Sovereign Debt and Structural Reforms*

Andreas Müller[†] Kjetil Storesletten[‡] Fabrizio Zilibotti[§]

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Abstract

Motivated by the European debt crisis, we construct a tractable theory of sovereign debt and structural reforms under limited commitment. The government of a sovereign country which has fallen into a recession of an uncertain duration issues one-period debt and can renege on its obligations by suffering a stochastic default cost. When faced with a credible default threat, creditors can make a take-it-or-leave-it debt haircut offer to the sovereign. The risk of renegotiation is reflected in the price at which debt is sold. The sovereign government can also introduce structural policy reforms that speed up recovery from the recession. We characterize the competitive equilibrium and compare it with the constrained efficient allocation. The equilibrium features increasing debt, falling consumption, and a non-monotone reform effort during the recession. In contrast, the constrained optimum yields step-wise increasing consumption and step-wise decreasing reform effort. Markets for state-contingent debt alone do not restore efficiency. The constrained optimum can be implemented by a flexible assistance program enforced by an international institution that monitors the reform effort. The terms of the program are improved each time the country makes a credible threat to leave the program unilaterally without repaying the outstanding loans.

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1 Introduction

The European debt crisis has revamped the debate on sovereign debt crises. With the Great Recession hitting Southern European economies especially hard, it would seem a natural response for their governments to borrow from international financial markets to smooth consumption. A complicating

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[†]University of Oslo, Department of Economics, andreas.mueller@econ.uio.no.

[‡]University of Oslo, Department of Economics, kjetil.storesletten@econ.uio.no.

[§]University of Zurich, Department of Economics, fabrizio.zilibotti@econ.uzh.ch.

factor is that some countries – most notably Greece and Italy – had entered the recession with an already large outstanding debt, in excess of 100% of their annual GDP. In 2009, a confidence crisis triggered growing default premia. The wildly mounting cost of servicing debt triggered social and political unrest, including pressure not to honor the outstanding financial obligations.

Greece's story is emblematic. As the Greek debt-to-GDP ratio climbed from 107% in 2008 to 146% in 2010, international organizations stepped in to provide financial assistance and access to new loans, asking in exchange a commitment to economic reforms. In May 2010 the Troika – a three-part commission representing the IMF, the EU and the ECB – launched a 110 billion Euro bailout conditional on a set of austerity measures (to be followed by a similar intervention in 2012). However, austerity met with sharp political opposition. Amidst angry street protests, pressure escalated not to honor the outstanding debt, which had increased to over 170% of GDP. In October 2011, creditors had to agree to a debt haircut implying a 53% loss on its face value. After a temporary decline, debt (now consisting largely of institutional loans) soared again. The electoral victory of the radical party Syriza in early 2015 called into question both the austerity policy, and the prospect for when and to what extent the outstanding debt will be repaid. A further round of renegotiation took place in July 2015. As of today, the issue is not yet fully resolved.

Motivated by these recent events, in this paper we construct a theory of sovereign debt aimed to address the following set of questions. First, what does the optimal dynamic contract between a planner and a sovereign, temporarily impoverished, country prescribe in an environment where the country cannot commit to honoring its debt? Second, what policies and institutional arrangements can implement such a contract? In particular, does the market equilibrium attain the efficient level of structural reforms and debt dynamics? Or, otherwise, can an international institution (such as the IMF) introduce welfare-improving programs? Finally, how should this institution deal with pressure to renegotiate an existing assistance program?

The theory rests on three building blocks. The first is that sovereign debt is subject to limited enforcement, and that countries can renege on their obligations subject to real costs, as in e.g. Aguiar and Gopinath (2006) and Arellano (2008). Different from this literature, we assume the size of default costs to be stochastic, reflecting exogenous changes in the domestic and international situation. For instance, when "dovish" center-left governments took office in Italy and France, the pressure on Greece eased. The second building block is that whenever creditors face a credible default threat, they can make a take-it-or-leave-it renegotiation offer to the indebted country. This approach conforms with the empirical observations that unordered defaults are rare events, and that there is great heterogeneity in the terms at which debt is renegotiated, as documented by Tomz and Wright (2007) and Sturzenegger and Zettelmeyer (2008). The third building block is the possibility for the government of the indebted country to make "structural" policy reforms that speed up recovery from an existing recession.

More formally, we construct a dynamic stochastic general equilibrium model with an endogenous debt price. Productivity is subject to aggregate shocks following a two-state Markov process. A benevolent local government can issue sovereign debt to smooth consumption.[1] The sovereign country starts in a recession of an unknown duration. The probability that the recession ends is endogenous, and hinges on the government's reform effort. Debt issuance is subject to a limited-commitment problem: the government can, *ex-post,* repudiate its debt, based on the publicly observable realization of a stochastic default cost. When this realization is sufficiently low relative to the outstanding debt, the default threat is credible. In this case, a syndicate of creditors makes a take-it-or-leave-it debt

[1] In the benchmark model, sovereign debt is a non-state-contingent bond. Our results do not hinge on this assumption, and we show in an extension that if the government can issue state-contingent debt, i.e., securities whose pay-off is contingent to the realization of the aggregate productivity, the main results remain unchanged.

haircut offer, as in Bulow and Rogoff (1989). In equilibrium, there is no outright default, but repeated debt renegotiations. Haircuts are more frequent during recession, and the larger the outstanding sovereign debt is. Consumption increases after a renegotiation, which is in line with the empirical evidence that economic conditions improve in the aftermath of debt relief as documented in Reinhart and Trebesch (forthcoming).

We first characterize the laissez-faire equilibrium. We focus on the notion of Markov Perfect Equilibrium where agents' strategy only depend on a vector of pay-off relevant state variables. During recessions, the government would like to issue debt in order to smooth consumption. However, as debt accumulates, the probability of renegotiation increases, implying a growing risk premium. The equilibrium does not feature full insurance. In a recession, consumption falls as debt accumulates. The reform effort exhibits a non-monotonic pattern: it is increasing with debt at low levels of debt because of the disciplining effect of recession (the welfare cost of the recession is higher when the country must service a large debt). However, for sufficiently high debt levels the relationship is flipped. The reason is that some benefits of the reform accrue to creditors, and more so the higher the debt level. Thus, the theory features a version of the debt overhang problem highlighted by Krugman (1988): very high debt levels deter useful reforms. The moral hazard problem exacerbates the country's inability to achieve consumption smoothing: at high debt levels, creditors expect little reform effort, are pessimistic about the economic outlook, and request an even higher risk premium.

Next, to establish a normative benchmark, we characterize the optimal dynamic contract between a planner without enforcement power and a country that has fallen into recession. In contrast with the competitive equilibrium, the constrained optimal allocation features non-decreasing consumption and non-increasing reform effort during the recession. More precisely, consumption and effort remain constant whenever the country's participation constraint is not binding. However, when the constraint is binding (corresponding to a low realization of the default cost), the planner increases the country's promised utility and consumption, and reduces its reform effort.

Having characterized the constrained-efficient allocation, we consider its implementation in a decentralized environment. We first show that, in the absence of aggregate productivity shocks, the laissez-faire equilibrium attains the constrained-efficient allocation. However, the equilibrium is not constrained-efficient when the economy is in a recession. In the laissez-faire equilibrium, there is too little consumption smoothing, and the reform effort is inefficiently low. Interestingly, the inefficiency cannot be resolved by allowing the government to issue state-contingent debt. In standard models, state-contingent debt provides insurance against the continuation of a recession – i.e., Arrow securities paying off conditional on the aggregate state, recession or normal times. However, the better insured the country is, the more severe the moral hazard problem becomes. For instance, full insurance would destroy any incentive to exert reform effort. Since creditors would anticipate the moral hazard problem, this would be priced into the debt. Thus, the social value of markets for state-contingent debt is limited. In a calibrated version of the model, we show that having access to state-contingent debt yields only small welfare gains in this environment.

While the implementation of the constrained efficient allocation requires interim monitoring, which is a strong assumption, we show that a weaker concept of constrained efficient can be attained if the planner can observe the reform effort ex post, and condition the continuation of the program to the execution of the desired reform.

Although not implemented by the laissez-faire equilibrium, the constrained optimal allocation can be attained through the intervention of an independent institution (e.g., the IMF) that has the power (i) to control the country's fiscal policy (an *austerity program*); (ii) to monitor the reform effort (possibly ex post). During the recession the optimal program entails a persistent budget support through extending loans on favorable terms, combined with a larger reform effort than the borrower would choose on its own. Upon recovery from the recession, the sovereign is settled with a (large) debt on market terms. A common objection to schemes implying deferred repayment is that the country may refuse to repay its loans when the economy recovers. In our theory, this risks exists, but is taken into account *ex-ante* when the deal is agreed upon. The larger the probability of future non-repayment, the harsher conditions the country must accept upon entering the assistance program. The program can in principle be budget-neutral, in expected terms, for the international institution. Ex-post, it can result in either gains or losses depending on the evolution of the crisis.

The optimal program has the interesting feature that, whenever a credible default threat is on the table, the international institution should relent and improve the terms of the agreement for the debtor by granting her higher consumption and a lower reform effort. In other words, the austerity program is relaxed over time, whenever necessary to avert the breakdown of the program. Notably, the efficiency of the contract is not enhanced if the institution can credibly threaten to stop its financial support whenever the debtor tries to renegotiate the original terms. Intuitively, such a threat would increase the probability that the government honors its debt, but could not prevent default when its cost is very low. In the event of a default, the country would suffer a real cost, being then forced to revert to the competitive equilibrium, which is not efficient. The international institution, in turn, would lose all the resources invested in the assistance program.

These observations have interesting policy implications for the recent debate about the management of the European debt crisis. Greece's request to renegotiate the austerity conditions has met with fierce opposition (especially in Germany) based on the argument that accepting a renegotiation would have perverse incentive consequences on the reform process in Greece. Our theory predicts that, to the extent that Syriza's threat is credible, appeasement may be the optimal response for the European Union, so long as the alternative is outright default. Interestingly, a post-default scenario may entail less structural reforms than one where the demands of Greece are appeased and default is averted.

We provide a quantitative evaluation of the theory with the aid of a calibrated version of the model. The model matches realistic debt-to-GDP ratios, as well as default premia and recovery rates. We regard this as a contribution in itself. In the existing quantitative literature, it is difficult to sustain high debt levels, contrary both to the observation that many countries have managed to finance debt-GDP ratios above 100%, and to the estimates of a recent study by Collard, Habib, and Rochet (2015) showing that OECD countries can sustain debt-GDP ratios even in excess of 200%. We find that an assistance program implementing the constrained optimum yields large welfare gains, equivalent to a transfer of 63% of the initial GDP with a zero expected cost for the institution running the assistance program.

1.1 Literature review

Preliminary and incomplete. Our paper relates to several streams of the literature on sovereign debt. The seminal contribution to the analysis of debt repudiation is Eaton and Gersovitz (1981). In their paper, debt is sustained through reputation and threat of future exclusion from credit markets. Pioneer contributions along this line include Grossman and Van Huyck (1988), and Fernandez and Rosenthal (1989).

Our theory is instead in the spirit of the direct-punishment approach proposed by Bulow and Rogoff (1989).[2] Our work is also related to the more recent quantitative models of sovereign default

[2] The distiction between the reputation approach and the punishment approach as the two main conceptual frameworks in the literature on sovereign debt crisis follows Bulow and Rogoff (2015).

such as Aguiar and Gopinath (2006), Arellano (2008) and Yue (2010). These papers assume that default is subject to a real cost (e.g., trade loss), and studies how the probability of default varies with the severity of recession. Yue (2010) considers, as we do, the possibility of renegotiation, although in her model renegotiation is costly and is determined by Nash bargaining between creditors and debtors - with no stochastic shocks to outside options.[3] In her model, ex-post inefficient restructuring helps ex-ante discipline and provides incentives to honor the debt. This literature has not studied the efficient allocation and its implementation through an assistance program. Moreover, we pursue an analytical characterization of the properties of the model, whereas their main focus is quantitative. One problem in the quantitative literature is that the equilibrium can sustain debt levels that are much lower than what is observed in the data. Our model, by assuming shocks to the outside option, can sustain higher and more realistic debt-GDP ratios.

Another recent paper complementary to ours is Conesa and Kehoe (2015). In their theory, under some circumstances, the government of the indebted country may opt to "gamble for redemption." Namely, it runs an irresponsible fiscal policy that sends the economy into the default zone if the recovery does not happen soon enough. While this resembles the debt overhang feature of our theory, the source and the mechanism of the crisis is different. Their model is based on the framework of Cole and Kehoe (1996, 2000) inducing multiple equilibria and sunspots. Our model features instead a unique equilibrium, owing to a different assumption about the timing of default and the issuance of new debt.

Hopenhayn and Werning (2008) study the optimal contract between a bank and a risk-neutral borrowing firm. As we do, they assume that the borrower has a stochastic default cost. However, they focus on the case when this outside option is not observable to the lender and show that this implies that default can occur in equilibrium. However, they do not study reform effort nor do they analyze the case of sovereign debt issued by a country in recession.

Our paper is related also to the literature on endogenous incomplete markets due to limited enforcement. This includes Alvarez and Jermann (2000) and Kehoe and Perri (2002). The analysis of constrained efficiency is related to the literature on competitive risk sharing contracts with limited commitment, including Thomas and Worrall (1988), Kocherlakota (1996), and Krueger and Uhlig (2006). An application of this methodology to the optimal design of a Financial Stability Fund is provided by Abraham, Carceles-Poveda, and Marimon (2014). In our model all debt is held by foreign lenders. In recent papers Broner, Martin, and Ventura (2010), Broner and Ventura (2011), and Brutti and Sauré (2012) study the implications for the incentives to default of having part of the government debt held by domestic residents. An excellent review of the literature on sovereign debt with limited enforcement can be found in Aguiar and Amador (2014).

In terms of reform effort, our paper is related to Krugman (1988), Atkeson (1991) and Jeanne (2009). Krugman (1988) showed that when a borrower has too large debt, productive investments might not be undertaken (the "debt overhang"). His is a static model where debt is exogenous. In our model debt is accumulated endogenously, despite the debt overhang issues. Atkeson (1991) studied a setting where a borrowing country (the agent) can use funds to invest in productive future capacity or to consume the funds. However, the principal cannot observe the allocation to investment or consumption, and the optimal contract involves capital outflow from the borrower during the worst aggregate state. In our model the lender can observe reform effort. This seems a more plausible abstraction in the context of for example the European debt crisis. Moreover, the focus of our paper is very different from that of Atkeson (1991). Finally, Jeanne (2009) studies how the debt term structure

[3] Other papers focusing on the restructuring of sovereign debt include Asonuma and Trebesch (forthcoming), Benjamin and Wright (2009), Bolton and Jeanne (2007), Dovis (2014), Hopenayn and Werning (2008), and Mendoza and Yue (2012).

affects a country's ex-post incentives to implement investor-friendly policies.

In the large empirical literature, our paper is related to the finding of Tomz and Wright (2007). Using a dataset for the period 1820–2004, they find a negative but weak relationship between economic output in the borrowing country and default on loans from private foreign creditors. While countries default more often during recessions, there are many cases of default in good times and many instances in which countries have maintained debt service during times of very bad macroeconomic conditions. They argue that these findings are at odds with the existing theories of international debt. Our theory is consistent with the pattern they document. In our model, due to the stochastic default cost, countries may default during booms (though this is less likely, consistent with the data) and can conversely fail to renegotiate their debt during very bad times. Their findings are reinforced by Sturzenegger and Zettelmeyer (2008) who document that even within a relatively short period (1998-2005) there are very large differences between average investor losses across different episodes of debt restructuring.[4] The observation of such a large variability in outcomes is in line with our theory, insofar as the bargaining outcome hinges on an outside option that is subject to stochastic shocks. Borensztein and Panizza (2009) evaluate empirically the costs that may result from an international sovereign default, including reputation costs, international trade exclusion, costs to the domestic economy through the financial system, and political costs to the authorities. They find that the economic costs are generally short-lived. Finally, the relationship between consumption and renegotiations is in line with the evidence documented by Reinhart and Trebesch (forthcoming), as discussed above. For a more thorough review of the evidence, see also Panizza, Sturzenegger, and Zettelmeyer (2009).

2 The model environment

The model economy is a small open endowment economy populated by an infinitely-lived representative agent. The endowments follow a two-state Markov switching process, with realizations $w \in \{\underline{w}, \bar{w}\}$ which we label, respectively *normal times*, $\bar{w}$, and *recession*. Normal times is an absorbing state. If the economy starts in a recession ($\underline{w}$), it leaves the recession with probability p and remains in the recession with probability $1-p$. These assumptions allow us to focus sharply on anomalous single events such as the Great Recession.

A benevolent government can issue a one-period bond (sovereign debt) to smooth consumption, and when in recession, it can implement a costly reform policy to increase the probability of a recovery. In our notation, p is both the *reform effort* and the probability that the recession ends. At the beginning of each period, before issuing new debt, the government also decides whether to honor or to repudiate the outstanding debt that reaches maturity.

The preferences of the representative household are represented by the following expected utility function:

$$E_0 \sum \beta^t \left[u\left(c_t\right) - \phi I_{\{\text{default in } t\}} - X\left(p_t\right) \right].$$

The utility function u is twice continuously differentiable and satisfies $u'(c) > 0$ and $u''(c) < 0$. $I \in \{0,1\}$ is an indicator variable switching on when the economy is in a default state; ϕ is a stochastic default cost assumed to be i.i.d. over time and drawn from the p.d.f. $f(\phi)$ with an associated c.d.f. $F(\phi)$. We assume that $f(\phi)$ has no mass points, and denote the support of the p.d.f. by $\aleph \equiv [\phi_{\min}, \phi_{\max}] \subset \mathbb{R}^+$. We denote the bounds of $\aleph$ by $0 \le \phi_{\min} < \phi_{\max} < \infty$. The assumption that

[4] In their data, losses range from 13 percent (Uruguay, 2003) to 73 percent (Argentina, 2005).

shocks are independent is inessential, but aids tractability. X is the cost of reform, assumed to be an increasing convex function of the probability of exiting recession. More formally, we assume that $p \in [\underline{p}, \bar{p}] \subseteq [0,1]$, $X(\underline{p}) = 0$, $X'(p) > 0$ and $X''(p) > 0$. In normal times, $X = 0$.

In a frictionless complete market economy, the country would obtain full consumption insurance. As we show in Section 4.1, if $\beta R = 1$ consumption is constant throughout, and effort is constant until the recession ends.

3 Competitive equilibrium

In this section, we characterize the laissez-faire equilibrium. The only asset is a one-period bond, b. This is a claim to one unit of next-period consumption good, which sells today at the price $Q(b,w)$. The bonds are purchased by a representative foreign creditor assumed to be risk neutral and to have access to international risk-free assets paying the world interest rate R. After issuing debt, the country decides its reform effort.

The key assumptions are that (i) the country cannot commit to repay its sovereign debt, and (ii) the reform effort is exerted after the debt is issued and is not contractible. At the beginning of each period, the government decides whether to repay the debt that reaches maturity or to announce default on all its debt. Default is subject to a stochastic cost, denoted by ϕ, capturing in a reduced form a variety of shocks including both taste shocks (e.g., the sentiments of the public opinion about defaulting on foreign debt) and institutional shocks (e.g., the election of a new prime minister, a new central bank governor taking office, the attitude of foreign governments, etc.).[5] ϕ is publicly observed. If a country defaults, no debt is reimbursed.[6]

After observing the realization of ϕ, creditors can make a take-it-or-leave-it renegotiation offer. By accepting the renegotiation offer, the government averts the default cost. In equilibrium, a haircut is offered only if the default threat is credible, i.e., if the realization of ϕ is sufficiently low to make the country prefer default to full repayment. When they offer renegotiation, creditors make the debtor indifferent between an outright default and the proposed haircut.

More formally, the timing is as follows: The government enters the period with the pledged debt b, then observes the realization of w and ϕ, and then decides whether to announce default. If a threat is on the table, the creditors offer a haircut. Next, the country decides whether to accept or decline the offer. Then, the government issues new debt and consumption is realized. Finally, the government decides its reform effort.

After repaying in part or in full the outstanding debt, the government sets the new debt level, b', subject to the following period budget constraint:

$$Q(b', w) \times b' = b + c - w. \tag{1}$$

If the country could commit, it would sell bonds at the price $Q(b,w) = 1/R$. However, due to the risk of default or renegotiation, it must generally sell at a discount, $Q(b,w) \leq 1/R$.

[5] Alternatively, ϕ could be given a politico-economic interpretation, as reflecting special interests of the groups in power. For instance, the government may care about the cost of default to its constituency rather than to the population at large. In the welfare analysis, we stick to the interpretation of a benevolent government and abstract from politico-economic factors, although the model could be extended in this direction.

[6] For simplicity, we assume that ϕ captures all costs associated with default. In an earlier version of this paper, we assumed that the government cannot issue new debt in the default period, but can start issuing bonds already in the following period. The results are unchanged. One could even consider richer post-default dynamics, such as prolonged or stochastic exclusion from debt markets. Since outright default does not occur in equilibrium, the details of the post-default dynamics are immaterial.

We restrict attention to Markov-perfect equilibria where the set of equilibrium functions only depend on the pay-off relevant state variables, b and ϕ. This rules out, among other things, history-dependent equilibria where the sovereign foregoes the opportunity to renegotiate debt even in the presence of a low ϕ in order to establish a reputation.

Definition 1 *A Markov-perfect equilibrium (MPE) is a set of value functions $\{V, W\}$, an equilibrium debt price function Q, a set of optimal decision rules $\{\digamma, B, C, \Psi\}$, and an equilibrium law of motion of debt such that, conditional on the state vector $(b, \phi, w) \in \left(\left[\underline{b}, \bar{b}\right] \times [\phi_{\min}, \phi_{\max}] \times \{\underline{w}, \bar{w}\}\right)$, the sovereign and the international creditors maximize utility, and markets clear. More formally:*

- *The value function V satisfies*

$$V(b, \phi, w) = \max\{W(b, w), W(0, w) - \phi\}, \tag{2}$$

where $W(b, w)$ is the value function conditional on the debt level b being honored,

$$W(b, w) = u(Q(B(b, w), w) \times B(b, w) + w - b) + Z(B(b, w), w),$$

having defined

$$\begin{aligned} Z(B(b, \underline{w}), \underline{w}) &= -X(\Psi(B(b, \underline{w}))) + \beta \left(\begin{array}{c} \Psi(B(b, \underline{w})) \times E\left[V\left(B(b, \underline{w}), \phi', \bar{w}\right)\right] \\ +(1 - \Psi(B(b, \underline{w}))) \times E\left[V\left(B(b, \underline{w}), \phi', \underline{w}\right)\right] \end{array} \right) & (3) \\ Z(B(b, \bar{w}), \bar{w}) &= \beta E\left[V\left(B(b, \bar{w}), \phi', \bar{w}\right)\right] & (4) \end{aligned}$$

and $E\left[V\left(x, \phi', w\right)\right] = \int_{\aleph} V(x, \phi, w)\, dF(\phi)$.

- *The debt price function satisfies the following arbitrage conditions:*

$$Q(b, \bar{w}) = \hat{Q}(b, \bar{w}) \tag{5}$$

$$Q(b, \underline{w}) = \Psi(b) \times \hat{Q}(b, \bar{w}) + [1 - \Psi(b)] \times \hat{Q}(b, \underline{w}), \tag{6}$$

where

$$\hat{Q}(b, w) = \frac{1}{R}\left((1 - F(\Phi(b, w))) + \frac{1}{b}\int_0^{\Phi(b,w)} \left(\hat{b}(\phi, w) \times f(\phi)\ d\phi\right)\right) \tag{7}$$

is the price of the debt conditional on the realization of the state w, $\hat{b}(\phi, w)$ is such that $W\left(\hat{b}(\phi, w), w\right) = W(0, w) - \phi$, and $\Phi(b, w) = W(0, w) - W(b, w)$.

- *The set of optimal decision rules comprises:*

  1. *A take-it-or-leave-it debt renegotiation offer:*

$$\digamma(b, \phi, w) = \begin{cases} \hat{b}(\phi, w) & \text{if} \quad \phi \leq \Phi(b, w), \\ b & \qquad \phi > \Phi(b, w). \end{cases} \tag{8}$$

  2. *An optimal debt accumulation and an associated consumption decision rule:*

$$B(\digamma(b, \phi, w), w) = \arg\max_{b'} \left\{u\left(Q\left(b', w\right) \times b' + w - \digamma(b, \phi, w)\right) + Z\left(b', w\right)\right\}, \tag{9}$$

$$C(\digamma(b, \phi, w), w) = Q(B(\digamma(b, \phi, w), w), w) \times B(\digamma(b, \phi, w), w) + w - \digamma(b, \phi, w). \tag{10}$$

3. An optimal effort decision rule:

$$\Psi\left(B\left(\digamma\left(b,\phi,w\right),\underline{w}\right)\right)=\arg\max_{p}\left\{-X\left(p\right)+\beta\left(\begin{array}{c}p\times E\left[V\left(B\left(\digamma\left(b,\phi,w\right),\underline{w}\right),\phi',\bar{w}\right)\right]\\+\left(1-p\right)\times E\left[V\left(B\left(\digamma\left(b,\phi,w\right),\underline{w}\right),\phi',\underline{w}\right)\right]\end{array}\right)\right\} \tag{11}$$

- *The equilibrium law of motion of debt is* $b'=B\left(\digamma\left(b,\phi,w\right),w\right)$.

V and W denote the value functions of the benevolent government. Equation (2) states that when ϕ is large there is no renegotiation, whereas for a range of low ϕ's the government renegotiates its debt. Since, *ex-post*, creditors have all the bargaining power, the present value utility accruing to the sovereign equals the value that she would get under outright default. A standard application of the envelope theorem ensures that $W\left(b,w\right)$ is decreasing in b. In particular, $\frac{\partial}{\partial b}W\left(b,w\right)=-u'\left(C\left(\digamma\left(b,\phi,w\right),w\right)\right)<0$. The value function $V\left(b,\phi,w\right)$ inherits the properties of $W\left(b,w\right)$. More formally, conditional on ϕ, we have:

$$V\left(b,\phi,w\right)=\begin{cases}W\left(b,w\right) & \text{if} \quad b\leq\hat{b}\left(\phi,w\right),\\ W\left(0,w\right)-\phi & \quad\quad b>\hat{b}\left(\phi,w\right).\end{cases}$$

$V\left(b,\phi,w\right)$ is decreasing in b for $b\leq\hat{b}\left(\phi,w\right)$, and constant in b thereafter. Conversely, conditional on b, $V\left(b,\phi,w\right)$ is decreasing in ϕ for $\phi\leq\Phi\left(b,w\right)$ and constant in ϕ thereafter.

Consider, next, the equilibrium debt price function. $Q\left(b,w\right)$ and $\hat{Q}\left(b,w\right)$ denote, respectively, the price of debt before and after the realization of the state w is observed.[7] Since creditors are risk neutral, the expected rate of return on the sovereign debt equals the risk-free rate of return. Suppose the realization of the aggregate state is known. Then, the arbitrage condition (7) ensures market clearing in the bond market and pins down the equilibrium bond price. With probability $1-F\left(\Phi\left(b,w\right)\right)$ debt is honored, where $\Phi\left(b,w\right)$ denotes the threshold default shock realization such that, conditional on the debt b, the government cannot credibly threaten to default for all $\phi\geq\Phi\left(b,w\right)$. With the complement probability, debt is renegotiated at the level $\hat{b}\left(\phi,w\right)$, which denotes the renegotiated debt level that keeps the government indifferent between accepting the creditors' offer and defaulting. In the rest of the paper, it is convenient to use the more compact notation $EV\left(b,w\right)\equiv E\left[V\left(b,\phi,w\right)\right]$ and $EV\left(b',w\right)\equiv E\left[V\left(b',\phi',w\right)\right]$.

Consider, finally, the set of decision rules. (8) stipulates that creditors always extract the entire surplus at the renegotiation stage. Equations (9)-(10) yield the optimal consumption-saving decisions subject to a resource constraint. Equation (11) yields the optimal effort decision. Note that the effort choice depends on b', reflecting the assumption that effort is determined after the new debt effort is issued.

The following Lemma establishes a relationship between the equilibrium functions $\hat{b}$ and Φ:

Lemma 1 *$\hat{b}\left(\Phi\left(b,w\right),w\right)=b$. Hence, $\hat{b}\left(\phi,\bar{w}\right)=\bar{\Phi}^{-1}\left(\phi\right)$ and $\hat{b}\left(\phi,\underline{w}\right)=\underline{\Phi}^{-1}\left(\phi\right)$, where $\bar{\Phi}\left(b\right)\equiv\Phi\left(b,\bar{w}\right)$, and $\underline{\Phi}\left(b\right)\equiv\Phi\left(b,\underline{w}\right)$. Moreover, the functions Φ and $\hat{b}$ are monotone increasing in b and ϕ, respectively.*

[7] Note that in the function $Q\left(b,w\right)$ w is the state in the period in which debt is issued. In contrast, in the function $Q\left(b',w\right)$ w is the state in the period in which debt reaches maturity. When $w=\bar{w}$, next-period state is known to be $\bar{w}$, and thus $Q\left(b',\bar{w}\right)=\hat{Q}\left(b',\bar{w}\right)$. Instead, when $w=\underline{w}$, next period state is uncertain, and thus $Q\left(b',\underline{w}\right)=\hat{Q}\left(b',\bar{w}\right)$ with probability $\Psi\left(b\right)$, and $Q\left(b',\underline{w}\right)=\hat{Q}\left(b',\underline{w}\right)$ with probability $1-\Psi\left(b\right)$.

The Lemma follows from the definitions of $\hat{b}$ and Φ : recall that $\hat{b}(\phi, w)$ is the debt level that, conditional on ϕ, makes the debtor indifferent between honoring and defaulting. In turn, $\Phi(b, w)$ is the realization of ϕ that, conditional on b, makes the debtor indifferent between honoring and defaulting.

3.1 Equilibrium in normal times

In this section, we characterize the equilibrium when the economy is in normal times ($w = \bar{w}$), in which case there is no aggregate uncertainty. We start by claiming that the value function $V(b, \phi, \bar{w})$ is concave, non-increasing in b, and continuously differentiable in b.

Claim 1 *$V(b, \phi, \bar{w})$ is concave, non-increasing in b, and continuously differentiable in b*

Next, we establish a useful property of the debt revenue function.

Lemma 2 *The debt revenue function, $Q(b, \bar{w}) \times b$, is continuously differentiable with derivative being given by $\frac{d}{db}(Q(b, \bar{w}) \times b) = \frac{1}{R}(1 - F(\Phi(b, \bar{w})))$.*

Intuitively, if there were no risk of repudiation, the marginal revenue from issuing debt would be $1/R$. However, the default risk reduces the revenue by a factor $(1 - F(\Phi(b, \bar{w})))$. An immediate implication of the Lemma is that if we define $\bar{b}$ to be the lowest debt inducing renegotiation almost surely (i.e., such that $\lim_{b \to \bar{b}} F(\bar{\Phi}(b)) = 1$), then, $\bar{b}$ is also the top of the Laffer curve, i.e., the endogenous debt limit. More formally, $\bar{b} = \arg\max_b \{Q(b, \bar{w}) b\}$. Note that Lemmas 1 and 2 imply that the revenue function, $Q(b, w) \times b$, is decreasing and concave in b for all $b \leq \bar{b}$.

We can now move to the consumption decisions and the associated debt dynamics. We introduce a definition that will be useful throughout the paper.

Definition 2 *A Conditional Euler Equation (CEE) is the equation describing the (expected) ratio of the marginal utility of consumption in all states of nature such that ϕ' induces the government to honor its debt.*

Next, we characterize the CEE. The sovereign government solves the following consumption-saving problem.

$$B(\digamma(b, \phi, \bar{w}), \bar{w}) = \arg\max_{b'} \left\{ u\left(Q\left(b', \bar{w}\right) \times b' + \bar{w} - \digamma(b, \phi, \bar{w})\right) + \beta E V\left(b', \bar{w}\right) \right\}. \tag{12}$$

Proposition 1 *Suppose that the government issues the debt level $b' = B(\digamma(b, \phi, \bar{w}), \bar{w})$. Then, if the realization of ϕ' induces no renegotiation, the following CEE holds true:*

$$\beta R \frac{u'(c'|_{H,\bar{w}})}{u'(c)} = 1, \tag{13}$$

where $c = C(\digamma(b, \phi, \bar{w}), \bar{w})$ is current consumption and $c'|_{H,\bar{w}} = C(b', \bar{w}) = C(B(\digamma(b, \phi, \bar{w}), \bar{w}), \bar{w})$ is next-period consumption conditional on no renegotiation.

The CEE (13) states that, in all states in which debt is fully honored, the marginal rate of substitution equals βR. Although the CEE resembles a standard Euler equation under full commitment, the similarity is deceiving: R is *not* the ex-post interest rate when debt is fully honored; the realized interest rate is in fact higher due to the default premium. Note that, when debt is renegotiated, consumption increases discretely, hence $u'(c_t)/u'(c_{t+1}) > \beta R$. This is not surprising, since the country benefits from a reduction in the repayment to creditors.

Henceforth, we simplify the analysis by assuming that $\beta R = 1$.[8] In this case, consumption and debt are respectively increasing and decreasing step functions of debt: they remain constant in every period in which the country honors its debt, while changing discretely upon every episode of renegotiation. Conversely, debt is a decreasing step function. Figure 1 illustrates a simulation of the consumption and debt dynamics. Note that the sequence of renegotiations eventually brings the debt to a sufficiently low level where the risk of renegotiation vanishes.[9] This consumption path is different from the first best where, as we shall see, consumption and debt are constant for ever. Interestingly, in the long run, consumption is higher the market equilibrium with risk of repudiation than in the first best.

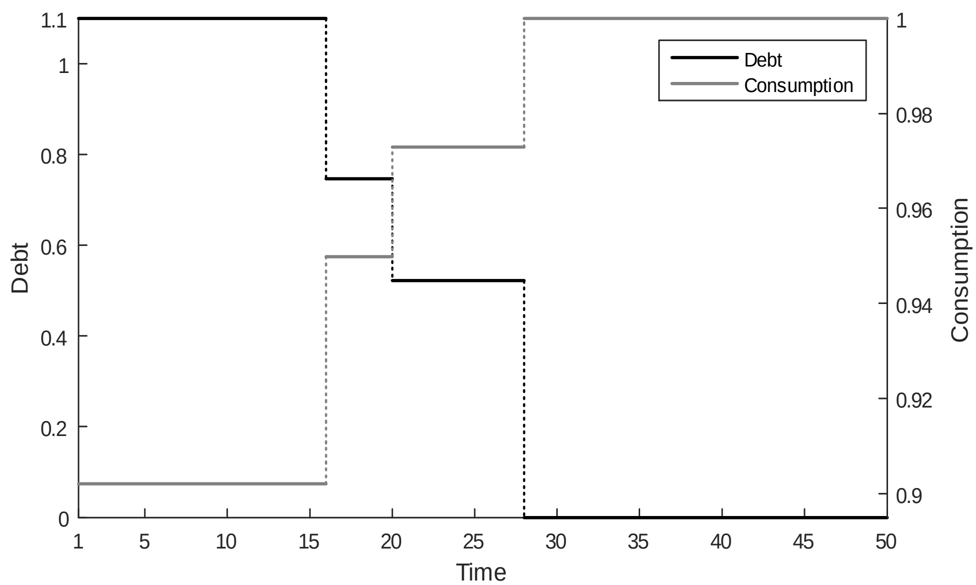

Figure 1: Simulation of debt and consumption for a particular sequence of ϕ's during normal times.

The prediction that whenever debt is renegotiated consumption increases permanently is extreme, and hinges on the assumption that ϕ is i.i.d. with a known distribution. In Section 6 we extend the

[8] Note that the value function $W(b,\bar{w})$ is concave in b :

$$\begin{aligned}
\frac{\partial^2}{\partial b^2} W(b,\bar{w}) &= u''\left(Q\left(B(b,\bar{w}),\bar{w}\right) \times B(b,\bar{w}) + \bar{w} - b\right) \times \frac{\partial}{\partial b}\left(Q\left(B(b,\bar{w}),\bar{w}\right) \times B(b,\bar{w})\right), \\
&= u''\left(C\left(F(b,\phi,w),w\right)\right) \times \frac{1}{R}\left(1 - F\left(\bar{\Phi}\left(B(b,w)\right)\right)\right) \times b < 0.
\end{aligned}$$

[9] Generalizing the results to the case of $\beta R \neq 1$ is straightforward under the assumption that utility features constant relative risk aversion. If $\beta R < 1$. the economy would accumulates debt even under full commitment. Thus, debt would increase and consumption would fall in periods in which debt is honored. After each round of renegotiation the economy would be pushed back into the range of debt where the default risk is positive. In a world comprising economies with different β, e.g., some with $\beta R = 1$ and some with $\beta R < 1$, economies with low β (e.g., due to a shorter-sighted political process) would experience recurrent debt crises.

model to a setting where there is uncertainty about the true distribution of ϕ and the market learns about this distribution by observing the sequence of ϕ's. In this case, a low realization of ϕ has two opposing effects on consumption: on the one hand, a low ϕ triggers debt renegotiation which on its own would increase consumption; on the other hand, a low ϕ affects the beliefs about the distribution of ϕ, inducing the market to regard the country as less creditworthy (namely, the country draws from a distribution where low ϕ is more likely). This tends on its own to increase the default premium on bonds and to lower consumption.

3.2 Equilibrium under recession

When the economy is in recession the government chooses, sequentially, whether to honor the current debt, how much new debt to issue, and how much reform effort to exert. In this section, we assume that the government cannot issue state-contingent debt, i.e., securities whose payment is contingent on the aggregate state of the economy. In Section 5.1 below we relax this restriction.

The analysis of the MPE is more complex when the future productivity is uncertain. To make progress towards an analytical characterization, we introduce an educated guess about a property of the equilibrium consumption function.

Guess 1 ( ) *$C(b, \underline{w}) < C(b, \bar{w})$ for all $b \leq \bar{b}$.*

That $C(b, \underline{w}) < C(b, \bar{w})$ is an intuitive property of the MPE allocation: *ceteris paribus*, consumption is higher in normal times than in a recession. Unfortunately, deriving a general proof of this property is difficult since the decision rule of consumption and effort and debt price function are determined simultaneously, rendering the problem involved. However, one can verify numerically the guess in equilibrium − we have considered a broad range of cases, involving different parameters and different distributions $f(\phi)$ and never found any violation. We will maintain this guess throughout the rest of the paper. All formal statements are subject to the *caveat* that the guess must be verified numerically.

We start by extending the properties of the value function to economies starting in recession.

Claim 2 *The value function $V(b, \phi, \underline{w})$ is concave, non-increasing in b, and continuously differentiable.*

Another general property of the MPE is that, conditional on the debt level, renegotiation is more likely in recession than in normal times. More formally, for all $b < \bar{b}$, $\underline{\Phi}(b) > \bar{\Phi}(b)$.[10] This property is standard in the theoretical literature, and is in line with the empirical evidence (see, e.g., Tomz and Wright 2007). An important corollary is that bonds become more valuable if the recession ends than if it continues, since the end of the recession reduces the probability of renegotiation. More formally, $\hat{Q}(b, \bar{w}) \geq \hat{Q}(b, \underline{w})$, where, recall, $\hat{Q}$ is the bond price conditional on the state being w in the beginning of the period when the debt reaches maturity. Thus, by equations (5)-(6), $Q(b, \bar{w}) \geq Q(b, \underline{w})$: the cost of issuing new debt is higher in recession than in normal times.

The property that $\underline{\Phi}(b) > \bar{\Phi}(b)$ also implies that one can partition the state space into three regions:

[10] To see this, note that $\frac{\partial}{\partial b}\Phi(b, w) = -\frac{\partial}{\partial b}W(b, w) = u'(C(b, w))$. Thus, $C(b, \underline{w}) < C(b, \bar{w}) \Leftrightarrow u'(C(b, \underline{w})) > u'(C(b, \bar{w})) \Leftrightarrow \underline{\Phi}'(b) > \bar{\Phi}'(b)$. Since $\underline{\Phi}(0) = \bar{\Phi}(0)$, $\underline{\Phi}'(b) > \bar{\Phi}'(b)$ implies that $\underline{\Phi}(b) > \bar{\Phi}(b)$ for all $b > 0$.

- in the low range, $b < b^-$, the country honors the debt with a positive probability, irrespective of the aggregate state (the probability of renegotiation being higher if the recession continues than if it ends);[11]

- in the intermediate range $b \in \left[b^-, \bar{b}\right)$, the country renegotiates with probability one if the recession continues, while it honors the debt with a positive probability if the recession ends;

- in the high range $b \geq \bar{b}$, the country renegotiates its debt with probability one, irrespective of the aggregate state.

Note that the risk of repudiation introduces some elements of state contingency, since debt is repaid with different probabilities under recession and normal times. This property is particularly stark when debt is renegotiated with probability one if the recession continues.

3.2.1 Reform effort in equilibrium

The reform effort is chosen after new debt is issued, and is assumed to be non-contractible. The equilibrium price of debt incorporates the rational expectations of lenders about the reform effort. We denote by $\Psi(b')$ the equilibrium policy function for effort, i.e., the probability that the recession ends next period, as a function of the newly-issued debt. More formally, the first-order condition from (11) yields:

$$X'\left(\Psi\left(b'\right)\right) = \beta\left[\int_0^\infty V\left(b', \phi', \bar{w}\right) dF(\phi) - \int_0^\infty V\left(b', \phi', \underline{w}\right) dF(\phi)\right]. \tag{14}$$

Intuitively, the government has an incentive to exert some reform effort because a recovery will increase its expected utility. The larger the difference in expected utility between being, next period, in recession and in normal times, the larger will the reform effort be in the current period. Although individually rational, the effort choice is not efficient. To see why, recall that the bond price increases upon economic recovery. Thus, the creditors reap part of the gain from the reform, whereas the country bears the full burden of the associated effort.

We can prove that effort is inefficiently provided with the aid of a simple one-period deviation argument. Consider an equilibrium effort choice path consistent with (14) – corresponding to the case of non-contractible effort. Next, suppose that, only in the initial period, the country can contract effort before issuing new debt. As it turns out, the country would choose a higher reform effort in the first period than in the equilibrium with non-contractible effort. We state this result as a lemma.

Lemma 3 *If $b' > 0$ and the borrower can, in the initial period, commit to an effort level upon issuing new debt, then the reform effort is strictly larger than in the case in which effort is never contractible.*

If the government could commit to reform, its reform effort would be monotone increasing in the debt level, since a high debt increases the hardship of a recession.[12] However, under moral hazard, the equilibrium reform effort exhibits a non-monotonic behavior. More precisely, $\Psi(b)$ is increasing at low levels of debt, and decreasing in a range of high debt levels, including the entire region $\left[b^-, \bar{b}\right]$. Proposition 2 establishes this result more formally.

Proposition 2 *There exist three ranges, $[0, b_1] \subseteq [0, b^-]$, $[b_2, \bar{b}] \supseteq [b^-, \bar{b}]$, and $\left[\bar{b}, \infty\right)$ such that:*

[11] b^- is implicitly determined by the equation $W^H\left(b^-, \underline{w}\right) = \inf_\phi W^D\left(\phi, \underline{w}\right)$.

[12] In Section 4.1 below, we show that even in the first best the reform effort would be monotone increasing in b.

1. *If* $b \in [0, b_1)$, $\Psi'(b) > 0$*;*
2. *If* $b \in \left(b_2, \bar{b}\right)$, $\Psi'(b) < 0$*;*
3. *If* $b \in \left[\bar{b}, \infty\right)$, $\Psi'(b) = 0$.

The following argument establishes the result. Consider a low debt range where the probability of renegotiation is zero. In this range, there is no moral hazard.[13] Thus, a higher debt level has a disciplining effect, i.e., it strengthens the incentive for economic reforms: due to the concavity of the utility function, the discounted gain of leaving the recession is an increasing function of debt. As one moves to a larger initial debt, however, moral hazard becomes more prominent, since the reform effort decreases the probability of default, and shifts some of the gains to the creditors. This is reminiscent of the *debt overhang* effect in Krugman (1988).

The debt overhang dominates over the disciplining effect in the region $[b^-, \bar{b}]$. In this range, if the economy remains in recession, debt is renegotiated for sure, so the continuation utility $\int_0^\infty V\left(b', \phi', \underline{w}\right) dF(\phi)$ is independent of b. In contrast, if the recession ends, the continuation utility is decreasing in b, so in this range debt is essentially state contingent. Thus, the value of reform effort necessarily declines in b. By continuity, the same argument extends to a range of debt below b^-. Finally, when $b > \bar{b}$, the economy renegotiates with probability one, and the gain from leaving the recession is independent of b. In a variety of numerical simulations, we have always found Ψ to be hump-shaped with a unique peak (see Figure 3), although we could not prove that hump-shapedness is a general property of the economy.

Note that the debt-overhang effect hinges on the presence of some renegotiation risk and an associated renegotiation premium on debt. If the borrower instead could commit to repay the debt, the price of debt would be $1/R$ regardless of the aggregate state, so an economic recovery would not yield any benefits to the lenders. Consequently, there would be no moral hazard in the effort choice and the effort function would be monotone increasing in debt.

3.2.2 Debt issuance and consumption dynamics

In this section we characterize the dynamics of consumption and debt. We proceed in two steps. First, we derive the properties of the CEE. Then we summarize its characterization in a formal proposition.

The government solves the following problem:

$$\begin{aligned} B\left(\digamma(b, \phi, \underline{w}), \underline{w}\right) &= \arg\max_{b'} \left\{u\left(Q\left(b', \underline{w}\right) \times b' + \underline{w} - \digamma(b, \phi, \underline{w})\right) - X\left(\Psi\left(b'\right)\right)\right. \\ &\quad \left. + \beta\Psi\left(b'\right) \times EV\left(b', \bar{w}\right) + \beta\left(1 - \Psi\left(b'\right)\right) \times EV\left(b', \underline{w}\right)\right\}. \end{aligned} \tag{15}$$

Using the first order condition together with the envelope condition (and recalling that $\beta R = 1$), yields the following CEE:

$$\begin{aligned} & E\left\{\frac{MU_{t+1}}{MU_t} | \text{debt is honored at } t+1\right\} \\ = \quad & \Psi(b_{t+1}) \frac{u'\left(c'|_{H,\bar{w}}\right)}{u(c)} + \left[1 - \Psi(b_{t+1})\right] \frac{u'\left(c'|_{H,\underline{w}}\right)}{u(c)} \\ = \quad & 1 + \frac{\Psi'(b_{t+1})}{\Pr(\text{debt is honored at } t+1)} R\left[\hat{Q}(b_{t+1}, \bar{w}) - \hat{Q}(b_{t+1}, \underline{w})\right] b_{t+1}, \end{aligned} \tag{16}$$

[13] If zero were in the support of the distribution of ϕ, the probability of renegotiation would be positive for all positive debt levels. However, a limit argument along the same lines applies as $b \to 0$ (see the proof of the proposition).

Equation (16) is the analogue of (13).[14] There are two differences. First, the expected ratio between the marginal utilities replaces the plain ratio between the marginal utilities, due to the uncertainty about the future aggregate state (recession or normal times). Second, there is a new term on the right-hand side capturing the effect of debt on reform effort.

For expositional purposes, it is useful to highlight first the properties of the case in which the probability that the recession ends is exogenous, so $\Psi'(.) = 0$. In this case, the CEE requires that the expected marginal utility be constant. For this to be true, it must be that $u'(c'|_{H,\bar{w}}) < u(c) < u'\left(c'|_{H,\underline{w}}\right)$, so consumption growth must be positive if the recession ends, and negative if the recession continues, namely,

$$c'|_{H,\bar{w}} > c > c'|_{H,\underline{w}}.$$

The lack of consumption insurance stems from the incompleteness of financial markets, and would disappear if the government could issue state-contingent bonds. However, this conclusion does not carry over to the economy with moral hazard, as we discuss in more detail in Section 5.1 below.

Consider, next, the general case. Moral hazard introduces a new strategic motive. By changing the level of newly-issued debt, the government manipulates its own *ex-post* incentive to make reforms. The sign of this strategic effect is ambiguous, and hinges on the sign of Ψ' (see Proposition 2). When the outstanding debt is low, $\Psi' > 0$. In this case, more debt strengthens the *ex-post* incentive to reform, thereby increasing the price of the newly-issued debt. The right-hand side of (16) is in this case larger than unity, and the CEE implies a lower consumption fall (hence, higher debt accumulation) than in the absence of moral hazard. In contrast, in the region of high initial debt, $\Psi' < 0$. In this case, there is lower debt accumulation than in the absence of moral hazard. The reason is that the market anticipates that a larger debt reduces the reform effort. In response, the government restrains its debt issuance strategically in order to mitigate the ensuing fall in the debt price. Thus, when the recession continues, a highly indebted country will suffer a deeper fall in consumption when the reform is endogenous than when the probability that the recession ends is exogenous.

We summarize the results in a formal proposition.[15]

Proposition 3 *If the economy starts in a recession, the following CEE holds true:*

$$\beta R \left(\begin{array}{c} \dfrac{\overbrace{\left[1-\Psi\left(b'\right)\right] \times \left(1-F\left(\underline{\Phi}\left(b'\right)\right)\right)}^{\text{prob. of repayment and continuing recession}}}{\underbrace{\Pr\left(H|b'\right)}_{\text{unconditional prob. of repayment}}} \times \underbrace{\dfrac{u'\left(c'|_{H,\underline{w}}\right)}{u(c)}}_{\text{MRS if cont. rec.}} \\ + \dfrac{\overbrace{\Psi\left(b'\right) \times \left(1-F\left(\bar{\Phi}\left(b'\right)\right)\right)}^{\text{prob. of repayment and end of recession}}}{\underbrace{\Pr\left(H|b'\right)}_{\text{unconditional prob. of repayment}}} \times \underbrace{\dfrac{u'\left(c'|_{H,\bar{w}}\right)}{u(c)}}_{\text{MRS if rec. ends}} \end{array} \right) = 1 + \frac{\Psi'\left(b'\right)}{\Pr\left(H|b'\right)} \times \underbrace{R\left(\hat{Q}\left(b',\bar{w}\right) - \hat{Q}\left(b',\underline{w}\right)\right) b'}_{\text{gain to lenders if recession ends}} \tag{17}$$

where $c = C\left(\digamma\left(b,\phi,\underline{w}\right),\underline{w}\right)$ *is current consumption and* $c'|_{H,\underline{w}} = C\left(b',\underline{w}\right) = C\left(B\left(\digamma\left(b,\phi,\underline{w}\right),\underline{w}\right),\underline{w}\right)$ *is next-period consumption conditional on no renegotiation, and* $\Pr\left(H|b'\right)$ *is the unconditional proba-*

[14] In particular, we use the notation $c = C\left(\digamma\left(b,\phi,\underline{w}\right),\underline{w}\right)$ denotes current consumption and $c'|_{H,\underline{w}} = C\left(b',\underline{w}\right) = C\left(B\left(\digamma\left(b,\phi,\underline{w}\right),\underline{w}\right),\underline{w}\right)$.

[15] Following Definition 2, the CEE here describes the expected ratio of the marginal utility of consumption in all states of nature such that ϕ' induces the government to honor its debt. Note that this set of realizations depends on the aggregate state.

bility that the debt b' be honored,

$$\Pr\left(H|b'\right)=\left[1-\Psi\left(b'\right)\right]\times\left(1-F\left(\bar{\Phi}\left(b'\right)\right)\right)+\Psi\left(b'\right)\times\left(1-F\left(\underline{\Phi}\left(b'\right)\right)\right).$$

We end this section by establishing that the top of the Laffer curve of debt is lower in recession than during normal times.

Lemma 4 *Let $\bar{b}=\arg\max_b\{Q(b,\bar{w})\,b\}$ and $\bar{b}^R=\arg\max_b\{Q(b,\underline{w})\,b\}$ denote the top of the Laffer curve during normal times and recession, respectively. Then, $\bar{b}^R\leq\bar{b}$. In particular, if the probability of staying in a recession is exogenous (i.e., $\Psi(b)=p$), then $\bar{b}^R=\bar{b}$, otherwise, $\bar{b}^R<\bar{b}$.*

The reason why the top of the Laffer curve under recession is located strictly to the left of $\bar{b}$ when effort is endogenous is that the reform effort is decreasing in debt (i.e., $\Psi'<0$) for b close to $\bar{b}$, as established in Proposition 2. This implies that in a neighborhood to the left of $\bar{b}$ bond revenue is strictly decreasing in b when effort is endogenous. This property illustrates the strategic use of debt: By reducing the newly-issued debt, the borrower can increase the subsequent reform effort, which in turn increases the current bond price and debt revenue.

3.2.3 Taking stock

The previous sections have established the main properties of the competitive equilibrium. The first property is that moral hazard induces an inefficient provision of reform effort in equilibrium, especially for high debt levels. Figure 2 shows the effort function, $\Psi(b)$, in a calibrated economy. Note in particular that the reform effort plunges for high debt levels. The second property is that the possibility of renegotiating a non-state-contingent debt may improve risk sharing, especially when debt is large. In particular, in the high-debt range $[b^-,\bar{b}]$, issuing renegotiable non-state-contingent debt goes in the direction of issuing state-contingent debt paying a higher return if the recession ends than if it continues. Risk sharing would *per se* be welfare-enhancing. However, it exacerbates the moral hazard in reform effort.

The third property is that in periods when debt is fully honored, the equilibrium features positive debt accumulation if the economy remains in recession, and constant debt when the economy returns to normal times. An implication of the first and third property is that, as the recession persists, the reform effort initially increases, but then, for high debt levels, declines over time. Figure 3 shows a time path for debt and consumption (left panel) and of the corresponding reform effort (right panel) for a particular simulated sequence of ϕ's. The recession ends at time $T=10$.

The fourth property concerns post-renegotiation dynamics. Consumption may increase after a sufficiently large haircut, even though the recession does not end. However, in this case debt accumulation resumes immediately after the haircut. This prediction is consistent with the debt dynamics of Greece after the 2011 haircut discussed in the introduction. Interestingly, a large haircut may in some cases increase the reform effort, contrary to the common view that pardoning debt has perverse effects on incentives.

For simplicity we assume that the government can only issue one-period non-state-contingent debt. Issuing debt at multiple maturities could in principle allow the borrower to obtain some additional insurance. In a world without moral hazard, this could complete the markets (cf. Angeletos 2002). However, as we show in Section 5.1 below, in our model even an economy with a full set of state-contingent assets would fail to attain efficiency due to the moral hazard problem associated with

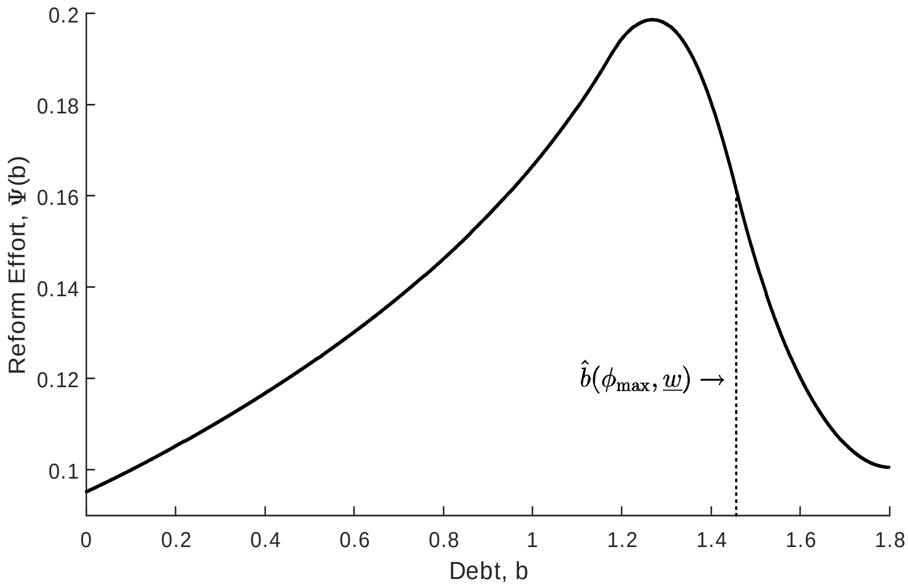

Figure 2: Reform effort function $\Psi(b)$. The parameter values correspond to those of the calibrated economy of Section 7.

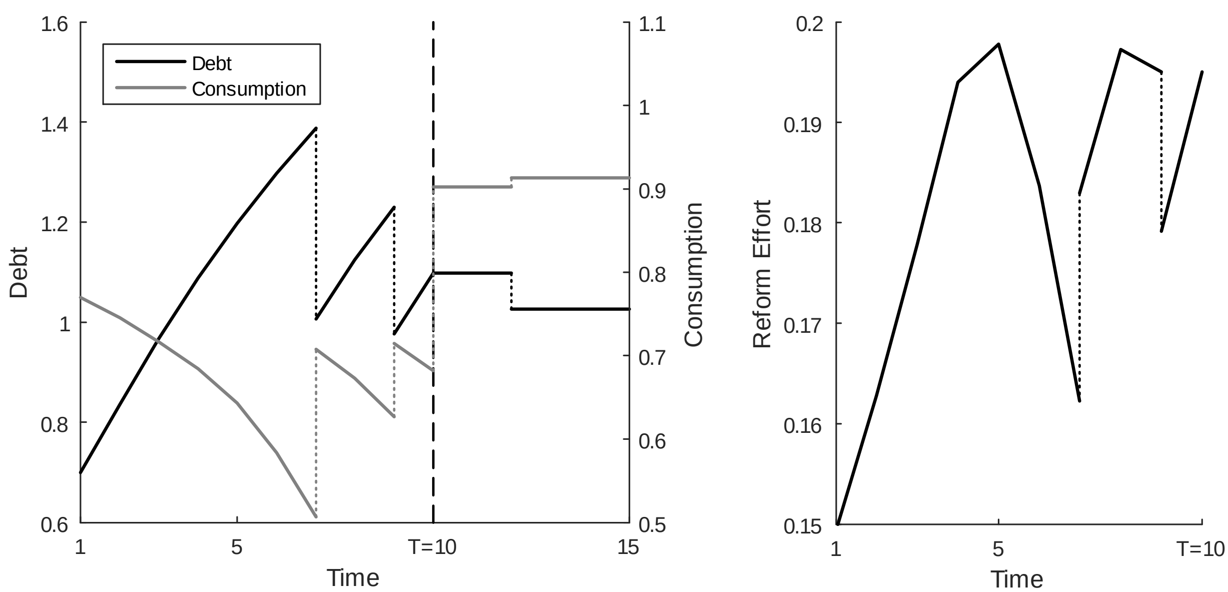

Figure 3: Simulation of debt, consumption and effort for a particular sequence of ϕ's in the competitive equilibrium. Here, the recession ends at time $T = 10$.

structural reforms. Thus, restricting the market structure to one-period debt is not an important restriction.[16]

It is interesting to compare two economies entering a surprise recession with different debt levels. Initially, both economies experience a falling consumption and a growing debt. However, the low-debt country may stay (at least temporarily) in the region where debt is repaid with probability one. Then, in the high-debt country, the effect of the recession is aggravated by a soaring interest rate, while this does not happen in the low-debt country. Consequently, unless there is renegotiation, consumption falls faster in the country with a high initial debt. This is consistent with the observation that the European debt crisis has hit consumption especially hard in countries which entered the recession with an already high debt.

4 Efficiency

In this section, we study the efficient allocation and compare it with the competitive equilibrium. We start by characterizing the first-best allocation. Then, we characterize the constrained efficient allocation in an environment where the planner cannot overrule the limited commitment constraint. This is a useful benchmark, since in reality international agencies (e.g., the IMF) can observe and possibly monitor countries' reforms but have limited instruments to prevent sovereign debt renegotiation.

4.1 First Best

The first best entails perfect insurance: the country enjoys a constant stream of consumption and exerts a constant reform effort during recession. For comparison with the constrained efficient allocation studied below, it is useful to write the problem in terms of a dynamic principle-agent framework. To this aim, let ν^{FB} denote the discounted utility that the planner is committed to deliver to the country (i.e., the "promised utility") and let p^{FB} denote the probability that the recession ends. The superscript FB refers to "first best." Then:

$$\nu^{FB} = \frac{u\left(c^{FB}\right)}{1-\beta} - \frac{1}{1-\beta\left(1-p^{FB}\right)} X\left(p^{FB}\right). \tag{18}$$

The planner maximizes the principal's profit,

$$\underline{P}^{FB} = \underline{w} - c + \beta\left(1-p\right)\underline{P}^{FB} + \beta p \bar{P}^{FB}, \tag{19}$$

subject to the promise-keeping constraint that $\nu \geq \nu^{FB}$. Here, $\bar{P}^{FB}(\nu)$ and $\underline{P}^{FB}(\nu)$ denote the expected present value of profits accruing to the (risk-neutral) principal in normal times and recession, respectively, conditional on delivering the promised utility ν in the most efficient way. In normal times, $\bar{P}^{FB} = \left(\bar{w} - c^{FB}\right)/\left(1-\beta\right)$. Writing the Lagrangian and applying standard methods yields the following lemma.

[16] We conjecture that if the borrower could issue debt at multiple maturities, it would only issue one-period debt in steady state in order to limit the extent of the moral hazard issue. Aguiar and Amador (2013) reach a similar conclusion in a different model. From an empirical standpoint, Broner, Lorenzoni, and Schmukler (2013) document that in emerging markets governments issue mostly short term debt.

Lemma 5 *Consider an economy starting in recession. The optimal contract (first best) satisfies the following trade-off between consumption and reform effort:*

$$\frac{\beta}{1-\beta\left(1-p^{FB}\right)}\left(\underbrace{\left(\bar{w}-\underline{w}\right)\times u'\left(c^{FB}\right)}_{\text{increase in profits if econ. recovers}}+\underbrace{X\left(p^{FB}\right)}_{\text{saved effort cost if econ. recovers}}\right)=X'\left(p^{FB}\right). \tag{20}$$

$X'\left(p^{FB}\right)$ is the marginal cost of increasing the probability of recovery. The marginal benefit (left-hand side) comprises two terms. The first term is the discounted value of the extra profit accruing to the principal if the recession ends, expressed in units of consumers' utils. The second term is the discounted gain accruing to the agent from dispensing with the reform effort. Perfect insurance implies that no consumption gain accrues to consumers when the recession ends.

Combining (20) and (18) yields the complete characterization of the first best. After rearranging terms, one obtains:

$$X'\left(p^{FB}\right)\left(\frac{1-\beta}{\beta}+p^{FB}\right)-X\left(p^{FB}\right) = \left(\bar{w}-\underline{w}\right)\times u'\left(c^{FB}\right) \tag{21}$$

$$\frac{1}{1-\beta\left(1-p^{FB}\right)}X\left(p^{FB}\right) = \frac{u\left(c^{FB}\right)}{1-\beta}-\nu^{FB}. \tag{22}$$

Equation (21) defines a negatively sloped locus in the plane $\left(p^{FB},c^{FB}\right)$, while equation (22) defines a positively sloped locus in the same plane. Under appropriate conditions, the two equations pin down a unique interior solution for p and c (otherwise, the optimal effort is zero). The comparative statics with respect to ν^{FB} is especially interesting. An increase in ν^{FB} yields an increase in c^{FB} and a reduction in p^{FB}, i.e., more consumption and less effort.

Note that ν^{FB} can be mapped into an initial debt level: a highly indebted country has a low ν^{FB} and, hence, a low consumption and a high reform effort. This finding contrasts with the competitive equilibrium where the relationship between debt and reform effort is hump-shaped.

4.2 Constrained Pareto optimum

In this section, we characterize the optimal dynamic contract, subject to limited commitment: the country can quit the contract, suffer the default cost, and resort to market financing. The problem is formulated as a one-sided commitment with lack of enforcement, following Ljungqvist and Sargent (2012) and based on a promised-utility approach in the vein of Spear and Srivastava (1987), Thomas and Worrall (1988) and Kocherlakota (1996).[17]

Here, ν denotes the promised utility to the risk-averse agent in the beginning of the period, before the realization of ϕ. ν is the key state variable of the problem. We denote by $\bar{\omega}_{\phi}$ and $\underline{\omega}_{\phi}$ the promised continuation utilities conditional on the realization ϕ and on the aggregate state $\bar{w}$ and $\underline{w}$, respectively.[18] $\underline{P}(\nu)$ and $\bar{P}(\nu)$ denote the expected present value of profits accruing to the principal conditional on delivering the promised utility ν in the most cost-effective way in recession and in normal times, respectively. The planning problem is evaluated after the uncertainty about the aggregate state has been resolved (i.e., the economy is either in recession or in normal times in the current period), but before the realization of ϕ is known.

[17] Adding a participation constraint for the planner would not affect the solution, since such a constraint is never binding. Thus, the problem can be interpreted also as a two-sided limited commitment program.

[18] The states of nature in this problem correspond to the realizations of ϕ in the decentralized equilibrium.

4.2.1 Constrained efficiency in normal times

In normal times, the optimal value $\bar{P}(v)$ satisfies the following functional equation:

$$\bar{P}(v) = \max_{\left\{\omega_{\phi},\bar{w},c_{\phi}\right\}_{\phi\in\aleph}} \int_{\aleph} \left[\bar{w} - c_{\phi} + \beta\bar{P}(\bar{\omega}_{\phi})\right] dF(\phi), \tag{23}$$

where the maximization is subject to the constraints

$$\int_{\aleph} \left[u(c_{\phi}) + \beta\bar{\omega}_{\phi}\right] dF(\phi) \quad \geq \quad v, \tag{24}$$

$$u(c_{\phi}) + \beta\bar{\omega}_{\phi} \quad \geq \quad \bar{v} - \phi, \tag{25}$$

where $\bar{v}$ is the value of "autarky" for the agent ($\bar{\nu} = W^{H}(0,\bar{w})$). The former is a promise-keeping constraint, whereas the latter is a participation constraint (PC). In addition, the problem must satisfy the constraints that $0 \leq c_{\phi} \leq \bar{w}$ and $\bar{\omega}_{\phi} \leq \bar{v}$. The problem has standard properties: the constraint set is convex, while the one-period return function in (23) is concave. In the online appendix, we prove that the profit function $\bar{P}(v)$ (and its analogue under recession, $\underline{P}(v)$) is decreasing, strictly concave and twice differentiable. The application of recursive methods allows us to establish the following proposition.

Proposition 4 *Assume the economy is in normal times. (I) For all states s such that the PC of the agent, (25), is binding, $\bar{\omega}_{\phi} > \nu$ and the solution for $(c_{\phi}, \bar{\omega}_{\phi})$ is determined by the following conditions:*

$$u'(c_{\phi}) = -\frac{1}{\bar{P}'(\bar{\omega}_{\phi})}, \tag{26}$$

$$u(c_{\phi}) + \beta\bar{\omega}_{\phi} = \bar{v} - \phi. \tag{27}$$

The solution is not history-dependent, i.e., the initial promise, v, does not matter. (II) For all realizations ϕ such that the PC of the agent, (25), is not binding, $\bar{\omega}_{\phi} = \nu$ and $c_{\phi} = c(\nu)$. The solution is history-dependent.

The efficient allocation has standard properties. Whenever the agent's PC is not binding, consumption and promised utility remain constant over time. Whenever the PC binds, the planner increases the agent's consumption and promised utility in order to meet her PC.

In normal times, the constrained efficient allocation of Proposition 4 is identical to the competitive equilibrium. To establish this result we return, first, to the competitive equilibrium. Let

$$\bar{\Pi}(b) = \left(1 - F\left(\bar{\Phi}(b)\right)\right) b + \int_{0}^{\bar{\Phi}(b)} \hat{b}(\phi,\bar{w})\, dF(\phi) \tag{28}$$

denote the expected value for the creditors of an outstanding debt b before the current-period uncertainty is resolved. Note that $\bar{\Pi}(b)$ yields the expected debt repayment, which is lower than the face value of debt, since in some states of nature debt is renegotiated. Recall that $EV(b,\bar{w}) = \int_{\aleph} V(b,\phi,\bar{w})\, dF(\phi)$ denotes the discounted utility accruing to a country with the debt level b in the competitive equilibrium. To prove the equivalence, we postulate that $\bar{\Pi}(b) = \bar{P}(\nu)$, and show that in this case $v = EV(b,\bar{w})$. If the equilibrium were not constrained efficient, the planner could do better, and we would find that $v > EV(b,\bar{w})$.

Proposition 5 *Assume that the economy is in normal times. The competitive equilibrium is constrained Pareto efficient, namely,* $\bar{\Pi}(b) = \bar{P}(\nu) \Leftrightarrow v = EV(b, \bar{w})$.

Intuitively, renegotiation provides the market economy with sufficiently many state contingencies to attain second-best efficiency. This result hinges on two features of the renegotiation protocol. First, renegotiation averts any real loss associated with unordered default. Second, creditors have all the bargaining power in the renegotiation game.[19]

4.2.2 Constrained efficiency in recession

Next, we consider an economy in recession. The principal's profit obeys the following functional equation:[20]

$$\underline{P}(\nu) = \max_{\left\{\bar{\omega}_\phi, \underline{\omega}_\phi, c_\phi, p_\phi\right\}_{\phi\in\aleph}} \int_\aleph \left[\underline{w} - c_\phi + \beta\left((1-p_\phi)\underline{P}\left(\underline{\omega}_\phi\right) + p_\phi \bar{P}(\bar{\omega}_\phi)\right)\right] dF(\phi), \tag{29}$$

where the maximization is subject to the constraints

$$\begin{aligned} \int_\aleph \left(u(c_\phi) - X(p_\phi) + \beta\left((1-p_\phi)\underline{\omega}_\phi + p_\phi\bar{\omega}_\phi\right)\right) dF(\phi) &\geq v, & (30)\\ u(c_\phi) - X(p_s) + \beta\left((1-p_\phi)\underline{\omega}_\phi + p_\phi\bar{\omega}_\phi\right) &\geq \underline{\nu} - \phi, & (31)\end{aligned}$$

and where $\underline{\nu} = W^H(0, \underline{w})$ is the value for the agent of breaking the contract when the economy is in recession. Note that there are two separate promised utilities, $\underline{\omega}_\phi$ and $\bar{\omega}_\phi$, associated with the two possible realizations of the aggregate state in the next period. The following proposition can be established.

Proposition 6 *Assume the economy is in recession. (I) For all realizations ϕ such that the PC of the agent, (25), is binding, the optimal choice vector $\left(c_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right)$ satisfies the following conditions:*

$$\begin{aligned} u'(c_\phi) &= -\frac{1}{\underline{P}'\left(\underline{\omega}_\phi\right)}, & (32)\\ u(c_\phi) - X(p_\phi) + \beta\left((1-p_\phi)\underline{\omega}_\phi + p_\phi\bar{\omega}_\phi\right) &= \underline{\nu} - \phi, & (33)\\ \underline{P}'\left(\underline{\omega}_\phi\right) &= \underline{P}'(\bar{\omega}_\phi), & (34)\\ X'(p_\phi) &= \beta\left(u'(c_\phi)\left(\bar{P}(\bar{\omega}_\phi) - \underline{P}\left(\underline{\omega}_\phi\right)\right) + \left(\bar{\omega}_\phi - \underline{\omega}_\phi\right)\right). & (35)\end{aligned}$$

The solution is not history-dependent, i.e., the promised utility ν does not affect the solution.
(II) For all realizations ϕ such that the agent's PC, (25), is not binding, $\underline{\omega}_\phi = \nu$, $\bar{\omega}_\phi = \bar{\omega}(\nu)$, $c_\phi = \underline{c}(\nu)$ and $p_\phi = p(\nu)$. The solution is history-dependent. The reform effort is decreasing in the promised utility level. (III) For all $\phi \in \aleph$, $\bar{\omega}_\phi > \underline{\omega}_\phi$.

[19] Both assumptions may be violated in the real world. For instance, renegotiations may entail costs associated with legal proceeds and lawsuits, trade retaliation, temporary market exclusion, etc. These would affect the strong efficiency results in a fairly obvious way. Also, creditors may be unable to force the country to its reservation utility in the renegotiation stage. This may reduce the amount of loans creditors can recover, increasing the ex-ante risk premium. In this case, the competitive equilibrium would fail to implement the second best.

[20] In this section, we assume that the planner can control directly the agents' effort subject to a promise-keeping and a set of participation constraints (equations (30) and (31), respectively). In Section 5.3 we relax this assumption and assume, instead, that effort provision is subject, in addition, to an incentive compatibility constraint.

When the agent's PC is slack, consumption, reform effort, and promised utilities remain constant over time. Every time the PC binds, the planner increases the promised utilities, and grants the agent an increase in consumption and a reduction in the reform effort. Relative to the first best, the agent is offered lower consumption and required to exercise higher effort as she enters the contract. The conditions the agent faces improve over time thereafter. Note that, if we compare two countries entering the contract with different initial promised utilities, the country with a lower promised utility earns a lower consumption and is asked to exercise higher effort. Thus, the country with the lower promised utility (i.e., a higher initial debt) is expected to recover faster from the recession.

Consider, next, the period in which the recession ends (part III of Proposition 6). As the recession ends, the promised utility increases and effort goes to zero. Consumption may either remain constant or increase depending on whether the PC binds. Interestingly, the set of states such that the PC binds expands. Namely, there are realizations of ϕ such that consumption rises and effort falls only if the recession ends. In contrast, for sufficiently large ϕ's, the agent's PC is binding irrespective of whether the recession continues or ends. In this case, consumption remains constant. In other words, because of limited commitment, the agent is offered some partial, but not perfect, insurance against the continuation of the recession.

4.2.3 Comparison between constrained optimum and competitive equilibrium

The competitive equilibrium is not constrained Pareto efficient. In the competitive equilibrium, consumption falls over time during recession even when the country honors its debt. In contrast, the planner would insure the agent's consumption by keeping it constant. Therefore, the market under-provides insurance. The dynamics of the reform effort also are sharply different. In the constrained efficient allocation, effort is a monotone decreasing function of promised utility. Since promised utility is an increasing step function over time, effort is step-wise decreasing. In contrast, in the competitive equilibrium the reform effort is hump-shaped in debt. Since debt increases over time (unless it is renegotiated), effort is also hump-shaped over time conditional on no renegotiation.

Figure 4 displays the time path of consumption and effort (left panel) and of the corresponding promised utilities (right panel) for a particular sequence of ϕ's in the constrained efficient allocation. The dynamics contrast sharply with those of the competitive equilibrium in Figure 3.

5 Decentralization

In this section, we discuss policies and institutions that decentralize the constrained efficient allocation.

5.1 Laissez-faire equilibrium with state-contingent debt

The analysis of the laissez-faire equilibrium in Section 3 was carried out under the assumption that the government can issue only one non-contingent asset. In this section, we show that a laissez-faire equilibrium with state-contingent debt would attain constrained efficiency if and only if there were no moral hazard.[21] However, when the reform effort is endogenous, the combination of moral hazard and limited commitment curtails the insurance that markets can provide. Consequently, the laissez-faire equilibrium with state-contingent debt is not constrained efficient. In the quantitative analysis of

[21] Even in the absence of moral hazard, state-contingent debt does not yield full insurance, due to the risk of renegotiation. However, it attains the second best.

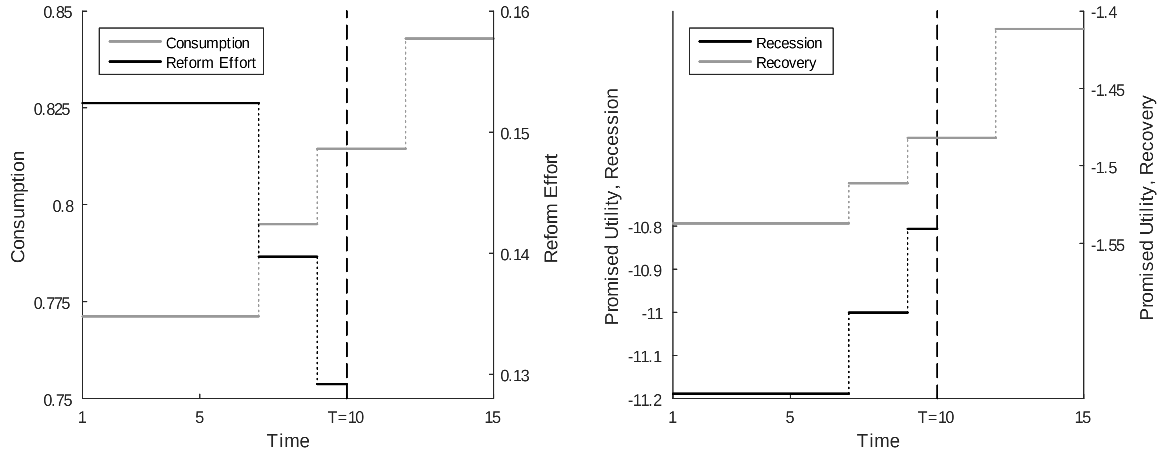

Figure 4: Simulation of consumption, effort, and promised utilities for a particular sequence of ϕ's in the constrained optimum. Here, the recession ends at time $T = 10$.

Section 7 below, we show that markets for state-contingent debt yield only small quantitative welfare gains relative to the benchmark economy.

Let $b_{\underline{w}}$ and $b_{\bar{w}}$ denote Arrow securities paying one unit of output if the economy is in a recession or in normal times, respectively. We label these securities *recession-contingent debt* and *recovery-contingent debt*, respectively, and denote by $Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ and $Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ their corresponding prices. The budget constraint in a recession is given by:

$$Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} = b_{\underline{w}} + c - \underline{w}.$$

To establish a benchmark, consider first a complete market environment in which there is neither moral hazard nor limited commitment. In this case, the security $b'_{\underline{w}}$ sells at the price $Q_{\underline{w}} = (1-p)/R$ whereas the security $b'_{\bar{w}}$ sells at the price $Q_{\bar{w}} = p/R$. In equilibrium, consumption is constant over time and across states. The equilibrium attains the first best.[22]

Under limited commitment, the price of each security depends on both outstanding debt levels, as both affect the reform effort and the probability of renegotiation.[23] The value function of the

[22] Given an outstanding recession-contingent debt $b_{\underline{w}}$, the equilibrium features:

$$\begin{aligned} b'_{\underline{w}} &= b_{\underline{w}} \\ b'_{\bar{w}} &= b_{\underline{w}} + \frac{R}{R-(1-p)}(\bar{w} - \underline{w}) \\ c &= \underline{w} + \frac{p}{R-(1-p)}(\bar{w} - \underline{w}) - \frac{R-1}{R} b_{\underline{w}} \end{aligned}$$

[23] Note that these assets are not Arrow-Debreu assets since their payoffs are conditional on the realization of ϕ. An alternative approach would have been to follow Alvarez and Jermann (2000) and issue an Arrow-Debreu asset for each state (w, ϕ) and let the default-driven participation constraint serve as an endogenous borrowing constraint.

benevolent government can be written as:

$$\begin{aligned} V\left(b_{\underline{w}},\phi,w\right) &= \max_{\left\{b'_{\underline{w}},b'_{\bar{w}}\right\}} \left\{u\left[Q_{\underline{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\times b'_{\underline{w}}+Q_{\bar{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\times b'_{\bar{w}}+\underline{w}-F\left(b,\phi,\underline{w}\right)\right]\right. \\ &\quad \left.-X\left(\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)\right)+\beta\left[1-\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)\right]EV\left(b'_{\underline{w}},\underline{w}\right)+\beta\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)EV\left(b'_{\bar{w}},\bar{w}\right)\right\}. \end{aligned} \tag{36}$$

Mirroring the analysis in the case of non-state-contingent debt, we proceed in two steps. First, we characterize the optimal reform effort. This is determined by the difference between the discounted utility conditional on the recession ending and continuing, respectively (cf. equation (14)):

$$X'\left(\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)\right)=\beta\left[\int_0^\infty V\left(b'_{\bar{w}},\phi',\bar{w}\right)dF\left(\phi\right)-\int_0^\infty V\left(b'_{\underline{w}},\phi',\underline{w}\right)\,dF\left(\phi\right)\right].$$

Note that the incentive to reform would vanish under full insurance.

Next, we characterize the consumption and debt policy. To this aim, consider first the equilibrium asset prices. The prices of the recession- and recovery-contingent debt are given by, respectively:

$$Q_{\underline{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right) = \frac{1-\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{R}\left(\left(1-F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right)+\frac{1}{b'_{\underline{w}}}\int_0^{\underline{\Phi}\left(b'_{\underline{w}}\right)}\left(\underline{\Phi}^{-1}\left(\phi\right)\,dF\left(\phi\right)\right)\right), \tag{37}$$

$$Q_{\bar{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right) = \frac{\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{R}\left(\left(1-F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right)+\frac{1}{b'_{\bar{w}}}\int_0^{\bar{\Phi}\left(b'_{\bar{w}}\right)}\left(\bar{\Phi}^{-1}\left(\phi\right)\,dF\left(\phi\right)\right)\right). \tag{38}$$

Operating as in Section 3, we determine the consumption and debt dynamics conditional on the continuation and on the end of the recession. The next proposition characterizes the CEE with state-contingent debt.

Proposition 7 *Assume that there exist markets for two Arrow securities delivering one unit of output if the economy is in recession and in normal times, respectively, and subject to the risk of renegotiation. Moreover, assume that* $c|_{H,\underline{w}} < c|_{H,\bar{w}}$*. Suppose that the economy initially is in recession. The following CEEs are satisfied in the competitive equilibrium:*
(I) If the recession continues and debt is honored next period, consumption growth is given by:

$$\underbrace{\frac{u'\left(c'|_{H,\underline{w}}\right)}{u\left(c\right)}}_{MRS\text{ if rec. continues}} = 1+\underbrace{\frac{\partial}{\partial b'_{\underline{w}}}\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}_{>0}\times\underbrace{\frac{R\times\Delta\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{\left(1-F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right)\left(1-\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)\right)}}_{>0}. \tag{39}$$

(II) If the recession ends and debt is honored next period, consumption growth is given by:

$$\underbrace{\frac{u'\left(c'|_{H,\bar{w}}\right)}{u\left(c\right)}}_{MRS\text{ if rec. ends}} = 1+\underbrace{\frac{\partial}{\partial b'_{\bar{w}}}\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}_{<0}\times\underbrace{\frac{R\times\Delta\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{\left(1-F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right)\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}}_{>0}. \tag{40}$$

The term $\Delta\left(b'_{\underline{w}},b'_{\bar{w}}\right)$ *is non-negative and is given by*

$$\Delta\left(b'_{\underline{w}},b'_{\bar{w}}\right)\equiv\frac{Q_{\bar{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\times b'_{\bar{w}}}{\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}-\frac{Q_{\underline{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\times b'_{\underline{w}}}{1-\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}\geq 0. \tag{41}$$

Moreover,

$$\begin{aligned}
c &= Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}} + \underline{w} - \digamma\left(b, \phi, \underline{w}\right), \\
c'|_{H,\underline{w}} &= Q_{\underline{w}}\left(B_{\bar{w}}\left(b'_{\underline{w}}\right), B_{\bar{w}}\left(b'_{\underline{w}}\right)\right) \times B_{\underline{w}}\left(b'_{\underline{w}}\right) + Q_{\bar{w}}\left(B_{\underline{w}}\left(b'_{\underline{w}}\right), B_{\bar{w}}\left(b'_{\underline{w}}\right)\right) \times B_{\bar{w}}\left(b'_{\underline{w}}\right) + \underline{w} - b'_{\underline{w}}, \\
c'|_{H,\bar{w}} &= Q\left(B\left(b'_{\bar{w}}, \bar{w}\right), \bar{w}\right) \times B\left(b'_{\bar{w}}, \bar{w}\right) + \bar{w} - b'_{\bar{w}},
\end{aligned}$$

$B_{\underline{w}}\left(b_{\underline{w}}\right)$ and $B_{\bar{w}}\left(b_{\underline{w}}\right)$ denote the optimal level of newly-issued recession- and recovery-contingent debt when the recession continues, debt is honored, and the outstanding debt level is $b_{\underline{w}}$.

Without moral hazard (i.e., if the probability that the recession ends is exogenous, and $\partial\Psi/\partial b'_{\underline{w}} = \partial\Psi/\partial b'_{\bar{w}} = 0$), consumption would be independent of the realization of the aggregate state as long as the government honors its debt. In this case, the CEEs imply constant consumption $c'|_{H,\underline{w}} = c'|_{H,\bar{w}} = c$ when the debt is honored. The solution has the same properties as the constrained Pareto optimum without moral hazard: consumption is constant when debt is honored, and increases discretely when it is renegotiated. The next proposition establishes formally that the two allocations are equivalent. To this aim, define $\underline{\Pi}\left(b_{\underline{w}}\right)$ to be the expected value of debt conditional on staying in recession but before the realization of ϕ.

Proposition 8 *If the probability that the recession ends is independent of the reform effort (i.e., $\Psi = p$), then the competitive equilibrium with state-contingent debt is constrained Pareto efficient, namely, $\underline{\Pi}\left(b_{\underline{w}}\right) = \underline{P}\left(\nu\right) \Leftrightarrow v = EV\left(b_{\underline{w}}, \underline{w}\right)$.*

This equivalence breaks down if there is moral hazard. In this case, the consumption and effort dynamics of the competitive equilibrium are qualitatively different from those of the constrained optimum. In the competitive equilibrium, consumption falls (and recession-contingent debt increases) whenever the economy remains in recession and debt is honored, as shown by equation (39). On the contrary, consumption increases whenever the recession ends, as shown by equation (40). Therefore, different from the efficient allocation (which, recall, features constant consumption when the outside option is not binding), the competitive equilibrium features imperfect insurance, even conditional on honoring the debt.

The intuition is as follows. By issuing more recession-contingent debt, the country strengthens its incentive to make reforms, since $\partial\Psi/\partial b'_{\underline{w}} > 0$. This induces the government to issue more recession-contingent debt than in the absence of moral hazard. The effect is stronger the larger is the term $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ defined in the proposition. This term can be interpreted as the net expected gain accruing to the lenders from a marginal increase in the probability that the recession ends. On the contrary, issuing more recovery-contingent debt weakens the incentives to do reform. As a result, consumption increases if the recession ends and falls if the recession continues (and debt is honored). This result highlights the trade-off between insurance and incentives: the country must give up insurance in order to gain credibility about its willingness to do reforms.

The behavior of effort is also different between the equilibrium and the constrained efficient allocation. In the planning allocation, effort is constant whenever the outside option is not binding. In contrast, in the decentralized allocation, changes in debt will generally influence the reform effort, which is increasing in the newly-issued recession-contingent debt and decreasing in the newly-issued the recovery-contingent debt. In summary, the decentralized equilibrium is inefficient and provides less smoothing of consumption and reform effort than does the planner.

5.2 An austerity program

The market failure in the previous section stems from the moral hazard in reform effort. The constrained efficient allocation would be decentralized by the competitive equilibrium if, in addition, effort were contractible. In reality, it seems unlikely that a country could issue state-contingent bonds in the market while committing credibly to future reforms. In this section, we discuss an institutional arrangement that implements the efficient allocation through enforcement by an international institution that can monitor the reform effort, but not overrule the limited commitment problem.

Consider a stand-by program implemented by an international institution (e.g., the IMF). The indebted country can decide to quit the stand-by program unilaterally. We show that a combination of transfers (or loans), repayment schedule and renegotiation strategy can implement the constrained optimal allocation. This program has two key features. First, the country cannot run an independent fiscal policy, i.e., it is not allowed to issue additional debt in the market. Second, the program is subject to renegotiation. More precisely, whenever the country can credibly threaten to abandon the program, the international institution should sweeten the deal by increasing the transfers and reducing the required effort, and reducing the debt the country owes when the recession ends. When no credible threat of default is on the table, consumption and reform effort should be held constant as long as the recession lasts. When the recession ends, the international institution receives a payment from the country.

Let ν denote the present discounted utility guaranteed to the country when the program is first agreed upon. Let $c^*(\nu)$ and $p^*(\nu)$ be the consumption and reform effort associated with the promised utility in the planning problem. Upon entering the program, the country receives a transfer equal to $T(\nu) + b_0$, where $T(\nu) = c^*(\nu) - \underline{w}$ (note that $T(\nu)$ could be negative). In the subsequent periods, the country is guaranteed the transfer flow $T(\nu)$ so long as the recession lasts and there is no credible request of renegotiating the terms of the austerity program. In other words, the international institution first bails out the country from its obligations to creditors, and then becomes the sole residual claimant of the country's sovereign debt. The country is also asked to exercise a reform effort $p^*(\nu)$. If the country faces a low realization of ϕ and threatens to leave the program, the institution improves the terms of the program so as to match the country's outside option. Thereafter, consumption and effort are held constant at new higher and lower levels, respectively, as in the planner's allocation. And so on, for as long as the recession continues.

As soon as the recession ends, the country owes a debt b_N to the international institution, determined by the equation

$$Q(b_N, \bar{w}) \times b_N = c^*(\nu_N) - \bar{w} + b_N.$$

Here ν_N is the expected utility granted to the country after the most recent round of renegotiation. After receiving this payment, the international institution terminates the program and lets the country finance its debt in the market.

This program resembles an *austerity* program, in the sense that the country is prevented from running an independent fiscal policy. In particular, the country would like to issue extra debt after entering the stand-by agreement, so austerity is a binding constraint. In addition, the country would like to shirk on the reform effort prescribed by the agreement. Thus, the government would like to deviate from the optimal plan, and an external enforcement power is an essential feature of the program. This conflict of interest may be a reason behind the tense relationship between the Greek government and the Troika since the stipulation of the stand-by agreement.

A distinctive feature of the assistance program is that the international institution sets "harsh" entry conditions in anticipation of future renegotiations. How harsh these conditions are depends on

ν. In turn, ν may reflect a political decision about how many (if any) own resources the international institution wishes to commit to rescuing the indebted country. A natural benchmark is to set ν such that the international institution makes zero profits (and zero losses) in expected discounted value. Whether, ex-post, the international institution makes net gains or losses hinges on the duration of the recession and on the realized sequence of ϕ's.

Another result that has important policy implications is that there would be no welfare gain if the international institution committed never to accept any renegotiation. On the contrary, such a policy would lead to welfare losses because, on the one hand, there would be inefficient default in equilibrium; on the other hand, the country could not expect future improvements, and therefore would not accept a very low initial consumption, or a very high reform effort. If one fixes the expected profit of the international institution to zero, the country would receive a lower expected utility from the alternative program.

In summary, our theory prescribes a pragmatic approach to debt renegotiation. Any credible threat of default should be appeased by reducing the debt and softening the austerity program. Such approach is often criticized for creating bad incentives. In our model, it is rather the optimal policy under the reasonable assumption that penalties on sovereign countries for breaking an agreement are limited.

5.3 Self-enforcing reform effort

Thus far we have assumed that the planner – or the international institution in the decentralized environment – can dictate the reform effort as long as the country stays within the contract. Assuming that reforms are observable seems natural to us. It is possible, for instance, to verify whether Greece introduces labor market reforms, cuts employment in the public sector, or passes legislative measures to curb tax evasion (e.g., by intensifying tax audits and enforcing penalties). Nevertheless, it may be difficult for international institutions to prevent deviations such as delays, lack of implementation, or weak enforcement of agreed-upon reforms. In other words, the borrower may try to cash-in the transfer agreed in the assistance program in exchange for promises of structural reforms, but indefinitely defer their execution.

In this section, we consider an alternative environment where reform effort can be verified only *ex-post* at the end of each period. If shirking is detected, the program is terminated irreversibly.[24] A new incentive-compatibility constraint (IC) arises from the inability of the country to commit to reforms. In particular, the country could behave opportunistically by cashing-in the loan at the beginning of the period and exercising a discretionary effort level. In this case, the government would be forced to revert to the market equilibrium with debt obligations restored to the level prior to the start of the assistance program. The appeal of such a deviation is greater the stronger is the required reform effort.

More formally, the allocation is identical to the solution to the planning problem (29)–(31) subject to the additional IC stipulating that, for all $\phi \in \aleph$,

$$-X(p_\phi) + \beta\left((1-p_\phi)\,\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right) \geq Z(b_0), \tag{42}$$

where b_0 is the debt of the country when it enters the contract, and Z is the continuation utility if the economy reverts to the competitive equilibrium, i.e.,[25]

[24] One could consider less severe punishments, such as a temporary suspension of the loans, or a reduction in the future loans. We leave these extensions to future research.

[25] The assumption that the deviating country is settled with the debt level b_0 irrespective of the history of the contract

$$Z\left(b_0\right) \equiv -X\left(\Psi\left(b_0\right)\right) + \beta\left[\Psi\left(b_0\right) \times E\left[V\left(b_0, \phi', \bar{w}\right)\right] + \left(1 - \Psi\left(b_0\right)\right) \times E\left[V\left(b_0, \phi', \underline{w}\right)\right]\right].$$

When the IC (42) is binding, the allocation of Proposition 6 is susceptible to profitable deviations.[26] The following Lemma establishes properties of the constrained allocation whenever the IC is binding.[27]

Lemma 6 *When the IC is binding, effort and promised utilities are constant at the levels $(p^*, \underline{\omega}^*, \bar{\omega}^*)$, where the triplet is uniquely determined by the equations*

$$\begin{aligned} Z\left(b_0\right) &= -X\left(p^*\right) + \beta\left(\left(1 - p^*\right)\underline{\omega}^* + p^*\bar{\omega}^*\right), & (43)\\ \bar{P}'\left(\bar{\omega}^*\right) &= \underline{P}'\left(\underline{\omega}^*\right), & (44)\\ X'\left(p^*\right) &= \beta\left(\bar{\omega}^* - \underline{\omega}^*\right) - \frac{\beta}{\underline{P}'\left(\underline{\omega}^*\right)}\left(\bar{P}\left(\bar{\omega}^*\right) - \underline{P}\left(\underline{\omega}^*\right)\right), & (45) \end{aligned}$$

where the profit functions $\bar{P}$ and $\underline{P}$ are defined as in Section 4.2.

Equation (43) yields the IC when it holds with equality. Equations (44) and (45) then follow from the FOCs stated in equations (77)-(79). These two conditions hold true regardless of whether the IC constraint is binding or not.[28] Note that the profit functions of the problem with an IC constraint differ in general from those of the problem without IC constraint studied in Section 4.2 (where there is no IC). However, we prove that the profit functions coincide when evaluated at the promised utilities $\underline{\omega}^*$ and $\bar{\omega}^*$.

The following proposition characterizes the equilibrium dynamics.

Proposition 9 *Suppose that the country starts in a recession, and is endowed with the initial promised utility ν.*

1. *If $\nu \geq \underline{\omega}^*$, then the IC is never binding, and the constrained optimal allocation, $\left(c_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right)$, is identical to that in Proposition 6.*

2. *If $\nu < \underline{\omega}^*$, then there exist two thresholds, ϕ^* and $\tilde{\phi}(\nu)$, where $\phi^* = \tilde{\phi}(\underline{\omega}^*)$ (expressions in the proof in the appendix) such that:*

   (a) *If $\phi < \phi^*$, the PC is binding while the IC is not binding. The solution is not history-dependent and is determined as in Proposition 6 (in particular, $\underline{\omega}_\phi > \underline{\omega}^*$ and $p_\phi < p^*$).*

   (b) *If $\phi \in [\phi^*, \tilde{\phi}(\nu)]$, both the PC and the IC are binding. Effort and promised utilities are equal to $(p^*, \underline{\omega}^*, \bar{\omega}^*)$ as given by Lemma 6. Consumption is determined by equations (31) and (42) which yield:*

   $$c_\phi^* = u^{-1}\left(\underline{\nu} - \phi - Z\left(b_0\right)\right). \quad (46)$$

   Consumption and effort are lower than in the allocation of Proposition 6.

is made for simplicity. This keeps the outside option of the IC constant, avoiding the technical complications arising from an endogenous outside option.

[26] If the IC constraint (42) is not binding, the solution is as in Proposition 6. Note that, if the IC constraint is not binding at t, it will never bind in future, since the allocation of Proposition 6 entails a non-decreasing promised utility and a non-increasing effort path.

[27] The proof of the lemma is merged with the proof of Proposition 9.

[28] To see why the solution to (42)–(45) is unique, note that the concavity and monotonicity of $\underline{P}$ and $\bar{P}$ imply that Equation (44) determines a positive relationship between $\underline{\omega}$ and $\bar{\omega}$. Thus, Equation (45) yields an implicit decreasing relationship between p and $\underline{\omega}$, while (42) yields an implicit increasing relationship between p and $\underline{\omega}$.

(c) If $\phi > \tilde{\phi}(\nu)$, the IC is binding, while the PC is not binding. Effort and promised utilities are equal to $(p^*, \underline{\omega}^*, \bar{\omega}^*)$. The consumption level is determined by the promise-keeping constraint (30). In particular, consumption is constant across ϕ and given by:

$$c^*_{\tilde{\phi}(\nu)} = u^{-1}\left(\underline{\nu} - \tilde{\phi}(\nu) - Z(b_0)\right). \tag{47}$$

For given ν and ϕ, consumption and effort are lower than in the allocation of Proposition 6.

Consider an economy where, initially, $\nu < \underline{\omega}^*$ (recall that a low ν corresponds to a high initial debt in the decentralized equilibrium). If the first realization of ϕ is sufficiently low (case 2.a of Proposition 9), the IC is not binding, the allocation is not history-dependent, and the characterization of Proposition 6 applies. If the first realization of ϕ is larger than ϕ^*, the IC is binding, and Lemma 6 implies that effort and promised utility are equal to $(p^*, \underline{\omega}^*, \bar{\omega}^*)$. If $\phi \in [\phi^*, \tilde{\phi}(\nu)]$ (case 2.b), consumption is pinned down jointly by the PC and the promise-keeping constraint (consumption will then be decreasing in ϕ). Finally, if $\phi > \tilde{\phi}(\nu)$ (case 2.c) the PC imposes no constraint, and the initial consumption is determined only by the promise-keeping constraints. When the IC is binding (cases 2.b and 2.c), both consumption and effort are lower than in the second-best solution of Proposition 6. Intuitively, the planner cannot set effort at the efficient level due to the IC, and adjust optimally to the constraint by reducing current consumption and increasing promised utilities. Thus, the contract provides less consumption insurance than does the second-best constrained-efficient allocation. When the IC is binding, the promise utility increases from ν to $\underline{\omega}^*$. Thereafter, consumption, effort and promised utilities remain constant until a realization of ϕ lower than ϕ^* is observed. In summary, after one period the equilibrium is characterized as in the second best of Proposition 6.

Figure 5 is the analogue of Figure 4 in an economy in which the IC is binding in the initial period, i.e., $\nu < \underline{\omega}^*$. The left panel shows the dynamics of consumption and effort, whereas the right panel shows the dynamics of expected utility. The initial promised utility (ν) is consistent with a break-even condition for the planner, namely, $\underline{\Pi}(b) = \underline{P}(\nu)$. The dash-dotted line in Figure 5 is for comparison, and shows the second-best constrained-efficient allocation of Proposition 6 corresponding to the same sequence of ϕ's. In the first period, the realization of the stochastic process is in the range $\phi > \phi^*$. Thus, the IC is binding, and consumption and effort are below the second-best constrained-efficient level. After one period, consumption increases to meet the promise-keeping constraint, and remains constant (as do effort and promised utilities) thereafter until period seven, when the first realization in the range $\phi < \phi^*$ is observed. From that period onwards, the IC never binds again and the economy settles down to the (ex-post) constrained-efficient allocation. Note that the constraint that the reform effort must be self-enforcing reduces the country's ex-ante welfare. This is because until period seven, the principal cannot extract the efficient reform effort level, and must offer the agent a lower consumption (compensated by a larger promised utility) to break even.[29]

Figure 5 yields simulated paths of consumption, effort and promised utility in the constrained optimal allocation for two otherwise identical economies where one economy (solid lines) is subject to the IC constraint, while the other economy (dashed lines) has no such constraints. The initial promised utility ν (not displayed) is lower than $\underline{\omega}^*$ implying that the IC is binding. In the first period, consumption and effort are lower in the economy with an IC constraint. In contrast, promised utility is

[29] The left-hand figure does not show the initial ν. For instance, at time t=1, one can see $\underline{\omega}^*$ and $\bar{\omega}^*$, where, recall, $\nu < \underline{\omega}^*$. The initial ν is lower than the dashed black line. However, after one period, the promised utility is higher in the economy with the IC constraint than in the second-best.

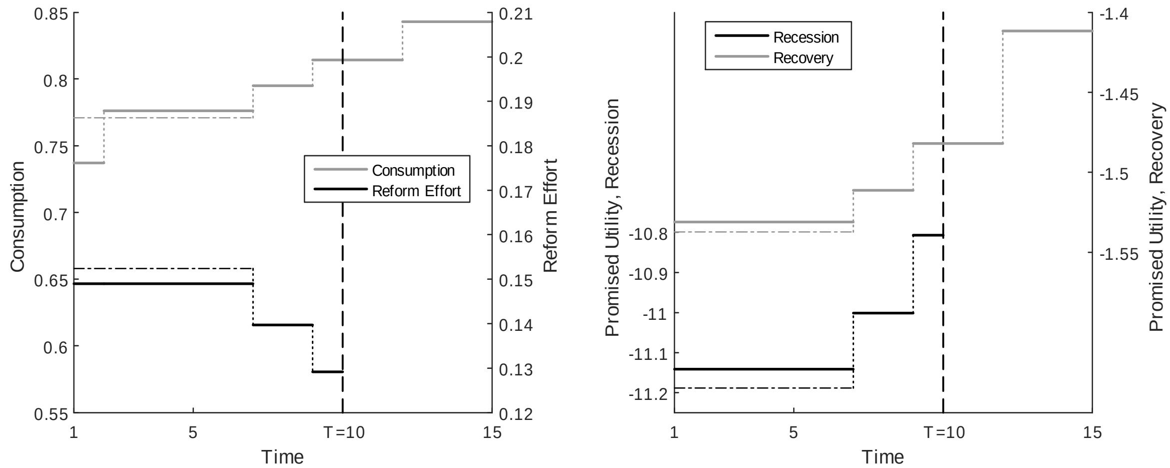

Figure 5: Simulation of consumption, effort, and promised utilities for a particular sequence of ϕ's where the IC is initially binding. Solid lines refer to an economy with an IC constraint. Dashed lines refer to the economy without an IC constraint.

higher. In other words, the planner provides less insurance by making consumption and effort initially lower, but growing at a higher speed. As of the second period, the dynamics of both economies are the same as in Figure 4.

6 Extension: reputation

In our theory, renegotiation is unambiguously good for the borrower. On the one hand, consumption always increases upon renegotiation, in line with the empirical evidence documented by Reinhart and Trebesch (forthcoming). On the other, renegotiations do not harm the country's reputation. In particular, conditional on the debt level, the risk premium is independent of the country's credit history. In this section, we sketch an extension showing that these are not inherent implications of the theory. We assume that there is imperfect information about the distribution from which countries draw their realizations of ϕ. In particular, there are two types of countries, creditworthy (CW) and not creditworthy (NC), that draw from different distributions.[30] In particular, $F_{NC}(\phi) \geq F_{CW}(\phi)$, with strict inequality holding for some ϕ, implying that the NC country is more likely to have lower realizations of the default cost. We assume that priors are common knowledge, and denote by π the belief that the borrower is CW. Beliefs are updated according to Bayes' rule:

$$\pi' = \frac{f_{CW}(\phi)}{f_{CW}(\phi) \times \pi + f_{NC}(\phi) \times (1-\pi)} \pi \equiv \Gamma(\phi, \pi).$$

We restrict attention to competitive equilibria during normal times ($w = \bar{w}$).

In the new environment, the price of debt depends on the country's reputation, i.e., $Q = \bar{Q}(b', \pi)$, where we drop the argument w since we focus on normal times. No arbitrage implies the following

[30] One can assume that the distributions have a common support in order to rule out perfectly revealing realizations. However, this is not essential.

bond price:

$$\bar{Q}\left(b',\pi\right)=\frac{1}{R}\left(\begin{array}{c}\left(1-F\left(\Phi^{*}\left(b',\pi\right)|\Gamma\left(\Phi^{*}\left(b',\pi\right),\pi\right)\right)\right)+\\ \frac{\pi}{b'}\int_{0}^{\Phi^{*}\left(b',\pi\right)}\hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right)\,dF_{CW}\left(\phi\right)+\frac{1-\pi}{b'}\int_{0}^{\Phi^{*}\left(b',\pi\right)}\hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right)dF_{NC}\left(\phi\right)\end{array}\right)$$

where $\Phi^{*}\left(b',\pi\right)$ denotes the threshold ϕ' such that debt will be honored next period if and only if $\phi'\geq\Phi^{*}\left(b',\pi\right)$. More formally, Φ^{*} is the unique fixed point of the following equation

$$\Phi^{*}=\bar{\Phi}\left(b',\Gamma\left(\Phi^{*},\pi\right)\right).$$

The function Φ^{*} takes into account that the realization of ϕ' will itself alter next-period beliefs, which in turn affect the country's incentive to renegotiate.[31] The bond price is falling in b' and increasing in π.

Consider, next, the consumption-savings decision. The CEE yields:[32]

$$\begin{aligned}1-F\left(\Phi^{*}\left(\bar{B}\left(b,\pi\right),\pi\right)|\Gamma\left(\Phi^{*},\pi\right)\right) &= \pi\int_{\Phi^{*}(b,\pi)}^{\infty}\frac{u'\left[C\left(\Gamma\left(\phi',\pi\right),\bar{B}\left(b,\pi\right)\right)\right]}{u'\left[C\left(\pi,b\right)\right]}dF_{CW}\left(\phi'\right)\\ &\quad+\left(1-\pi\right)\int_{\Phi^{*}(b,\pi)}^{\infty}\frac{u'\left[C\left(\Gamma\left(\phi',\pi\right),\bar{B}\left(b,\pi\right)\right)\right]}{u'\left[C\left(\pi,b\right)\right]}dF_{NC}\left(\phi'\right).\end{aligned}$$

If next-period consumption conditional on honoring the debt did not depend on ϕ', then the CEE would boil down to equation (13). However, in this extension, the realized consumption growth depends on ϕ' because of its effect on reputation. For example, take two realization of ϕ', say ϕ'_h and ϕ'_l, such that $\phi'_h>\phi'_l$, neither inducing renegotiation. Here, consumption will be larger under ϕ'_h because the larger realization has a stronger positive effect on the belief that the country is CW. This improves the terms of borrowing, and hence consumption.

Note that renegotiation might be associated with a fall in consumption – for example if the realized ϕ is just below $\Phi^{*}\left(b',\pi\right)$ the effect of a very small renegotiation is more than offset by that of a large reputational loss. Conversely, a country experiencing a sequence of large ϕ's which induces it to honor debt for a long time will enjoy a growing reputation and an increasing consumption.

In summary, this simple extension shows that our theory can incorporate reputational effects through which countries prone to renegotiation are punished by the market with high interest rates, because of their bad reputation. The extension could be enriched, for instance, by assuming that types are not fixed effects, but a borrower may switches back and forth between being CW and NC according to a persistent Markov process. A more substantial extension would be to consider asymmetric information. This is left to future research.

7 Quantitative Analysis

In this section, we study the quantitative properties of the model. To this end, we calibrate the model economy to match salient facts on default premia, investor losses, and debt-to-GDP ratios. Our

[31] Note that some functions must be redefined to take into account their dependence on public beliefs. Apart from $\bar{Q}\left(b',\pi\right)$, defined in the text, $\hat{b}\left(\phi,\pi\right)$ is the renegotiated debt given ϕ and π. Moreover, $\bar{\Phi}\left(b',\pi\right)$ denotes the threshold that makes the country indifferent between honoring the debt level b' and defaulting, conditional on the realized belief π.

[32] See the online appendix for technical details.

main purpose here is to evaluate the welfare gain of going from the competitive equilibrium to the constrained optimal allocation. We also assess the quantitative importance of limited commitment, non-contractible effort, and incomplete financial markets. Finally, we evaluate the welfare effect of various austerity programs.

7.1 Calibration

A model period corresponds to one year. We normalize the GDP during normal times to $\bar{w} = 1$ and assume that the recession causes a drop in income of 38%, i.e., $\underline{w} = 0.62 \times \bar{w}$. This corresponds to the fall of GDP per capita for Greece between 2007 and 2013, relative to trend.[33] Since we focus on the return on government debt, the annual real gross interest rate is set to $R = 1.02$, implying $\beta = 1/R = 0.98$. The utility function is assumed to be CRRA with a relative risk aversion (RRA) of 2 (we also do some sensitivity analysis on this parameter).

We assume a standard constant elasticity version of the effort cost function; $X(p) = \frac{\xi}{1+1/\varphi}(p)^{1+1/\varphi}$, where ξ regulates the average level of effort and φ regulates the elasticity of reform effort to changes in the return to reforms. We set the two parameters, φ and ξ, so as to match two points on the equilibrium effort function $\Psi(b)$. In particular, we assume that the effort at the debt limit is $\Psi\left(\bar{b}\right) = 10\%$, so that a country with a debt at the debt limit chooses an effort inducing an expected duration of the recession of one decade (we have Greece in mind). Moreover, we assume that the maximum effort is $\max_b \Psi(b) = 20\%$, inducing an expected recession duration of five years (we have Iceland and Ireland in mind). This implies setting $\varphi = 22.1$ and $\xi = 24.45$.

Finally, we determine the distribution of the default cost ϕ, assumed to have bounded support $\left[0, \bar{\phi}\right]$. We calibrate the distribution $f(\phi)$ so that the model matches key moments of quantities and prices of sovereign debt. One common problem in the quantitative literature on sovereign debt is that models fail to match observed values of debt-to-GDP ratios under realistic parametrization (Arellano 2008; Yue 2010). This is not a problem in our model. In fact, the maximum default cost realization $\bar{\phi}$ is set so that the debt limit during normal times is $\bar{b}/\bar{w} = 180\%$. Moreover, the distribution f is chosen so that the model matches two moments: an average post-renegotiation recovery rate of 62% (Tomz and Wright 2007) and an average default premium of 4% for a country which has a debt-output ratio of 100% during recession.[34] This was the average debt and average default premium for Greece, Ireland, Italy, Portugal, and Spain (GIIPS) during 2008-2012. To match these moments we assume that ϕ is distributed according to a generalized Beta, with c.d.f. given by $F(\phi;\eta_1,\eta_2) = \mathcal{B}(\phi/\bar{\phi},\eta_1,\eta_2)/\mathcal{B}(1,\eta_1,\eta_2)$, where $\mathcal{B}(x,\eta_1,\eta_2)$ denotes the incomplete Beta function $\mathcal{B}(x,\eta_1,\eta_2) = \int_0^x t^{\eta_1-1}(1-t)^{\eta_2-1}dt$. By setting $\eta_1 = 0.8$ and $\eta_2 = 0.105$ the targets are met.

7.2 Welfare Comparison

We use the calibrated economy to evaluate the welfare gains of different policy arrangements. The welfare gains are measured as the equivalent variation in terms of the market value of an initial debt to output reduction in the market economy, namely, the market value of a reduction in initial

[33] GDP per capita of Greece fell from 18,924 to 14,551 Euro between 2007 and 2013 (Eurostat). The annualized growth rate between 1997 and 2007 was 3.8%. The fall in output between 2007 and 2013 relative to trend is therefore 38%.

[34] Our calibration is also in line with the findings of Reinhart and Trebesch (forthcoming). They document an average debt relief of 40% of external government debt across both the 1930s and the 1980s/1990s. Moreover, the average debt relief is reported to be 21% of GDP for advanced economies in the 1930s, and 16% of GDP for emerging market economies in the 1980s/1990s. Even though we do not target this moment of the data in the calibration, our simulations yield an average debt relief of 22% of GDP for the benchmark calibration, which is in the ball park of the estimates.

debt required to make the borrower indifferent between staying in the market arrangement (with the reduction in debt) and moving to an alternative allocation.

We assume that the economy starts in a recession and has an initial debt-output ratio of 100%, corresponding to $b_0 = 0.62$. Table 1 reports the welfare gains of the policy arrangements relative to the market allocation. The results for the benchmark calibration with a relative risk aversion of two, are listed in the middle column of the table. Given an initial debt-output ratio of 100% and the

	RRA		
	1	2	3
Market Equilibrium	0%	0%	0%
First-best	74.9%	151.6%	228.2%
Second-best	22.8%	46.9%	58.7%
State-contingent	1.4%	5.3%	7.9%
Grexit	6.2%	-5.2%	-5.3%

Table 1: Welfare equivalent transfer (market value, % of GDP)

country being in a recession, the welfare gain of going from the market allocation to the first best is equivalent to a one-time transfer of 152% of GDP. Similarly, the gain of going to the second best is equivalent to a one-time transfer of 47% of GDP. The big welfare difference between the first- and the second-best allocation shows the large losses associated with the limited enforcement.

Allowing for state-contingent debt, on the other hand, yields a gain equivalent to a mere 5% one-time transfer, substantially less than the second-best gains. As discussed above, this illustrates that the trade-off between moral hazard and insurance renders access to full insurance not very useful. Moreover, it shows that the large gains of the second best must be originating from the planner's ability to mitigate the moral hazard problem.

To understand the role of risk aversion, we vary the relative risk aversion from 1 to 3 (see Table 1).[35] Not surprisingly, the welfare gains relative to the market allocation are increasing in the degree of risk aversion, as the excess consumption volatility in the market equilibrium becomes more painful and insurance becomes more important. Still, even with a high RRA of 3, the welfare gains of having access to state-contingent debt remain small relative to the constrained efficient allocation.

7.3 Consumption volatility and recession duration

In this section we report some important moments of our simulation results. Recall that for each considered policy arrangement, we start the simulation at the initial debt level (or a level of promised utility) corresponding to a zero profit intervention in the market equilibrium at a debt-output ratio of 100%.

Table 2 shows the variance of log consumption over the first ten simulation periods. The second best yields substantially smoother consumption than the market allocation. The consumption volatility is not much affected by the degree of relative risk aversion.

Table 3 reports the expected duration (the cross-sectional mean duration resulting from a large number of stochastic simulations) of the recession which is inversely related to the average level of reform effort over the recession period. For example, in the benchmark calibration the first-best

[35] When changing the relative risk aversion, the internally calibrated parameters, φ, ξ, $\bar{\phi}$, η_1, and η_2, are recalibrated so the model meets the stated empirical moments.

	RRA		
	1	2	3
Market Equilibrium	0.0197	0.0185	0.0224
First-best	0	0	0
Second-best	0.0077	0.0070	0.0084
State-contingent	0.0175	0.0144	0.0161
Grexit	0.0285	0.0327	0.0334

Table 2: Consumption volatility over the first 10 years

allocation implies an expected recession of 17.8 years, which corresponds to a constant recovery probability (reform effort level) of 5.6%. Note that for the market equilibrium, the length of the recession remains rather stable across different degrees of risk aversion. This property is by construction, as we are recalibrating the economy to match the peak and the end point of the reform effort function. Interestingly, the expected duration of the recession is roughly equal for the market equilibrium and

	RRA		
	1	2	3
Market Equilibrium	6.5	6.9	6.4
First-best	15.7	17.8	20.6
Second-best	7.4	7.2	5.9
State-contingent	7.9	9.2	9.4
Grexit	8.9	7.1	5.9

Table 3: Expected duration of the recession (in years)

the second best (where the ranking depends on the relative risk aversion). In contrast, the allocation with state-contingent debt yields significantly lower reform effort. This illustrates a main point of our paper: access to better insurance magnifies the moral hazard problem.

Figure 6 plots the reform effort as a function of debt revenue (planner profits) for the market allocation and the planner allocations.For the market equilibrium, the figure plots last period's reform effort against the market value of the outstanding debt at the beginning of the current recession period. For the first- and second-best allocation it plots the same reform effort against the planner's profit in a recession (thus, the comparison corresponds to a zero profit intervention so all the gain go by assumption to the borrower). The reform effort in the first-best allocation is generally lower, independent of the initial debt level.

7.4 An austerity program: "Grexit"

In the last row of Table 1 we consider a simple version of an "austerity program" with the following features: (1) the external institution guarantees the debt of the borrower (so the market price is always $1/R$), (2) the borrower cannot issue additional debt in the private market (i.e., "fiscal austerity"), (3) the institution commits to terminate the arrangement and induce an outright default ("Grexit") if the borrower attempts to renegotiate the debt obligations (in which case the institution steps in an honors the issued guarantees) or increase the debt level, (4) once the recession is over, no new guarantees will be issued, and (5) the initial debt is increased to a level such that the institution has zero expected

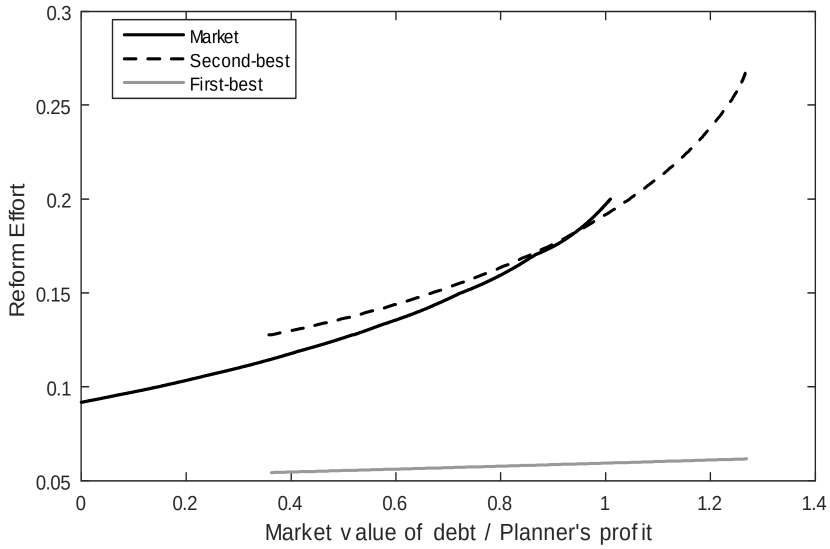

Figure 6: Comparison of initial reform effort for a zero profit intervention, RRA=2.

profits (i.e., the austerity program breaks even in expectation). However, there are no conditions imposed on the reform effort.

One the one hand, this program could potentially be good because it mitigates the moral hazard program. To see this, note that with a debt guarantee the bond price does not change when the recession ends, so more of the gains from reform effort will accrue to the borrower. On the other hand, the program will induce inefficient default. For the benchmark calibration, we find that such an austerity program yields a loss equivalent to a one-time tax of 5.2%, so the borrower would prefer the market to this austerity program. However, when the risk aversion is sufficiently low, the austerity program is actually better than the market allocation. With a lower risk aversion, the insurance value is smaller and the improvements in the moral hazard problem outweighs the inefficiencies.

8 Conclusions

This paper presents a theory of sovereign debt dynamics under limited commitment. A sovereign country issues debt to smooth consumption during a recession whose duration is uncertain and endogenous. The expected duration of the recession depends on the intensity of (costly) structural reforms. Both elements – the risk of repudiation and the need for structural reforms – are salient features of the recent European debt crisis.

The competitive equilibrium features repeated debt renegotiations. Renegotiations are more likely to occur during recessions and when the country has accumulated a high level of debt. As a recession drags on, the country has an incentive to go deeper into debt. A higher level of debt in turn may deter rather than stimulate economic reforms.

The theory bears normative predictions that are relevant for the management of the European crisis. The market equilibrium is inefficient for two reasons. On the one hand, the government of the sovereign country underinvests in structural reforms. The intuitive reason is that the short-run cost of reforms is borne entirely by the country, while future benefits of reforms accrue in part to the creditors in the form of an *ex-post* increased price of debt, due to a reduction in the probability of renegotiation. On the other hand, the limited commitment to honor debt induces high risk premia and excess consumption volatility. A well-designed intervention by an international institution can improve welfare, as long as the institution can monitor the reform process. While we assume, for tractability, that the international institution can monitor reforms perfectly, our results carry over to a more realistic scenario where reforms are only imperfectly monitored. The optimal policy also entails an assistance program whereby an international organization provides the country with a constant transfer flow, deferring the repayment of debt to the time when the recession ends. The optimal contract factors in that this payment is itself subject to renegotiation risk.

A second implication is that, when the government of the indebted country credibly threatens to renege on an existing agreement, concessions should be made to avoid an outright repudiation. Contrary to a common perception among policy makers, a rigid commitment to enforce the terms of the original agreement is not optimal. Rather, the optimal policy entails the possibility of multiple renegotiations, which are reflected in the terms of the initial agreement.

To retain tractability, we make important assumptions that we plan to relax in future research. First, in our theory the default cost follows an exogenous stochastic process. In a richer model, this would be part of the equilibrium dynamics. Strategic delegation is a potentially important extension. In the case of Greece, voters may have an incentive to elect a radical government with the aim of delegating the negotiation power to an agent that has or is perceived to have a lower default cost than voters do (cf. Rogoff 1985). In our current model, however, the stochastic process governing the creditor's outside option is exogenous, and is outside of the control of the government and creditors.

Second, again for simplicity, we assume that renegotiation is costless, that creditors can perfectly coordinate and that they have full bargaining power in the renegotiation game. Each of these assumptions could be relaxed. For instance, in reality the process of negotiation may entail costs. Moreover, as in the recent contention between Argentina and the so-called vulture funds, some creditors may hold out and refuse to accept a restructuring plan signed by a syndicate of lenders. Finally, the country may retain some bargaining power in the renegotiation. All these extensions would introduce interesting additional dimensions, and invalidate some of the strong efficiency results (for instance, the result that the competitive economy attains the second best in the absence of income fluctuations). However, we are confident that the gist of the results is robust to these extensions.

Finally, by focusing on a representative agent, we abstract from conflicts of interest between different groups of agents within the country. Studying the political economy of sovereign debt would be an interesting extension. We leave the exploration of these and other avenues to future work.

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10 Appendix

10.1 Proofs of lemmas, propositions, and corollaries.

Note: This appendix section is preliminary and incomplete. The proofs of Claim 1 and Claim 2 are not contained in this version.

Proof of Lemma 1. The first part follows from the definitions of Φ and $\hat{b}$. For the second part, note that $\frac{\partial}{\partial b}\Phi(b,w) = -\frac{\partial}{\partial b}W(b,w) = u'(Q(B(b,w),w)\times B(b,w)+w-b) > 0$. Thus, $\bar{\Phi}'(b) > 0$ and $\underline{\Phi}'(b) > 0$. Moreover, differentiating the condition $W\left(\hat{b}(\phi,w),w\right) = W(0,w)-\phi$ with respect to ϕ yields $\frac{\partial}{\partial \phi}\hat{b}(\phi,w) = -\frac{1}{\frac{\partial}{\partial b}W\left(\hat{b}(\phi,w),w\right)} > 0$. ∎

Proof of Lemma 2. Differentiating $b\times Q(b,\bar{w})$ with respect to b yields:

$$\begin{aligned}
\frac{d}{db}\left\{b\times Q(b,\bar{w})\right\} &= Q(b,\bar{w}) + b\times\frac{d}{db}\left(\frac{1}{R}\left(1-F\left(\bar{\Phi}(b)\right)\right)+\frac{1}{R}\frac{1}{b}\int_0^{\bar{\Phi}(b)}\left(\bar{\Phi}^{-1}(\phi)\times f(\phi)\ d\phi\right)\right)\\
&= Q(b,\bar{w}) - \frac{b}{R}f\left(\bar{\Phi}(b)\right)\bar{\Phi}'(b) + \\
&\quad \frac{b}{R}\frac{1}{b}\bar{\Phi}^{-1}\left(\bar{\Phi}(b)\right)\times f\left(\bar{\Phi}(b)\right)\times\bar{\Phi}'(b) - \underbrace{\frac{1}{R}\frac{1}{b}\int_0^{\bar{\Phi}(b)}\left(\bar{\Phi}^{-1}(\phi)\times f(\phi)\ d\phi\right)}_{=Q(b,\bar{w})-\frac{1}{R}\left(1-F\left(\bar{\Phi}(b)\right)\right)}\\
&= \frac{1}{R}\left(1-F\left(\bar{\Phi}(b)\right)\right).
\end{aligned}$$

∎

Proof of Proposition 1. The first order condition of (12) yields:

$$\frac{d}{db'}\left\{b' \times Q\left(b', \bar{w}\right)\right\} \times u'(c) + \frac{d}{db'}\beta EV\left(b', \bar{w}\right) = 0,$$

The value function has a kink at $b = \hat{b}(\phi, \bar{w})$. Consider, first, the range of realizations $\phi \in \left[\bar{\Phi}(b), \infty\right)$, implying that $b < \hat{b}(\phi, \bar{w})$. Differentiating the value function yields:

$$\frac{d}{db}V(b, \phi, \bar{w}) = -u'\left[Q\left(B(b, \bar{w}), \bar{w}\right) \times B(b, \bar{w}) + \bar{w} - b\right].$$

Next, consider the region of renegotiation, $\phi \in [0, \Phi(b)]$, implying that $b > \hat{b}(\phi, \bar{w})$. In this case, $\frac{d}{db}V(b, \phi, w) = 0$. Using the results above one obtains:

$$\begin{aligned}
\frac{d}{db}EV(b, \bar{w}) &= \int_0^{\bar{\Phi}(b)} \frac{d}{db}V(b, \phi, \bar{w})\, dF(\phi) + \int_{\bar{\Phi}(b)}^{\infty} \frac{d}{db}V(b, \phi, \bar{w})\, dF(\phi) \\
&= -\int_{\bar{\Phi}(b)}^{\infty} u'\left[Q\left(B(b, \bar{w}), \bar{w}\right) \times B(b, \bar{w}) + \bar{w} - b\right] dF(\phi) \\
&= -\left[1 - F\left(\bar{\Phi}(b)\right)\right] \times u'\left[Q\left(B(b, \bar{w}), \bar{w}\right) \times B(b, \bar{w}) + \bar{w} - b\right] \qquad (48)
\end{aligned}$$

Plugging this expression back into the FOC, and leading the expression by one period, yields

$$0 = \frac{d}{db'}\left\{b' \times Q\left(b', \bar{w}\right)\right\} \times u'(c) - \beta\left[1 - F\left(\bar{\Phi}\left(b'\right)\right)\right] \times u'\left(c'|_{H,\bar{w}}\right) \qquad (49)$$

Finally, recall that $\frac{d}{db}\left\{b \times Q(b, \bar{w})\right\} = \frac{1}{R}\left(1 - F\left(\bar{\Phi}(b)\right)\right)$, as established in the proof of Lemma 2. Thus, equation (49) is equivalent to the CEE (13) in the proposition. ■

Proof of Lemma 3. If in the initial period (but not later) the country can contract on effort while issuing new debt, the problem becomes

$$\max_{b', p^*}\left\{u(c) - X(p^*) + \beta p^* \times EV\left(b', \bar{w}\right) + \beta(1 - p^*) \times EV\left(b', \underline{w}\right)\right\}.$$

Note that the next-period value function V is the same as in the benchmark problem with non-contractible effort, since we are considering a one-period deviation. The first-order condition with respect to p yields

$$\begin{aligned}
0 &= \frac{d}{dp^*}\left\{Q\left(b', \underline{w}\right) b'\right\} \times u'(c) - X'(p^*) + \beta\left(EV\left(b', \bar{w}\right) - EV\left(b', \underline{w}\right)\right) \\
&\Rightarrow X'(p^*) = \left[\hat{Q}\left(b', \bar{w}\right) - \hat{Q}\left(b', \underline{w}\right)\right] b' \times u'(c) \\
&\quad +\beta\left[\int_0^{\infty} V\left(b', \phi', \bar{w}\right) dF(\phi) - \int_0^{\infty} V\left(b', \phi', \underline{w}\right) dF(\phi)\right] \\
&> \beta\left[\int_0^{\infty} V\left(b', \phi', \bar{w}\right) dF(\phi) - \int_0^{\infty} V\left(b', \phi', \underline{w}\right) dF(\phi)\right], \qquad (50)
\end{aligned}$$

where the last equation follows from the facts that $\hat{Q}(b', \bar{w}) > \hat{Q}(b', \underline{w})$, and that

$$\begin{aligned}
\frac{d}{dp}\left\{Q\left(b', \underline{w}\right) b'\right\} &= \frac{d}{dp}\left\{\left[p\hat{Q}\left(b', \bar{w}\right) + (1 - p)\hat{Q}\left(b', \underline{w}\right)\right] b'\right\} \\
&= \left[\hat{Q}\left(b', \bar{w}\right) - \hat{Q}\left(b', \underline{w}\right)\right] b'.
\end{aligned}$$

The right-hand side of the inequality in equation (50) is the optimal effort in the benchmark case with non-contractible effort, given in equation (14). This establishes the lemma. ■

Proof of Proposition 2. Consider, first, the range $b \in [0, b_1)$. Differentiate equation (14) with respect to b',

$$\begin{aligned} X''\left(\Psi\left(b'\right)\right)\Psi'\left(b'\right) &= \beta\left[\int_0^\infty \frac{\partial}{\partial b'}V\left(b',\phi',\bar{w}\right)dF\left(\phi\right) - \int_0^\infty \frac{\partial}{\partial b'}V\left(b',\phi',\underline{w}\right)dF\left(\phi\right)\right] \\ &= -\beta\left[1-F\left(\bar{\Phi}\left(b'\right)\right)\right]\times u'\left[Q\left(B\left(b',\bar{w}\right),\bar{w}\right)\times B\left(b',\bar{w}\right)+\bar{w}-b'\right] \\ &\quad +\beta\left[1-F\left(\underline{\Phi}\left(b'\right)\right)\right]\times\left[Q\left(B\left(b',\underline{w}\right)\right)\times B\left(b',\underline{w}\right)+\underline{w}-b'\right] \end{aligned} \tag{51}$$

Taking the limit of equation (51) as $b' \to 0$ yields

$$\begin{aligned} X''\left(\Psi\left(0\right)\right)\Psi'\left(0\right) &= \beta\left[1-F\left(\underline{\Phi}\left(0\right)\right)\right]\times\left[Q\left(B\left(0,\underline{w}\right)\right)\times B\left(0,\underline{w}\right)+\underline{w}-0\right] \\ &\quad -\beta\left[1-F\left(\bar{\Phi}\left(0\right)\right)\right]\times u'\left[Q\left(B\left(0,\bar{w}\right),\bar{w}\right)\times B\left(0,\bar{w}\right)+\bar{w}-0\right] \\ &= \beta\left[u'\left(Q\left(B\left(0,\underline{w}\right)\right)\times B\left(0,\underline{w}\right)+\underline{w}-0\right)-u'\left(\bar{w}\right)\right]>0, \end{aligned} \tag{52}$$

where the last equation uses the facts that $\bar{\Phi}(0) = \underline{\Phi}(0) = F(0) = 0$ and that during normal times $c = \bar{w}$ if $b = 0$. Note that during recession, the annualized present value of income is strictly smaller than $\bar{w}$. Therefore, it can never be optimal to choose consumption during recession larger than or equal to $\bar{w}$ when $b = 0$. Since the marginal utility of consumption is larger in a recession than during normal times, the right-hand side of equation (52) is strictly positive. Since $X'' > 0$, then $\lim_{b\to 0}\Psi'(b) = \Psi'(0) > 0$. By continuity, it follows then that $\Psi'(b)$ will be positive for a range of b close to $b = 0$, so there must exist a $b_1 > 0$ such that $\Psi'(b) > 0$ for all $b \in [0, b_1)$.
Consider, next, the range $b \in \left[\bar{b}^R, \bar{b}\right)$, in which case $F(\underline{\Phi}(b)) = 1$ and $F\left(\bar{\Phi}(b)\right) < 1$. This implies that equation (51) can be written as

$$X''\left(\Psi\left(b\right)\right)\Psi'\left(b\right) = -\beta\left[1-F\left(\bar{\Phi}\left(b\right)\right)\right]\times u'\left[Q\left(B\left(b,\bar{w}\right),\bar{w}\right)\times B\left(b,\bar{w}\right)+\bar{w}-b\right]<0,$$

which establishes that $\Psi'(b) < 0$ for all $b \in \left[\bar{b}^R, \bar{b}\right)$ and with strict inequality also for $b = \bar{b}^R$. By continuity, it follows then that there exists a $b_2 < \bar{b}^R$ such that $\Psi'(b) < 0$ for all $b \in \left(b_2, \bar{b}\right)$. Finally, in the range where $b \geq \bar{b}$, $F(\underline{\Phi}(b)) = F\left(\bar{\Phi}(b)\right) = 1$ so the right-hand side of equation (51) becomes zero, implying that $\Psi'(b) = 0$. ■

Proof of Lemma 4. Differentiating the bond revenue with respect to b yields

$$\begin{aligned} \frac{d}{db}\left\{Q\left(b,\underline{w}\right)b\right\} &= \frac{d}{db}\left\{pb\hat{Q}\left(b,\bar{w}\right)+\left(1-p\right)b\hat{Q}\left(b,\underline{w}\right)\right\}+\Psi'\left(b\right)\times\left(\hat{Q}\left(b,\bar{w}\right)-\hat{Q}\left(b,\underline{w}\right)\right)b \\ &= \Psi\left(b\right)\times\frac{1}{R}\left(1-F\left(\bar{\Phi}\left(b\right)\right)\right)+\left(1-\Psi\left(b\right)\right)\times\frac{1}{R}\left(1-F\left(\underline{\Phi}\left(b\right)\right)\right) \\ &\quad +\Psi'\left(b\right)\times\left(\hat{Q}\left(b,\bar{w}\right)-\hat{Q}\left(b,\underline{w}\right)\right)b, \end{aligned} \tag{53}$$

where the second equality can be derived as following:

$$\begin{aligned} \frac{d}{db}\left\{pb\hat{Q}\left(b,\bar{w}\right)+\left(1-p\right)b\hat{Q}\left(b,\underline{w}\right)\right\} &= p\frac{1}{R}\left(1-F\left(\bar{\Phi}\left(b\right)\right)\right)+\left(1-p\right)SEZ\left(b,\underline{w}\right) \\ &\quad +\left(1-p\right)\left[\begin{array}{c} -\frac{b}{R}f\left(\underline{\Phi}\left(b\right)\right)\times\underline{\Phi}'\left(b\right)- \\ \frac{1}{R}\frac{1}{b}\int_0^{\underline{\Phi}(b)}\left(\underline{\Phi}^{-1}\left(\phi\right)\times f\left(\phi\right)\ d\phi\right)+\frac{1}{R}bf\left(\underline{\Phi}\left(b\right)\right)\underline{\Phi}'\left(b\right) \end{array}\right] \\ &= p\frac{1}{R}\left(1-F\left(\bar{\Phi}\left(b\right)\right)\right)+\left(1-p\right)\frac{1}{R}\left(1-F\left(\underline{\Phi}\left(b\right)\right)\right) \end{aligned}$$

Consider, first, the case in which $\Psi(b)$ is constant, $\Psi(b) = p$. In this case, debt revenue is increasing for all $b < \bar{b}$, since, then, $p/R \times \left(1 - F\left(\bar{\Phi}(b)\right)\right) + (1-p)/R \times (1 - F(\underline{\Phi}(b))) > 0$. Moreover, it reaches a maximum at $b = \bar{b}$ (recall that $F\left(\bar{\Phi}(b)\right) < F(\underline{\Phi}(b))$ for all $b < \bar{b}$). This establishes that, if Ψ is constant, then $\bar{b}^R = \bar{b}$.

Consider, next, the general case. Proposition 2 implies that, in the range where $b \in [b_2, \bar{b}]$, $\Psi'(b) < 0$. Since $\hat{Q}(b, \bar{w}) > \hat{Q}(b, \underline{w})$, then, in a left neighborhood of $\bar{b}$, $\Psi'(b) \times \left[\hat{Q}(b, \bar{w}) - \hat{Q}(b, \underline{w})\right] b < 0$. This means that, starting from $\bar{b}$, one can increase the debt revenue by reducing debt, i.e., $\bar{b}^R < \bar{b}$. ■

Proof of Proposition 3. The procedure is analogous to the derivation of the CEE in normal times. The first order condition of (15) yields

$$0 = \frac{d}{db'}\left\{Q\left(b', \underline{w}\right) \times b'\right\} \times u'(c) + \beta\left[1 - \Psi\left(b'\right)\right]\frac{d}{db'}EV\left(b', \underline{w}\right) + \beta\Psi\left(b'\right)\frac{d}{db'}EV\left(b', \bar{w}\right),$$

where a term has been cancelled by the envelope theorem. Using the same argument as in the proof of Proposition 1 we can write:

$$\frac{d}{db}EV(b, \underline{w}) = -\left[1 - F(\underline{\Phi}(b))\right] \times u'\left[Q\left(B(b, \underline{w}), \underline{w}\right) \times B(b, \underline{w}) + \underline{w} - b\right].$$

Plugging this back into the FOC (after leading the expression by one period) yields the CEE

$$\begin{aligned}
0 \;\; = \;\; & u'(c) \times \left\{\Psi'\left(b'\right) \times R\left[\hat{Q}\left(b', \bar{w}\right) - \hat{Q}\left(b', \underline{w}\right)\right] b' + \right. \\
& \left. + \Psi\left(b'\right) \times \left(1 - F\left(\bar{\Phi}\left(b'\right)\right)\right) + \left[1 - \Psi\left(b'\right)\right] \times \left(1 - F\left(\underline{\Phi}\left(b'\right)\right)\right)\right\} \\
& -\beta R\left(\left[1 - \Psi\left(b'\right)\right] \times \left[1 - F\left(\underline{\Phi}\left(b'\right)\right)\right] u'\left(c'|_{H,\underline{w}}\right) + \Psi\left(b'\right) \times \left[1 - F\left(\bar{\Phi}\left(b'\right)\right)\right] u'\left(c'|_{H,\bar{w}}\right)\right),
\end{aligned}$$

where the equality follows from Lemma 4. Rearranging terms yields equation (17). ■

Proof of Lemma 5. Write the Lagrangian (with the associated multiplier, λ):

$$\max_{c,p} \frac{1}{1 - \beta(1-p)}(\underline{w} - c) + \frac{\beta}{1-\beta}\frac{p}{1 - \beta(1-p)}(\bar{w} - c) + \lambda\left(\frac{u(c)}{1-\beta} - \frac{1}{1 - \beta(1-p)}X(p) - \nu\right)$$

Differentiating with respect to c yields the standard condition, $u'(c) = 1/\lambda$., which can be substituted back into the program to eliminate λ. This yields:

$$\max_{p} \frac{1}{1 - \beta(1-p)}(\underline{w} - c) + \frac{\beta}{1-\beta}\frac{p}{1 - \beta(1-p)}(\bar{w} - c) + \frac{1}{u'(c)}\left(\frac{u(c)}{1-\beta} - \frac{1}{1 - \beta(1-p)}X(p) - \nu\right).$$

The first-order condition yields

$$\begin{aligned}
0 \;\; = \;\; & -\frac{\beta}{\left[1 - \beta(1-p)\right]^2}(\underline{w} - c) + \frac{\beta}{1-\beta}\frac{1-\beta}{\left[1 - \beta(1-p)\right]^2}(\bar{w} - c) \\
& + \frac{1}{u'(c)}\frac{\beta}{\left[1 - \beta(1-p)\right]^2}X(p) - \frac{1}{u'(c)}\frac{1}{1 - \beta(1-p)}X'(p).
\end{aligned}$$

Simplifying terms yields equation (20). ■

Proof of Proposition 4. We write the Lagrangian,

$$\begin{aligned}\bar{\Lambda} &= \int_{\aleph} \left[\bar{w} - c_\phi + \beta \bar{P}\left(\bar{\omega}_\phi, \bar{w}\right)\right] dF\left(\phi\right) + \bar{\mu}\left(\int_{\aleph} \left[u\left(c_\phi\right) + \beta \bar{\omega}_\phi\right] dF\left(\phi\right) - v\right) \\ &\quad + \int_{\aleph} \bar{\lambda}_\phi \left[u\left(c_\phi\right) + \beta \bar{\omega}_\phi - \bar{v} + \phi\right] d\phi,\end{aligned}$$

with the associated multipliers $\bar{\mu}$ and $\bar{\lambda}_\phi$. The first-order conditions yield

$$f\left(\phi\right) = u'\left(c_\phi\right)\left(\bar{\mu} f\left(\phi\right) + \bar{\lambda}_\phi\right), \tag{54}$$

$$\bar{\lambda}_\phi + \bar{\mu} f\left(\phi\right) = -\bar{P}'\left(\bar{\omega}_\phi\right) f\left(\phi\right). \tag{55}$$

The envelope condition yields

$$-\bar{P}'\left(\nu\right) = \bar{\mu} \tag{56}$$

The two first-order conditions and the envelope condition jointly imply that

$$u'\left(c_\phi\right) = -\frac{1}{\bar{P}'\left(\bar{\omega}_\phi\right)} \tag{57}$$

$$\bar{P}'\left(\bar{\omega}_\phi\right) = \bar{P}'\left(\nu\right) - \frac{\bar{\lambda}_\phi}{f\left(\phi\right)}. \tag{58}$$

Note that (57) is equivalent to (26) in the text. Consider, next, two cases, namely, when the PC is binding and when it is not binding.

When the PC is binding, $\bar{\lambda}_\phi > 0$. (58) implies then that $\bar{\omega}_\phi > \nu$. Then, (57) and (27) determine jointly the solution for $\left(c_\phi, \bar{\omega}_\phi\right)$. When the PC is not binding, $\bar{\lambda}_\phi = 0$. (58) implies then that $\bar{\omega}_\phi = \nu$ and $c_\phi = c\left(\nu\right)$. ■

Proof of Proposition 5. We prove the proposition by deriving a contradiction. To this aim, suppose that, for $\bar{\Pi}\left(b\right) = \bar{P}\left(\nu\right)$, the planner can deliver more utility to the agent than can the competitive equilibrium. Namely, $\nu > EV\left(b, \bar{w}\right)$. Then, since $\bar{P}$ is a decreasing strictly concave function, we must have that $\bar{P}\left(EV\left(b, \bar{w}\right)\right) > \bar{P}\left(\nu\right)$ and $\bar{P}'\left(EV\left(b, \bar{w}\right)\right) > \bar{P}'\left(\nu\right)$. We show that this inequality, along with the set of optimality conditions, induces a contradiction.

First, recall, that equation (5) implies that $\bar{\Pi}\left(b\right) = RQ\left(b, \bar{w}\right) b$. Thus,

$$\bar{P}\left(EV\left(b, \bar{w}\right)\right) > \bar{P}\left(\nu\right) = RQ\left(b, \bar{w}\right) b, \tag{59}$$

where $EV\left(b, \bar{w}\right)$ is decreasing in b. Differentiating the two sides of the inequality (59) with respect to b yields

$$\bar{P}'\left(EV\left(b, \bar{w}\right)\right) \times \frac{d}{db} EV\left(b, \bar{w}\right) > \frac{d}{db}\left[Q\left(b, \bar{w}\right) b\right] \times R = 1 - F\left(\bar{\Phi}\left(b\right)\right), \tag{60}$$

where the right-hand side equality follows from the proof of Lemma 2. Next, equation (48) implies that

$$\frac{d}{db} EV\left(b, \bar{w}\right) = -\left[1 - F\left(\bar{\Phi}\left(b\right)\right)\right] \times u'\left[C\left(b, \bar{w}\right)\right],$$

where $C\left(b, \bar{w}\right) = Q\left(B\left(b, \bar{w}\right), \bar{w}\right) \times B\left(b, \bar{w}\right) + \bar{w} - b$ is the consumption level in the competitive equilibrium when the debt b is honored. Plugging the expression of $\frac{d}{db} EV\left(b, \bar{w}\right)$ into (60), and simplifying terms, yields

$$u'\left(C\left(b, \bar{w}\right)\right) > -\frac{1}{\bar{P}'\left(EV\left(\bar{b}, \bar{w}\right)\right)}. \tag{61}$$

Next, note that $C(b, \bar{w}) = c(\nu)$. Equation (61) yields $u'(c(\nu)) > -1/\bar{P}'\left(EV\left(\bar{b}, \bar{w}\right)\right)$, while (57) yields that $u'(c(\nu)) = -1/\bar{P}'(\nu)$. Thus, the two conditions jointly imply that $\bar{P}'(\nu) > \bar{P}'\left(EV\left(\bar{b}, \bar{w}\right)\right)$ which in turn implies that $\nu < EV\left(\bar{b}, \bar{w}\right)$, since $\bar{P}$ is decreasing and concave. This contradicts the assumption that $\nu > EV\left(\bar{b}, \bar{w}\right)$.

The analysis thus far implies that $\nu \leq EV\left(\bar{b}, \bar{w}\right)$. We can also rule out that $\nu < EV(b, \bar{w})$, because it would contradict that the allocation chosen by the planner is constrained efficient. Therefore, $\nu = EV(b, \bar{w})$. ■

Proof of Proposition 6. We write the Lagrangian,

$$\begin{aligned}
\underline{\Lambda} \quad = \quad & \int_{\aleph} \left[\underline{w} - c_\phi + \beta\left((1-p_\phi)\,\underline{P}\left(\underline{\omega}_\phi\right) + p_\phi \bar{P}\left(\bar{\omega}_\phi\right)\right)\right] dF(s) \\
& + \underline{\mu}\left(\int_{\aleph}\left(u(c_\phi) - X(p_\phi) + \beta\left((1-p_\phi)\,\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right)\right) dF(\phi) - v\right) \\
& + \int_{\aleph} \underline{\lambda}_\phi \left(u(c_\phi) - X(p_\phi) + \beta\left((1-p_\phi)\,\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right) - \underline{\nu} + \phi\right) ds,
\end{aligned}$$

where $\underline{\mu}$ and $\underline{\lambda}_\phi$ denote the multipliers in recession. The first-order conditions yield:

$$f(\phi) = u'(c_s)\left(\underline{\mu}\, f(\phi) + \underline{\lambda}_\phi\right) \tag{62}$$
$$\underline{\lambda}_\phi + \underline{\mu}\, f(\phi) = -\underline{P}'\left(\underline{\omega}_\phi\right) f(\phi) \tag{63}$$
$$\underline{\lambda}_\phi + \underline{\mu}\, f(\phi) = -\bar{P}'\left(\bar{\omega}_\phi\right) f(\phi) \tag{64}$$
$$\beta\left(\bar{P}\left(\bar{\omega}_\phi\right) - \underline{P}\left(\underline{\omega}_\phi\right)\right) f(\phi) = \left(\underline{\lambda}_\phi + \underline{\mu}\, f(\phi)\right)\left(X'(p_\phi) - \beta\left(\bar{\omega}_\phi - \underline{\omega}_\phi\right)\right) \tag{65}$$

The envelope condition yields:

$$-\underline{P}'(\nu) = -\underline{\mu} \tag{66}$$

Combining the first-order conditions and the envelope condition yields:

$$\begin{aligned}
u'(c_s) &= -\frac{1}{\underline{P}'\left(\underline{\omega}_\phi\right)} \qquad (67) \\
\underline{P}'\left(\underline{\omega}_\phi\right) &= \underline{P}'(\nu) - \frac{\underline{\lambda}_\phi}{f(\phi)} \\
\underline{P}'\left(\underline{\omega}_\phi\right) &= \bar{P}'\left(\bar{\omega}_\phi\right) \\
X'(p_\phi) &= \beta\left(u'(c_\phi)\left(\bar{P}\left(\bar{\omega}_\phi\right) - \underline{P}\left(\underline{\omega}_\phi\right)\right) + \left(\bar{\omega}_\phi - \underline{\omega}_\phi\right)\right)
\end{aligned}$$

We distinguish two cases, namely, when the PC is binding and when it is not binding.

(I) When the PC is binding and the recession continues, $\underline{\lambda}_\phi > 0$, $\underline{\omega}_\phi > \nu$, and

$$u(c_\phi) - X(p_\phi) + \beta\left((1-p_\phi)\,\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right) = \underline{\nu} - \phi_\phi$$

Then, (32), (34), (35) and (33) determine jointly the solution for $\left(c_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi, p_\phi\right)$. In this case, there is no history dependence, i.e., v does not matter.

(II) When the PC is not binding, $\underline{\lambda}_\phi = 0$. Then, $\underline{\omega}_\phi = \nu$ and $c_\phi = \underline{c}(\nu)$. The solution is history dependent. Moreover, (35) implies that

$$\beta \times \left\{u'\left(\underline{c}(\nu)\right)\left[\bar{P}\left(\bar{\omega}(\nu)\right) - \underline{P}(\nu)\right] + \left[\bar{\omega}(\nu) - \nu\right]\right\} = X'(p(\nu)), \tag{68}$$

namely, the planner requires constant effort over the set of states for which the constraint is not binding: $p_\phi = p(\nu)$. Differentiating the left-hand side yields

$$\begin{aligned} & \underbrace{u''(\underline{c}(\nu))\,\underline{c}'(\nu) \times \left[\bar{P}(\bar{\omega}(\nu)) - \underline{P}(\nu)\right]}_{<0} + \left[u'(\underline{c}(\nu))\,\underline{P}'(\nu) + 1\right]\left(\bar{\omega}'(\nu) - 1\right) \\ = \; & u''(\underline{c}(\nu))\,\underline{c}'(\nu) \times \left(\bar{P}(\bar{\omega}(\nu)) - \underline{P}(\nu)\right) < 0 \end{aligned}$$

since, recall, (32) implies that $\bar{P}'(\nu) = -1/u'(\underline{c}(\nu))$. This implies that the right-hand side must also be decreasing in ν. Since X is concave and increasing, this implies in turn that $p(\nu)$ must be increasing in ν.

(III) Finally, we prove that $\underline{\omega}_\phi < \bar{\omega}_\phi$. To this aim, note that $\underline{\mu} > \bar{\mu}$ since it is more expensive to deliver a given promised utility during recession than in normal times. Hence, the respective envelope conditions, (56) and (66), imply that, for any x, $\underline{P}'(x) < \bar{P}'(x)$. Next, note that since both $\underline{P}(x)$ and $\bar{P}(x)$ are decreasing concave functions, then $\underline{P}'(x_1) = \bar{P}'(x_2) \Leftrightarrow x_1 < x_2$. Therefore, the condition $\underline{P}'\left(\underline{\omega}_\phi\right) = \bar{P}'(\bar{\omega}_\phi)$ (see equation (67)) implies that $\underline{\omega}_\phi < \bar{\omega}_\phi$. ∎

Proof of Proposition 7. We proceed in two steps: first, we derive the CEEs (step A), and then we show that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$ (step B).

Step A: The first order conditions with respect to $b'_{\bar{w}}$ and $b'_{\underline{w}}$ in problem (36) yields

$$0 = u'(c) \times \frac{d}{db'_{\bar{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) + \beta\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \frac{d}{db'_{\bar{w}}} EV\left(b'_{\bar{w}}, \bar{w}\right),$$

$$0 = u'(c) \times \frac{d}{db'_{\underline{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) + \beta\left[1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right] \frac{d}{db'_{\underline{w}}} EV\left(b'_{\underline{w}}, \underline{w}\right),$$

where $REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \equiv b'_{\underline{w}} \times Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) + b'_{\bar{w}} \times Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ is the bond revenue and c is defined in the Proposition. Note that both equations have been simplified using an envelope condition. The value function has a kink at $b_{\underline{w}} = \hat{b}(\phi, \underline{w})$. Consider, first, the range of realizations $\phi \in [\underline{\Phi}(b), \infty)$, implying that $b_{\underline{w}} < \hat{b}(\phi, \underline{w})$. Differentiating the value function yields:

$$\begin{aligned} \frac{d}{db} V(b, \phi, \bar{w}) &= -u'\left[Q(B(b, \bar{w}), \bar{w}) \times B(b, \bar{w}) + \bar{w} - b\right], \\ \frac{d}{db} V(b, \phi, \underline{w}) &= -u'\left[Q_{\underline{w}}\left(B_{\underline{w}}(b), B_{\bar{w}}(b)\right) \times B_{\underline{w}}(b) + Q_{\bar{w}}\left(B_{\underline{w}}(b), B_{\bar{w}}(b)\right) \times B_{\bar{w}}(b) + \underline{w} - b\right], \end{aligned}$$

where $B_{\underline{w}}$ and $B_{\bar{w}}$ denote the optimal issuance of the two assets, respectively. Next, consider the range of realizations $\phi < \underline{\Phi}(b)$, implying that $b_{\underline{w}} \geq \hat{b}(\phi, \underline{w})$. In this case, $\frac{d}{db} V(b, \phi, \underline{w}) = 0$.

In analogy with equation (48), we obtain:

$$\frac{d}{db_{\underline{w}}} EV\left(b_{\underline{w}}, \underline{w}\right) = -\left[1 - F\left(\underline{\Phi}\left(b_{\underline{w}}\right)\right)\right] \times u'\left(c'|_{H,\underline{w}}\right). \tag{69}$$

Plugging (48) and (69) into the respective first-order conditions, and leading by one period, yields

$$u'(c) \times \frac{d}{db'_{\bar{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \beta\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times \left[1 - F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right] \times u'\left(c'|_{H,\bar{w}}\right),$$

$$u'(c) \times \frac{d}{db'_{\underline{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \beta\left[1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right] \times \left[1 - F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right] \times u'\left(c'|_{H,\underline{w}}\right).$$

The marginal revenues from issuing recession-contingent debt is given by:

$$\begin{aligned}
&\frac{d}{db'_{\underline{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \\
= \quad &\frac{1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{R}\left(1 - F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right) - \frac{\frac{\partial \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\partial b'_{\underline{w}}}}{1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + b'_{\bar{w}} \times \frac{\partial}{\partial b'_{\underline{w}}} Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \\
= \quad &\frac{1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{R}\left(1 - F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right) + \frac{\partial \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\partial b'_{\underline{w}}} \times \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right), \qquad (70)
\end{aligned}$$

where, note, $\frac{\partial}{\partial b'_{\underline{w}}} Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \frac{\partial \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\partial b'_{\underline{w}}} \frac{Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}$ follows from applying standard differentiation to the definition of $Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ in equation (38). Applying the same methodology to the recovery-contingent debt, we obtain:

$$\frac{d}{db'_{\bar{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \frac{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{R}\left(1 - F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right) + \frac{\partial \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\partial b'_{\bar{w}}} \times \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right).$$

The CEEs conditional on the recession continuing and ending, respectively, are then:

$$\begin{aligned}
\beta \frac{u'\left(c'|_{H,\underline{w}}\right)}{u'(c)} &= \frac{1}{R} + \frac{\partial}{\partial b'_{\underline{w}}} \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times \frac{\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\left(1 - F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right)\left[1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right]} \\
\beta \frac{u'\left(c'|_{H,\bar{w}}\right)}{u'(c)} &= \frac{1}{R} + \frac{\partial}{\partial b'_{\bar{w}}} \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times \frac{\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\left(1 - F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right) \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}.
\end{aligned}$$

Setting $\beta R = 1$ yields equations (39)-(40).

Step B: Next, we prove that, in equilibrium, $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$. To prove the claim, it is useful to define the two functions

$$\begin{aligned}
\theta_{\bar{w}}\left(b'_{\bar{w}}\right) &\equiv \frac{Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}}}{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} = \frac{1}{R}\left(\left(1 - F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right) b'_{\bar{w}} + \int_0^{\bar{\Phi}\left(b'_{\bar{w}}\right)} \bar{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right), \\
\theta_{\underline{w}}\left(b'_{\underline{w}}\right) &\equiv \frac{Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}}}{1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} = \frac{1}{R}\left(\left(1 - F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right) b'_{\underline{w}} + \int_0^{\underline{\Phi}\left(b'_{\underline{w}}\right)} \underline{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right),
\end{aligned}$$

where, recall, $\underline{\Phi}(x) > \bar{\Phi}(x)$ and $F\left(\underline{\Phi}(x)\right) > F\left(\bar{\Phi}(x)\right)$. Note that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \theta_{\bar{w}}\left(b'_{\bar{w}}\right) - \theta_{\underline{w}}\left(b'_{\underline{w}}\right)$, where both $\theta_{\bar{w}}$ and $\theta_{\underline{w}}$ are increasing functions in the relevant range, i.e., $b'_{\bar{w}} \leq \bar{b}$ and $b'_{\underline{w}} \leq \bar{b}$. We proceed in two steps. First, we show that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0 \Rightarrow b'_{\bar{w}} > b'_{\underline{w}}$ (step B1). Next, we show that $b'_{\bar{w}} > b'_{\underline{w}} \Rightarrow \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$ (step B2). Steps B1 and B2 establish jointly a contradiction ruling out that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0$ (step B3).

Step B1: Suppose that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0$. Then, the CEEs (39)–(40) and the assumption that $u'' < 0$, imply that

$$c'|_{H,\bar{w}} \leq c \leq c'|_{H,\underline{w}}. \qquad (71)$$

Suppose, to derive a contradiction, that $b'_{\underline{w}} \geq b'_{\bar{w}}$. Recall that, if the recession ends and debt is honored, debt remains constant, i.e., $b'' = B(b'_{\bar{w}}) = b'_{\bar{w}}$. Moreover, $Q(b'_{\bar{w}}, \bar{w}) = Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)/\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$. Thus, $c'|_{H,\bar{w}} = Q(b'_{\bar{w}}, \bar{w})\, b'_{\bar{w}} + \bar{w} - b'_{\bar{w}} = \theta_{\bar{w}}(b'_{\bar{w}}) + \bar{w} - b'_{\bar{w}}$.

$$\begin{aligned}
c'|_{H,\bar{w}} &= \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \left(1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right)\theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \bar{w} - b'_{\bar{w}} \\
&\geq \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \left(1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right)\theta_{\bar{w}}\left(b'_{\underline{w}}\right) + \bar{w} - b'_{\underline{w}} \\
&\geq \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \left(1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right)\theta_{\underline{w}}\left(b'_{\underline{w}}\right) + \bar{w} - b'_{\underline{w}} \\
&> \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \left(1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right)\theta_{\underline{w}}\left(b'_{\underline{w}}\right) + \underline{w} - b'_{\underline{w}} = c'|_{H,\underline{w}}.
\end{aligned}$$

The first inequality follows from the assumption that $b'_{\underline{w}} \geq b'_{\bar{w}}$ and the fact that $(1-p)\,\theta_{\bar{w}}(x) - x < 0$ for any $p \in [0,1]$, which is due to the fact that $\theta_{\bar{w}}(x) \leq x/R < x$ for any x. The second inequality follows from the fact that $\theta_{\bar{w}}\left(b'_{\underline{w}}\right) \geq \theta_{\underline{w}}\left(b'_{\underline{w}}\right)$, see equation (72) below. The last inequality follows from the maintained assumption that $\bar{w} > \underline{w}$. We have therefore proven that if $b'_{\underline{w}} \geq b'_{\bar{w}}$ then $c'|_{H,\bar{w}} > c'|_{H,\underline{w}}$, which contradicts (71) and, hence, implies that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \geq 0$. We conclude from Step B1 that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0 \Rightarrow b'_{\bar{w}} > b'_{\underline{w}}$.

Step B2: Suppose that $b'_{\underline{w}} = b'_{\bar{w}} = x$. Then, for any x:

$$\begin{aligned}
\Delta(x,x) &= \theta_{\bar{w}}(x) - \theta_{\underline{w}}(x) = \frac{1}{R}\underbrace{\int_{\bar{\Phi}(x)}^{\underline{\Phi}(x)} \left(\left(x - \underline{\Phi}^{-1}(\phi)\right) \times f(\phi)\ d\phi\right)}_{>0} \qquad (72)\\
&\quad + \frac{1}{R}\underbrace{\int_{0}^{\bar{\Phi}(x)} \left(\left(\bar{\Phi}^{-1}(\phi) - \underline{\Phi}^{-1}(\phi)\right) \times f(\phi)\ d\phi\right)}_{>0} > 0.
\end{aligned}$$

Since $\theta_{\bar{w}}(x)$ is an increasing function for $x \leq \bar{b}$, equation (72) implies that $\theta_{\bar{w}}(b'_{\bar{w}}) > \theta_{\underline{w}}\left(b'_{\underline{w}}\right)$, for all $b'_{\underline{w}} < b'_{\bar{w}} \leq \bar{b}$. We conclude from Step B2 that $b'_{\bar{w}} > b'_{\underline{w}} \Rightarrow \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$.

Step B3: Putting together the conclusions of Step B1 and Step B2, we derive a contradition: $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0 \Rightarrow b'_{\bar{w}} > b'_{\underline{w}} \Rightarrow \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$. Therefore, we must have that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$. ■

Proof of Proposition 8. The strategy of the proof is the same as that of Proposition 5. We prove the proposition by deriving a contradiction. To this aim, suppose that, for $\underline{\Pi}\left(b_{\underline{w}}\right) = \underline{P}(\nu)$, the planner can deliver more utility than the agent gets in the competitive equilibrium. Namely, $\nu > EV\left(b_{\underline{w}}, \underline{w}\right)$. Then, since $\underline{P}$ is a decreasing strictly concave function, we must have that $\underline{P}\left(EV\left(b_{\underline{w}}, \underline{w}\right)\right) > \underline{P}(\nu)$ and $\underline{P}'\left(EV\left(b_{\underline{w}}, \underline{w}\right)\right) > \underline{P}'(\nu)$. Note that, absent moral hazard, the price of recession-contingent debt is independent of the amount of recovery-contingent debt. It is therefore legitimate to define $\tilde{Q}_{\underline{w}}\left(b_{\underline{w}}\right) \equiv Q_{\underline{w}}\left(b_{\underline{w}}, b_{\bar{w}}\right)$.

First, the same argument invoked in the proof of Proposition 5 implies that $\underline{\Pi}\left(b_{\underline{w}}\right) = \frac{R}{1-p}\tilde{Q}_{\underline{w}}\left(b_{\underline{w}}\right) b_{\underline{w}}$. Hence,

$$\underline{P}\left(EV\left(b_{\underline{w}}, \underline{w}\right)\right) > \underline{P}(\nu) = \frac{R}{1-p}\tilde{Q}_{\underline{w}}\left(b_{\underline{w}}\right) b_{\underline{w}}. \qquad (73)$$

where $EV\left(b_{\underline{w}}, \underline{w}\right)$ is decreasing in $b_{\underline{w}}$. Differentiating the two sides of the inequality (73) with respect to $b_{\underline{w}}$ yields:

$$\underline{P}'\left(EV\left(b_{\underline{w}},\underline{w}\right)\right)\times\frac{d}{db_{\underline{w}}}EV\left(b_{\underline{w}},\underline{w}\right) \tag{74}$$
$$> \frac{R}{1-p}\frac{d}{db_{\underline{w}}}\left(\tilde{Q}_{\underline{w}}\left(b_{\underline{w}}\right)b_{\underline{w}}\right)=-\left[1-F\left(\underline{\Phi}\left(b_{\underline{w}}\right)\right)\right],$$

where the right-hand side equality follows from equation (70). Next, equation (48) implies that

$$\frac{d}{db}EV\left(b,\bar{w}\right)=-\left[1-F\left(\bar{\Phi}\left(b\right)\right)\right]\times u'\left(C\left(b,\bar{w}\right)\right),$$

where $C\left(b_{\underline{w}},\underline{w}\right)$ is the consumption level assuming that the recession-contingent debt $b_{\underline{w}}$ is honored. Plugging in the expression of $\frac{d}{db_{\underline{w}}}EV\left(b_{\underline{w}},\bar{w}\right)$ allows us to simplify (74) as follows:

$$u'\left(C\left(b_{\underline{w}},\underline{w}\right)\right)>-\frac{1}{\underline{P}'\left(EV\left(b_{\underline{w}},\underline{w}\right)\right)}. \tag{75}$$

Next, note that $C\left(b_{\underline{w}},\underline{w}\right)=c\left(\nu\right)$. Equation (75) yields $u'\left(c\left(\nu\right)\right)>-\frac{1}{\bar{P}'\left(EV\left(b_{\bar{w}},\bar{w}\right)\right)}$, while (67) yields that $u'\left(c\left(\nu\right)\right)=-\frac{1}{\underline{P}'\left(\nu\right)}$. Thus, the two conditions jointly imply that $-\frac{1}{\underline{P}'\left(\nu\right)}>-\frac{1}{\underline{P}'\left(EV\left(b_{\bar{w}},\underline{w}\right)\right)}$ which in turn implies that $\nu<EV\left(b_{\bar{w}},\underline{w}\right)$, since $\underline{P}$ is decreasing and concave. This contradicts the assumption that $\nu>EV\left(b_{\bar{w}},\underline{w}\right)$.

The analysis thus far establishes that $\nu\leq EV\left(b_{\bar{w}},\underline{w}\right)$. We can also rule out that $\nu<EV\left(b_{\bar{w}},\underline{w}\right)$ because it would contradict that the allocation chosen by the planner is constrained efficient. Therefore, $\nu=EV\left(b_{\bar{w}},\underline{w}\right)$. ■

Proof of Lemma 6. The Lagrangian of the planner's problem reads as

$$\begin{aligned}
\underline{\Lambda} \;=\; & \int_{\aleph}\left[\underline{w}-c_{\phi}+\beta\left(\left(1-p_{\phi}\right)\underline{P}\left(\underline{\omega}_{\phi}\right)+p_{\phi}\bar{P}\left(\bar{\omega}_{\phi}\right)\right)\right]dF\left(\phi\right)\\
& +\underline{\mu}\left(\int_{\aleph}\left(u\left(c_{\phi}\right)-X\left(p_{\phi}\right)+\beta\left(\left(1-p_{\phi}\right)\underline{\omega}_{\phi}+p_{\phi}\bar{\omega}_{\phi}\right)\right)dF\left(\phi\right)-\nu\right)\\
& +\int_{\aleph}\underline{\lambda}_{\phi}\left(u\left(c_{\phi}\right)-X\left(p_{\phi}\right)+\beta\left(\left(1-p_{\phi}\right)\underline{\omega}_{\phi}+p_{\phi}\bar{\omega}_{\phi}\right)-\underline{\nu}+\phi\right)d\phi\\
& +\int_{\aleph}\gamma_{\phi}\left(Z\left(b_{0}\right)-X\left(p_{\phi}\right)+\beta\left(\left(1-p_{\phi}\right)\underline{\omega}_{\phi}+p_{\phi}\bar{\omega}_{\phi}\right)\right)d\phi,
\end{aligned}$$

where the Lagrange multipliers of the PC and IC must be non-negative for all ϕ, $\underline{\lambda}_{\phi}\geq0,\gamma_{\phi}\geq0$. The first-order conditions yield:

$$f\left(\phi\right) = u'\left(c_{\phi}\right)\left(\underline{\mu}\,f\left(\phi\right)+\underline{\lambda}_{\phi}\right), \tag{76}$$
$$\underline{\lambda}_{\phi}+\underline{\mu}\,f\left(\phi\right)+\gamma_{\phi} = -\underline{P}'\left(\underline{\omega}_{\phi}\right)\,f\left(\phi\right), \tag{77}$$
$$\underline{\lambda}_{\phi}+\underline{\mu}\,f\left(\phi\right)+\gamma_{\phi} = -\bar{P}'\left(\bar{\omega}_{\phi}\right)\,f\left(\phi\right), \tag{78}$$
$$\beta\left(\bar{P}\left(\bar{\omega}_{\phi}\right)-\underline{P}\left(\underline{\omega}_{\phi}\right)\right)\,f\left(\phi\right) = \left(\underline{\lambda}_{\phi}+\underline{\mu}\,f\left(\phi\right)+\gamma_{\phi}\right)\left(X'\left(p_{\phi}\right)-\beta\left(\bar{\omega}_{\phi}-\underline{\omega}_{\phi}\right)\right), \tag{79}$$

while the envelope condition yields $\underline{P}'\left(\nu\right)=\underline{\mu}$.

The first order conditions (77)–(79) imply equations (44)-(45) in the text. Since $\underline{P}$ and $\bar{P}$ are monotonic and concave, equation (44) implies a positive relationship between $\underline{\omega}_\phi$ and $\bar{\omega}_\phi$. Equation (45) yields then a negative relationship between p_ϕ and $\underline{\omega}_\phi$. Consider, next, the IC constraint. When the IC constraint is binding, equations (43), (44), and (45) pin down a unique solution for p_ϕ, $\underline{\omega}_\phi$ and $\bar{\omega}_\phi$, denoted by $(p^*, \underline{\omega}^*, \bar{\omega}^*)$. ■

Proof of Proposition 9. This proof builds on the proof of Lemma 6. Suppose $\nu > \underline{\omega}^*$ (case 1). Then the IC is not binding in the initial period. Moreover, by Proposition 6, the promised utility is non-decreasing over time. Thus, the IC will never bind in the future, and can be ignored altogether. Suppose, next, that $\nu \le \underline{\omega}^*$ (case 2). We first determine the upper bound on ϕ, denoted by ϕ^*, such that the PC is binding while the IC is not binding. Let $\left(c_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right)$ denote the solution characterized in Proposition 6 when the IC is not binding and $(c^*_\phi, p^*, \underline{\omega}^*, \bar{\omega}^*)$ the solution characterized in Proposition 9 when the IC is binding. Note that c^*_ϕ is defined in (46). At the threshold realization ϕ^*, the two allocations must be equivalent, i.e.,

$$(c_{\phi^*}, p_{\phi^*}, \underline{\omega}_{\phi^*}, \bar{\omega}_{\phi^*}) = (c^*_{\phi^*}, p^*, \underline{\omega}^*, \bar{\omega}^*).$$

Given a promised utility of $\underline{\omega}^*$, the promise-keeping constraint implies:

$$\begin{aligned}
\underline{\omega}^* \;&=\; \underline{\omega}_{\phi^*} = \int_0^{\phi^*} \left[u(c_\phi) - X(p_\phi) + \beta\left[(1-p_\phi)\underline{\omega}_\phi + p_\phi\bar{\omega}_\phi\right]\right] dF(\phi) \\
&\quad + \int_{\phi^*}^{\phi_{\max}} \left[u(c_{\phi^*}) - X(p_{\phi^*}) + \beta\left[(1-p_{\phi^*})\underline{\omega}_{\phi^*} + p_{\phi^*}\bar{\omega}_{\phi^*}\right]\right] dF(\phi) \\
&= \int_0^{\phi^*} (\underline{v} - \phi)\, dF(\phi) + (\underline{v} - \phi^*)\left[1 - F(\phi^*)\right] = \underline{v} - \int_0^{\phi^*} \phi dF(\phi) - \phi^*\left[1 - F(\phi^*)\right]. \qquad (80)
\end{aligned}$$

Since $\underline{\omega}_\phi$ is decreasing in ϕ, then ϕ^* is unique. Moreover, if $\phi < \phi^*$, then $\underline{\omega}_\phi > \underline{\omega}^*$. In this case, the solution is not history-dependent and is determined as in Proposition 6 (case 2.a). If, to the opposite, $\phi \ge \phi^*$, then $\underline{\omega}_\phi = \underline{\omega}^*$. Two subcases must be distinguished here. First, if $\phi \ge \phi^*$ and $\nu = \underline{\omega}^*$, then the multipliers of both the IC and PC must be zero, $\underline{\lambda}_\phi = \gamma_\phi = 0$, because the planner keeps the triplet $(p^*, \underline{\omega}^*, \bar{\omega}^*)$ constant. More formally, the envelope condition together with equation (77) implies that

$$\underline{P}'(\nu) = \underline{P}'\left(\underline{\omega}_\phi\right) + \frac{\underline{\lambda}_\phi + \gamma_\phi}{f(\phi)}.$$

Thus, $\nu = \underline{\omega}_\phi = \underline{\omega}^*$, and both the multiplier of the IC and that of the PC must be zero. In other words, as long as the IC was binding in the previous period, and continues to bind in the current period, the planner keeps consumption, effort and promised utilities constant.

Second, if $\phi \ge \phi^*$ and $\nu < \underline{\omega}^*$, then the planner must adjust promised utility, $\underline{\omega}_\phi = \underline{\omega}^*$, to satisfy the IC. In this case, the multiplier of the IC must be strictly positive, $\gamma_\phi > 0$.

For the determination of consumption, two separate cases must be distinguished. In the first case (2.b), $\phi \le \tilde{\phi}(\nu)$ (where the expression for $\tilde{\phi}(\nu)$ is given below), so that both the IC and the PC bind. In this case, $\underline{\lambda}_\phi > 0$, and the IC and the PC determine jointly the consumption level, whose level is given by c^*_ϕ as defined in equation (46). In the second case (2.c), $\phi > \tilde{\phi}(\nu)$, so that the PC does not bind. In this region, $\underline{\lambda}_\phi = 0$, and the consumption level provided by the planner is pinned down by the promise-keeping constraint (30), and given by equation (47).

Next, we determine the unique threshold, $\tilde{\phi}(\nu)$, that sets apart case (2.b) from case (2.c) and show that $\tilde{\phi}(\nu) \geq \phi^*$. Because the PC holds with equality at the threshold realization $\tilde{\phi}(\nu)$, the consumption level for all realizations of $\phi > \tilde{\phi}(\nu)$ where the PC is not binding must be given by

$$c^*_{\tilde{\phi}(\nu)} = u^{-1}\left(\underline{\nu} - \tilde{\phi}(\nu) - Z\left(b_0\right)\right).$$

This condition, together with the promise-keeping constraint, fully characterizes the threshold $\tilde{\phi}(\nu)$:

$$\begin{aligned}
\nu &= \int_0^{\phi^*} \left[u(c_\phi) - X(p_\phi) + \beta\left[(1-p_\phi)\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right]\right] dF(\phi) \\
&\quad + \int_{\phi^*}^{\tilde{\phi}(\nu)} \left[u(c^*_\phi) - X(p^*) + \beta\left[(1-p^*)\underline{\omega}^* + p^* \bar{\omega}^*\right]\right] dF(\phi) \\
&\quad + \int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \left[u(c^*_{\tilde{\phi}(\nu)}) - X(p_{\phi^*}) + \beta\left[(1-p_{\phi^*})\underline{\omega}_{\phi^*} + p_{\phi^*} \bar{\omega}_{\phi^*}\right]\right] dF(\phi) \\
&= \int_0^{\phi^*} (\underline{v} - \phi)\, dF(\phi) + \int_{\phi^*}^{\tilde{\phi}(\nu)} (\underline{v} - \phi)\, dF(\phi) + \int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \left[u(c^*_{\tilde{\phi}(\nu)}) + Z(b_0)\right] dF(\phi) \\
&= F(\tilde{\phi}(\nu))\underline{v} - \int_0^{\tilde{\phi}(\nu)} \phi dF(\phi) + \left(\underline{v} - \tilde{\phi}(\nu)\right)\left[1 - F(\tilde{\phi}(\nu))\right] = \underline{v} - \int_0^{\tilde{\phi}(\nu)} \phi dF(\phi) - \tilde{\phi}(\nu)\left[1 - F(\tilde{\phi}(\nu))\right]
\end{aligned} \tag{81}$$

If $\nu = \underline{\omega}^*$ then equations (80) and (81) imply that $\phi^* = \tilde{\phi}(\nu)$. The right-hand side of (81) is falling in $\tilde{\phi}$. It follows that if $\nu < \underline{\omega}^*$, then $\tilde{\phi}(\nu) \geq \tilde{\phi}(\underline{\omega}^*) = \phi^*$. ■

10.2 Strict concavity and differentiability of the profit function

In this appendix, we prove that the functions $\bar{P}(\nu)$ and $\underline{P}(\nu)$ are strictly concave and differentiable. The proof strategy follows Thomas and Worrall (1990, Proof of Proposition 1) and Ljungqvist and Sargent (2012, Section 20.5.3).

10.2.1 Normal times

Let $\nu \in [\bar{\omega}_{\min}, \bar{\omega}_{\max}] \equiv \bar{\Omega} \subset \mathbb{R}$, where the bounds of the interval are given by

$$[\bar{\omega}_{\min}, \bar{\omega}_{\max}] = [\bar{\nu} - E[\phi], u(\bar{w})/(1-\beta)].$$

Let d_∞ be the supremum norm and $\bar{\Gamma} \equiv C[\bar{\Omega}]$ the set of continuous bounded functions that maps the interval $\bar{\Omega}$ into $\mathbb{R}$, such that the pair $(\bar{\Gamma}, d_\infty)$ is a complete metric space. Let T denote the maximum operator associated with the right-hand side of the Bellman equation

$$\begin{aligned} \bar{P}(\nu) &= \max_{\{c_\phi \in [0,\bar{w}], \bar{\omega}_\phi \in \bar{\Omega}\}_{\phi \in \aleph}} \int_{\phi \in \aleph} \left[\bar{w} - c_\phi + \beta \bar{P}(\bar{\omega}_\phi)\right] dF(\phi) \\ &\equiv T(\bar{P})(\nu), \end{aligned}$$

where the maximization is subject to the constraints

$$\begin{aligned} \int_{\phi \in \aleph} \left[u(c_\phi) + \beta \bar{\omega}_\phi\right] dF(\phi) &\geq \nu \\ u(c_\phi) + \beta \bar{\omega}_\phi &\geq \bar{\nu} - \phi, \quad \forall \phi \in \aleph. \end{aligned}$$

The operator T maps the metric space $(\bar{\Gamma}, d_\infty)$ into itself. According to Blackwell's sufficient conditions, T is a contraction mapping (see Ljungqvist and Sargent (2012, Appendix A.1) if: (i) T is monotone, (ii) T discounts.

1. Monotonicity: Let $x(\nu) \geq y(\nu)$, $\nu \in \bar{\Omega}$, $x, z \in \bar{\Gamma}$. Then

$$\begin{aligned} T(x) &= \max_{\{c_\phi \in [0,\bar{w}], \bar{\omega}_\phi \in \bar{\Omega}\}_{\phi \in \aleph}} \int_{\phi \in \aleph} \left[\bar{w} - c_\phi + \beta x(\bar{\omega}_\phi)\right] dF(\phi) \\ &\geq \max_{\{c_\phi \in [0,\bar{w}], \bar{\omega}_\phi \in \bar{\Omega}\}_{\phi \in \aleph}} \int_{\phi \in \aleph} \left[\bar{w} - c_\phi + \beta y(\bar{\omega}_\phi)\right] dF(\phi) \\ &= T(y). \end{aligned}$$

2. Discounting: Let z be a real constant. Then

$$\begin{aligned} T(\bar{P} + z) &= \max_{\{c_\phi \in [0,\bar{w}], \bar{\omega}_\phi \in \bar{\Omega}\}_{\phi \in \aleph}} \int_{\phi \in \aleph} \left[\bar{w} - c_\phi + \beta \bar{P}(\bar{\omega}_\phi) + \beta z\right] dF(\phi) \\ &= T(\bar{P}) + \beta z \\ &< T(\bar{P}) + z, \end{aligned}$$

since $\beta < 1$.

Thus, T is indeed a contraction mapping and $\bar{P}$ is the unique fixed-point in $\bar{\Gamma}$ satisfying

$$\bar{P}(\nu) = T(\bar{P})(\nu).$$

We will show in the next step that T maps strictly concave functions into strictly concave functions. Let $\nu' \neq \nu'' \in \bar{\Omega}$, $\delta \in (0,1)$, $\nu^o = \delta\nu' + (1-\delta)\nu''$, and $\bar{P}_{k-1} \in \bar{\Gamma}$ strictly concave

$$\bar{P}_{k-1}(\delta\nu' + (1-\delta)\nu'') > \delta\bar{P}_{k-1}(\nu') + (1-\delta)\bar{P}_{k-1}(\nu'').$$

Furthermore, let $\bar{P}_k(\nu) = T(\bar{P}_{k-1})(\nu)$. Then, the profit function evaluated at the optimal contract implies

$$\begin{aligned}
\delta\bar{P}_k(\nu') + (1-\delta)\bar{P}_k(\nu'') &= \int_{\phi\in\aleph} \left[\begin{array}{c} \bar{w} - [\delta c_\phi(\nu') + (1-\delta)c_\phi(\nu'')] + \\ \beta\left[\delta\bar{P}_{k-1}(\bar{\omega}_\phi(\nu')) + (1-\delta)\bar{P}_{k-1}(\bar{\omega}_\phi(\nu''))\right] \end{array} \right] dF(\phi) \\
&< \int_{\phi\in\aleph} \left[\begin{array}{c} \bar{w} - [\delta c_\phi(\nu') + (1-\delta)c_\phi(\nu'')] + \\ \beta\bar{P}_{k-1}(\delta\bar{\omega}_\phi(\nu') + (1-\delta)\bar{\omega}_\phi(\nu'')) \end{array} \right] dF(\phi) \\
&< \int_{\phi\in\aleph} \left[\begin{array}{c} \bar{w} - u^{-1}\left(\delta u(c_\phi(\nu')) + (1-\delta)u(c_\phi(\nu''))\right) + \\ \beta\bar{P}_{k-1}\left(\delta\bar{\omega}_\phi(\nu') + (1-\delta)\bar{\omega}_\phi(\nu'')\right) \end{array} \right] dF(\phi) \\
&\equiv \int_{\phi\in\aleph} \left[\bar{w} - c^o_\phi(\nu^o) + \beta\bar{P}_{k-1}(\bar{\omega}^o_\phi(\nu^o))\right] dF(\phi) \\
&\equiv \bar{P}^o_k(\nu^o),
\end{aligned}$$

where the functions $\bar{\omega}^o_\phi$ and c^o_ϕ are defined as $\bar{\omega}^o_\phi(\nu^o) \equiv \delta\bar{\omega}_\phi(\nu') + (1-\delta)\bar{\omega}_\phi(\nu'')$ and $c^o_\phi(\nu^o) = u^{-1}\left(\delta u(c_\phi(\nu')) + (1-\delta)u(c_\phi(\nu''))\right)$. We then show that the allocation $\left(c^o_\phi(\nu^o), \bar{\omega}^o_\phi(\nu^o)\right)$ is feasible. Given the promised-utility ν^o, the possibly suboptimal allocation $\left(c^o_\phi(\nu^o), \bar{\omega}^o_\phi(\nu^o)\right)$ satisfies the promise-keeping constraint

$$\begin{aligned}
& \int_{\phi\in\aleph} \left[u\left(c^o_\phi(\nu^o)\right) + \beta\bar{\omega}^o_\phi(\nu^o)\right] dF(\phi) \\
= & \int_{\phi\in\aleph} \left[\delta u\left(c_\phi(\nu')\right) + (1-\delta)u\left(c_\phi(v'')\right) + \beta\left[\delta\bar{\omega}_\phi(\nu') + (1-\delta)\bar{\omega}_\phi(\nu'')\right]\right] dF(\phi) \\
= & \ \delta\nu' + (1-\delta)\nu'' = v^o,
\end{aligned}$$

and the participation constraint

$$\begin{aligned}
& u\left(\bar{c}^o_\phi(\nu^o)\right) + \beta\bar{\omega}^o_\phi(\nu^o) \\
= & \ \delta u\left(\bar{c}_\phi(\nu')\right) + (1-\delta)u\left(\bar{c}_\phi(v'')\right) + \beta\left[\delta\bar{\omega}_\phi(\nu') + (1-\delta)\bar{\omega}_\phi(\nu'')\right] \\
\geq & \ \bar{\nu} - \phi,
\end{aligned}$$

which ensures feasibility. As the profits at the optimal allocation must be weakly higher than $\bar{P}^o_k$, i.e., $\bar{P}_k(\nu^o) \geq \bar{P}^o_k(\nu^o)$, we conclude that $\bar{P}_k$ is strictly concave,

$$\bar{P}_k(\delta\nu' + (1-\delta)\nu'') \geq \bar{P}^o_k(\nu^o) > \delta\bar{P}_k(\nu') + (1-\delta)\bar{P}_k(\nu'').$$

Let $\bar{P}_0(\nu)$ be any strictly concave element of $\bar{\Gamma}$, then by induction and the Contraction Mapping Theorem, $\bar{P}(\nu) \equiv \bar{P}_\infty(\nu)$ is strictly concave.

To prove the profit function $\bar{P}_\infty(\nu)$ is continuously differentiable we apply Lemma 1 of Benveniste and Scheinkman (1979). Consider the function

$$\begin{aligned}\bar{W}(\tilde{\nu};\nu) &= \int_0^{\tilde{\phi}(\tilde{\nu})} \left[\bar{w} - \tilde{c}_\phi(\tilde{\nu}) + \beta \bar{P}(\bar{\omega}_\phi(\nu))\right] dF(\phi) \\ &\quad + \int_{\tilde{\phi}(\tilde{\nu})}^{\infty} \left[\bar{w} - \tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu}) + \beta \bar{P}(\bar{\omega}_\phi(\nu))\right] dF(\phi)\end{aligned}$$

where $\bar{\omega}_\phi(\nu)$ is the promise of the optimal contract given an initial promise ν and the consumption function $\tilde{c}_\phi(\tilde{\nu})$ is chosen to satisfy (for $\tilde{\nu} = \nu$ this is equivalent to the promise-keeping constraint)

$$\begin{aligned}\tilde{\nu} &= \int_0^{\tilde{\phi}(\tilde{\nu})} \left[u\left(\tilde{c}_\phi(\tilde{\nu})\right) + \beta \bar{\omega}_\phi(\nu)\right] dF(\phi) \\ &\quad + \int_0^{\tilde{\phi}(\tilde{\nu})} \left[u\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right) + \beta \bar{\omega}_\phi(\nu)\right] dF(\phi), \end{aligned} \tag{82}$$

and,

$$\begin{aligned} u\left(\tilde{c}_\phi(\tilde{\nu})\right) + \beta \bar{\omega}_\phi(\nu) &= \bar{\nu} - \phi, \quad \forall \phi \leq \tilde{\phi}(\tilde{\nu}) \\ u\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right) + \beta \bar{\omega}_\phi(\nu) &= \bar{\nu} - \tilde{\phi}(\tilde{\nu}), \quad \forall \phi > \tilde{\phi}(\tilde{\nu}), \end{aligned} \tag{83}$$

where $\tilde{\phi}\left(\tilde{\nu}\right)$ is defined through the equation

$$\tilde{\nu} = \bar{\nu} - \int_0^{\tilde{\phi}(\tilde{\nu})} \phi dF(\phi) - \left(1 - F\left(\tilde{\phi}\left(\tilde{\nu}\right)\right)\right) \times \tilde{\phi}\left(\tilde{\nu}\right).$$

For $\tilde{\nu} = \nu$, $\tilde{\phi}\left(\tilde{\nu}\right)$ corresponds to the threshold realization of the default cost ϕ that separates states with binding from states with non-binding participation constraints for the optimal contract. Differentiating equation (**??**) with respect to $\tilde{\nu}$ shows that for low realizations of ϕ the derivative of the consumption function must be zero:

$$u'\left(\tilde{c}_\phi(\tilde{\nu})\right) \tilde{c}'_\phi(\tilde{\nu}) = 0, \quad \forall \phi \leq \tilde{\phi}(\tilde{\nu}).$$

On the other hand, differentiating equation (82) shows that the consumption function, $\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})$, is also continuously differentiable for the remaining realizations, $\phi > \tilde{\phi}(\nu)$:

$$1 = (1 - F(\tilde{\phi}(\tilde{v})))u'\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right) \tilde{c}'_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu}).$$

By construction the function $\bar{W}$ concides with the profit function at $\tilde{\nu} = \nu$, $\bar{W}(\nu;\nu) = \bar{P}(\nu)$, and is strictly lower otherwise, $\bar{W}(\tilde{\nu};\nu) < \bar{P}(\tilde{\nu})$ for $\tilde{\nu} \neq \nu$. Furthermore, $\bar{W}$ is continuously differentiable,

$$\bar{W}'(\tilde{\nu};\nu) = -(1 - F(\tilde{\phi}(\tilde{v})))\tilde{c}'_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu}) = -u'\left[\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right)\right]^{-1} < 0,$$

and strictly concave

$$\bar{W}''(\tilde{\nu};\nu) = u''\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right) \tilde{c}'_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu}) \left[u'\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right)\right]^{-2} < 0.$$

Thus, Lemma 1 in Benveniste and Scheinkman (1979) applies and $\bar{P}$ is continuously differentiable at ν.

10.2.2 Recession

Let $\nu \in [\underline{\omega}_{\min}, \underline{\omega}_{\max}] \equiv \underline{\Omega} \subset \mathbb{R}$, where the bounds of the interval are given by

$$[\underline{\omega}_{\min}, \underline{\omega}_{\max}] = [\underline{\nu} - E[\phi], u(c^{FB})/(1-\beta)].$$

Let d_∞ be the supremum norm and $\underline{\Gamma} \equiv C[\bar{\Omega}]$ the set of continuous bounded functions that maps the interval $\underline{\Omega}$ into $\mathbb{R}$, such that the pair $(\bar{\Gamma}, d_\infty)$ is a complete metric space. Let T denote the maximum operator associated with the right-hand side of the Bellman equation

$$\begin{aligned}
\underline{P}(\nu) &= \max_{\{p_\phi \in [0,1], c_\phi \in [0, c^{FB}], \underline{\omega}_\phi \in \underline{\Omega}, \bar{\omega}_\phi \in \bar{\Omega}\}_{\phi \in \aleph}} \int_{\phi \in \aleph} \left[\underline{w} - c_\phi + \beta \left[(1-p_\phi)\underline{P}(\underline{\omega}_\phi) + p_\phi \bar{P}(\bar{\omega}_\phi)\right]\right] dF(\phi) \\
&\equiv T(\underline{P})(\nu),
\end{aligned}$$

where the maximization is subject to the constraints

$$\begin{aligned}
\int_{\phi \in \aleph} \left[u(c_\phi) - X(p_\phi) + \beta \left[(1-p_\phi)\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right]\right] dF(\phi) &\geq \nu \\
u(\underline{c}_\phi) - X(p_\phi) + \beta \left[(1-p_\phi)\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right] &\geq \underline{\nu} - \phi, \quad \forall \phi \in \aleph.
\end{aligned}$$

The operator T maps functions from the metric space $(\underline{\Gamma}, d_\infty)$ into itself. According to Blackwell's sufficient conditions T is a contraction mapping (see Ljungqvist and Sargent (2012, Appendix A.1) if: (i) T is monotone, (ii) T discounts.

1. Monotonicity: Let $x(\nu) \geq y(\nu)$, $\nu \in \underline{\Omega}$, $x, z \in \Gamma(\underline{\Omega}, d_\infty)$. Then

$$\begin{aligned}
T(x) &= \max_{\{p_\phi \in [0,1], c_\phi \in [0, c^{FB}], \underline{\omega}_\phi \in \underline{\Omega}, \bar{\omega}_\phi \in \bar{\Omega}\}_{\phi \in \aleph}} \int_{\phi \in \aleph} \left[\underline{w} - c_\phi + \beta \left[(1-p_\phi)x(\underline{\omega}_\phi) + p_\phi \bar{P}(\bar{\omega}_\phi)\right]\right] dF(\phi) \\
&\geq \max_{\{p_\phi \in [0,1], c_\phi \in [0, c^{FB}], \underline{\omega}_\phi \in \underline{\Omega}, \bar{\omega}_\phi \in \bar{\Omega}\}_{\phi \in \aleph}} \int_{\phi \in \aleph} \left[\underline{w} - c_\phi + \beta \left[(1-p_\phi)y(\underline{\omega}_\phi) + p_\phi \bar{P}(\bar{\omega}_\phi)\right]\right] dF(\phi) \\
&= T(y).
\end{aligned}$$

2. Discounting: Let z be a real constant. Then

$$\begin{aligned}
T(\underline{P} + z) &= \max_{\{p_\phi \in [0,1], c_\phi \in [0, c^{FB}], \underline{\omega}_\phi \in \underline{\Omega}, \bar{\omega}_\phi \in \bar{\Omega}\}_{\phi \in \aleph}} \int_{\phi \in \aleph} \\
&\quad \left[\underline{w} - c_\phi + \beta \left[(1-p_\phi)\underline{P}(\underline{\omega}_\phi) + p_\phi \bar{P}(\bar{\omega}_\phi)\right] + \beta(1-p_\phi)x\right] dF(\phi) \\
&\leq T(\underline{P}) + \beta z \\
&< T(\underline{P}) + z,
\end{aligned}$$

since $\beta < 1$ and $p_\phi < 1$.

Thus, T is indeed a contraction mapping and $\underline{P}$ is the unique fixed-point in $\underline{\Gamma}$ satisfying

$$\underline{P}(\nu) = T(\underline{P})(\nu).$$

We will show in the next step that T maps strictly concave functions into strictly concave functions. Let $\nu' \neq \nu'' \in \underline{\Omega}$, $\delta \in (0,1)$, $\nu^o = \delta\nu' + (1-\delta)\nu''$, and $\underline{P}_{k-1} \in \underline{\Gamma}$ strictly concave

$$\underline{P}_{k-1}(\delta\nu' + (1-\delta)\nu'') > \delta\underline{P}_{k-1}(\nu') + (1-\delta)\underline{P}_{k-1}(\nu'').$$

Furthermore, let $\underline{P}_k(\nu) = T(\underline{P}_{k-1})(\nu)$ and define the weights $\underline{\delta}, \bar{\delta} \in (0,1)$ and the triplet $(p^o_\phi, \underline{\omega}^o_\phi, \bar{\omega}^o_\phi)$ according to

$$
\begin{aligned}
\underline{\delta} &\equiv \delta \frac{1-p_\phi(\nu')}{\delta(1-p_\phi(\nu'))+(1-\delta)(1-p_\phi(\nu''))} \equiv \delta \frac{1-p_\phi(\nu')}{1-p^o_\phi(\nu^o)} \\
\bar{\delta} &\equiv \delta \frac{p_\phi(\nu')}{\delta p_\phi(\nu')+(1-\delta)p_\phi(\nu'')} \equiv \delta \frac{p_\phi(\nu')}{p^o_\phi(\nu^o)},
\end{aligned}
$$

and let

$$
\begin{aligned}
\underline{\omega}^o_\phi(\nu^o) &= \underline{\delta}\underline{\omega}_\phi(\nu') + (1-\underline{\delta})\underline{\omega}_\phi(\nu'') \\
\bar{\omega}^o_\phi(\nu^o) &= \bar{\delta}\bar{\omega}_\phi(\nu') + (1-\bar{\delta})\bar{\omega}_\phi(\nu'') \\
c^o_\phi(\nu^o) &= u^{-1}\left(\delta u(c_\phi(\nu')) + (1-\delta)u(c_\phi(\nu''))\right)
\end{aligned}
$$

such that

$$
\begin{aligned}
(1-p^o_\phi(\nu^o))\underline{\omega}^o_\phi(\nu^o) &= \delta(1-p_\phi(\nu'))\underline{\omega}_\phi(\nu') + (1-\underline{\delta})(1-p_\phi(\nu''))\underline{\omega}_\phi(\nu'') \\
p^o_\phi(\nu^o)\bar{\omega}^o_\phi(\nu^o) &= \delta p_\phi(\nu')\underline{\omega}_\phi(\nu') + (1-\underline{\delta})p_\phi(\nu'')\underline{\omega}_\phi(\nu'').
\end{aligned}
$$

Then the profit function evaluated at the optimal contract $(c_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi)$ implies

$$
\begin{aligned}
& \delta\underline{P}_k(\nu') + (1-\delta)\underline{P}_k(\nu'') \\
= & \int_{\phi\in\aleph} \left[\begin{array}{c} \underline{w} - [\delta c_\phi(\nu') + (1-\delta)c_\phi(\nu'')] + \\ \beta\left[\delta(1-p_\phi(\nu'))\underline{P}_{k-1}(\underline{\omega}_\phi(\nu')) + (1-\delta)(1-p_\phi(\nu''))\underline{P}_{k-1}(\underline{\omega}_\phi(\nu''))\right] \\ \beta\left[\delta p_\phi(\nu')\bar{P}_{k-1}(\bar{\omega}_\phi(\nu')) + (1-\delta)p(\nu'')\bar{P}_{k-1}(\bar{\omega}_\phi(\nu''))\right] \end{array} \right] dF(\phi) \\
= & \int_{\phi\in\aleph} \left[\begin{array}{c} \underline{w} - [\delta c_\phi(\nu') + (1-\delta)c_\phi(\nu'')] + \\ \beta(1-p^o_\phi(\nu^o))\left[\underline{\delta}\underline{P}_{k-1}(\underline{\omega}_\phi(\nu')) + (1-\underline{\delta})\underline{P}_{k-1}(\underline{\omega}_\phi(\nu''))\right] \\ \beta p^o_\phi(\nu^o)\left[\bar{\delta}\bar{P}_{k-1}(\bar{\omega}_\phi(\nu')) + (1-\bar{\delta})\bar{P}_{k-1}(\bar{\omega}_\phi(\nu''))\right] \end{array} \right] dF(\phi) \\
< & \int_{\phi\in\aleph} \left[\begin{array}{c} \underline{w} - u^{-1}\left(\delta u(c_\phi(\nu')) + (1-\delta)u(c_\phi(\nu''))\right) + \\ \beta(1-p^o_\phi(\nu^o))\underline{P}_{k-1}(\underline{\delta}\underline{\omega}_\phi(\nu') + (1-\underline{\delta})\underline{\omega}_\phi(\nu'')) \\ \beta p^o_\phi(\nu^o)\bar{P}_{k-1}(\bar{\delta}\bar{\omega}_\phi(\nu') + (1-\bar{\delta})\bar{\omega}_\phi(\nu'')) \end{array} \right] dF(\phi) \\
\equiv & \int_{\phi\in\aleph} \left[\underline{w} - c^o_\phi(\nu^o) + \beta\left[(1-p^o_\phi(\nu^o))\underline{P}_{k-1}(\underline{\omega}^o_\phi(\nu^o)) + p^o_\phi(\nu^o)\bar{P}(\bar{\omega}^o_\phi(\nu^o))\right]\right] dF(\phi) \\
\equiv & \underline{P}^o_k(\nu^o).
\end{aligned}
$$

Given the promised-utility ν^o, the suboptimal allocation $(c^o_\phi, p^o_\phi, \underline{\omega}^o_\phi, \bar{\omega}^o_\phi)$ satisfies the promise-keeping constraint

$$
\begin{aligned}
& \int_{\phi\in\aleph} \left[u\left(\underline{c}^o_\phi(\nu^o)\right) - X(p^o_\phi(\nu^o)) + \beta\left[(1-p^o_\phi(\nu^o))\underline{\omega}^o_\phi(\nu^o) + p^o_\phi(\nu^o)\bar{\omega}^o_\phi(\nu^o)\right]\right] dF(\phi) \\
= & \int_{\phi\in\aleph} \left[\begin{array}{c} \delta u\left(\underline{c}_\phi(\nu')\right) + (1-\delta)\left(\underline{c}_\phi(\nu'')\right) - X(\delta p_\phi(\nu') + (1-\delta)p_\phi(\nu'')) \\ +\beta\left[\delta(1-p_\phi(\nu'))\underline{\omega}_\phi(\nu') + (1-\delta)(1-p_\phi(\nu''))\underline{\omega}_\phi(\nu'')\right] \\ +\beta\left[\delta p_\phi(\nu')\bar{\omega}_\phi(\nu') + (1-\delta)p_\phi(\nu''))\bar{\omega}_\phi(\nu'')\right] \end{array} \right] dF(\phi) \\
> & \int_{\phi\in\aleph} \left[\begin{array}{c} \delta u\left(\underline{c}_\phi(\nu')\right) + (1-\delta)\left(\underline{c}_\phi(\nu'')\right) - [\delta X(p_\phi(\nu') + (1-\delta)X(p_\phi(\nu''))] \\ +\beta\left[\delta(1-p_\phi(\nu'))\underline{\omega}_\phi(\nu') + (1-\delta)(1-p_\phi(\nu''))\underline{\omega}_\phi(\nu'')\right] \\ +\beta\left[\delta p_\phi(\nu')\bar{\omega}_\phi(\nu') + (1-\delta)p_\phi(\nu''))\bar{\omega}_\phi(\nu'')\right] \end{array} \right] dF(\phi) \\
= & \delta\nu' + (1-\delta)\nu'' = \nu^o,
\end{aligned}
$$

where we used the strict convexity of the cost function:

$$X(\delta p_\phi(\nu') + (1-\delta)p_\phi(\nu'')) < \left[\delta X(p_\phi(\nu') + (1-\delta)X(p_\phi(\nu''))\right],$$

to derive the last inequality. The participation constraint is also satisfied

$$\begin{aligned}
& u\left(c^o_\phi(\nu^o)\right) - X(p^o_\phi(\nu^o)) + \beta\left[(1-p^o_\phi(\nu^o))\underline{\omega}^o_\phi(\nu^o) + p^o_\phi(\nu^o)\bar{\omega}^o_\phi(\nu^o)\right] \\
= & \left[\begin{array}{c} \delta u\left(c_\phi(\nu')\right) + (1-\delta)\left(c_\phi(\nu'')\right) - X(\delta p_\phi(\nu') + (1-\delta)p_\phi(\nu'')) \\ +\beta\left[\delta(1-p_\phi(\nu'))\underline{\omega}_\phi(\nu') + (1-\delta)(1-p_\phi(\nu''))\underline{\omega}_\phi(\nu'')\right] \\ +\beta\left[\delta p_\phi(\nu')\bar{\omega}_\phi(\nu') + (1-\delta)p_\phi(\nu''))\bar{\omega}_\phi(\nu'')\right] \end{array}\right] \\
> & \left[\begin{array}{c} \delta u\left(c_\phi(\nu')\right) + (1-\delta)\left(c_\phi(\nu'')\right) - \left[\delta X(p_\phi(\nu') + (1-\delta)X(p_\phi(\nu''))\right] \\ +\beta\left[\delta(1-p_\phi(\nu'))\underline{\omega}_\phi(\nu') + (1-\delta)(1-p_\phi(\nu''))\underline{\omega}_\phi(\nu'')\right] \\ +\beta\left[\delta p_\phi(\nu')\bar{\omega}_\phi(\nu') + (1-\delta)p_\phi(\nu''))\bar{\omega}_\phi(\nu'')\right] \end{array}\right] \\
\geq & \ \underline{\nu} - \phi.
\end{aligned}$$

As the profits at the optimal allocation must be weakly higher than at the suboptimal allocation, $\underline{P}_k(\nu^o) \geq \underline{P}^o_k(\nu^o)$, we conclude that $\underline{P}_k$ is strictly concave

$$\underline{P}_k(\delta\nu' + (1-\delta)v'') \geq \underline{P}^o_k(\nu^*) > \delta\underline{P}_k(\nu') + (1-\delta)\underline{P}_k(v'').$$

Let $\underline{P}_0(\nu)$ be any strictly concave element of $\underline{\Gamma}$, then by induction $\underline{P}(\nu) \equiv \underline{P}_\infty(\nu)$ is strictly concave.

To prove continuous differentiability we apply Lemma 1 of Benveniste and Scheinkman (1979). Consider the function

$$\begin{aligned}
\underline{W}(\tilde{\nu},\nu) \;=\; & \int_0^{\tilde{\phi}(\tilde{\nu})} \left[\underline{w} - \tilde{c}_\phi(\tilde{\nu}) + \beta\left[(1-p_\phi(\nu))\underline{P}(\underline{\omega}_\phi(\nu)) + p_\phi(\nu)\bar{P}(\bar{\omega}_\phi(\nu))\right]\right] dF(\phi) \\
& + \int_{\tilde{\phi}(\tilde{\nu})}^{\infty} \left[\underline{w} - \tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu}) + \beta\left[(1-p_\phi(\nu))\underline{P}(\underline{\omega}_\phi(\nu)) + p_\phi(\nu)\bar{P}(\bar{\omega}_\phi(\nu))\right]\right] dF(\phi)
\end{aligned}$$

where the triplet $(p_\phi(\nu), \underline{\omega}_\phi(\nu), \bar{\omega}_\phi(\nu))$ is the same as in the optimal contract given an initial promise ν and the consumption function, $\tilde{c}_\phi(\tilde{\nu})$, satisfies the equation (for $\tilde{\nu} = \nu$ this is equivalent to the promise-keeping constraint)

$$\begin{aligned}
\tilde{\nu} \;=\; & \int_0^{\tilde{\phi}(\tilde{\nu})} \left[u\left(\tilde{c}_\phi(\tilde{\nu})\right) - X(p_\phi(\nu)) + \beta\left[(1-p_\phi(\nu))\underline{\omega}_\phi(\nu) + p_\phi(\nu)\bar{\omega}_\phi(\nu)\right]\right] dF(\phi) \\
& + \int_{\tilde{\phi}(\tilde{\nu})}^{\infty} \left[u\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right) - X(p_\phi(\nu)) + \beta\left[(1-p_\phi(\nu))\underline{\omega}_\phi(\nu) + p_\phi(\nu)\bar{\omega}_\phi(\nu)\right]\right] dF(\phi),
\end{aligned} \tag{84}$$

and,

$$\begin{aligned}
u\left(\tilde{c}_\phi(\tilde{\nu})\right) - X(p_\phi(\nu)) + \beta\left[(1-p_\phi(\nu))\underline{\omega}_\phi(\nu) + p_\phi(\nu)\bar{\omega}_\phi(\nu)\right] \;&\geq\; \bar{\nu} - \phi, \quad \forall\phi \leq \tilde{\phi}(\tilde{\nu}) \qquad (85) \\
u\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right) - X(p_\phi(\nu)) + \beta\left[(1-p_\phi(\nu))\underline{\omega}_\phi(\nu) + p_\phi(\nu)\bar{\omega}_\phi(\nu)\right] \;&\geq\; \bar{\nu} - \tilde{\phi}(\tilde{\nu}), \quad \forall\phi > \tilde{\phi}(\tilde{\nu}),
\end{aligned}$$

where $\tilde{\phi}\left(\tilde{\nu}\right)$ is defined through the equation

$$\tilde{\nu} = \underline{\nu} - \int_0^{\tilde{\phi}(\tilde{\nu})} \phi dF(\phi) - \left(1 - F\left(\tilde{\phi}\left(\tilde{\nu}\right)\right)\right) \times \tilde{\phi}\left(\tilde{\nu}\right).$$

For $\tilde{\nu} = \nu$, $\tilde{\phi}(\tilde{\nu})$ corresponds to the threshold realization of the default cost ϕ that separates states with binding from states with non-binding participation constraints for the optimal contract. Differentiating equation (85) with respect to ν shows that for low realizations of ϕ the derivative of the consumption function must be zero:

$$u'\left(\tilde{c}_\phi(\tilde{\nu})\right)\tilde{c}'_\phi(\tilde{\nu}) = 0, \quad \forall \phi \leq \tilde{\phi}(\tilde{\nu}).$$

On the other hand, differentiating equation (84) shows that the consumption function, $\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})$, is also continuously differentiable for the remaining realizations, $\phi > \tilde{\phi}(\nu)$:

$$1 = (1 - F(\tilde{\phi}(\tilde{v})))u'\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right)\tilde{c}'_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu}).$$

Note that the function $\underline{W}$ was constructed such that it concides with the profit function at $\tilde{\nu} = \nu$, $\underline{W}(\nu;\nu) = \underline{P}(\nu)$, and is strictly lower otherwise, $\underline{W}(\tilde{\nu};\nu) < \underline{P}(\tilde{\nu})$ for $\tilde{\nu} \neq \nu$. Furthermore, $\underline{P}$ is continuously differentiable,

$$\underline{W}'(\tilde{\nu};\nu) = -(1 - F(\tilde{\phi}(\tilde{v})))\tilde{c}'_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu}) = -u'\left[\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right)\right]^{-1} < 0,$$

and strictly concave

$$\bar{W}''(\tilde{\nu};\nu) = u''\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right)\tilde{c}'_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\left[u'\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right)\right]^{-2} < 0.$$

Thus, Lemma 1 in Benveniste and Scheinkman (1979) applies such that $\underline{P}$ is continuously differentiable at ν.

10.2.3 Self-enforcing reform effort

The prove that the profit function is strict concavity and differentiability when reform effort is self-enforcing is by-and-large a corollary of the case without the additional incentive constraint. We know from Proposition 9 that for $\nu > \underline{\omega}^*$ the additional incentive constraint is never relevant, thus strict concavity and the differentiability of the profit function follows immediately from the recession case discussed above. On the other hand, if $\nu \leq \underline{\omega}^*$, than the profit function evaluated at the optimal contract reads as

$$\begin{aligned}
\underline{P}(\nu) \quad = \quad & \int_0^{\phi^*} \left[\underline{w} - c_\phi + \beta\left[(1-p_\phi)\underline{P}(\underline{\omega}_\phi) + p_\phi \bar{P}(\bar{\omega}_\phi)\right]\right] dF(\phi) \\
& + \int_{\phi^*}^{\tilde{\phi}(\nu)} \left[\underline{w} - c^*_\phi + \beta\left[(1-p^*)\underline{P}(\underline{\omega}^*) + p^* \bar{P}(\bar{\omega}^*)\right]\right] dF(\phi) \\
& + \int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \left[\underline{w} - c^*_{\tilde{\phi}(\nu)} + \beta\left[(1-p^*)\underline{P}(\underline{\omega}^*) + p^* \bar{P}(\bar{\omega}^*)\right]\right] dF(\phi).
\end{aligned}$$

Note that the promised-utility ν only enters the last two terms such that the first derivative of the profit function is given by (we will prove differentiability of the profit function below)

$$
\begin{aligned}
\underline{P}'(\nu) &= \left[\underline{w} - c^*_{\tilde{\phi}(\nu)} + \beta\left[(1-p^*)\underline{P}(\underline{\omega}^*) + p^*\bar{P}(\bar{\omega}^*)\right]\right] f(\tilde{\phi}(\nu))\tilde{\phi}'(\nu) \\
&\quad - \int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \frac{dc^*_{\tilde{\phi}(\nu)}}{d\tilde{\phi}(\nu)}\tilde{\phi}'(\nu)dF(\phi) \\
&\quad - \left[\underline{w} - c^*_{\tilde{\phi}(\nu)} + \beta\left[(1-p^*)\underline{P}(\underline{\omega}^*) + p^*\bar{P}(\bar{\omega}^*)\right]\right] f(\tilde{\phi}(\nu))\tilde{\phi}'(\nu) \\
&= - \int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \frac{dc^*_{\tilde{\phi}(\nu)}}{d\nu} dF(\phi) < 0,
\end{aligned}
$$

where equation (47) implies that $dc^*_{\tilde{\phi}(\nu)}/d\nu = -u'(\underline{\nu} - \tilde{\phi}(\nu) - Z(b_0))^{-1}\tilde{\phi}'(\nu) > 0$ as the threshold $\tilde{\phi}(\nu)$ is decreasing in the promised utility. The negative second derivative follows immediately

$$
\underline{P}''(\nu) = -\left[\int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \frac{d^2 c^*_{\tilde{\phi}(\nu)}}{d\nu^2} dF(\phi) + \frac{dc^*_{\tilde{\phi}(\nu)}}{d\nu} f(\tilde{\phi}(\nu))(-\tilde{\phi}'(\nu))\right] < 0,
$$

because

$$
\begin{aligned}
d^2 c^*_{\tilde{\phi}(\nu)}/d\nu^2 &= \left[-u''(\underline{\nu} - \tilde{\phi}(\nu) - Z(b_0))\tilde{\phi}'(\nu)^2 + u'(\underline{\nu} - \tilde{\phi}(\nu) - Z(b_0))\tilde{\phi}''(\nu)\right] \\
&\quad \times \left[u'(\underline{\nu} - \tilde{\phi}(\nu) - Z(b_0))^{-1}\tilde{\phi}'(\nu)\right]^{-2} \\
&> 0.
\end{aligned}
$$

The positive sign of the second derivative is based on the fact that $\tilde{\phi}''(\nu) > 0$. This can be verified from totally differentiating equation (81) with respect to ν :

$$
\begin{aligned}
1 &= -\left[\tilde{\phi}(\nu) f(\tilde{\phi}(\nu))\tilde{\phi}'(\nu)\right] - \left[\tilde{\phi}'(\nu)\left[1 - F(\tilde{\phi}(\nu))\right] - \tilde{\phi}(\nu) f(\tilde{\phi}(\nu))\tilde{\phi}'(\nu)\right] \\
&= -\tilde{\phi}'(\nu)\left[1 - F(\tilde{\phi}(\nu))\right] \quad \Rightarrow \tilde{\phi}'(\nu) = -\left[1 - F(\tilde{\phi}(\nu))\right]^{-1} < 0. \\
\tilde{\phi}''(\nu) &= -f(\tilde{\phi}(\nu))\tilde{\phi}'(\nu)/\left[1 - F(\tilde{\phi}(\nu))\right]^2 > 0.
\end{aligned}
$$

Finally, as $\tilde{\phi}(\nu)$ is continuously differentiable, so is consumption, $c^*_{\tilde{\phi}(\nu)}$, and the profit function, $\underline{P}(\nu)$. This concludes the argument.

10.3 Reputation (Section 6): technical details

We assume that $\beta R = 1$. We assume that ϕ can be drawn from two alternative distributions. This captures the notion that the borrower can be of two types: creditworthy (CW) and not creditworthy (NC). The associated p.d.f. and c.d.f. are $f_{CW}(\phi)$ and $f_{NC}(\phi)$, and $F_{CW}(\phi)$ and $F_{NC}(\phi)$, respectively. For simplicity, we assume that the two distribution have the same support, $\phi \in [\phi_{\min}, \phi_{\max}]$. This ensures that no realization of ϕ is perfectly revealing. We assume that, for all ϕ, $F_{NC}(\phi) \geq F_{CW}(\phi)$ (with strict inequality holding for some ϕ). There is no asymmetric information, and at the outset there is some prior belief that the borrower is creditworthy. We denote this belief by π. The belief evolves over time following Bayes' rule. Intuitively, a low realization of ϕ lowers π.

$$\begin{aligned}
\pi' &= \frac{f_{CW}(\phi)}{f_{CW}(\phi) \times \pi + f_{NC}(\phi) \times (1-\pi)} \pi \\
&\equiv \Gamma(\phi, \pi).
\end{aligned}$$

Define

$$f(\phi|\pi) = \pi f_{CW}(\phi) + (1-\pi) f_{NC}(\phi),$$

$$\begin{aligned}
F(\phi|\pi) &= \int_0^{\phi} (\pi f_{CW}(z) + (1-\pi) f_{NC}(z))\, dz \\
&= \pi \int_0^{\phi} f_{CW}(z)\, dz + (1-\pi) \int_0^{\phi} f_{NC}(z)\, dz \\
&= \pi F_{CW}(\phi) + (1-\pi) F_{NC}(\phi).
\end{aligned}$$

We focus on the competitive equilibrium during normal times ($w = \bar{w}$). Now, the price of debt depends on the reputation of the country, i.e., $Q = \bar{Q}(b', \pi)$, where we dropped the argument w since we focus on good times. The value function of the government will be denoted by $\bar{V}(b, \phi, \pi)$.

Since outright default is never observed in equilibrium, the value functions simplify to:

$$\bar{V}(b, \phi, \pi) = \max_{b'} \left\{ u\left(\bar{Q}(b', \pi) \times b' + \bar{w} - \digamma(b, \phi, \bar{w}, \pi)\right) + \beta \times EV\left(b', \pi\right) \right\}. \tag{86}$$

where $EV(b', \pi) \equiv E\bar{V}\left(b', \phi', \Gamma\left(\phi', \pi\right)\right)$.

The function $\bar{\Phi}$ is such that $\bar{W}^H(b, \pi) = \bar{W}^D\left(\bar{\Phi}(b, \pi), \pi\right)$. This implies that:

$$\bar{\Phi}(b, \pi) = u(\bar{w}) + \beta \times \bar{W}^H(0, \pi) - \bar{W}^H(b, \pi) = \frac{u(\bar{w})}{1-\beta} - \bar{W}^H(b, \pi)$$

(note that $\bar{W}^H(0, \pi)$ is in fact independent of π under the assumption that $\beta R = 1$).

Given an issuance of b' and a current prior of π, debt wll be honored next period id $\phi' \geq \Phi^*(b', \pi)$ where Φ^* is the unique fixed point of the following equation

$$\Phi^* = \bar{\Phi}\left(b', \Gamma(\Phi^*, \pi)\right).$$

The probability the debt is renegotiated is:

$$\begin{aligned} E\left\{F\left(\bar{\Phi}\left(b',\pi'\right)|\pi'\right),\pi\right\} &= \pi F_{CW}\left(\bar{\Phi}\left(b',\Gamma\left(\Phi^*,\pi\right)\right)\right)+(1-\pi)F_{NC}\left(\bar{\Phi}\left(b',\Gamma\left(\Phi^*,\pi\right)\right)\right) \\ &= \pi F_{CW}\left(\Phi^*\left(b',\pi\right)\right)+(1-\pi)F_{NC}\left(\Phi^*\left(b',\pi\right)\right) \\ &= F\left(\Phi^*\left(b',\pi\right),\Gamma\left(\Phi^*,\pi\right)\right) \end{aligned}$$

Arbitrage implies the following bond price:

$$\bar{Q}\left(b',\pi\right)=\frac{1}{R}\left(\begin{array}{c} \left(1-F\left(\Phi^*\left(b',\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right)\right)+ \\ \frac{\pi}{b'}\int_0^{\Phi^*(b',\pi)}\hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right)\,dF_{CW}\left(\phi\right)+\frac{1-\pi}{b'}\int_0^{\Phi^*(b',\pi)}\hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right)dF_{NC}\left(\phi\right) \end{array}\right)$$

Assume $\bar{\Phi}(b,\pi)$ and $\Phi^*(b,\pi)$ are differentiable. Then, differentiating $b\times\bar{Q}(b,\pi)$, with respect to b yields:

$$\begin{aligned} \frac{d}{db}\left\{b\times\bar{Q}\left(b,\pi\right)\right\} &= \bar{Q}\left(b,\pi\right)+b\times\frac{d}{db}\left(\frac{1}{R}\left(\begin{array}{c} 1-F\left(\bar{\Phi}\left(b,\Gamma\left(\Phi^*,\pi\right)\right)|\Gamma\left(\Phi^*,\pi\right)\right)+ \\ \frac{\pi}{b}\int_0^{\Phi^*(b,\pi)}\hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right)\,dF_{CW}\left(\phi\right)+\frac{1-\pi}{b}\int_0^{\Phi^*(b,\pi)}\hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right)dF_{NC}\left(\phi\right) \end{array}\right)\right) \\ &= \bar{Q}\left(b,\pi\right)-\frac{b}{R}\left(\frac{\partial F\left(\Phi^*\left(b,\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right)}{\partial\phi}\frac{\partial\Phi^*\left(b,\pi\right)}{\partial b}\right)+ \\ &\quad +\frac{b}{R}\frac{\pi}{b}\hat{b}\left(\Phi^*\left(b,\pi\right),\Gamma\left(\Phi^*\left(b,\pi\right),\pi\right)\right)\times f_{CW}\left(\Phi^*\left(b,\pi\right)\right)\frac{\partial\Phi^*\left(b,\pi\right)}{\partial b} \\ &\quad -\frac{1}{R}\frac{\pi}{b}\int_0^{\Phi^*(b,\pi)}\hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right)\,dF_{CW}\left(\phi\right) \\ &\quad +\frac{b}{R}\frac{1-\pi}{b}\hat{b}\left(\Phi^*\left(b,\pi\right),\Gamma\left(\Phi^*\left(b,\pi\right),\pi\right)\right)\times f_{NC}\left(\Phi^*\left(b,\pi\right)\right)\frac{\partial\Phi^*\left(b,\pi\right)}{\partial b} \\ &\quad -\frac{1}{R}\frac{1-\pi}{b}\int_0^{\Phi^*(b,\pi)}\hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right)dF_{NC}\left(\phi\right) \\ &= \bar{Q}\left(b,\pi\right)-\underbrace{\frac{b}{R}\left(\frac{\partial F\left(\Phi^*\left(b,\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right)}{\partial\phi}\frac{\partial\Phi^*\left(b,\pi\right)}{\partial b}\right)}_{A+B}+ \\ &\quad \underbrace{+\pi\frac{b}{R}\times f_{CW}\left(\Phi^*\left(b,\pi\right)\right)\frac{\partial\Phi^*\left(b,\pi\right)}{\partial b}}_{A} \\ &\quad \underbrace{+(1-\pi)\frac{b}{R}\times f_{NC}\left(\Phi^*\left(b,\pi\right)\right)\frac{\partial\Phi^*\left(b,\pi\right)}{\partial b}}_{B} \\ &\quad -\bar{Q}\left(b,\pi\right)+\frac{1}{R}\left(1-F\left(\Phi^*\left(b,\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right)\right) \\ &= \frac{1}{R}\left(1-F\left(\Phi^*\left(b,\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right)\right) \end{aligned}$$

so

$$\frac{d}{db}\left\{b\times\bar{Q}\left(b,\pi\right)\right\}=\frac{1}{R}\left(1-F\left(\Phi^*\left(b,\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right)\right).$$

Next, consider the consumption-savings decision:

$$\bar{V}(b,\phi,\pi)=\max_{b'}\left\{u\left(\bar{Q}\left(b',\pi\right)\times b'+\bar{w}-\digamma\left(b,\phi,\bar{w},\pi\right)\right)+\beta\times EV\left(b',\pi\right)\right\}.$$

where $EV(b',\pi)\equiv E\bar{V}\left(b',\phi',\Gamma\left(\phi',\pi\right)\right)$

The first order condition of (86) yields:

$$\frac{d}{db'}\left\{\bar{Q}\left(b',\pi\right)b'\right\}\times u'\left[\bar{Q}\left(b',\pi\right)\times b'+\bar{w}-\digamma\left(b,\phi,\bar{w},\pi\right)\right]+\frac{d}{db'}\beta EV\left(b',\pi\right)=0$$

Proof. The value function has a kink at $b=\hat{b}(\phi,\pi)$. Consider, first, the range where $b<\hat{b}(\phi,\pi)$. Differentiating the value function yields:

$$\frac{d}{db}\bar{V}(b,\phi,\pi)=-u'\left[\bar{Q}\left(\bar{B}(b,\pi),\pi\right)\times\bar{B}(b,\pi)+\bar{w}-b\right],$$

where $\bar{B}$ denotes the optimal issuance of new bonds. Next, consider the region of renegotiation, $b>\hat{b}(\phi,\pi)$. In this case, $\frac{d}{db}\bar{V}(b,\phi,\pi)=0$. ■

Using the results above one obtains:

$$\begin{aligned}
\frac{d}{db}E\bar{V}(b,\phi,\Gamma(\phi,\pi)) &= \pi\int_{\aleph}\frac{d}{db}\bar{V}(b,\phi,\Gamma(\phi,\pi))\,dF_{CW}(\phi)+(1-\pi)\int_{\aleph}\frac{d}{db}\bar{V}(b,\phi,\Gamma(\phi,\pi))\,dF_{NC}(\phi)\\
&= \pi\left(\int_0^{\Phi^*(b,\pi)}\frac{d}{db}\bar{V}(b,\phi,\Gamma(\phi,\pi))\,dF_{CW}(\phi)+\int_{\Phi^*(b,\pi)}^{\infty}\frac{d}{db}\bar{V}(b,\phi,\Gamma(\phi,\pi))\,dF_{CW}(\phi)\right)\\
&\quad+(1-\pi)\left(\int_0^{\Phi^*(b,\pi)}\frac{d}{db}\bar{V}(b,\phi,\Gamma(\phi,\pi))\,dF_{NC}(\phi)+\int_{\Phi^*(b,\pi)}^{\infty}\frac{d}{db}\bar{V}(b,\phi,\Gamma(\phi,\pi))\,dF_{NC}(\phi)\right)\\
&= \pi\int_{\Phi^*(b,\pi)}^{\infty}\frac{d}{db}\bar{V}(b,\phi,\Gamma(\phi,\pi))\,dF_{CW}(\phi)+(1-\pi)\int_{\Phi^*(b,\pi)}^{\infty}\frac{d}{db}\bar{V}(b,\phi,\Gamma(\phi,\pi))\,dF_{NC}(\phi)\\
&= \int_{\Phi^*(b,\pi)}^{\infty}\frac{d}{db}\bar{V}(b,\phi,\Gamma(\phi,\pi))\,dF(\phi|\pi)\\
&= -\int_{\Phi^*(b,\pi)}^{\infty}u'\left[\bar{Q}\left(\bar{B}(b,\Gamma(\phi,\pi)),\Gamma(\phi,\pi)\right)\times\bar{B}(b,\Gamma(\phi,\pi))+\bar{w}-b\right]dF(\phi|\pi)
\end{aligned}$$

Plugging this expression back into the FOC, and leading the expression by one period, yields

$$\begin{aligned}
0 &= \frac{1}{R}\left(1-F\left(\Phi^*\left(b',\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right)\right)\times u'\left[\bar{Q}\left(b',\pi\right)\times b'+\bar{w}-\digamma(b,\phi,\bar{w},\pi)\right]\\
&\quad-\beta\int_{\Phi^*(b,\pi)}^{\infty}u'\left[\bar{Q}\left(\bar{B}\left(b',\Gamma(\phi,\pi)\right),\Gamma(\phi,\pi)\right)\times\bar{B}\left(b',\Gamma(\phi,\pi)\right)+\bar{w}-b\right]dF(\phi|\pi)\\
&\Rightarrow\\
&\quad\left(1-F\left(\Phi^*\left(b',\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right)\right)\left(\int_{\Phi^*(b,\pi)}^{\infty}\frac{u'\left[\bar{Q}\left(\bar{B}(b',\Gamma(\phi,\pi)),\Gamma(\phi,\pi)\right)\times\bar{B}(b',\Gamma(\phi,\pi))+\bar{w}-b\right]}{u'\left[\bar{Q}(b',\pi)\times b'+\bar{w}-\digamma(b,\phi,\bar{w},\pi)\right]}dF(\phi|\pi)\right)\\
&= \beta R
\end{aligned}$$

where the last step uses the fact that $\frac{d}{db'}\left\{\bar{Q}\left(b',\pi\right)b'\right\} = \frac{1}{R}\left(1 - F\left(\Phi^{*}\left(b',\pi\right)|\Gamma\left(\Phi^{*},\pi\right)\right)\right)$, as shown above. If $\beta R = 1$, then:

$$\begin{aligned}
& 1 - F\left(\Phi^{*}\left(\bar{B}\left(b,\pi\right),\pi\right)|\Gamma\left(\Phi^{*},\pi\right)\right) \\
= & \left(\int_{\Phi^{*}(b,\pi)}^{\infty}\frac{u'\left[C\left(\Gamma\left(\phi,\pi\right),\bar{B}\left(b,\pi\right)\right)\right]}{u'\left[C\left(\pi,b\right)\right]}dF\left(\phi|\pi\right)\right) \\
= & \pi\int_{\Phi^{*}(b,\pi)}^{\infty}\frac{u'\left[C\left(\Gamma\left(\phi,\pi\right),\bar{B}\left(b,\pi\right)\right)\right]}{u'\left[C\left(\pi,b\right)\right]}dF_{CW}\left(\phi\right) + \left(1-\pi\right)\int_{\Phi^{*}(b,\pi)}^{\infty}\frac{u'\left[C\left(\Gamma\left(\phi,\pi\right),\bar{B}\left(b,\pi\right)\right)\right]}{u'\left[C\left(\pi,b\right)\right]}dF_{NC}\left(\phi\right).
\end{aligned}$$

10.4 Numerical Algorithm

10.4.1 Competitive Equilibrium

We solve for the competitive equilibrium described in Section 3 with an augmented value function iteration algorithm. Let $b = (b_1, b_2, ..., b_N)$ denote the equally spaced and inreasingly ordered grid for sovereign debt. Let $\phi = (\phi_1, \phi_2, ...\phi_S)$ be the increasingly ordered grid for the default cost, where the location of any grid point, ϕ_s, is chosen such that the cumulative weighted sum, $\widetilde{F}(\phi_s) \equiv \sum_{k=1}^{s} 1/S$, approximates the CDF of the default cost shock $F(\phi_s)$. We choose $N = 5000$ and $S = 600$ to get a solution with high accuracy.

1. Guess the default threshold, $\Phi_0(b, w) \in \phi$, for both aggregate states w and the reform effort, $\Psi_0(b)$, over the debt grid b. Compute the associated bond revenue,

$$Q_0(b,w)b = (1 - \widetilde{F}(\Phi_0(b,w)))b + \sum_{\phi_s \in \phi_1, ..., \Phi_0(b,w)} \hat{b}_0(\phi_s, w)/S,$$

   where $\hat{b}_0(\phi, w) \in b$ is the inverse function of $\Phi_0(b, w)$.

2. Guess the value functions conditional on honoring the debt, $W^H_{0,0}(b, w)$. For any given debt level, $b_n \leq \hat{b}_0(\phi_S, \bar{w})$, on the debt grid update the value function in normal times according to

$$\begin{aligned} W^H_{i+1,0}(b_n, \bar{w}) &= \max_{b' \in (b_1, ..., \hat{b}_0(\phi_S, \bar{w}))} u\left(Q_0(b', \bar{w})b' + \bar{w} - b_n\right) \\ &\quad + \beta \left[(1 - \widetilde{F}(\Phi_0(b', \bar{w})))W^H_{i,0}(b', \bar{w}) + \sum_{\phi_s \in \phi_1, ..., \Phi_0(b,w)} W^H_{i,0}\left(\hat{b}_0(\phi_s, \bar{w}), \bar{w}\right)/S \right], \end{aligned}$$

   until convergence. For the remaining grid points, $b_n \geq \hat{b}_0(\phi_S, \bar{w})$, set $W^H_{i+1,0}(b_n, \bar{w}) = W^H_{i+1,0}(\hat{b}_0(\phi_S, \bar{w}), \bar{w})$. In recession, for any given debt level, $b_n \leq \hat{b}_0(\phi_S, \underline{w})$, on the debt grid, update the value function according to

$$\begin{aligned} W^H_{i+1,0}(b_n, \underline{w}) &= \max_{b' \in (b_1, ..., \hat{b}_0(\phi_S, \bar{w}))} u\left(Q_0(b', \underline{w})b' + \underline{w} - b_n\right) \\ &\quad + \beta\left(1 - \Psi_0(b')\right) \left[\begin{array}{c} (1 - \widetilde{F}(\Phi_0(b', \underline{w})))W^H_{i,0}(b', \underline{w}) \\ + \sum_{\phi_s \in \phi_1, ..., \Phi_0(b,w)} W^H_{i,0}\left(\hat{b}_0(\phi_s, \underline{w}), \underline{w}\right)/S \end{array} \right] \\ &\quad + \beta\Psi_0(b') \left[\begin{array}{c} (1 - \widetilde{F}(\Phi_0(b', \bar{w})))W^H_{\infty,0}(b', \bar{w}) \\ + \sum_{\phi_s \in \phi_1, ..., \Phi_0(b,w)} W^H_{\infty,0}\left(\hat{b}_0(\phi_s, \bar{w}), \bar{w}\right)/S \end{array} \right], \end{aligned}$$

   until convergence. For the remaining grid points, $b_n \geq \hat{b}_0(\phi_S, \underline{w})$, set $W^H_{i+1,0}(b_n, \underline{w}) = W^H_{i+1,0}(\hat{b}_0(\phi_S, \underline{w}), \underline{w})$. $W^H_{\infty,0}(b', w)$ denotes the converged value function conditional on the guess for the threshold and the reform effort.

3. Update the default threshold and the reform effort according to

$$\Phi_{j+1}(b, \bar{w}) = u(\bar{w}) + \beta W^H_{\infty,j}(0, \bar{w}) - W^H_{\infty,j}(b, \bar{w}),$$

   Equation (**??**) and Equation (14). Go back to step 1 and iterate until convergence.

10.4.2 Constrained Pareto Optimum

We solve for the second-best allocation with an augmented function iteration algorithm. Consider the same grid $\phi = (\phi_1, \phi_2, ...\phi_S)$ for the default cost that we used above. Let $\nu_w = (\nu_w(\phi_1), ..., \nu_w(\phi_S))$ denote the grid for promised utility, where

$$\begin{aligned}\nu_{\bar{w}}(\phi_s) &= \bar{\nu} - \sum_{k=1}^{s} \phi_k/S - \phi_s \left(1 - \widetilde{F}(\phi_s)\right) \\ \nu_{\underline{w}}(\phi_s) &= \underline{\nu} - \sum_{k=1}^{s} \phi_k/S - \phi_s \left(1 - \widetilde{F}(\phi_s)\right).\end{aligned}$$

Note that given a promised utility, $\nu_w(\phi_s)$, the default cost realization $\phi_s = \tilde{\phi}(\nu_w(\phi_s))$ corresponds to the state s where the participation constraint of the debtor starts binding. It turns out to be convenient to set the promised utility for a continued recession, $\bar{\omega}_{\bar{w}} = \nu_{\bar{w}}$, and the promised utility for a continued recession, $\underline{\omega}_{\underline{w}} = \nu_{\underline{w}}$.

1. Guess the reform effort, $p_{\underline{w},0}(\nu_{\underline{w}})$, over the grid $\nu_{\underline{w}}$.

2. Guess the future consumption, $c'_{w,0,0}(\nu_w)$, and the promised utilitiy, $\bar{\omega}_{\underline{w},0,0}(\nu_{\underline{w}})$, over the grids $\nu_{\bar{w}}$ and $\nu_{\underline{w}}$.

3. Compute current consumption from the Euler equations (which holds for all states s where the participation constraint is not strictly binding)
$$c_{w,0,0}(\nu_w) = (u')^{-1} \left[u'\left(c'_{w,0,0}(\nu_w)\right) \beta R\right],$$
and the initial promised utility, $\tilde{\nu}_{w,0}(\nu_w)$, implicitly defined by
$$\begin{aligned}\bar{\nu} - \tilde{\phi}(\tilde{\nu}_{\bar{w},0,0}(\nu_{\bar{w}})) &= u(c_{\bar{w},0,0}(\nu_{\bar{w}})) + \beta\nu_{\bar{w}} \\ \underline{\nu} - \tilde{\phi}(\tilde{\nu}_{\underline{w},0,0}(\nu_{\underline{w}})) &= u(c_{\underline{w},0,0}(\nu_{\underline{w}})) - X(p_{\underline{w},0}(\nu_{\underline{w}})) + \beta\left[p_{\underline{w},0}(\nu_{\underline{w}})\bar{\omega}_{\underline{w},0,0}(\nu_{\underline{w}}) + (1 - p_{\underline{w},0}(\nu_{\underline{w}}))\nu_{\underline{w}}\right],\end{aligned}$$

4. Update the guess for the future consumption function by interpolating ν_w on the pairs $(\tilde{\nu}_{w,0,0}, c_{w,0,0})$ to yield $c'_{w,1,0}(\nu_w)$ Update the guess for promised utility by interpolating $c'_{\underline{w},1,0}(\nu_{\underline{w}})$ on the pairs $\left(c_{\underline{w},0,0}, \tilde{\nu}_{\underline{w},0,0}\right)$ to yield $\bar{\omega}_{\underline{w},1,0}(\nu_{\underline{w}})$. Go back to step 3 and iterate until convergence. Let $c_{w,\infty,0}(\nu_w)$ and $\bar{\omega}_{\underline{w},\infty,0}(\nu_{\underline{w}})$ denote the converged functions given the guess on the reform effort.

5. Guess the profit functions, $\bar{P}_0(\nu_{\bar{w}})$ and $\bar{P}_0(\nu_{\underline{w}})$. Update the profit function in normal times according to
$$\begin{aligned}\bar{P}_{i+1,0}(\tilde{\nu}_{\bar{w},\infty,0}(\nu_{\bar{w}})) &= \left(1 - \widetilde{F}(\tilde{\phi}(\tilde{\nu}_{\bar{w},\infty,0}(\nu_{\bar{w}})))\right)\left[\bar{w} - c_{\bar{w},\infty,0}(\nu_{\bar{w}}) + R^{-1}\bar{P}_{i,0}(\nu_{\bar{w}})\right] \\ &+ \sum_{\phi_s \in \phi_1, ..., \tilde{\phi}(\tilde{\nu}_{\bar{w},\infty,0}(\nu_{\bar{w}}))} \left[\bar{w} - c_{\bar{w},\infty,0}(\nu_{\bar{w}}(\phi_s)) + R^{-1}\bar{P}_{i,0}(\nu_{\bar{w}}(\phi_s))\right]/S,\end{aligned}$$
until convergence, $\bar{P}_{\infty,0}(\nu_{\bar{w}})$. In recession, update according to
$$\begin{aligned}\underline{P}_{i+1,0}(\tilde{\nu}_{\underline{w},\infty,0}(\nu_{\underline{w}})) &= \left(1 - \widetilde{F}(\tilde{\phi}(\tilde{\nu}_{\underline{w},\infty,0}(\nu_{\underline{w}})))\right)\left[\begin{array}{c}\underline{w} - c_{\underline{w},\infty,0}(\nu_{\underline{w}}) \\ +R^{-1}\left[\begin{array}{c}p_{\underline{w},0}(\nu_{\underline{w}})\bar{P}_{i,0}(\bar{\omega}_{\underline{w},\infty,0}(\nu_{\underline{w}})) \\ +(1 - p_{\underline{w},0}(\nu_{\underline{w}}))\underline{P}_{i,0}(\nu_{\underline{w}})\end{array}\right]\end{array}\right] \\ &+ \sum_{\phi_s \in \phi_1, ..., \tilde{\phi}(\tilde{\nu}_{\underline{w},\infty,0}(\nu_{\underline{w}}))} \left[\begin{array}{c}\underline{w} - c_{\underline{w},\infty,0}(\nu_{\underline{w}}(\phi_s)) \\ +R^{-1}\left[\begin{array}{c}p_{\underline{w},0}(\nu_{\underline{w}}(\phi_s)\bar{P}_{i,0}(\bar{\omega}_{\underline{w},\infty,0}(\nu_{\underline{w}}(\phi_s))) \\ +(1 - p_{\underline{w},0}(\nu_{\underline{w}}(\phi_s)))\underline{P}_{i,0}(\nu_{\underline{w}}(\phi_s))\end{array}\right]\end{array}\right]/S.\end{aligned}$$

6. Update the reform effort function according to

$$p_{\underline{w},j+1}(\nu_{\underline{w}}) = \left(X'\right)^{-1} \left[\begin{array}{c} u'(c_{\underline{w},\infty,j}(\nu_{\underline{w}}))R^{-1}\left(\bar{P}_{\infty,j}(\bar{\omega}_{\underline{w},\infty,j}(\nu_{\underline{w}})) - \underline{P}_{\infty,j}(\nu_{\underline{w}})\right) \\ +\beta\left(\bar{\omega}_{\underline{w},\infty,j}(\nu_{\underline{w}}) - \nu_{\underline{w}}\right) \end{array} \right].$$

Go back to step 3 and iterate until convergence.