Word level reading	.00	.44***	.02***	.60***	.03**	.32***
Panel 3. Each predictor entered separately, after controlling for prior ability, IQ, and reading						
3a. Part-whole	.05***	.49***	.06***	.67***	.07***	.37***
3a. Size of fraction	.02*	.46***	.02**	.63***	.08***	.38***
3a. Picture computation	.08***	.52***	.05***	.66***	.13***	.43***
3a. Arithmetic-SP	.02*	.46***	.03**	.64***	.02*	.32***
3a. Arithmetic-Latency	.02**	.46***	.01**	.62***	.02***	.32***
3a. Working memory	.05***	.49***	.01**	.62***	.04***	.34***
3a. Attentive behavior	.08***	.52***	.06***	.67***	.12***	.42***
Panel 4. Each predictor entered together, after controlling for prior ability, full scale IQ, and reading						
3a. Part-whole	.01		.00		.00	
3b. Size of fraction	.00		.00		.03**	
3c. Picture computation	.03***		.01*		.02*	
3d. Arithmetic-SP	.00		.01*		.00	
3e. Arithmetic-Latency	.01		.00		.00	
3f. Working memory	.00		.00		.00	
3g. Attentive behavior	.02*	.59***	.01***	.71***	.02*	.50***

Table 4

Tests of Mediation Effects (standard errors in parentheses)

		Fraction Computation		Word Problems		Estimation	
Mediator	**a**	**b**	**(aXb)**	**b**	**(aXb)**	**b**	**(aXb)**
Panel 1. Each predictor entered separately into the regression equation, after controlling for the autoregressor, full scale IQ, and word level reading							
1. Part-whole	1.11(.12)	1.12(.38)	1.24(.44)	.68(.23)	.75(.27)	.63(.25)	.70(.29)
2. Measurement	4.00(.84)	.17(.07)	.68(.23)	.11(.04)	.44(.19)	.16(.04)	.64(.21)
3. Picture computation	3.46(.51)	.45(.08)	1.56(.36)	.28(.06)	.97(.25)	.30(.06)	1.04(.26)
4. Arithmetic-SP	1.13(.17)	.28(.31)^{ns}	.32(.35)^{NM}	.49(.20)	.55(.24)	.06(.28)^{ns}	.07(.32)^{NM}
5. Arithmetic latency	1.10 (.15)	.10(.21)^{ns}	.11(.23)^{NM}	.20(.28)^{ns}	.22(.31)^{NM}	.08(.30)^{ns}	.09(.33)^{NM}
6. Working memory	1.49(.33)	.42(.20)	.63(.33)^{NM}	.16(.08)	.24(.13)^{NM}	.16(.12)^{ns}	.24(.19)^{NM}
7. Attentive behavior	6.06(.74)	.27(.06)	1.64(.42)	.18(.03)	1.09(.23)	.18(.04)	1.09(.28)

	Fraction Computation		Word Problems		Estimation	
Mediator	**b**	**(aXb)**	**b**	**(aXb)**	**b**	**(aXb)**
Panel 2. Each of the mediators entered simultaneously into the regression equation after controlling for the autoregressor, full scale IQ, and word level reading						
1. Part-whole	−.03(.42)^{ns}	−.03(.42)^{NM}	−.08(.34)^{ns}	−.09(.38)^{NM}	−.24(.34)^{ns}	−.27(.38)^{NM}
2. Measurement	.08(.06)^{ns}	.32(.25)^{NM}	.03(.03)^{ns}	.12(.11)^{NM}	.09(.04)	.36(.18)
3. Picture computation	.31(.11)	1.07(.41)	.21(.10)	.73(.36)	.21(.10)	.73(.35)
4. Arithmetic-SP	−.09(.38)^{ns}	.10(.42)^{NM}	.28(.18)^{ns}	.32(.21)^{NM}	−.29(.31)^{ns}	.33(.35)^{NM}
5. Arithmetic latency	.33(.27)^{ns}	.36(.30)^{NM}	.17(.24)^{ns}	.19(.27)^{NM}	.19(.27)^{ns}	.21(.44)^{NM}
6. Working memory	.35(.17)	.52(.28)^{NM}	.10(.07)	.15(.11)^{NM}	.12(.11)^{ns}	.18(.17)^{NM}
7. Attentive behavior	.16(.07)	.96(.44)	.10(.03)	.61(.20)	.12(.05)	.73(.32)

a = values represent the path from the group dummy variable to the mediator. b values represent the path from the mediating variable to the fraction problem. For the b path, the group contrast dummy variable is also entered as covariates. aXb = the mediating effect. All values significant ($p <$ at most .05) unless otherwise specified. ns = not significant. NM stands for no mediation. + $p < .10$.

NIH-PA Author Manuscript
NIH-PA Author Manuscript

Table 5

Proportion of variance in conceptual knowledge explained by fraction computation, word problem, and estimation skills.

	Part-whole		Measurement		Picture Computation	
Step and measure	**Unique R^2**	**Total R^2**	**Unique R^2**	**Total R^2**	**Unique R^2**	**Total R^2**
Panel 1. Prior fourth grade math entered						
1a. Prior ability	.50***	.50***	.34***	.34***	.60***	.60***
Panel 2. Full scale IQ and reading entered after prior fourth grade math						
2b. FS IQ	.00		.06***		.00	
2c. Word level reading	.02*	.52***	.00	.40***	.03***	.63***
Panel 3. Each predictor entered separately, after controlling for prior ability, IQ, and reading						
3a. Computation	.00	.52***	.02*	.42***	.02**	.65***
3a. Word problem	.02***	.54***	.05***	.45***	.01*	.64***
3a. Estimation	.00	.52***	.02*	.42***	.01*	.64***
Panel 4. Each predictor entered together, after controlling for prior ability, full scale IQ, and reading						
3a. Computation	.00		.00		.01*	
3b. Word problem	.02*		.02**		.00	
3c. Estimation	.00	.54***	.00	.45***	.00	.66***