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RADICALS AND SEMISIMPLE CLASSES OF Ω-GROUPS

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In this paper radicals in the sense of Kuroš and Amitsur (KA-radicals) for Ω-groups will be studied. For the sake of simplicity these radicals will be considered on varieties, the results remaining valid for more general classes.

A KA-radical can be defined in the following way (see for example (**1**)):

a property P of Ω-groups is called a radical property if the following conditions hold:

(α) every homomorphic image of an Ω-group having P has P;

(β) every Ω-group G in the considered variety contains a unique maximal ideal having $P: P(G)$;

(γ) $P(P/P(G))$ is the zero ideal.

It is well known that a subclass $\mathfrak{R}$ of a variety $\mathfrak{V}$ is a radical class (i.e. the class of all Ω-groups in $\mathfrak{V}$ with $P(G) = G$ for some radical property P), iff it satisfies the conditions

(1) $\mathfrak{R}$ is homomorphically closed;

(2) any sum of ideals in $\mathfrak{R}$ of some Ω-group G belongs to $\mathfrak{R}$;

(3) $\mathfrak{R}$ is closed under extensions (i.e.: $I \lhd G$, $I \in \mathfrak{R}$, $G/I \in \mathfrak{R}$ implies $G \in \mathfrak{R}$).

(see for example (**8**)—the proofs mentioned there make use only of isomorphism theorems and are therefore valid for Ω-groups). Good characterizations of the semisimple classes (i.e. the classes of all Ω-groups in $\mathfrak{V}$ with $P(G) = \{0\}$ for some radical property P) have been given by v. Leeuwen, Roos and Wiegandt in (**5**) for associative and alternative rings. Unfortunately their methods—using the hereditariness of the radicals—do not apply to arbitrary rings or to Ω-groups (see **4**).

The aim of this paper is threefold: (1) to describe the semisimple classes of Ω-groups; (2) to describe the pairs of corresponding radical and semisimple classes (in analogy to Dickson's torsion theory—see for example (**6**)); (3) to describe the mappings assigning to each Ω-group its radical. The proof methods will be essentially based on Hoehnke's theory of M-radicals (see **2**).

Let $\mathfrak{V}$ be any variety of Ω-groups.

Definition 1. For any subclass $\mathfrak{C}$ of $\mathfrak{V}$ define the classes $S\mathfrak{C}$ and $R\mathfrak{C}$ by:

$$S\mathfrak{C} = \{G \in \mathfrak{V} / I \lhd G, I \in \mathfrak{C} \Rightarrow I = \{0\}\}$$

$$R\mathfrak{C} = \{G \in \mathfrak{V} \mid G/I \in \mathfrak{C} \Rightarrow I = G\}$$

($S\mathfrak{C} = \mathfrak{C}^f$ and $R\mathfrak{C} = \mathfrak{C}^r$ in Hoehnke's terminology).

Definition 2. A *radical in the sense of Hoehnke* (2) (*H-radical*) on $\mathfrak{B}$ is defined by a mapping ρ assigning to each $G \in \mathfrak{B}$ an ideal $\rho(G)$ (the radical) of G and satisfying

(i) $\varphi(\rho G) \subseteq \rho(\varphi G)$ for any homomorphism φ defined on G
(ii) $\rho(G/\rho G) = \{0\}$.

Definition 3. Given a binary relation M on $\mathfrak{B}$, a H-radical is called an *M-radical* if the corresponding semisimple class $\mathfrak{S}$ satisfies the conditions:

(M_1) $G \in \mathfrak{S}$, $(I, G) \in M \Rightarrow I \notin R\mathfrak{S}$
(M_2) If $I \notin R\mathfrak{S}$ for every I with $(I, G) \in M$, then G belongs to $\mathfrak{S}$.

The following results of Hoehnke (2, p. 365–366) will be used:

For M-radicals, the radical class $\mathfrak{R}$ and the semisimple class $\mathfrak{S}$ are uniquely determined each by the other: $\mathfrak{S} = S\mathfrak{R}$ and $\mathfrak{R} = R\mathfrak{S}$. It follows that $\mathfrak{S} = SR\mathfrak{S}$ and $\mathfrak{R} = RS\mathfrak{R}$ hold. A subclass $\mathfrak{S}$ of $\mathfrak{B}$ is the semisimple class of an M-radical iff $\mathfrak{S}$ is closed under subdirect products and satisfies (M_1) and (M_2); a subclass $\mathfrak{R}$ of $\mathfrak{B}$ is the radical class of an M-radical iff it satisfies the conditions
(N_1) $\mathfrak{R}$ is homomorphically closed;
(N_2) If for every homomorphic image $\varphi G \neq \{0\}$ of G there exists $I \in \mathfrak{R}$ with $(I, \varphi G) \in M$, then G belongs to $\mathfrak{R}$;
(N_3) $S\mathfrak{R}$ is closed under subdirect products;
(N_4) For every $G \in \mathfrak{R}$, $G \neq \{0\}$, there is an element I in $\mathfrak{R}$ with $(I, G) \in M$.

In the following, we will consider the relation M defined by:

$$(I, G) \in M \quad \text{iff} \quad \{0\} \neq I \lhd G.$$

For this relation M, one easily checks that every subclass of $\mathfrak{B}$ satisfying (M_1) and (M_2) is closed under subdirect products and that the conditions (N_1) and (N_2) imply (N_3) and (N_4). Therefore our M-radicals are then exactly the radicals in the sense of Kuroš and Amitsur (see (8), p. 10).

Notation. For any subclass $\mathfrak{C}$ of $\mathfrak{B}$ containing $\{0\}$ and any $G \in \mathfrak{B}$ define $(G)C = (\cap I : I \lhd G, G/I \in \mathfrak{C})$ and $\mathfrak{C}(G) = (\Sigma\, I : I \lhd G, I \in \mathfrak{C})$.

Theorem 1. *For all pairs $(\mathfrak{R}, \mathfrak{S})$ of corresponding KA-radical and semisimple classes in $\mathfrak{B}$, $(G)\mathfrak{S}$ equals $\mathfrak{R}(G)$ for every $G \in \mathfrak{B}$.*

Proof. $\mathfrak{R}$ satisfies the conditions (1)–(3), $\mathfrak{S}$ the conditions (M_1) and (M_2). Let I be any ideal of $G/\mathfrak{R}(G)$ with $I \in \mathfrak{R}$ and let K be the complete preimage of I in G. (2) and (3) for $\mathfrak{R}$ then imply $K \in \mathfrak{R}$, i.e. $K \subseteq \mathfrak{R}(G)$. Hence I is the zero-ideal and $G/\mathfrak{R}(G)$ belongs to $\mathfrak{S}$ by (M_2). It follows that $(G)\mathfrak{S}$ is contained in $\mathfrak{R}(G)$.

Conversely it follows from (2) that the radical defined by $\mathfrak{S}$ is $(G)\mathfrak{S}$ for every $G \in \mathfrak{B}$. (1) and (2) for $\mathfrak{R}$ imply that $\mathfrak{R}(G)/(G)\mathfrak{S}$ belongs to $\mathfrak{R}$. But $\mathfrak{R}(G)/(G)\mathfrak{S}$ is an ideal in $G/(G)\mathfrak{S} \in \mathfrak{S}$, hence by ($M_1$)$\mathfrak{R}(G)$ must be contained in $(G)\mathfrak{S}$.

Theorem 2. *For a pair* $(\mathfrak{F}, \mathfrak{T})$ *of subclasses of* $\mathfrak{V}$ *the following assertions are equivalent*:

(2.1) $(\mathfrak{F}, \mathfrak{T})$ *is the pair of corresponding semisimple class* $(\mathfrak{F})$ *and radical class* $(\mathfrak{T})$ *for some KA-radical on* $\mathfrak{V}$;

(2.2) $(\mathfrak{F}, \mathfrak{T})$ *satisfies the conditions*

(A) $\mathfrak{F} \cap \mathfrak{T} = \{0\}$,

(B) $\mathfrak{T}$ *is homomorphically closed,*

(C) $G \in \mathfrak{F}$, $(I, G) \in M$ *implies* $I \notin \mathfrak{T}$,

(D) *For any* $G \in \mathfrak{V}$, *there is an ideal* $T(G)$ *of* G *satisfying* $T(G) \in \mathfrak{T}$ *and* $G/T(G) \in \mathfrak{F}$;

(2.3) $(\mathfrak{F}, \mathfrak{T})$ *satisfies the conditions* (A), (C), (D) *and the dual to* (C):

(E) $G \in \mathfrak{T}$, $G/I \neq \{0\}$ *implies* $G/I \notin \mathfrak{F}$.

(Assertion 2.3 is more general than 2.2 and gives a self-dual characterisation of the pairs of semisimple and radical classes; it is due to a remark of R. Wiegandt.)

Proof. Equivalence of 2.1 and 2.2:

Let $(\mathfrak{F}, \mathfrak{T})$ be a pair of corresponding semisimple and radical classes. Then $\mathfrak{F} = S\mathfrak{T}$ and $\mathfrak{T} = R\mathfrak{F}$; (A), (B) and (C) are trivially fulfilled. To prove (D) take $T(G) = (G)\mathfrak{F} = \mathfrak{T}(G)$. Then by (2) $T(G)$ belongs to T and $G/T(G)$ belongs to $\mathfrak{F}$ as $\mathfrak{F}$ is closed under subdirect products.

Assume conversely that $(\mathfrak{F}, \mathfrak{T})$ satisfies (A)–(D). Then (C) and (D) imply $S\mathfrak{T} = \mathfrak{F}$ and (A), (B) and (D) imply $R\mathfrak{F} = \mathfrak{T}$. Condition (M_1) for $\mathfrak{F}$ is exactly (C); to prove (M_2) assume that for some $G \in \mathfrak{V}$, $(I, G) \in M$ implies $I \notin R\mathfrak{F} = \mathfrak{T}$; then by (D) $T(G) = \{0\}$ and G belongs to $\mathfrak{F}$. Hence $\mathfrak{F}$ is a semisimple class and $\mathfrak{T} = R\mathfrak{F}$ the corresponding radical class of some KA-radical.

Equivalence of (B) and (E) if (A), (C) and (D) are satisfied:

(B) implies (E) by (A).

(E) implies (B) by the following argument: $G \in \mathfrak{T}$, $G/I \neq \{0\}$ implies $G/I \notin \mathfrak{F}$ by (E); $G/I/\,T(G/I)$ belongs to $\mathfrak{F}$ by (D); hence $G/I/\,T(G/I)$ is equal to $\{0\}$; it follows that $T(G/I) = G/I$ and therefore that G/I belongs to $\mathfrak{T}$.

Corollary. *A subclass R of* $\mathfrak{V}$ *is the radical class for some KA-radical on* $\mathfrak{V}$ *iff it satisfies the conditions*:

(R_1) $\mathfrak{R}(G) \in \mathfrak{R}$ for every $G \in \mathfrak{V}$;

(R_2) $\mathfrak{R}(G) = (G)S\mathfrak{R}$ for every $G \in \mathfrak{V}$.

(This corollary is due to a remark of R. Wiegandt.)

Proof. (R_1) for a KA-radical class follows from (2), (R_2) is easily deduced from Theorem 1. Conversely, if (R_1) and (R_2) hold for a class $\mathfrak{R}$, take $\mathfrak{T} = \mathfrak{R}$ and $\mathfrak{F} = S\mathfrak{R}$; then (A) holds for $(\mathfrak{F}, \mathfrak{T})$ by construction; $G \in \mathfrak{T}$ implies $(G)S\mathfrak{R} = \mathfrak{R}(G) = G$, i.e. (E) holds; $G \in \mathfrak{F}$ implies $\mathfrak{R}(G) = (G)S\mathfrak{R} = (G)\mathfrak{F} = 0$, hence (C) and (D) are fulfilled (with $T(G) = \mathfrak{R}(G)$).

Remark. If $(\mathfrak{F}, \mathfrak{T})$ satisfies (A)–(D), $T(G)$ is uniquely determined for every $G \in \mathfrak{V}$: if we assume that (D) is satisfied for both $T(G)$ and $U(G)$, by (B) we get that $U(G)/T(G)$ belongs to $\mathfrak{T}$; but $U(G)/T(G)$ is an ideal of $G/T(G) \in \mathfrak{F}$ and hence belongs to $\mathfrak{T}$ iff it is the zero-ideal. It follows $U(G) \subseteq T(G)$. By the same argument we get $T(G) \subseteq U(G)$.

From this remark we can deduce that for every pair $(\mathfrak{F}, \mathfrak{T})$ of subclasses of $\mathfrak{V}$ satisfying (A)–(D) the corresponding KA-radical is given for every $G \in \mathfrak{V}$ by: $T(G) = (G)\mathfrak{F} = \mathfrak{T}(G)$.

Theorem 3. *A mapping ρ assigning to each $G \in \mathfrak{V}$ an ideal $\rho(G)$ of G defines a KA-radical iff it satisfies the conditions* (i) *and* (ii) (*i.e. defines a H-radical*) *and the conditions*

(iii) *idempotence*

(iv) *$I \lhd G$, $\rho(I) = I$ implies $I \subseteq \rho(G)$.*

Proof. Assume that ρ satisfies (i)–(iv) and take $\mathfrak{T}_\rho = \{G \in \mathfrak{V} \mid \rho(G) = G\}$ and $\mathcal{F}_\rho = \{G \in \mathfrak{V} \mid \rho(G) = \{0\}\}$. Then for the pair $(\mathfrak{F}_\rho, \mathfrak{T}_\rho)$ condition (A) is fulfilled by construction, (B) follows directly from (i), (C) can be deduced from (iv) and (D) holds for $T(G) = \rho(G)$ by (ii) and (iii). Hence by Theorem 2 and the remark above follows that $(\mathfrak{F}_\rho, \mathfrak{T}_\rho)$ is the pair of semisimple and radical class for KA-radical and that this radical is given by $T(G) = \rho(G)$ on every $G \in \mathfrak{V}$.

Another proof of this implication can be found in (**3**).

Conversely, take any KA-radical on $\mathfrak{V}$; the pair of the corresponding semisimple class $\mathfrak{F}$ and radical class $\mathfrak{T}$ satisfies (A)–(D) and the radical is given by $T(G)$ for every $G \in \mathfrak{V}$. Since the KA-radicals are M-radicals, the mapping $G \to T(G)$ satisfies (i) and (ii); (iii) follows directly from (D) and the fact that $T(G)$ is uniquely determined. To prove (iv) consider $I \lhd G$ with $T(I) = I$, i.e. with $I \in \mathfrak{T}$; from (B) follows that $I/T(G)$ belongs to $\mathfrak{T}$, but $I/T(G)$ is an ideal in $G/T(G) \in \mathfrak{F}$; hence by (C) we get $I \subseteq T(G)$.

Theorem 4. *For a subclass $\mathfrak{S}$ of $\mathfrak{V}$ the following assertions are equivalent*:

(4.1) *$\mathfrak{S}$ is the semisimple class of some KA-radical on $\mathfrak{V}$*

(4.2) *$\mathfrak{S}$ has the properties*

(a) *closed under subdirect products*

(b) (M_1)

(c) *$((G)\mathfrak{S})\mathfrak{S} = (G)\mathfrak{S}$ for every $G \in \mathfrak{V}$.*

(4.3) *$\mathfrak{S}$ has the properties* (a), (b),

(d) *closed under extensions, i.e.: $I \lhd G$, $I \in \mathfrak{S}$, $G/I \in \mathfrak{S}$ implies $G \in \mathfrak{S}$*

(e) *$((G)\mathfrak{S})\mathfrak{S}$ is an ideal in G for every $G \in \mathfrak{V}$.*

Proof. (4.1)$\Rightarrow$(4.2): As the KA-radicals are M-radicals, (a) and (b) hold; by Theorem 2 the considered radical is given by the mapping $G \to (G)\mathfrak{S}$ and by Theorem 3 this mapping is idempotent; hence (c) holds.

(4.2)$\Rightarrow$(4.3): (e) is a trivial consequence of (c); (d) follows from the following argument: for every ideal I of G with $G/I \in \mathfrak{S}$, $(G)\mathfrak{S}$ is contained in I by construction and hence is an ideal of I; $I \in \mathfrak{S}$ then implies $(G)\mathfrak{S} = \{0\}$ or $(G)\mathfrak{S} \notin RS$ by (M_1). Since the second assertion is a contradiction to (c), we get $(G)\mathfrak{S} = \{0\}$, i.e. $G \in \mathfrak{S}$ by (a).

(4.3)$\Rightarrow$(4.1): (M_1) is satisfied by (b), hence we only have to prove (M_2) for $\mathfrak{S}$. Assume that for G $(I, G) \in M$ implies $I \notin R\mathfrak{S}$, i.e. $(I)\mathfrak{S} \neq I$; it follows that for $G \notin \mathfrak{S}$ (i.e.

$(G)\mathfrak{S} \neq \{0\})$, $((G)\mathfrak{S})\mathfrak{S} \neq (G)\mathfrak{S}$; by (e) $((G)\mathfrak{S})\mathfrak{S}$ is an ideal of G and we get

$$G/((G)\mathfrak{S})\mathfrak{S} \,/\, (G)\mathfrak{S} \,/\, ((G)\mathfrak{S})\mathfrak{S} - G/(G)\mathfrak{S};$$

since by (a) $G/(G)\mathfrak{S}$ and $(G)\mathfrak{S} \,/\, ((G)\mathfrak{S})\mathfrak{S}$ are elements of $\mathfrak{S}$, we can apply (d) to get $G/((G)\mathfrak{S})\mathfrak{S} \in \mathfrak{S}$ and hence $(G)\mathfrak{S} = ((G)\mathfrak{S})\mathfrak{S}$ in contradiction to our assumptions for $G \notin \mathfrak{S}$.

Remark. The equivalence of (4.1) and (4.3) of Theorem 4 has been independently shown by a slightly different approach by v. Leeuwen and Wiegandt (7).

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