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SPACE-CHARGE LIMITS FOR LINEAR ACCELERATORS

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ACCELERATOR DEPARTMENT
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ABSTRACT

Equations are developed which determine the space-charge limits for a generalized linear accelerator. A single parameter k, the ratio of space charge to restoring forces, is the only unknown parameter. Experience and computer simulations indicate that $k \approx 0.5$.

I. THE GENERAL FORMULATION

The field in a sphere of charge is given by*

$$\vec{E}_{sc} = \frac{N_{ze}}{4\pi\varepsilon_o\ r_o^3}\vec{r} \quad , \tag{1}$$

where we assume a uniform density distribution of N particles of charge ze. r_o is the radius of the sphere.

We assume that there are linear restoring (focusing) forces $\vec{E} = \overline{E}'\ \vec{r}$, which keep the ball of charge together.

The equations of motion for each of the three dimensions give the results:

$$X = X_o \sin \omega t,$$

$$\dot{X} = \omega X_o \cos \omega t,$$

where

$$\omega = \sqrt{\frac{ez\ \overline{E}'\ (1-k)}{Am_p}} \ ,$$

and

The maximum normalized emittance is given by

$$\varepsilon_N = r_o^2\ \frac{\omega}{c} . \tag{2}$$

Now the maximum electric field is $\overline{E}_{max} = \overline{E}' r_o$, so we can write

$$\varepsilon_N = r_o^{3/2} \sqrt{\frac{ez\ \overline{E}_{max}\ (1-k)}{Am_p\ c^2}} . \tag{3}$$

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*See Appendix for notation and definitions.

If we keep ε_N and $\overline{E}_{max}$ fixed, there will be a limiting number of particles in the ball for some maximum value of k. Therefore, we obtain the following expression for the space-charge limit:

$$\frac{ez\, N_{sc}}{r_o} = \frac{4\pi\varepsilon_o\, k}{(1-k)^{1/3}} \left(\frac{A}{z}\right)^{1/3} \varepsilon_N^{2/3}\, \overline{E}_{max}^{\,2/3} \cdot \left(\frac{m_p\, c^2}{e}\right)^{1/3} . \qquad (4)$$

If we were to view this ball of charge moving by with velocity βc, the space-charge-limited maximum current would be given by

$$i_{max} = \frac{k\; 3\pi\varepsilon_o c}{(1-k)^{1/3}} \left(\frac{m_p\, c^2}{e}\right)^{1/3} \left(\frac{A}{z}\right)^{1/3} \varepsilon_N^{2/3}\, \overline{E}_{max}^{\,2/3}\, \beta . \qquad (5)$$

II. LONGITUDINAL SPACE-CHARGE LIMITS

Now it is appropriate to consider the longitudinal motion separately.

If we suppose a linac with stable phase angle ϕ_s, frequency f, and electric field $\overline{E}_p$, the average accelerating field is $\overline{E} = E_p \sin\phi_s$. The longitudinal or "synchrotron" frequency is given by

$$\omega_s = \sqrt{\frac{\overline{E}\cot\phi_s\, 2\pi f\, ez\, (1-k)}{Am_p\, \beta c}} . \qquad (6)$$

The "phase" radius of the bunch is given by

$$\phi_o = \frac{2\pi f\, r_o}{\beta c} .$$

Then

$$\varepsilon_{NL} = \left(\frac{\beta c}{2\pi f}\right)^{3/2} \phi_o^2 \sqrt{\frac{\bar{E} \cot \phi_s \, (1-k) \, ez}{A m_p c^2}} \; . \qquad (7)$$

Using $\bar{E}_{max} = \bar{E} \cot \phi_s \, \phi_o$, and substituting into Eq. (5), we obtain

$$i_{max_L} = 1.2 \times 10^6 \; k \; \phi_o^2 \; \frac{\beta^2 \bar{E}}{f} \cot \phi_s \; . \qquad (8)$$

Typical values are $\phi_o \sim 2\pi/10$ and $\cot \phi_s \sim 1$,

$$i_{max_L} = 4.73 \times 10^5 \; k \; \frac{\beta^2 \bar{E}}{f} \; . \qquad (9)$$

The average linac current, assuming all the buckets are filled, is given by the relation:

$$\bar{i}_{m_L} = \frac{4}{3} \frac{\phi_o}{2\pi} \, i_{max} \cong 0.1333 \; i_{max_L} \; .$$

III. TRANSVERSE SPACE-CHARGE LIMITS

Now we can consider the transverse current limit by considering a quadrupole-focusing system. For our purposes here, we will consider electrostatic quadrupoles. We will formulate the space-charge limit in terms of the pole-trip field $\bar{E}_{Q_{max}}$. This can be converted to an equivalent magnetic channel by the relationship:

$$B_{Q_{max}} = \frac{E_{Q_{max}}}{\beta c} \; .$$

Now we define a focusing cell as two drift tubes, each containing one quad. We can write the acceptance as a function of the phase shift by noting

$$\mu_{cell} = \frac{\omega n}{f} ,$$

where f is the linac frequency and n is the number of half wavelengths per quad, i.e., n = 1 for $\beta\lambda/2$, n = 2 for $\beta\lambda$, etc. Then we can write

$$\varepsilon_{NT} = r_0^2 \ \frac{f \ \mu_{cell}}{cn} . \tag{10}$$

The phase advance/cell is related to the zero space-charge advance by the following relation:

$$\mu_{cell} = \mu_o \ (1-k)^{1/2}.$$

Now if we describe the channel in thin-lens approximation, we have a simple relationship for the phase advance.

We use two parameters to describe the lattice. The length of each quad is $l_Q = k_4 \ \beta c/f$. The radius of the quads is given by $r_0 = k_3 \ \beta c/f$.

Since the quad must fit in the drift tube, we have typically $k_4 \sim 0.25n$. k_3 cannot be very large or the transit time factor will be too small. Typical values for k_3 might be ~ 0.1.

Using the thin-lens approximation for the transport matrices gives a simple expression for the phase shift/cell in terms of the linac parameters. We obtain

$$E_{Q_{max}} = \left\{ \mu_o \ \frac{k_3}{k_4} \cdot \frac{6.26}{n} \right\} \ \frac{fA\beta}{z} . \tag{11}$$

Similarly, the acceptance can be determined in terms of these parameters.

$$\varepsilon_{NT} = \left\{ \eta^2 \, k_3^2 \, c \, \frac{\mu_o}{n} \, (1-k)^{1/2} \right\} \frac{\beta^2}{f} , \qquad (12)$$

where $\eta = r_{max}/\bar{r}$ to account for the fluctuation of the β function inside the transport system. η is a weak function of μ_o and for typical values of $\mu_o \sim 1$, $\eta \sim 0.707$.

Now we can obtain an expression for the maximum current that can be transported due to transverse considerations*:

$$i_{max_T} = 2.34 \times 10^7 \left\{ \frac{k \, \mu_o^2 \, k_3^2 \, n^{4/3}}{\eta^2} \right\} \frac{A}{z} \left(\beta\gamma\right)^3 . \qquad (13)$$

For typical values, $\mu_o \sim 1.5$, $k_3 \sim 0.1$, $\eta = 0.707$, we get

$$i_{max_T} = 3.3 \times 10^5 \; \frac{k}{n^2} \frac{A}{z} \left(\beta\gamma\right)^3 . \qquad (14)$$

Note that the current limit given by Eq. (13) does not depend upon the linac parameters. If we interpret k_3 in terms of the ratio of quadrupole radius to focusing cell length we get a maximum current limit for any transport system, since k_3 cannot become greater than a few tenths, and μ_o will not be greater than $\sim$1.5. The correction for a cylinder of beam, rather than a ball, is a factor of 2/3.

*The γ^3 term has been added to make the expression relativistically correct.

The current limit for any quadrupole-focusing system, then, is

$$i_{max} \sim 1.1 \times 10^5 \frac{A}{z} \left(\beta\gamma\right)^3 . \qquad (14a)$$

IV. OPTIMIZING SIX-DIMENSIONAL PHASE-SPACE DENSITY

Now comparing the transverse and longitudinal current limits, we can equate them to arrive at the maximum brightness condition. We have

$$1.2 \times 10^6 \, k\phi_o^2 \, \beta^2 \, \frac{\bar{E} \cot \phi_s}{f} = 2.34 \times 10^7 \, \frac{k \, \mu_o^2 \, k_3^2 \, \eta^{4/3}}{n^2} \, \frac{A}{z} \, \beta^3$$

Strictly speaking, the space-charge factors k for the longitudinal and transverse cases need not be equal. Putting $\phi_o{}^2 = k_3^2 \times (2\pi)^2$, we get

$$\bar{E} \cot \phi_s = \frac{0.5 \, \mu_o^2 \eta^{4/3}}{n^2} \, \frac{fA}{z} \, \beta \, . \qquad (15)$$

Typical values for μ_o are ~ 1.5, $\eta \sim 0.707$. For a Wideroe, typically n = 1. Then,

$$\bar{E} \cot \phi_s = \frac{0.709}{n^2} \, \frac{fA\beta}{z} \, .$$

Recalling (11) and putting in typical values:

$$k_3 = 0.1, \quad k_4 = 0.25 \, n,$$

we have

$$E_{0_{max}} = \frac{3.75}{n^2} \, \frac{fA\beta}{z} \, .$$

Therefore $\overline{E} \sim 0.2\ E_{Q_{max}}$ is the condition for getting the maximum density in 6-D phase space. Also, $\varepsilon_{NL} \sim \varepsilon_{NT}$.

The expression for the 6-D phase-space density is given by the following expression:

$$\rho_{6D} = \frac{\overline{i}/fez}{\varepsilon_{NL}\ \varepsilon_{NT}^2} \tag{16}$$

where

$$\overline{i} = \frac{2}{3}\frac{\phi_o}{\pi}\ i_{max} \ .$$

Using the previously derived relation for $\overline{i}$ and ε_{NL}, ε_{NT}, we obtain

$$\rho_{6D} \propto \frac{f^2}{\beta^3}\ \frac{A}{z} \ . \tag{17}$$

This indicates to us that if we want a super-bright beam we should start with an injection energy as low as is practicable, into a linac with as high a frequency as possible.

V. SINGLE-BUNCH SPACE-CHARGE LIMITS

These same methods can be used to obtain the space-charge limit of an induction accelerator. If we assume a parabolic distribution of charge, the space charge electric field gradient is given by

$$E'_{sc} = z\left(\frac{e}{4\pi\varepsilon_0}\right)\cdot\frac{3}{2}\left(1 + 2\ \ln\frac{b}{a}\right)\cdot\frac{N_{sc}}{l_o^3} \ . \tag{18}$$

The restoring force applied by the induction linac voltage slope can be described in terms of a tilt coefficient, k_T, where the restoring field gradient is

$$E' = \frac{k_T \bar{E}}{l_o}$$

Here, $\bar{E} \equiv$ average accelerating field. Then we obtain

$$\varepsilon_{NL} = l_o^{3/2} \sqrt{\frac{e z k_T \bar{E} (1-k)}{m_p c^2 A}} . \tag{19}$$

Substituting into (18), we obtain

$$N_{sc} = \frac{4.44 \times 10^{14}}{\left(1 + 2 \ln \frac{b}{a}\right)} \quad \frac{k (k_T \bar{E})^{1/3}}{(1-k)^{2/3}} \quad \frac{A^{2/3}}{z^{5/3}} \quad \varepsilon_{NL}^{4/3} . \tag{20}$$

If we apply the same methods to a single bunch (nonrelativistic) circulating in an accumulator ring, we can derive an expression for the space-charge limit of a bunch. We obtain

$$N_{sc} = \frac{1.88 \times 10^{13}}{\left(1 + 2 \ln \frac{b}{a}\right) z} \cdot \frac{k}{(1-k)^{3/4}} \cdot \left(\frac{A}{z}\right)^{3/4} \varepsilon_{NL}^{3/2} \left(\frac{Vf}{\beta\gamma R}\right)^{1/4} .$$

Caution must be taken in applying this formula. That is because $\varepsilon_{NL} = [(\Delta p/p)\, l_o]\beta\gamma$. For fixed ε_{NL}, one might be tempted to increase the space-charge limit by increasing V, or f, say. However, this will generally increase $\Delta p/p$. A value of $\Delta p/p$ much greater than about 1% is usually a limit for practical reasons associated with the accumulator.

For a single bunch in a linac, the longitudinal charge limit for a single bunch is

$$N_{sc} = \frac{4.5 \times 10^{13}}{z} \frac{k}{(1-k)^{3/4}} \left(\frac{A}{z}\right)^{3/4} \left(\frac{\overline{E} \cot \phi_s f}{\beta\gamma}\right)^{1/4} \epsilon_{NL}^{3/2} . \quad (21)$$

It is interesting to note that if $\overline{E}$ and f remain fixed, the space-charge limit <u>decreases</u> with increasing energy. If the linac was limited at injection, then we will get an increase of ϵ_{NL} such as to maintain N_{sc} as constant.

Another caveat. In attempting to design for a certain space-charge limit, one is not free to increase ϵ_{NL}. For a given $\overline{E}$, f, and β, ϵ_{NLmax} is readily computed. This can easily be seen by substituting (7) into (22).

$$N_{sc} = \frac{1.58 \times 10^{24}}{z} k \phi_0^3 \overline{E} \cot \phi_s \frac{\beta^2}{f^2} .$$

The limits of ϕ_0 typically are $\sim \pi/5$.

APPENDIX

The normalized emittances, ϵ_{NL} and ϵ_{NT}, are defined as follows:

$$(\Delta X \Delta p) = \epsilon_{NT} Am_p c = (X X') \beta\gamma Am_p c ,$$

$$(\Delta E \Delta t) = \epsilon_{NL} Am_p c = \left(\frac{\Delta p}{p} l_0\right) \beta\gamma Am_p c .$$

The units used throughout are mks, with E in volts/meter and B in tesla.