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SPACE SHUTTLE GN & C EQUATION DOCUMENT

No. 4

(Revision 2)

Precision State and Filter Weighting Matrix Extrapolation

by

William M. Robertson

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Systems Analysis Branch
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Manned Spacecraft Center, Houston, Texas

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REVISION 2 CHANGES

This document is a revised version of the previous one dated October 1971, and supersedes that document. The following is a summary of the major technical changes included in this revision:

1. The filter-weighting matrix extrapolation equations have been expanded to provide the (optional) capability of including process noise in the filter.
2. The detailed flow diagrams have been modified so that the routine may be re-called to continue an extrapolation already started on a previous call without rerectification.
3. The rows and columns of the state covariance matrix which pertain to the additionally estimated quantities such as landmark locations or instrument biases have been re-arranged to be between those rows and columns pertaining to the two position and velocity variables. This engenders some changes, mostly notational, in the filter-weighting matrix extrapolation equations.

FOREWORD

This document is one of a series of candidates for inclusion in a future revision of MSC-04217, "Space Shuttle Guidance Navigation and Control Design Equations". The enclosed has been prepared under NAS9-10268, Task No. 15-A, "GN & C Flight Equation Specification Support", and applies to function 1 of the Orbital Navigation Module (ON2) and function 1 of the Co-orbiting Vehicle Navigation Module (ON3) as defined in MSC-03690 Rev. B, "Space Shuttle Orbiter Guidance, Navigation and Control Software Functional Requirements - Vertical Flight Operations", dated 15 December 1971.

Gerald M. Levine

Gerald M. Levine, Director
APOLLO Space Guidance Analysis Division

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NOMENCLATURE

$\underline{a}_d(t)$	Perturbing acceleration at time t
b	Number of additional quantities, such as landmark locations or instrument biases, being estimated
c_{nom}	Constant for adjustment of nominal step-size
d	Number of columns in the filter weighting sub-matrix W
E_P	Primary vehicle covariance matrix (6 x 6)
E_T	Target vehicle covariance matrix (6 x 6)
$f(q)$	Special function of q defined in text
$G(t)$	Gravity gradient matrix
$\underline{i}_{pole}$	Unit vector of earth's north polar axis expressed in reference coordinates
$\underline{i}_r$	Unit vector in the direction of the position vector $\underline{r}$
I_3	Three-dimensional identity matrix
J_2	Constant describing dominant term of earth's oblateness
q	Special function of $\underline{r}$ and $\underline{\delta}$ defined in text
$\underline{q}_i$	Three-dimensional column vector in the 3 x 3 process noise matrix for either the primary vehicle (i = 3, 4, 5) or the target vehicle (i = 9+b, 10+b, 11+b)

Q	Process noise matrix (3 x 3). Subscripts P or T refer to the process noise matrix associated with the primary or target vehicle state
$\underline{r}_0$	Geocentric position vector at time t_0
$\underline{r}(t)$	Geocentric position vector at time t
$r(t)$	Magnitude of geocentric position vector
$\underline{r}_{con}(t)$	Reference conic position vector at time t
$r_{con}(t)$	Magnitude of reference conic position vector at time t
r_E	Mean equatorial radius of the earth
$\underline{r}_F$	Geocentric position vector at time t_F
$\underline{r}_j$	Intermediate values of $\underline{r}$
s_{cont}	Switch indicating whether previous extrapolation is to be continued without re-rectification
s_{pert}	Switch indicating the perturbing accelerations to be included
s_q	Switch controlling whether process noise is to be included in the W-matrix extrapolation
s_{veh}	Switch indicating whether the filter-weighting sub-matrix being extrapolated is associated with the primary or target vehicle
s_W	Switch controlling whether state or filter-weighting matrix integration is being performed (used only internally in routine)

t_0	Initial time point. Also, time of last rectification
t_F	Time to which it is desired to extrapolate $(\underline{r}_0, \underline{v}_0)$ and optionally W_0
$\underline{v}_0$	Geocentric velocity vector at time t_0
$\underline{v}_F$	Geocentric velocity vector at time t_F
$\underline{v}_{con}(t)$	Reference conic velocity vector at time t
W_0	Filter-weighting matrix at time t_0
W_F	Filter-weighting matrix at time t_F
$w_{k,i}$	Elements of the filter-weighting matrix
$\underline{w}_{k,i}$	Three-dimensional column vectors into which the filter-weighting matrix is partitioned
x	Independent variable in Kepler routine
x'	Previous value of x
$\underline{\gamma}(t)$	Vector random variable of dimension b representing errors in the additionally estimated quantities such as landmark locations or instrument biases
$\underline{\delta}(t)$	Position deviation vector of true position from reference conic position at time t
δ'	Magnitude of position deviation vector (temporary variable used for rectification test)
δ_{max}	Maximum value of $\lvert\underline{\delta}\rvert$ permitted (used as rectification criterion)

Δt	Time-step in numerical integration of differential equation
Δt_{max}	Maximum permissible time-step size
Δt_{nom}	Nominal integration time-step size
ϵ_t	Time convergence tolerance criterion
$\underline{\epsilon}(t)$	Random variable representing error in estimate of position vector at time t
$\underline{\eta}(t)$	Random variable representing error in estimate of velocity vector at time t
μ	Earth's gravitational parameter
$\underline{\nu}(t)$	Velocity deviation vector of true velocity from reference conic velocity at time t
ν'	Magnitude of velocity deviation vector (temporary variable used for rectification test)
ν_{max}	Maximum value of $\lvert \underline{\nu} \rvert$ permitted (used as rectification criterion)
τ	Time interval since last rectification
τ'	Previous value of τ
ϕ	Geocentric latitude

1. INTRODUCTION

The Precision State and Filter Weighting Matrix Extrapolation Routine provides the capability to extrapolate any spacecraft geocentric state vector either backwards or forwards in time through a force field consisting of the earth's primary central-force gravitational attraction and a superimposed perturbing acceleration. The perturbing acceleration may be either the single dominant term (J_2) of the earth's oblateness or a more complete expression involving all significant perturbation effects. The Routine also provides the capability of extrapolating the filter-weighting matrix along the precision trajectory. This matrix, also known as the "W-matrix", is a square root form of the error covariance matrix and contains statistical information relative to the accuracies of the state vectors and certain other optionally estimated quantities.

On any one call, the routine extrapolates only one state vector and only those six rows of the filter-weighting matrix relating to this state vector. Two calls are required to extrapolate two separate state vectors and a complete filter-weighting matrix pertaining to two state vectors. The complete extrapolated filter-weighting matrix is obtained by properly adjoining the two separately extrapolated submatrices of six rows each.

The routine is merely a coded algorithm for the numerical solution of modified forms of the basic differential equations which are satisfied by the geocentric state vector of the spacecraft's center of mass and by the filter-weighting matrix, namely:

$$\frac{d^2}{dt^2}\,\underline{r}(t) + \frac{\mu}{r^3(t)}\,\underline{r}(t) = \underline{a}_d(t)$$

and

$$\frac{d}{dt}\,W(t) = F(t)\,W(t) + \left\{\frac{1}{2}\,Q[\,W^T(t)\,]^{-1}\right\}$$

where $\underline{a}_d(t)$ is the vector sum of all the desired perturbing accelerations, $F(t)$ is a matrix containing the gravity gradient matrix and the identity matrix in its off-diagonal sub-blocks, and Q is the process noise matrix. A simplified form of the term in braces is included only during phases when process noise is to be introduced into the navigation filter to improve the long-term navigation accuracy.

Because of its high accuracy and its capability of extrapolating the filter-weighting matrix, this routine serves as the computational foundation for precise space navigation. It suffers from a relatively slow computation speed in comparison with the Conic State Extrapolation Routine.

2. FUNCTIONAL FLOW DIAGRAM

The Precision State and Filter Weighting Matrix Extrapolation Routine performs its functions by integrating modified forms of the basic differential equations at a sequence of points separated by intervals known as time-steps, which are not necessarily of the same size. The routine automatically determines the size to be taken at each step.

As shown in Fig. 1, the state vector and (optionally) the filter-weighting sub-matrix are updated one step at a time along the precision trajectory until the specified overall transfer time interval is exactly attained. (The size of the last time-step is adjusted as necessary to make this possible.)

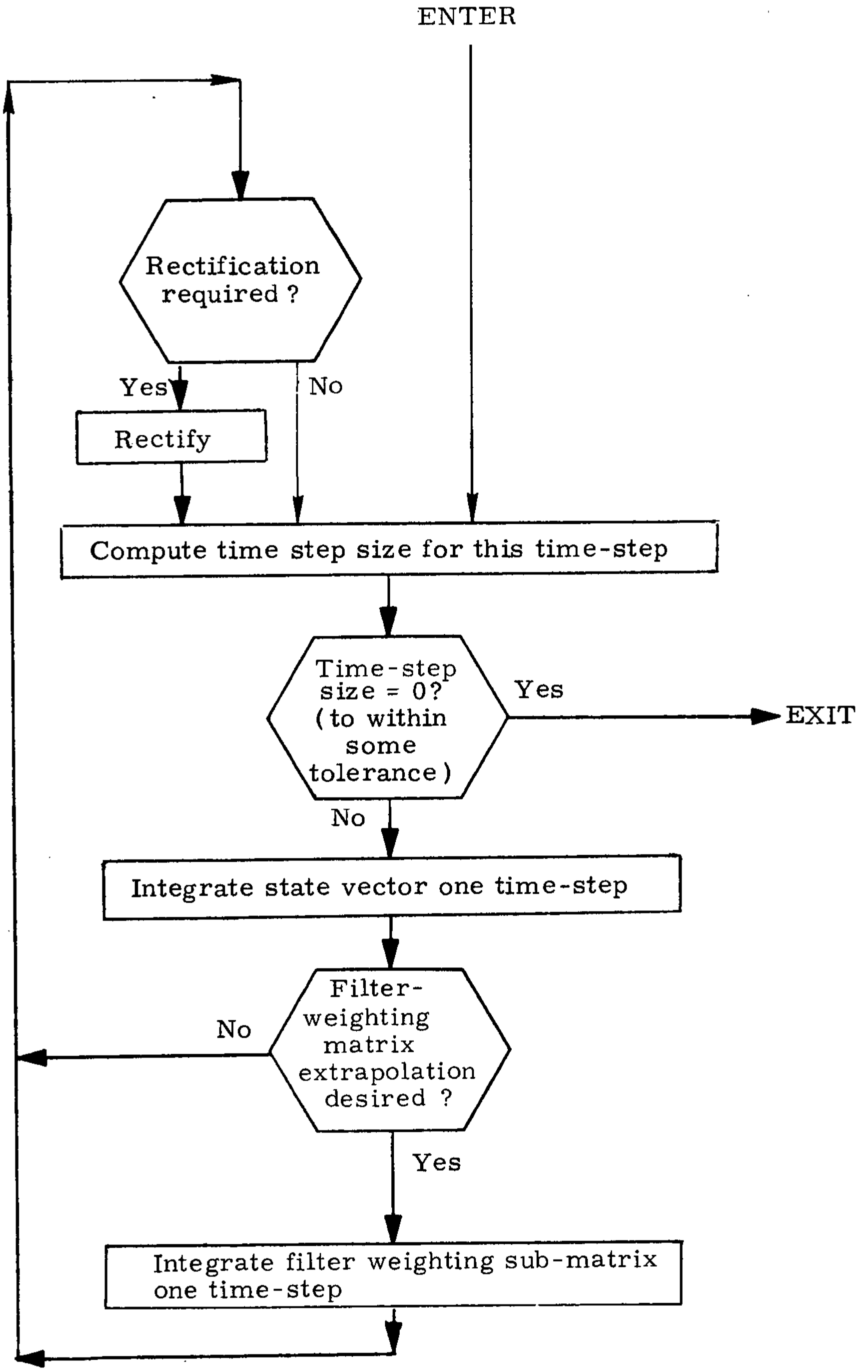

Figure 1. Functional Flow Diagram Precision State and Filter Weighting Matrix Extrapolation Routine

3. INPUT AND OUTPUT VARIABLES

The Precision State and Filter Weighting Matrix Extrapolation Routine has the following input and output variables:

Input Variables

$(\underline{r}_0, \underline{v}_0)$	Geocentric state vector to be extrapolated. [If $s_{cont} \neq 0$, $(\underline{r}_0, \underline{v}_0)$ is last rectified geocentric state vector]
t_0	Time associated with $(\underline{r}_0, \underline{v}_0)$ and W_0. [If $s_{cont} \neq 0$, t_0 is last rectification time]
t_F	Time to which it is desired to extrapolate $(\underline{r}_0, \underline{v}_0)$ and optionally W_0
s_{pert}	Switch indicating the perturbing accelerations to be included. ($s_{pert} = 1$ implies J_2 oblateness term only; $s_{pert} > 1$ implies a more complete perturbing acceleration model (or models).)
d	Number of columns in filter-weighting sub-matrix (d = 0, 6, 7, . . . , where 0 indicates no W-matrix extrapolation)
b	Number of additional quantities, such as landmark locations or instrument biases, being estimated
s_{veh}	Switch indicating whether the filter-weighting sub-matrix being extrapolated is associated with the primary ($s_{veh} = 0$) or target ($s_{veh} = 1$) vehicle
W_0	Filter-weighting sub-matrix to be extrapolated (optional) (W_0 has dimension 6 x d)
s_q	Switch indicating whether process noise is to be included ($s_q = 1$) or not ($s_q = 0$) in the W-matrix extrapolation
Q	Process noise matrix (3 x 3) associated with the state being extrapolated

s_{cont}	Switch indicating whether previous extrapolation is to be continued (s_{cont} = 1) or not (s_{cont} = 0) without re-rectification	
$(\underline{\delta}, \underline{\nu})$	Position and velocity deviation vectors	At end of previous extrapolation [used only if $s_{cont} = 1$]
$(\underline{r}_{con}, \underline{v}_{con})$	Conic position and velocity vectors	
τ	Time interval since rectification	
x'	Last value of independent variable in Kepler Routine	
τ'	Last value of dependent variable in Kepler Routine	

Output Variables

$(\underline{r}_F, \underline{v}_F)$	Extrapolated geocentric state vector	
t	Time associated with $(\underline{r}_F, \underline{v}_F)$ and W_F. [Will equal t_F within tolerance of ϵ_t]	
W_F	Extrapolated filter-weighting sub-matrix of dimension 6 x d	
$(\underline{\delta}, \underline{\nu})$	Position and velocity deviation vectors	For use as input if $s_{cont} = 1$ on a subsequent extrapolation
$(\underline{r}_{con}, \underline{v}_{con})$	Conic position and velocity vectors	
τ	Time interval since rectification	
$(\underline{r}_0, \underline{v}_0)$	Last rectified position and velocity vectors	
t_0	Time of last rectification	
x'	Last value of independent variable in Kepler Routine	
τ'	Last value of dependent variable in Kepler Routine	

4. DESCRIPTION OF EQUATIONS

4.1 Precision State Extrapolation Equations

Since the perturbing acceleration is small compared with the central force field, direct numerical integration of the basic differential equations of motion of the spacecraft state vector is inefficient. Instead, a technique due to Encke is utilized in which only the deviations of the state from a reference conic orbit are numerically integrated. The positions and velocities along the reference conic are obtained from the Kepler routine.

At time t_0 the position and velocity vectors, $\underline{r}_0$ and $\underline{v}_0$, define an osculating conic orbit. Because of the perturbing accelerations, the true position and velocity vectors $\underline{r}(t)$ and $\underline{v}(t)$ will deviate as time progresses from the conic position and velocity vectors $\underline{r}_{con}(t)$ and $\underline{v}_{con}(t)$ which have been conically extrapolated from $\underline{r}_0$ and $\underline{v}_0$. Let

$$\underline{\delta}(t) = \underline{r}(t) - \underline{r}_{con}(t)$$

$$\underline{\nu}(t) = \underline{v}(t) - \underline{v}_{con}(t)$$

be the vector deviations. It can be shown that the position deviation $\underline{\delta}(t)$ satisfies the differential equation

$$\frac{d^2}{dt^2}\underline{\delta}(t) + \frac{\mu}{r_{con}^3(t)}\left[f(q)\,\underline{r}(t) + \underline{\delta}(t) \right] = \underline{a}_d(t)$$

with the initial conditions

$$\underline{\delta}(t_0) = \underline{0}, \quad \underline{\nu}(t_0) = \underline{0}$$

where

$$q = \frac{(\underline{\delta} - 2\underline{r}) \cdot \underline{\delta}}{r^2}, \quad f(q) = q \, \frac{3 + 3q + q^2}{1 + (1 + q)^{3/2}},$$

and $\underline{a}_d(t)$ is the total perturbing acceleration. The above second order differential equation in the deviation vector $\underline{\delta}(t)$ is numerically integrated by a method described in a later sub-section.

The term

$$\frac{\mu}{r_{con}^{3}} \left[f(q)\, \underline{r}(t) + \underline{\delta}(t) \right]$$

must remain small, i.e. of the same order as $\underline{a}_d(t)$, if the method is to be efficient. As the deviation vector $\underline{\delta}(t)$ grows in magnitude, this term will eventually increase in size. When

$$|\underline{\delta}(t)| > 0.01 \, |\underline{r}_{con}(t)| \text{ or } |\underline{\nu}(t)| > 0.01 \, |\underline{v}_{con}(t)|$$

or when

$$|\underline{\delta}(t)| > \delta_{max} \text{ or } |\underline{\nu}(t)| > \nu_{max},$$

a new osculating conic orbit is established based on the latest precision position and velocity vectors $\underline{r}(t)$ and $\underline{v}(t)$, the deviations $\underline{\delta}(t)$ and $\underline{\nu}(t)$ are zeroed, and the numerical integration of $\underline{\delta}(t)$ and $\underline{\nu}(t)$ continues. The process of establishing a new conic orbit is called rectification.

The total perturbing acceleration $a_d(t)$ is in general the vector sum of all the desired individual perturbing accelerations comprising the total force field, such as those due to the earth's oblateness, the gravitational attractions of the sun and moon, and the earth's atmospheric drag. Since many Shuttle applications will require only the perturbing effect of the dominant term J_2 of the earth's oblateness, the use of only this term has been made a standard option in the routine diagrammed in Section 5. However, provision has been made for handling a completely general perturbing acceleration. The form of this perturbing acceleration will depend primarily upon the requirements of the Orbit Navigation function.

The explicit expression for the earth's J_2 oblateness acceleration alone is:

$$\underline{a}_d = -\frac{\mu}{r^2}\left\{\frac{3}{2} J_2\left[\frac{r_E}{r}\right]^2 \left[(1 - 5 \sin^2\phi)\,\underline{i}_r + 2 \sin\phi\, \underline{i}_{pole}\right]\right\}$$

where

$\underline{i}_r$ is the unit position vector in reference coordinates,

$\underline{i}_{pole}$ is the unit vector of the earth's north polar axis expressed in reference coordinates,

$\sin\phi = \underline{i}_r \cdot \underline{i}_{pole}$,

and

r_E is the mean equatorial radius of the earth.

4.2 Filter-Weighting (W) Matrix Extrapolation Equations

The position and velocity vectors which are maintained by the spacecraft's computer are only estimates of the actual values of these vectors. As part of the navigation technique it is also necessary for the computer to maintain statistical information about the position and velocity vectors. Furthermore, in particular applications it is necessary to include statistical data on various other quantities, such as landmark locations during Orbit Navigation and certain instrument biases during Co-orbiting Vehicle Navigation. The filter-weighting W-matrix is used for all these purposes.

If $\underline{\epsilon}(t)$ and $\underline{\eta}(t)$ are three dimensional vector random variables with zero mean which represent the errors in the estimates of a spacecraft's position and velocity at time t, then the six-dimensional state error covariance matrix $E(t)$ at time t is defined by:

$$E(t) = \begin{bmatrix} \overline{\underline{\epsilon}(t)\,\underline{\epsilon}(t)^T} & \overline{\underline{\epsilon}(t)\,\underline{\eta}(t)^T} \\ \overline{\underline{\eta}(t)\,\underline{\epsilon}(t)^T} & \overline{\underline{\eta}(t)\,\underline{\eta}(t)^T} \end{bmatrix},$$

where the bar represents the expected value or ensemble average at the fixed time t of each element of the matrix over which it appears.

If $\underline{\gamma}(t)$ is a b-dimensional vector random variable with zero mean which represents the errors in the estimates of the b additionally estimated quantities such as landmark locations or instrument biases, then a (6+b) - dimensional state and other parameter covariance matrix is defined by:

$$E(t) = \begin{bmatrix} \overline{\underline{\epsilon}(t)\,\underline{\epsilon}(t)^T} & \overline{\underline{\epsilon}(t)\,\underline{\eta}(t)^T} & \overline{\underline{\epsilon}(t)\,\underline{\gamma}(t)^T} \\ \overline{\underline{\eta}(t)\,\underline{\epsilon}(t)^T} & \overline{\underline{\eta}(t)\,\underline{\eta}(t)^T} & \overline{\underline{\eta}(t)\,\underline{\gamma}(t)^T} \\ \overline{\underline{\gamma}(t)\,\underline{\epsilon}(t)^T} & \overline{\underline{\gamma}(t)\,\underline{\eta}(t)^T} & \overline{\underline{\gamma}(t)\,\underline{\gamma}(t)^T} \end{bmatrix}$$

Further, if the statistical properties of the positions and velocities of two separate spacecraft are to be maintained, a twelve-dimensional state covariance matrix is defined by:

$$E(t) = \begin{bmatrix} \overline{\underline{\epsilon}_P\,\underline{\epsilon}_P^T} & \overline{\underline{\epsilon}_P\,\underline{\eta}_P^T} & \overline{\underline{\epsilon}_P\,\underline{\epsilon}_T^T} & \overline{\underline{\epsilon}_P\,\underline{\eta}_T^T} \\ \overline{\underline{\eta}_P\,\underline{\epsilon}_P^T} & \overline{\underline{\eta}_P\,\underline{\eta}_P^T} & \overline{\underline{\eta}_P\,\underline{\epsilon}_T^T} & \overline{\underline{\eta}_P\,\underline{\eta}_T^T} \\ \overline{\underline{\epsilon}_T\,\underline{\epsilon}_P^T} & \overline{\underline{\epsilon}_T\,\underline{\eta}_P^T} & \overline{\underline{\epsilon}_T\,\underline{\epsilon}_T^T} & \overline{\underline{\epsilon}_T\,\underline{\eta}_T^T} \\ \overline{\underline{\eta}_T\,\underline{\epsilon}_P^T} & \overline{\underline{\eta}_T\,\underline{\eta}_P^T} & \overline{\underline{\eta}_T\,\underline{\epsilon}_T^T} & \overline{\underline{\eta}_T\,\underline{\eta}_T^T} \end{bmatrix}$$

where the subscripts P and T refer to the primary and target vehicles, respectively.

And finally, if the statistical properties of the b additionally estimated quantities are also to be maintained along with the two state vectors, a (12 + b) state and other parameter covariance matrix is defined by:

$$E(t) = \begin{bmatrix} \overline{\underline{\epsilon}_P \underline{\epsilon}_P^T} & \overline{\underline{\epsilon}_P \underline{\eta}_P^T} & \overline{\underline{\epsilon}_P \underline{\gamma}^T} & \overline{\underline{\epsilon}_P \underline{\epsilon}_T^T} & \overline{\underline{\epsilon}_P \underline{\eta}_T^T} \\ \overline{\underline{\eta}_P \underline{\epsilon}_P^T} & \overline{\underline{\eta}_P \underline{\eta}_P^T} & \overline{\underline{\eta}_P \underline{\gamma}^T} & \overline{\underline{\eta}_P \underline{\epsilon}_T^T} & \overline{\underline{\epsilon}_P \underline{\eta}_T^T} \\ \overline{\underline{\gamma}\, \underline{\epsilon}_P^T} & \overline{\underline{\gamma}\, \underline{\eta}_P^T} & \overline{\underline{\gamma}\, \underline{\gamma}^T} & \overline{\underline{\gamma}\, \underline{\epsilon}_T^T} & \overline{\underline{\gamma}\, \underline{\eta}_T^T} \\ \overline{\underline{\epsilon}_T \underline{\epsilon}_P^T} & \overline{\underline{\epsilon}_T \underline{\eta}_P^T} & \overline{\underline{\epsilon}_T \underline{\gamma}^T} & \overline{\underline{\epsilon}_T \underline{\epsilon}_T^T} & \overline{\underline{\epsilon}_T \underline{\eta}_T^T} \\ \overline{\underline{\eta}_T \underline{\epsilon}_P^T} & \overline{\underline{\eta}_T \underline{\eta}_P^T} & \overline{\underline{\eta}_T \underline{\gamma}^T} & \overline{\underline{\eta}_T \underline{\epsilon}_T^T} & \overline{\underline{\eta}_T \underline{\eta}_T^T} \end{bmatrix}$$

Rather than use one of the above covariance matrices in the navigation procedure, it is more convenient to use a matrix $W(t)$ having the same dimension d as the covariance matrix $E(t)$ and defined by:

$$E(t) = W(t)\; W(t)^T$$

The matrix $W(t)$ is called the filter-weighting matrix, and is in a sense a square root of the covariance matrix.

Extrapolation of the $W(t)$ matrix in time may be made by direct numerical integration of the differential equation which it satisfies. In the one-spacecraft case, this is:

$$\frac{d}{dt} W(t) = \left[\begin{array}{cc|c} O & I_3 & O_{(6 \times b)} \\ G(t) & O & \\ \hline O_{(b \times 6)} & & O_{(b \times b)} \end{array}\right] W(t) + s_q^{\;1/2} \left[\begin{array}{cc|c} O & O & O_{(6 \times b)} \\ O & Q & \\ \hline O_{(b \times 6)} & & O_{(b \times b)} \end{array}\right] [\, W^T(t)]^{-1}$$

(where $b = 0, 1, 2, \ldots$ is the number of additionally estimated quantities).

In the two-spacecraft case, the differential equation is:

$$\frac{d}{dt} W(t) = \left[\begin{array}{cc|c|cc} O & I_3 & O_{(6 \times b)} & O & O \\ G_P(t) & O & & O & O \\ \hline O_{(b \times 6)} & & O_{(b \times b)} & O_{(b \times 6)} & \\ \hline O & O & O_{(6 \times b)} & O & I_3 \\ O & O & & G_S(t) & O \end{array}\right] W(t) +$$

$$
s_q^{\,1/2} \left[\begin{array}{cc|c|cc}
O & O & O_{(6 \times b)} & O & O \\
O & Q_P & & O & O \\
\hline
O_{(b \times 6)} & & O_{(b \times b)} & O_{(b \times 6)} & \\
\hline
O & O & O_{(6 \times b)} & O & O \\
O & O & & O & Q_S
\end{array}\right] \left[W^T(t) \right]^{-1} ,
$$

where I_3 is the 3 x 3 identity matrix, the O's are zero matrices of the required dimensions, the $G(t)$ are the 3 x 3 conic gravity gradient matrices

$$
G(t) = \frac{\mu}{r^5(t)} \left[3\, \underline{r}(t)\, \underline{r}(t)^T - r^2(t)\, I_3 \right]
$$

associated with the vehicle under consideration or with the primary (P) or target (T) vehicle, and the Q are the likewise associated 3 x 3 process noise matrices.

Extrapolation of the W matrix may also be made by the following technique, which is somewhat simpler to implement in an on-board computer since matrix mainpulations are reduced to more tractable vector manipulations, and matrix inversion is avoided.

Let the d x d filter-weighting matrix $W = [\, w_{k,i}]$ be partially partitioned into three-dimensional column vectors $\underline{w}_{k,i}$ which bear the subscripts of their first component:

$$
W = \left[\begin{array}{c|c|c}
\underline{w}_{0,0}\, \underline{w}_{0,1} \cdots \underline{w}_{0,5} & \underline{w}_{0,6} \cdots\, \underline{w}_{0,5+b} & \underline{w}_{0,6+b}\, \underline{w}_{0,7+b} \cdots\, \underline{w}_{0,d-1} \\
\underline{w}_{3,0}\, \underline{w}_{3,1} \cdot \underline{w}_{3,5} & \underline{w}_{3,6} \cdots\, \underline{w}_{3,5+b} & \underline{w}_{3,6+b}\, \underline{w}_{3,7+b} \cdots\, \underline{w}_{3,d-1} \\
\hline
w_{6,0}\, w_{6,1} \cdots w_{6,1} & w_{6,6} \cdots\, w_{6,5+b} & w_{6,6+b}\, w_{6,7+b} \cdots\, w_{6,d-1} \\
w_{7,0}\, w_{7,1} \cdots w_{7,5} & w_{7,6} \cdots\, w_{7,5+b} & w_{7,6+b}\, w_{7,7+b} \cdots\, w_{7,d-1} \\
\cdots & \cdots & \cdots \\
w_{5+b,0} \cdots w_{5+b,5} & w_{5+b,6} \cdots w_{5+b,5+b} & w_{5+b,6+b}\, w_{5+b,7+b} \cdots w_{5+b,d-1} \\
\hline
\underline{w}_{6+b,0} \cdots \underline{w}_{6+b,5} & \underline{w}_{6+b,6} \cdots \underline{w}_{6+b,5+b} & \underline{w}_{6+b,6+b}\, \underline{w}_{6+b,7+b} \cdots \underline{w}_{6+b,d-1} \\
\underline{w}_{9+b,0} \cdots \underline{w}_{9+b,5} & \underline{w}_{9+b,6} \cdots \underline{w}_{9+b,5+6} & \underline{w}_{9+b,6+b}\, \underline{w}_{9+b,8+b} \cdots \underline{w}_{9+b,d-1}
\end{array}\right]
$$

and let the 3 x 3 process noise matrices be partitioned into three-dimensional column vectors:

$$Q_P = [\underline{q}_3, \underline{q}_4, \underline{q}_5], \quad Q_T = [\underline{q}_{9+b}, \underline{q}_{10+b}, \underline{q}_{11+b}].$$

Furthermore let the inverse of the filter-weighting matrix be approximated by the diagonal matrix $W_D^{-1}(t)$ whose diagonal elements are the reciprocals of diagonal elements of the filter-weighting W-matrix:

$$[W^T(t)]^{-1} \approx [W_D(t)]^{-1} = \begin{bmatrix} 1/w_{0,0} & & & O \\ & 1/w_{1,1} & & \\ & & \ddots & \\ O & & & 1/w_{d-1,d-1} \end{bmatrix}$$

Then the previous first order differential equations are equivalent to:

$$\left.\begin{aligned} &\frac{d^2}{dt^2}\underline{w}_{0,i} = G(t)\,\underline{w}_{0,i} + s_q\,(1/2w_{i,i})\underline{q}_i \\ &\text{with} \qquad\qquad\qquad\qquad\qquad (i = 3, 4, 5 \text{ only}) \\ &\underline{w}_{3,i} = \frac{d}{dt}\underline{w}_{0,i} \\ &w_{k,i} = \text{constant for } k \geq 6 \end{aligned}\right\} \quad i = 0, 1, \ldots (d-1)$$

and

$$\left.\begin{aligned} &\frac{d^2}{dt^2}\underline{w}_{0,i} = G_P(t)\,\underline{w}_{0,i} + s_q(1/2\,w_{i,i})\underline{q}_i \\ &\qquad\qquad\qquad\qquad\qquad (i = 3, 4, 5 \text{ only}) \\ &\frac{d^2}{dt^2}\underline{w}_{6+b,i} = G_T(t)\,\underline{w}_{6+b,i} + s_q(1/2\,w_{i,i})\underline{q}_i \\ &\qquad\qquad\qquad\qquad\qquad (i = 9+b, 10+b, 11+b \text{ only}) \\ &\text{with} \\ &\underline{w}_{3,i} = \frac{d}{dt}\underline{w}_{0,i} \\ &\underline{w}_{9+b,i} = \frac{d}{dt}\underline{w}_{6+b,i} \\ &w_{k,i} = \text{constant for } 6 \leq k < 6+b \end{aligned}\right\} \quad i = 0, 1, \ldots (d-1)$$

When written out in full, the above equations are:

$i = 0, 1, \ldots (d-1)$

$$\frac{d^2}{dt^2}\,\underline{w}_{0,i} = \frac{\mu}{r^5(t)}\left\{3\left[\underline{r}(t)\cdot\underline{w}_{0,i}(t)\right]\underline{r}(t) - r^2(t)\,\underline{w}_{0,i}(t)\right\}$$

$$+\, s_q(1/2\, w_{i,i})\underline{q}_i \quad (i = 3, 4, 5 \text{ only})$$

with

$$\underline{w}_{3,i} = \frac{d}{dt}\,\underline{w}_0$$

$$w_{k,i} = \text{constant for } k \geq 6$$

and

$i = 0, 1, \ldots (d-1)$

$$\frac{d^2}{dt^2}\,\underline{w}_{0,i} = \frac{\mu}{r_P^{\,5}(t)}\left\{3\left[\underline{r}_P(t)\cdot\underline{w}_{0,i}(t)\right]\underline{r}_P(t) - r_P^{\,2}(t)\,\underline{w}_{0,i}(t)\right\}$$

$$+\, s_q(1/2\, w_{i,i})\underline{q}_i \quad (i = 3, 4, 5 \text{ only})$$

$$\frac{d^2}{dt^2}\,\underline{w}_{6+b,i} = \frac{\mu}{r_T^{\,5}(t)}\left\{3\left[\underline{r}_T(t)\cdot\underline{w}_{6+b,i}(t)\right]\underline{r}_T(t) - r_T^{\,2}(t)\,\underline{w}_{6+b,i}(t)\right.$$

$$+\, s_q(1/2\, w_{i,i})\,\underline{q}_i \quad (i = 9{+}b,\ 10{+}b,\ 11{+}b \text{ only})$$

with

$$\underline{w}_{3,i} = \frac{d}{dt}\,\underline{w}_{0,i}$$

$$\underline{w}_{9+b,i} = \frac{d}{dt}\,\underline{w}_{6+b,i}$$

$$w_{k,i} = \text{constant for } 6 \leq k < 6+b$$

These second-order differential equations may be integrated using the same numerical integration technique as is used for the spacecraft position vector. The vectors $\underline{w}_{3,i}$ and $\underline{w}_{9+b,i}$ bear the same relationship to the spacecraft velocity vector as the vectors $\underline{w}_{0,i}$ and $\underline{w}_{6+b,i}$ bear to the spacecraft position vector, and $\underline{w}_{3,i}$ and $\underline{w}_{9+b,i}$ are a by-product of the numerical integration of $\underline{w}_{0,i}$ and $\underline{w}_{6+b,i}$ just as the velocity vector is a by-product of the numerical integration of the position vector.

4.3 Numerical Integration Method

The extrapolation of inertial state vectors and filter weighting matrices requires the numerical solution of two second-order vector differential equations, which are special cases of the general form

$$\frac{d^2}{dt^2} \underline{y}(t) = \underline{f}(t, \underline{y}(t), \underline{z}(t))$$

where

$$\underline{z} = \frac{d}{dt} \underline{y} .$$

Nystrom's standard fourth-order method is utilized to numerically solve this equation. The algorithm for this method is:

$$\underline{y}_{n+1} = \underline{y}_n + \underline{z}_n \Delta t + \frac{1}{6}(\underline{k}_1 + \underline{k}_2 + \underline{k}_3)(\Delta t)^2$$

$$\underline{z}_{n+1} = \underline{z}_n + \frac{1}{6}(\underline{k}_1 + 2\underline{k}_2 + 2\underline{k}_3 + \underline{k}_4) \Delta t$$

$$\underline{k}_1 = \underline{f}(t_n, \underline{y}_n, \underline{z}_n)$$

$$\underline{k}_2 = \underline{f}\left(t_n + \frac{1}{2}\Delta t, \underline{y}_n + \frac{1}{2}\underline{z}_n \Delta t + \frac{1}{8}\underline{k}_1(\Delta t)^2, \underline{z}_n + \frac{1}{2}\underline{k}_1 \Delta t\right)$$

$$\underline{k}_3 = \underline{f}\left(t_n + \frac{1}{2}\Delta t, \underline{y}_n + \frac{1}{2}\underline{z}_n \Delta t + \frac{1}{8}\underline{k}_1(\Delta t)^2, \underline{z}_n + \frac{1}{2}\underline{k}_2 \Delta t\right)$$

$$\underline{k}_4 = \underline{f}\left(t_n + \Delta t, \underline{y}_n + \underline{z}_n \Delta t + \frac{1}{2}\underline{k}_3(\Delta t)^2, \underline{z}_n + \underline{k}_3 \Delta t\right)$$

where

$$\underline{y}_n = \underline{y}(t_n), \ \underline{z}_n = \underline{z}(t_n)$$

and

$$t_{n+1} = t_n + \Delta t$$

As can be seen, the method requires four evaluations of $\underline{f}(t, \underline{y}, \underline{z})$ per integration step Δt as does the classical fourth-order Runge-Kutta method when it is extended to second-order equations. However, if $\underline{f}$ is independent of $\underline{z}$, then Nystrom's method above only requires three evaluations per step since $\underline{k}_3 = \underline{k}_2$. (Runge-Kutta's method will still require four).

The integration time step Δt may be varied from step to step. The nominal integration step size is

$$\Delta t_{nom} = c_{nom} \, r_{con}{}^{3/2} / \sqrt{\mu}$$

where c_{nom} is a program constant. (The value $c_{nom} = 0.3$ is recommended and implies that about 21 steps will be taken per trajectory revolution). The actual step-size is however limited to a maximum of Δt_{max}, which is also a program constant. (A value of about 4000 seconds is suggested.) Also, in the last step, the actual step size is taken to be the interval between the end of the previous step and the desired integration endpoint, so that the extrapolated values of the state or W-matrix are immediately available. Thus the integration step-size Δt is given by the formula

$$\Delta t = \pm \text{ minimum } (|t_F - t|, \Delta t_{nom}, \Delta t_{max})$$

where t_F is the desired integration end-point and t is the time at the end of the previous step. The plus sign is used if forward extrapolation is being performed, while the negative sign is used in the backdating case.

5. DETAILED FLOW DIAGRAMS

This section contains detailed flow diagrams of the Precision State and Filter Weighting Matrix Extrapolation Routine.

Each input and output variable in the routine and subroutine call statements can be followed by a symbol in brackets. This symbol identifies the notation for the corresponding variable in the detailed description and flow diagrams of the called routine. When identical notation is used, the bracketed symbol is omitted.

5.1 A Note to Coders of the Detailed Flow Diagrams

The Precision State and Filter Weighting Matrix Extrapolation Routine does not require the input of the entire filter weighting matrix. However, for coding convenience and conservation of storage, it may be input if desired. If only the $6 \times d$ filter weighting submatrix is input, the vehicle switch s_{veh} is used only when process noise is included ($s_q \neq 0$). Its apparent use in Figure 2e in order to set the index k is merely so that the notation $\underline{w}_{k,i}$ in the flow diagram will be consistent with the same notation in the description of equations section. However, if the entire filter weighting matrix is input, some type of vehicle switch is necessary even when process noise is not included. The parameters s_{veh} and b could be combined into a single parameter k which is 0 for the primary vehicle and $6 + b$ for the target vehicle. For clarity, however, they have been kept separate.

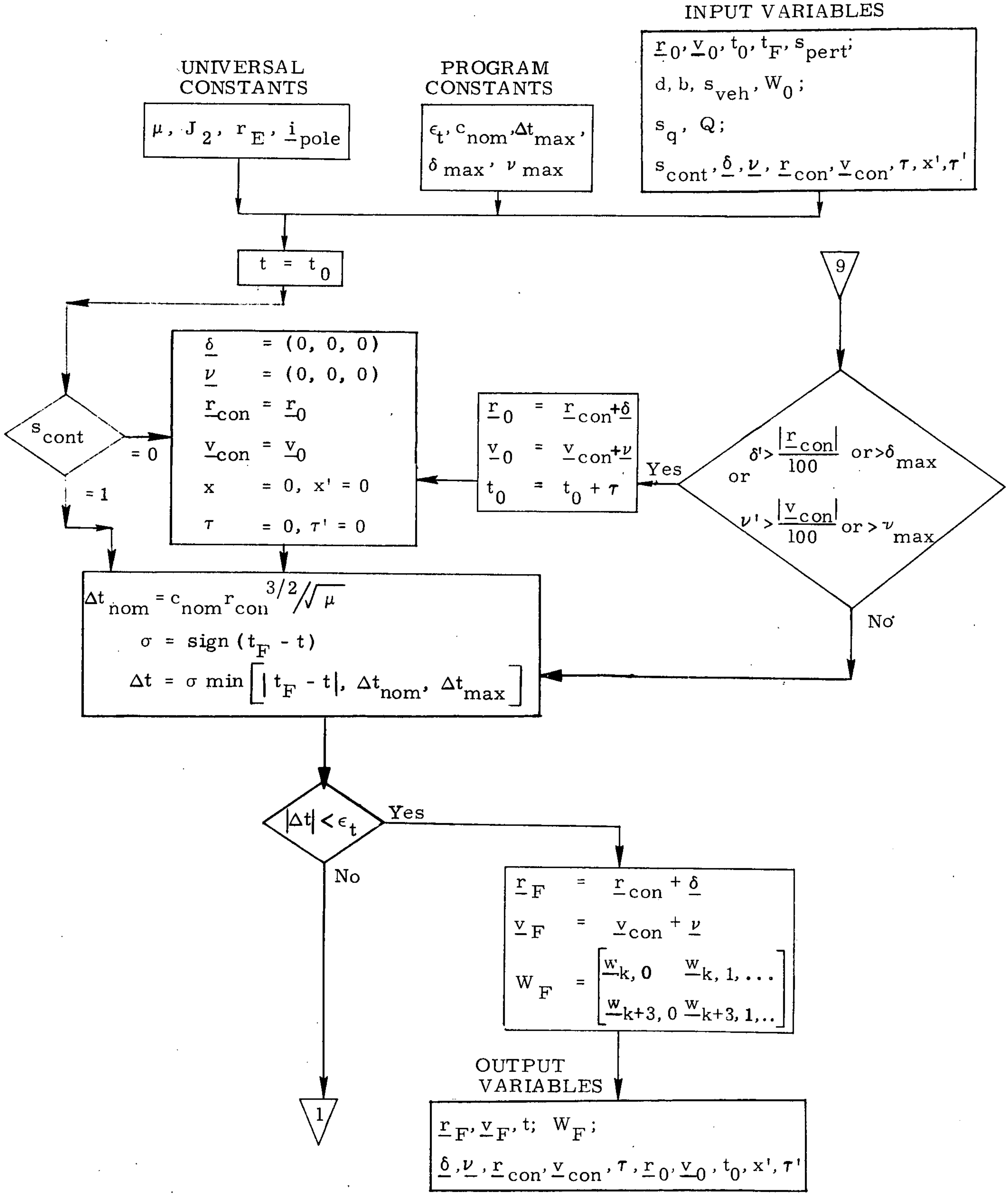

Figure 2a. Detailed Flow Diagram

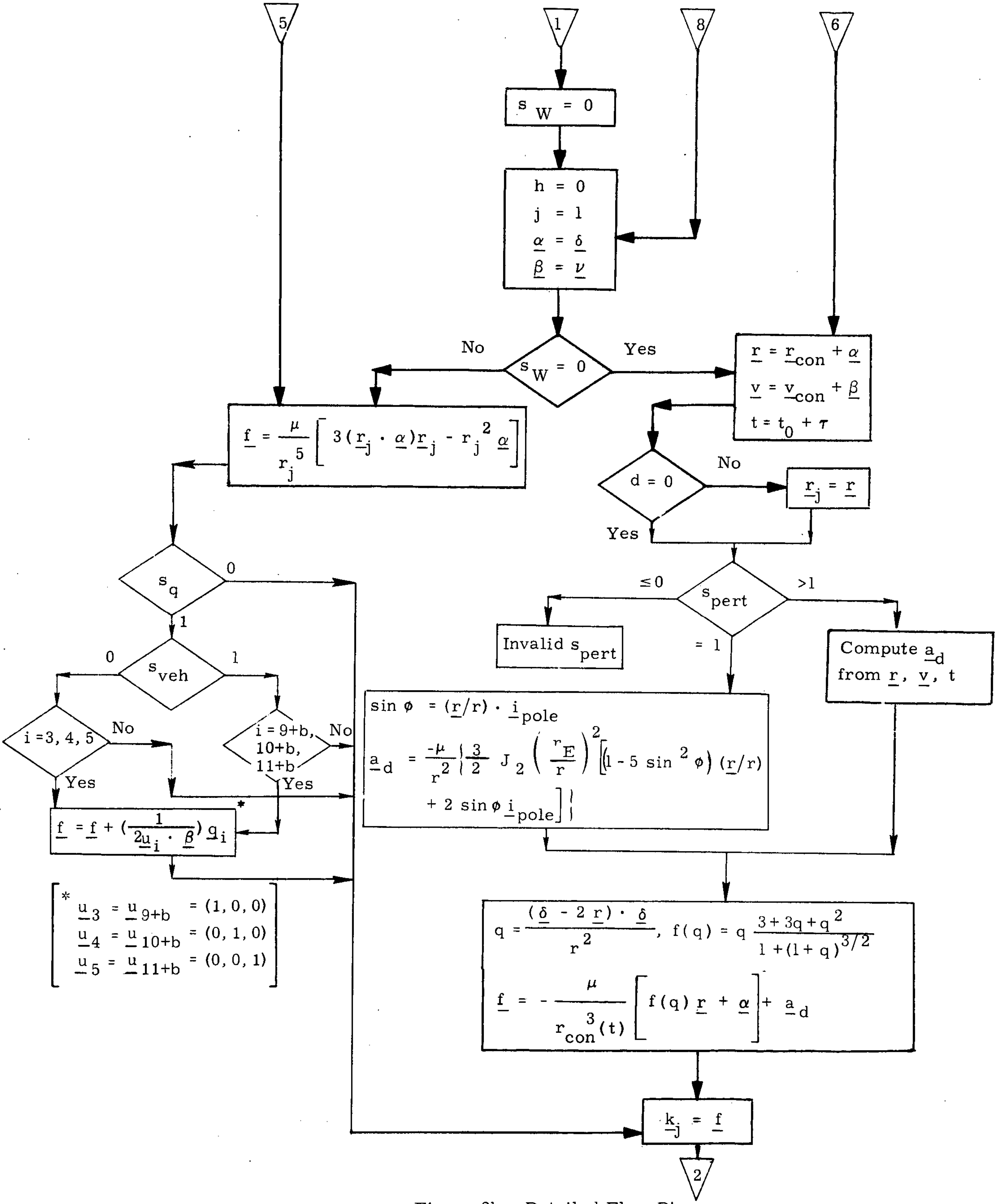

Figure 2b. Detailed Flow Diagram

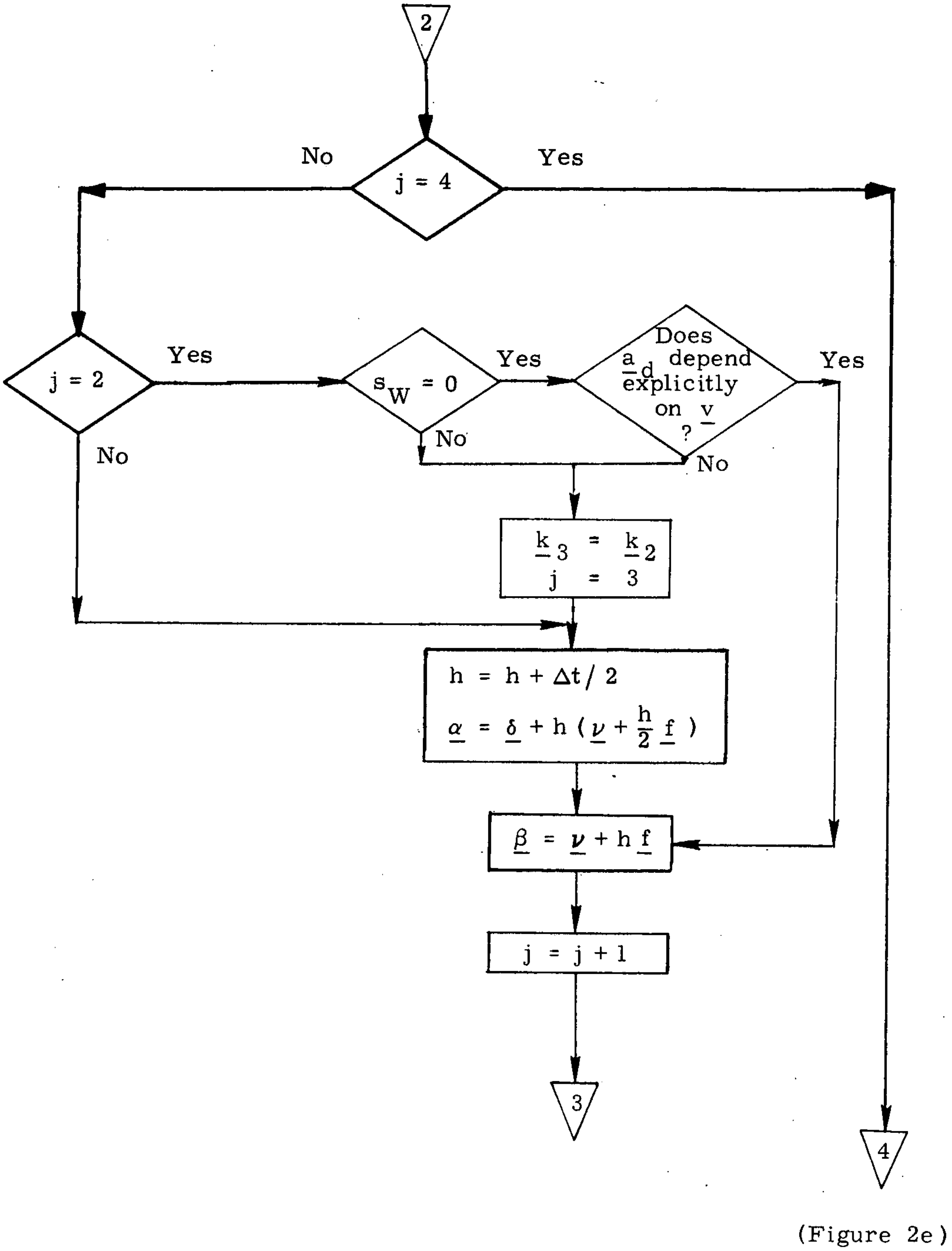

Figure 2c. Detailed Flow Diagram

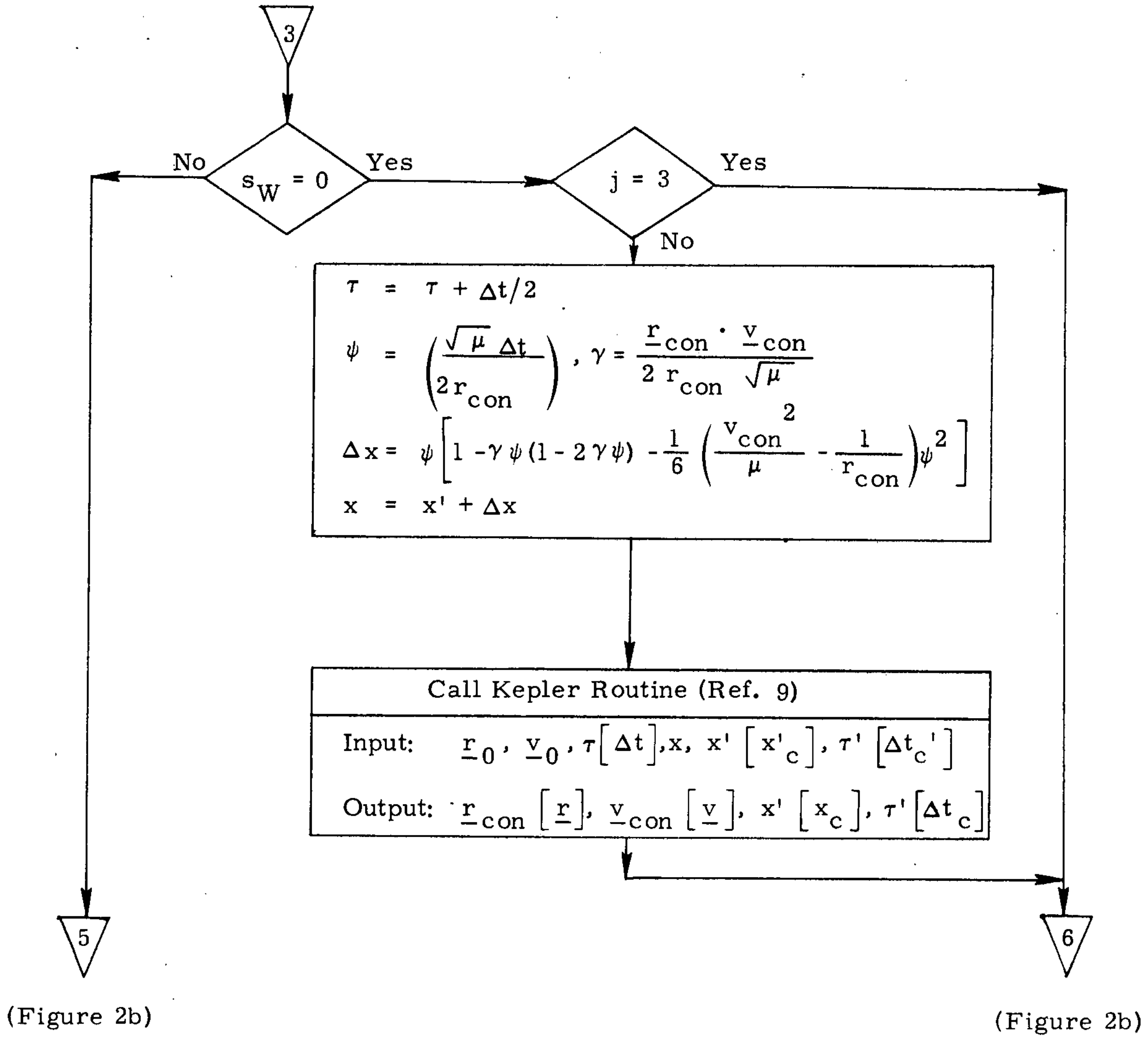

Figure 2d. Detailed Flow Diagram

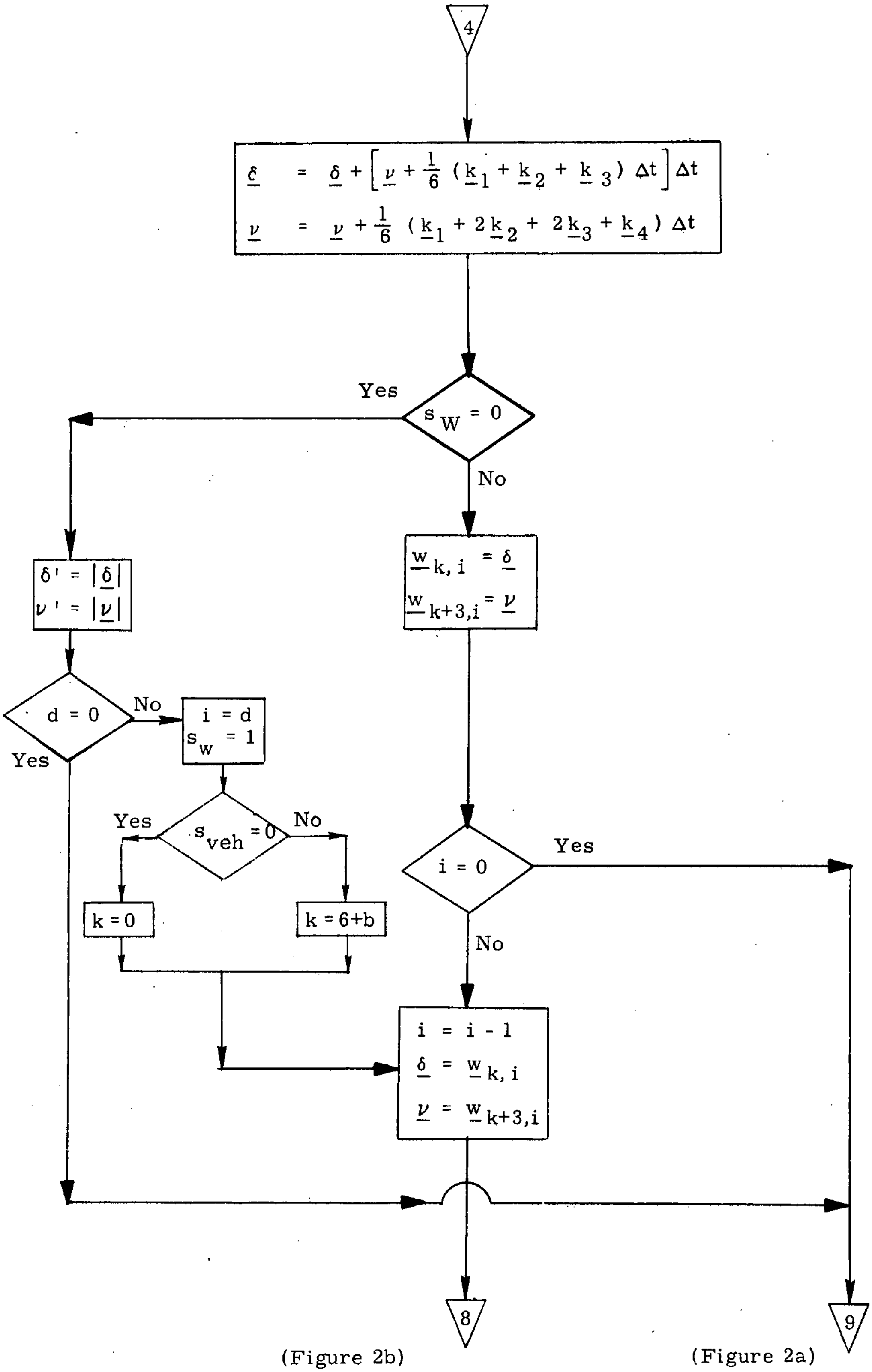

Figure 2e. Detailed Flow Diagram

6. SUPPLEMENTARY INFORMATION

Encke's technique is a classical method in astrodynamics and is described in all standard texts, for example Battin (1964). The f(q) function used in Encke's technique (and in the lunar-solar perturbing acceleration computations) has generally been evaluated by a power series expansion; the closed form expression given here was derived by Potter, and is described in Battin (1964).

The oblateness acceleration in terms of a general spherical harmonic expansion may be calculated in a variety of ways; three different recursive algorithms are given in Gulick (1970). For low order expansions, especially those involving mostly zonal terms, an explicit formulation is generally superior computation-time-wise, as only the non-zero terms enter into the calculation. The general expression for the zonal terms is given by Battin (1964), while Zeldin and Robertson (1970) give explicit analytic expressions for each of the tesseral terms up through fifth order; hence all combinations of terms may easily be included in the oblateness acceleration by consulting the formulations in these references.

A full discussion of the use of covariance matrices in space navigation is given in Battin (1964). Potter (1963) suggested the use of the W-matrix and developed several of its properties. It should be noted that strictly the gravity gradient matrix **G(t)** should also include the gradient of the perturbing acceleration; however, these terms are so small that they may be neglected for our purposes. The use of only the conic gravity gradient, however, does not imply the W-matrix is being extrapolated conically. (Conic extrapolation of the W-matrix can be performed by premultiplying the W-matrix by the conic state transition matrix, which can be expressed in closed form). Rather the W-matrix is here extrapolated along the precision (perturbed) trajectory, as can be seen from the detailed flow diagram of Section 5.

The expression for the inclusion of process noise in the differential equation satisfied by the filter-weighting matrix is taken from Gustafson and Kriegsman (1970), page 7.

The Nystrom numerical integration technique was first conceived by Nystrom (1925), and is described in all standard texts on the numerical integration of ordinary differential equations, such as Henrici (1962). Parametric studies carried out by Robertson (1970) on the general fourth-order Runge-Kutta and Nystrom integration techniques indicate that the "classic" techniques are the best overall techniques for a variety of earth orbiting trajectories in the sense of minimizing the terminal position error for all the trajectories, although for any one trajectory a special technique can generally be found which decreases the position error after ten steps by one or two orders of magnitude for only that trajectory. The classical fourth-order Runge-Kutta and Nystrom techniques are approximately equally accurate, but the latter possesses the computational advantage of requiring one less perturbing acceleration evaluation per step when the perturbing acceleration is independent of the velocity. This fact has been taken into account in the detailed flow diagram of Section 5, in that the extra evaluation is performed only when the perturbing acceleration depends explicitly on the velocity. Some past Apollo experience has suggested that extra evaluation effect with drag is so small as to be negligible; further analysis will confirm or deny this for the Space Shuttle. In regard to step-size, the constants and the functional form of the nominal and maximum time-step expressions have been determined by Marscher (1965).

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