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GANDER, Martin Jakob, STUART, A.M.

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SPACE-TIME CONTINUOUS ANALYSIS OF WAVEFORM RELAXATION FOR THE HEAT EQUATION

MARTIN J. GANDER, ANDREW M. STUART

Abstract. Waveform relaxation algorithms for partial differential equations (PDEs) are traditionally obtained by discretizing the PDE in space and then splitting the discrete operator using matrix splittings. For the semidiscrete heat equation one can show linear convergence on unbounded time intervals and superlinear convergence on bounded time intervals by this approach. However the bounds depend in general on the mesh parameter and convergence rates deteriorate as one refines the mesh.

Motivated by the original development of waveform relaxation in circuit simulation, where the circuits are split in the physical domain into subcircuits, we split the PDE by using overlapping domain decomposition. We prove linear convergence of the algorithm in the continuous case on an infinite time interval, at a rate depending on the size of the overlap. This result remains valid after discretization in space and the convergence rates are robust with respect to mesh refinement. The algorithm is in the class of waveform relaxation algorithms based on overlapping multi-splittings. Our analysis quantifies the empirical observation by Jeltsch and Pohl [SISC, 16 no. 1 (1995)] that the convergence rate of a multi-splitting algorithm depends on the overlap.

Numerical results are presented which support the convergence theory.

Key words. waveform relaxation, domain decomposition, overlapping Schwarz, multi-splitting

AMS subject classifications. 65M55, 65M12, 65M15, 65Y05

1. Introduction. The basic ideas of waveform relaxation were introduced in the late 19th century by Picard [18] and Lindelöf [11] to study initial value problems. There has been much recent interest in waveform relaxation as a practical parallel method for the solution of stiff ordinary differential equations (ODEs) after the publication of a paper by Lelarasmee and coworkers [10] in the area of circuit simulation.

There are two classical convergence results for waveform relaxation algorithms for ODEs: (i) for linear systems of ODEs on unbounded time intervals one can show linear convergence of the algorithm under some dissipation assumptions on the splitting ([15], [14], [4] and [9]); (ii) for nonlinear systems of ODEs (including linear ones) on bounded time intervals one can show superlinear convergence assuming a Lipschitz condition on the splitting function ([15], [1] and [3]). For classical relaxation methods (Jacobi, Gauss Seidel, SOR) the above convergence results depend on the discretization parameter if the ODE arises from a PDE which is discretized in space. The convergence rates deteriorate as one refines the mesh.

Jeltsch and Pohl propose in [9] a multi-splitting algorithm with overlap, generalizing the eliptic analysis of O'Leary and White in [17] to the parabolic case. They prove results (i) and (ii) for their algorithm, but the convergence rates are mesh dependent. However they show numerically that increasing the overlap accelerates the convergence of the waveform relaxation algorithm. We quantify their numerical results by formulating the waveform relaxation algorithm at the space-time continuous level using overlapping domain decomposition; this approach was motivated by the work of Bjørhus [2]. We show linear convergence of this algorithm on unbounded time intervals at a rate depending on the size of the overlap. This is an extension of the first classical convergence result (i) for waveform relaxation from ODEs to PDEs. Discretizing the algorithm, the size of the physical overlap corresponds to the overlap of the multi-splitting algorithm analyzed by Jeltsch and Pohl. We show furthermore that the convergence rate is robust with respect to mesh refinement, provided the physical overlap is hold constant during the refinement process.

Giladi and Keller [8] study superlinear convergence of domain decomposition algorithms for the convection diffusion equation on bounded time intervals, hence generalizing the second classical waveform relaxation result (ii) from ODEs to PDEs.

It is interesting to note that, using multigrid to formulate a waveform relaxation algorithm, Lubich and Osterman [13] prove linear convergence for the heat equation independent of the mesh parameter.

In section 2 we consider a decomposition of the domain into two subdomains. This section is mainly for illustrative purposes, since the analysis can be performed in great detail. In section 3 we generalize the results to an arbitrary number of subdomains. In section 4 we show numerical experiments which confirm the convergence results.

Although the analysis presented is restricted to the one dimensional heat equation, the techiques applied in the proofs are more general. Future work successfully applies these techniques to higher dimensional problems and to nonlinear parabolic equations.

2. Two Subdomains.

2.1. Continuous Case. Consider the one dimensional heat equation on the interval $[0, L]$,

$$\begin{array}{rcll} \dfrac{\partial u}{\partial t} & = & \dfrac{\partial^2 u}{\partial x^2} + f(x,t) & 0 < x < L,\ t > 0 \\ u(0,t) & = & g_1(t) & t > 0 \\ u(L,t) & = & g_2(t) & t > 0 \\ u(x,0) & = & u_0(x) & 0 \leq x \leq L, \end{array} \tag{2.1}$$

where we assume $f(x,t)$ to be bounded on the domain $[0, L] \times [0, \infty)$ and uniformly Hölder continuous on each compact subset of the domain. We assume furthermore that the initial data $u_0(x)$ and the boundary data $g_1(t)$, $g_2(t)$ are piecewise continuous. Then (2.1) has a unique bounded solution [5]. We consider in the following functions in $L^\infty := L^\infty(\mathbb{R}^+; \mathbb{R})$ with the infinity norm

$$||f(\cdot)||_\infty := \sup_{t>0} |f(t)|.$$

The maximum principle, and a corollary thereof, establishing the steady state solution as a bound on the solution of the heat equation are instrumental in our analysis.

THEOREM 2.1. **(Maximum Principle)** *The solution $u(x,t)$ of the heat equation (2.1) with $f(x,t) \equiv 0$ attains its maximum and minimum either on the initial line $t = 0$ or on the boundary at $x = 0$ or $x = L$. If $u(x,t)$ attains its maximum in the interior, then $u(x,t)$ must be constant.*

Proof. The proof can be found in [21]. □

COROLLARY 2.2. *The solution $u(x,t)$ of the heat equation (2.1) with $f(x,t) \equiv 0$ and $u_0 \equiv 0$ satisfies the inequality*

$$||u(x,\cdot)||_\infty \leq \frac{L-x}{L}||g_1(\cdot)||_\infty + \frac{x}{L}||g_2(\cdot)||_\infty,\ 0 \leq x \leq L. \tag{2.2}$$

Proof. Consider $\tilde{u}$ solving

$$\begin{array}{rcll} \dfrac{\partial \tilde{u}}{\partial t} & = & \dfrac{\partial^2 \tilde{u}}{\partial x^2} & 0 < x < L,\ t > 0 \\ \tilde{u}(0,t) & = & ||g_1(\cdot)||_\infty & t > 0 \\ \tilde{u}(L,t) & = & ||g_2(\cdot)||_\infty & t > 0 \\ \tilde{u}(x,0) & = & \frac{L-x}{L}||g_1(\cdot)||_\infty + \frac{x}{L}||g_2(\cdot)||_\infty & 0 \leq x \leq L \end{array} \tag{2.3}$$

The solution $\tilde{u}$ of (2.3) does not depend on t and is given by the steady state solution

$$\tilde{u}(x) = \frac{L-x}{L}||g_1(\cdot)||_\infty + \frac{x}{L}||g_2(\cdot)||_\infty$$

By construction we have $\tilde{u}(x) - u(x,t) \geq 0$ at $t = 0$ and on the boundary $x = 0$ and $x = L$. Since $\tilde{u} - u$ is in the kernel of the heat operator, we have by the maximum principle for the heat equation $\tilde{u}(x) - u(x,t) \geq 0$ on the whole domain $[0, L]$. Hence

$$u(x,t) \leq \frac{L-x}{L}||g_1(\cdot)||_\infty + \frac{x}{L}||g_2(\cdot)||_\infty.$$

Likewise $\tilde{u}(x) + u(x,t) \geq 0$ at $t = 0$, $x = 0$ and $x = L$ and is in the kernel of the heat operator. Hence

$$u(x,t) \geq -\left(\frac{L-x}{L}||g_1(\cdot)||_\infty + \frac{x}{L}||g_2(\cdot)||_\infty\right).$$

Therefore we have

$$|u(x,t)| \leq \frac{L-x}{L}||g_1(\cdot)||_\infty + \frac{x}{L}||g_2(\cdot)||_\infty.$$

Now the right hand side does not depend on t, so we can take the supremum over t, which leads to the desired result. □

To obtain a waveform relaxation algorithm, we decompose the domain $\Omega = [0, L] \times [0, \infty)$ into two overlapping subdomains $\Omega_1 = [0, \beta L] \times [0, \infty)$ and $\Omega_2 = [\alpha L, L] \times [0, \infty)$ where $0 < \alpha < \beta < 1$ as given in figure 2.1. The solution $u(x,t)$ of (2.1) can now

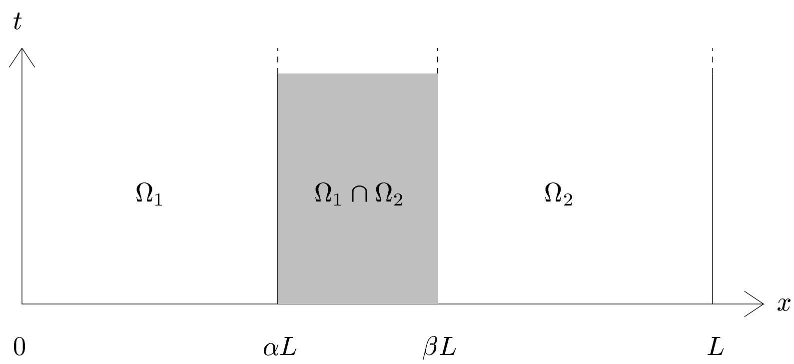

FIG. 2.1. *Decomposition into two overlapping subdomains.*

be obtained from the solutions $v(x,t)$ on Ω_1 and $w(x,t)$ on Ω_2, which satisfy the equations

$$\begin{array}{rcll} \frac{\partial v}{\partial t} &=& \frac{\partial^2 v}{\partial x^2} + f(x,t) & 0 < x < \beta L,\ t > 0 \\ v(0,t) &=& g_1(t) & t > 0 \\ v(\beta L, t) &=& w(\beta L, t) & t > 0 \\ v(x,0) &=& u_0(x) & 0 \leq x \leq \beta L \end{array} \tag{2.4}$$

and

$$\begin{array}{rcll} \frac{\partial w}{\partial t} &=& \frac{\partial^2 w}{\partial x^2} + f(x,t) & \alpha L < x < L,\ t > 0 \\ w(\alpha L,t) &=& v(\alpha L, t) & t > 0 \\ w(L, t) &=& g_2(t) & t > 0 \\ w(x,0) &=& u_0(x) & \alpha L \leq x \leq L. \end{array} \tag{2.5}$$

First note that $v = u$ on Ω_1 and $w = u$ on Ω_2 are solutions to (2.4) and (2.5). Uniqueness follows from our analysis of a Schwarz type iteration introduced for eliptic problems in [19] and further studied in [12] and [6]. We get

$$\begin{array}{rcll} \dfrac{\partial v^{k+1}}{\partial t} &=& \dfrac{\partial^2 v^{k+1}}{\partial x^2} + f(x,t) & 0 < x < \beta L,\ t > 0 \\ v^{k+1}(0,t) &=& g_1(t) & t > 0 \\ v^{k+1}(\beta L,t) &=& w^k(\beta L,t) & t > 0 \\ v^{k+1}(x,0) &=& u_0(x) & 0 \le x \le \beta L \end{array}$$

and

$$\begin{array}{rcll} \dfrac{\partial w^{k+1}}{\partial t} &=& \dfrac{\partial^2 w^{k+1}}{\partial x^2} + f(x,t) & \alpha L < x < L,\ t > 0 \\ w^{k+1}(\alpha L,t) &=& v^k(\alpha L,t) & t > 0 \\ w^{k+1}(L,t) &=& g_2(t) & t > 0 \\ w^{k+1}(x,0) &=& u_0(x) & \alpha L \le x \le L. \end{array}$$

Let $d^k(x,t) := v^k(x,t) - v(x,t)$ and $e^k(x,t) := w^k(x,t) - w(x,t)$ and consider the error equations

$$\begin{array}{rcll} \dfrac{\partial d^{k+1}}{\partial t} &=& \dfrac{\partial^2 d^{k+1}}{\partial x^2} & 0 < x < \beta L,\ t > 0 \\ d^{k+1}(0,t) &=& 0 & t > 0 \\ d^{k+1}(\beta L,t) &=& e^k(\beta L,t) & t > 0 \\ d^{k+1}(x,0) &=& 0 & 0 \le x \le \beta L \end{array} \tag{2.6}$$

and

$$\begin{array}{rcll} \dfrac{\partial e^{k+1}}{\partial t} &=& \dfrac{\partial^2 e^{k+1}}{\partial x^2} & \alpha L < x < L,\ t > 0 \\ e^{k+1}(\alpha L,t) &=& d^k(\alpha L,t) & t > 0 \\ e^{k+1}(L,t) &=& 0 & t > 0 \\ e^{k+1}(x,0) &=& 0 & \alpha L \le x \le L. \end{array} \tag{2.7}$$

The following Lemma establishes convergence of the Schwarz iteration on the interfaces of the subdomains in L^∞. Using the maximum principle convergence in the interior follows.

LEMMA 2.3. *On the interfaces $x = \alpha L$ and $x = \beta L$ the error of the Schwarz iteration decays at the rate*

$$||d^{k+2}(\alpha L,\cdot)||_\infty \le \frac{\alpha(1-\beta)}{\beta(1-\alpha)}||d^k(\alpha L,\cdot)||_\infty, \tag{2.8}$$

$$||e^{k+2}(\beta L,\cdot)||_\infty \le \frac{\alpha(1-\beta)}{\beta(1-\alpha)}||e^k(\beta L,\cdot)||_\infty. \tag{2.9}$$

Proof. By Corollary 2.2 we have

$$||d^{k+2}(x,\cdot)||_\infty \le \frac{x}{\beta L}||e^{k+1}(\beta L,\cdot)||_\infty \quad \forall x \in [0,\beta L]. \tag{2.10}$$

and

$$||e^{k+1}(x,\cdot)||_\infty \le \frac{L-x}{(1-\alpha)L}||d^k(\alpha L,\cdot)||_\infty \quad \forall x \in [\alpha L, L]. \tag{2.11}$$

Evaluating (2.11) at $x = \beta L$ and (2.10) at $x = \alpha L$ and combining the two we obtain inequality (2.8). Inequality (2.9) is obtained similarly. $\square$

For any function $g(\cdot, t)$ in $L^\infty([a, b], L^\infty)$ we introduce the norm

$$||g(\cdot, \cdot)||_{\infty,\infty} := \sup_{a \le x \le b} ||g(x, \cdot)||_\infty .$$

THEOREM 2.4. *The Schwarz iteration for the heat equation with two subdomains converges in $L^\infty([a, b], L^\infty)$ at the linear rate*

$$||d^{2k+1}(\cdot, \cdot)||_{\infty,\infty} \le \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||e^0(\beta L, \cdot)||_\infty \tag{2.12}$$

$$||e^{2k+1}(\cdot, \cdot)||_{\infty,\infty} \le \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||d^0(\alpha L, \cdot)||_\infty . \tag{2.13}$$

Proof. Since the errors d^k and e^k are both in the kernel of the heat operator they obtain, by the maximum principle, their maximum value on the boundary or on the initial line. On the initial line and the exterior boundary both d^k and e^k vanish. Hence

$$||d^{2k+1}(\cdot, \cdot)||_{\infty,\infty} \le ||e^{2k}(\beta L, \cdot)||_\infty , \quad ||e^{2k+1}(\cdot, \cdot)||_{\infty,\infty} \le ||d^{2k}(\alpha L, \cdot)||_\infty .$$

Using Lemma 2.3 the result follows. $\square$

2.2. Semi-Discrete Case. Consider the heat equation continuous in time, but discretized in space using a centered second order finite difference scheme on a grid with n grid points and $\Delta x = \frac{L}{n+1}$. This gives the linear system of ODEs

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} &= A_{(n)} \boldsymbol{u} + \boldsymbol{f}(t) \qquad t > 0 \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0 , \end{aligned} \tag{2.14}$$

where the $n \times n$ matrix $A_{(n)}$, the vector valued function $\boldsymbol{f}(t)$ and the initial condition $\boldsymbol{u}_0$ are given by

$$A_{(n)} = \frac{1}{(\Delta x)^2} \begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix}, \boldsymbol{f}(t) = \begin{pmatrix} f(\Delta x, t) + \frac{1}{(\Delta x)^2} g_1(t) \\ f(2\Delta x, t) \\ \vdots \\ f((n-1)\Delta x, t) \\ f(n\Delta x, t) + \frac{1}{(\Delta x)^2} g_2(t) \end{pmatrix}, \boldsymbol{u}_0 = \begin{pmatrix} u_0(\Delta x) \\ \vdots \\ u_0(n\Delta x) \end{pmatrix} . \tag{2.15}$$

We note the following property of $A_{(n)}$ for later use: let $\boldsymbol{p} := (p_1, \ldots, p_n)^T$ where $p_j := j$. Then

$$A_{(n)} \boldsymbol{p} = (0, \ldots, 0, \frac{-(n+1)}{(\Delta x)^2})^T . \tag{2.16}$$

Likewise let $\boldsymbol{q} := (q_1, \ldots, q_n)^T$ where $q_j := n + 1 - j$. Then

$$A_{(n)} \boldsymbol{q} = (\frac{-(n+1)}{(\Delta x)^2}, 0, \ldots, 0)^T \tag{2.17}$$

We denote the i-th component of a vector valued function $\boldsymbol{v}(t)$ by $\boldsymbol{v}(i,t)$ and $\boldsymbol{v}(t) \geq \boldsymbol{u}(t)$ is understood component wise. We establish now the discrete analogs of the Maximum Principle and Corollary 2.2:

THEOREM 2.5. *(**Semi-Discrete Maximum Principle**) Assume $\boldsymbol{u}(t)$ solves the semi-discrete heat equation (2.14) with $\boldsymbol{f}(t) = (f_1(t), 0, \ldots, 0, f_2(t))^T$ and $\boldsymbol{u}(0) = (u_1(0), \ldots, u_n(0))^T$. If $f_1(t)$ and $f_2(t)$ are non-negative for $t \geq 0$ and $\boldsymbol{u}(i,0) \geq 0$ for $i = 1, \ldots, n$ then*

$$\boldsymbol{u}(t) \geq 0, \ \forall t \geq 0.$$

Proof. We follow Varga's proof in [20]. By Duhamel's principle the solution $\boldsymbol{u}(t)$ is given by

$$\boldsymbol{u}(t) = e^{A_{(n)}t}\boldsymbol{u}(0) + \int_0^t e^{A_{(n)}(t-s)}\boldsymbol{f}(s)ds \tag{2.18}$$

The key is to note that the matrix $e^{A_{(n)}t}$ contains only non-negative entries. To see why write $A_{(n)} = -2I_{(n)} + J_{(n)}$ where $J_{(n)}$ contains only non-negative entries and $I_{(n)}$ is the identity matrix of size $n \times n$. We get

$$e^{A_{(n)}t} = e^{-2I_{(n)}t}e^{J_{(n)}t} = e^{-2t}e^{J_{(n)}t} = e^{-2t}\sum_{l=0}^{\infty}\frac{J_{(n)}^l t^l}{l!}$$

where the last expression has clearly only non-negative entries. Since the matrix exponential in (2.18) is applied only to vectors with non-negative entries, it follows that $\boldsymbol{u}(t)$ can not become negative. □

COROLLARY 2.6. *The solution $\boldsymbol{u}(t)$ of the semi-discrete heat equation (2.14) with $\boldsymbol{f}(t) = (\frac{1}{(\Delta x)^2}g_1(t), 0, \ldots, 0, \frac{1}{(\Delta x)^2}g_2(t))^T$ and $\boldsymbol{u}_0 \equiv 0$ satisfies the inequality*

$$||\boldsymbol{u}(j,\cdot)||_\infty \leq \frac{n+1-j}{n+1}||g_1(\cdot)||_\infty + \frac{j}{n+1}||g_2(\cdot)||_\infty, \ 1 \leq j \leq n. \tag{2.19}$$

Proof. Consider $\tilde{\boldsymbol{u}}(t)$ solving

$$\begin{array}{rcll} \dfrac{\partial \tilde{\boldsymbol{u}}}{\partial t} & = & A_{(n)}\tilde{\boldsymbol{u}} + \tilde{\boldsymbol{f}}, & t > 0 \\ \tilde{\boldsymbol{u}}(j,0) & = & \dfrac{n+1-j}{n+1}||g_1(\cdot)||_\infty + \dfrac{j}{n+1}||g_2(\cdot)||_\infty, & 1 \leq j \leq n, \end{array} \tag{2.20}$$

with $\tilde{\boldsymbol{f}} = (\frac{1}{(\Delta x)^2}||g_1(t)||_\infty, 0, \ldots, 0, \frac{1}{(\Delta x)^2}||g_2(t)||_\infty)^T$. Using the properties (2.16) and (2.17) of $A_{(n)}$ and the linearity of (2.20) we find that the solution $\tilde{\boldsymbol{u}}$ of (2.20) does not depend on t and is given by the steady state solution

$$\tilde{\boldsymbol{u}}(j) = \frac{n+1-j}{n+1}||g_1(\cdot)||_\infty + \frac{j}{n+1}||g_2(\cdot)||_\infty, \ 1 \leq j \leq n.$$

The difference $\boldsymbol{\phi}(j,t) := \tilde{\boldsymbol{u}}(j) - \boldsymbol{u}(j,t)$ satisfies the equation

$$\begin{array}{rcll} \dfrac{\partial \boldsymbol{\phi}}{\partial t} & = & A_{(n)}\boldsymbol{\phi} + \begin{pmatrix} \frac{1}{(\Delta x)^2}(||g_1(\cdot)||_\infty - g_1(t)) \\ 0 \\ \vdots \\ 0 \\ \frac{1}{(\Delta x)^2}(||g_2(\cdot)||_\infty - g_2(t)) \end{pmatrix}, & t > 0 \\ \boldsymbol{\phi}(j,0) & = & \dfrac{n+1-j}{n+1}||g_1(\cdot)||_\infty + \dfrac{j}{n+1}||g_2(\cdot)||_\infty, & 1 \leq j \leq n. \end{array}$$

and hence by the discrete maximum principle $\boldsymbol{\phi}(j,t) \geq 0$ for all $t > 0$ and $1 \leq j \leq n$. Thus

$$\boldsymbol{u}(j,t) \leq \frac{n+1-j}{n+1}||g_1(\cdot)||_\infty + \frac{j}{n+1}||g_2(\cdot)||_\infty,\ 1 \leq j \leq n.$$

Likewise from $\boldsymbol{\psi}(j,t) := \tilde{\boldsymbol{u}}(j) + \boldsymbol{u}(j,t)$ we get

$$\boldsymbol{u}(j,t) \geq -\left(\frac{n+1-j}{n+1}||g_1(\cdot)||_\infty + \frac{j}{n+1}||g_2(\cdot)||_\infty\right),\ 1 \leq j \leq n.$$

Hence we can bound the modulus of $\boldsymbol{u}$ by

$$|\boldsymbol{u}(j,t)| \leq \frac{n+1-j}{n+1}||g_1(\cdot)||_\infty + \frac{j}{n+1}||g_2(\cdot)||_\infty,\ 1 \leq j \leq n.$$

Now the right hand side does not depend on t, so we can take the supremum over t, which leads to the desired result. □

We decompose the domain into two overlapping subdomains Ω_1 and Ω_2 as in figure 2.2. We assume for simplicity that αL falls on the grid point $i = a$ and βL on

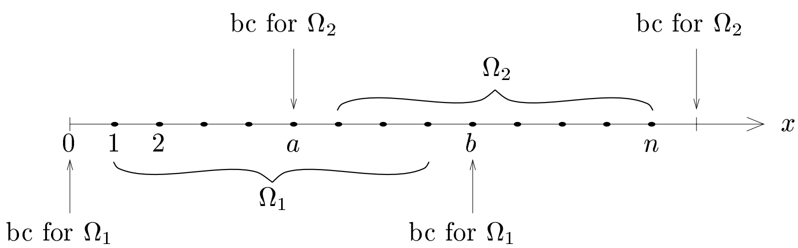

FIG. 2.2. *Decomposition in the semi-discrete case.*

the grid point $i = b$. We therefore have $a\Delta x = \alpha L$ and $b\Delta x = \beta L$. For notational convenience we define

$$\begin{aligned}
\boldsymbol{f}^1(\boldsymbol{x}, y, z) &:= (\boldsymbol{x}(1) + \frac{y}{(\Delta x)^2}, \boldsymbol{x}(2), \ldots, \boldsymbol{x}(b-2), \boldsymbol{x}(b-1) + \frac{z}{(\Delta x)^2})^T \\
\boldsymbol{f}^2(\boldsymbol{x}, y, z) &:= (\boldsymbol{x}(a+1) + \frac{y}{(\Delta x)^2}, \boldsymbol{x}(a+2), \ldots, \boldsymbol{x}(n-1), \boldsymbol{x}(n) + \frac{z}{(\Delta x)^2})^T.
\end{aligned}$$

As in the continuous case, the solution $\boldsymbol{u}(t)$ of (2.14) can be obtained from the solutions $\boldsymbol{v}(t)$ on Ω_1 and $\boldsymbol{w}(t)$ on Ω_2, which satisfy the equations

$$\begin{aligned}
\frac{\partial \boldsymbol{v}}{\partial t} &= A_{(b-1)}\boldsymbol{v} + \boldsymbol{f}^1(\boldsymbol{f}(t), g_1(t), \boldsymbol{w}(b-a,t)) && t > 0 \\
\boldsymbol{v}(j,0) &= \boldsymbol{u}_0(j), && 1 \leq j < b,
\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}
\frac{\partial \boldsymbol{w}}{\partial t} &= A_{(n-a)}\boldsymbol{w} + \boldsymbol{f}^2(\boldsymbol{f}(t), \boldsymbol{v}(a,t), g_2(t)) && t > 0 \\
\boldsymbol{w}(j-a,0) &= \boldsymbol{u}_0(j) && b \leq j \leq n.
\end{aligned} \tag{2.22}$$

Applying the Schwarz iteration to (2.21) and (2.22) we obtain

$$\begin{aligned}
\frac{\partial \boldsymbol{v}^{k+1}}{\partial t} &= A_{(b-1)}\boldsymbol{v}^{k+1} + \boldsymbol{f}^1(\boldsymbol{f}(t), g_1(t), \boldsymbol{w}^k(b-a,t)) && t > 0 \\
\boldsymbol{v}^{k+1}(j,0) &= \boldsymbol{u}_0(j) && 1 \leq j < b,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \boldsymbol{w}^{k+1}}{\partial t} &= A_{(n-a)}\boldsymbol{w}^{k+1} + \boldsymbol{f}^2(\boldsymbol{f}(t), \boldsymbol{v}^k(a,t), g_2(t)) & t > 0 \\
\boldsymbol{w}^{k+1}(j-a, 0) &= \boldsymbol{u}_0(j) & b \le j \le n.
\end{aligned}$$

Let $\boldsymbol{d}^k(t) := \boldsymbol{v}^k(t) - \boldsymbol{v}(t)$ and $\boldsymbol{e}^k(t) := \boldsymbol{w}^k(t) - \boldsymbol{w}(t)$ and consider the error equations

$$\begin{aligned}
\frac{\partial \boldsymbol{d}^{k+1}}{\partial t} &= A_{(b-1)}\boldsymbol{d}^{k+1} + \boldsymbol{f}^1(0, 0, \boldsymbol{e}^k(b-a, t)) & t > 0 \\
\boldsymbol{d}^{k+1}(0) &= \boldsymbol{0}
\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}
\frac{\partial \boldsymbol{e}^{k+1}}{\partial t} &= A_{(n-a)}\boldsymbol{e}^{k+1} + \boldsymbol{f}^2(0, \boldsymbol{d}^k(a,t), 0) & t > 0 \\
\boldsymbol{e}^{k+1}(0) &= \boldsymbol{0}.
\end{aligned} \tag{2.24}$$

The following Lemma establishes convergence of the Schwarz iteration on the interface nodes of the subdomains in L^∞ . Using the discrete maximum principle convergence in the interior then follows.

LEMMA 2.7. *On the interface gridpoints a and b the error of the Schwarz iteration decays at the rate*

$$||\boldsymbol{d}^{k+2}(a,\cdot)||_\infty \le \frac{\alpha(1-\beta)}{\beta(1-\alpha)}||\boldsymbol{d}^k(a,\cdot)||_\infty \tag{2.25}$$

$$||\boldsymbol{e}^{k+2}(b,\cdot)||_\infty \le \frac{\alpha(1-\beta)}{\beta(1-\alpha)}||\boldsymbol{e}^k(b,\cdot)||_\infty. \tag{2.26}$$

Proof. By Corollary 2.6 we have

$$||\boldsymbol{d}^{k+2}(j,\cdot)||_\infty \le \frac{j}{b}||\boldsymbol{e}^{k+1}(b-a,\cdot)||_\infty, \ 1 \le j < b \tag{2.27}$$

and

$$||\boldsymbol{e}^{k+1}(j,\cdot)||_\infty \le \frac{n+1-a-j}{n+1-a}||\boldsymbol{d}^k(a,\cdot)||_\infty, \ 1 \le j \le b-a. \tag{2.28}$$

Evaluating (2.28) at $j = b - a$ and (2.27) at $j = a$ and combining the two we get

$$||\boldsymbol{d}^{k+2}(a,\cdot)||_\infty \le \frac{a(n+1-b)}{b(n+1-a)}||\boldsymbol{d}^k(a,\cdot)||_\infty.$$

Now using $a\Delta x = \alpha L$, $b\Delta x = \beta L$ and $(n+1)\Delta x = L$ we get the desired result. The second inequality (2.26) is obtained similarly. □

For any vector valued function $\boldsymbol{h}(t)$ in $L^\infty(\mathbb{R}^+, \mathbb{R}^n)$ we define

$$||\boldsymbol{h}(\cdot,\cdot)||_{\infty,\infty} := \max_{1<j<n} ||\boldsymbol{h}(j,\cdot)||_\infty$$

THEOREM 2.8. *The Schwarz iteration for the semi-discrete heat equation with two subdomains converges in $L^\infty(\mathbb{R}^+, \mathbb{R}^n)$ at the linear rate*

$$||\boldsymbol{d}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} \le \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||\boldsymbol{e}^0(b-a,\cdot)||_\infty$$

$$||\boldsymbol{e}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} \le \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||\boldsymbol{d}^0(a,\cdot)||_\infty.$$

Proof. By Corollary 2.6 we have

$$||\boldsymbol{d}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} \leq ||\boldsymbol{e}^{2k}(b-a,\cdot)||_{\infty}, \quad ||\boldsymbol{e}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} \leq ||\boldsymbol{d}^{2k}(a,\cdot)||_{\infty}.$$

Using Lemma 2.7 the result follows. □

3. Arbitrary number of subdomains. We generalize the two subdomain case described in section 2 to an arbitrary number of subdomains N. This leads to an algorithm which can be run in parallel. Subdomains with even indices depend only on subdomains with odd indices. Hence one can solve on all the even subdomains in parallel in one sweep, and then on all the odd ones in the next one. Boundary information is propagated in between sweeps.

Consider N subdomains Ω_i of Ω, $i = 1, \ldots, N$ where $\Omega_i = [\alpha_i L, \beta_i L] \times [0, \infty)$ and $\alpha_1 = 0$, $\beta_N = 1$ and $\alpha_{i+1} < \beta_i$ for $i = 1, \ldots, N-1$ so that all the subdomains overlap, as in figure 3.1. We assume also that $\beta_i \leq \alpha_{i+2}$ for $i = 1, \ldots, N-2$ so that domains which are not adjacent do not overlap. The solution $u(x,t)$ of (2.1) can be obtained

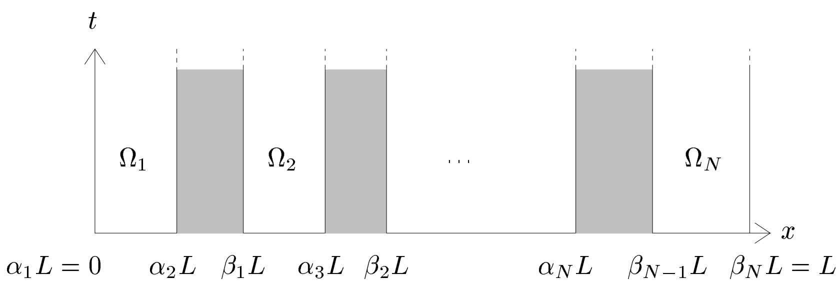

FIG. 3.1. *Decomposition into N overlapping subdomains.*

as in the case of two subdomains by composing the solutions $v_i(x,t)$, $i = 1, \ldots, N$, which satisfy the equations

$$\begin{array}{rrcll} & \dfrac{\partial v_i}{\partial t} & = & \dfrac{\partial^2 v_i}{\partial x^2} + f(x,t) & \alpha_i L < x < \beta_i L,\ t > 0 \\ (3.1) & v_i(\alpha_i L, t) & = & v_{i-1}(\alpha_i L, t) & t > 0 \\ & v_i(\beta_i L, t) & = & v_{i+1}(\beta_i L, t) & t > 0 \\ & v(x,0) & = & u_0(x) & \alpha_i L \leq x \leq \beta_i L, \end{array}$$

where we have introduced for convenience of notation the two functions v_0 and v_{N+1} which are constant in x and satisfy the given boundary conditions, namely $v_0(x,t) \equiv g_1(t)$ and $v_{N+1}(x,t) \equiv g_2(t)$. The system of equations (3.1), which is coupled through the boundary, can be solved using the Schwarz iteration. We get for $i = 1, \ldots, N$

$$\begin{array}{rrcll} & \dfrac{\partial v_i^{k+1}}{\partial t} & = & \dfrac{\partial^2 v_i^{k+1}}{\partial x^2} + f(x,t) & \alpha_i L < x < \beta_i L,\ t > 0 \\ (3.2) & v_i^{k+1}(\alpha_i L, t) & = & v_{i-1}^k(\alpha_i L, t) & t > 0 \\ & v_i^{k+1}(\beta_i L, t) & = & v_{i+1}^k(\beta_i L, t) & t > 0 \\ & v_i^{k+1}(x,0) & = & u_0(x) & \alpha_i L \leq x \leq \beta_i L, \end{array}$$

where again $v_0^k(t) \equiv g_1(t)$ and $v_{N+1}^k(t) \equiv g_2(t)$. Let $e_i^k := v_i^k(x,t) - v_i(x,t)$, $i = 1, \ldots, N$ and consider the error equations (compare figure 3.2)

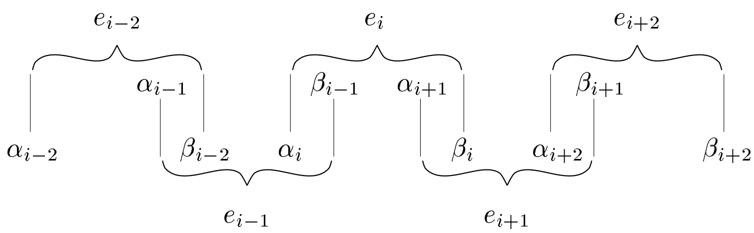

FIG. 3.2. *Overlapping subdomains and corresponding error functions e_i*

$$
\begin{array}{rcll}
\dfrac{\partial e_i^{k+1}}{\partial t} & = & \dfrac{\partial^2 e_i^{k+1}}{\partial x^2} & \alpha_i L < x < \beta_i L,\ t > 0 \\
e_i^{k+1}(\alpha_i L, t) & = & e_{i-1}^k(\alpha_i L, t) & t > 0 \\
e_i^{k+1}(\beta_i L, t) & = & e_{i+1}^k(\beta_i L, t) & t > 0 \\
e_i^{k+1}(x, 0) & = & 0 & \alpha_i L \leq x \leq \beta_i L,
\end{array}
\tag{3.3}
$$

with $e_0^k(t) \equiv 0$ and $e_{N+1}^k(t) \equiv 0$.

For the following Lemma, we need some additional definitions to facilitate the notation. We define $\alpha_0 = \beta_0 = 0$, $\alpha_{N+1} = \beta_{N+1} = 1$ and the constant functions $e_{-1} \equiv 0$ and $e_{N+2} \equiv 0$.

LEMMA 3.1. *The error e_i^{k+2} of the i-th subdomain of the Schwarz iteration (3.3) decays on the interfaces $x = \beta_{i-1}L$ and $x = \alpha_{i+1}L$ at the rate*

$$
\begin{aligned}
||e_i^{k+2}(\beta_{i-1}L, \cdot)||_\infty \leq\ & r_i r_{i+1} ||e_{i+2}^k(\beta_{i+1}L, \cdot)||_\infty + r_i p_{i+1} ||e_i^k(\alpha_{i+1}L, \cdot)||_\infty \\
& + p_i q_{i-1} ||e_i^k(\beta_{i-1}L, \cdot)||_\infty + p_i s_{i-1} ||e_{i-2}^k(\alpha_{i-1}L, \cdot)||_\infty,
\end{aligned}
\tag{3.4}
$$

for $i = 2, \ldots, N$ and

$$
\begin{aligned}
||e_i^{k+2}(\alpha_{i+1}L, \cdot)||_\infty \leq\ & q_i r_{i+1} ||e_{i+2}^k(\beta_{i+1}L, \cdot)||_\infty + q_i p_{i+1} ||e_i^k(\alpha_{i+1}L, \cdot)||_\infty \\
& + s_i q_{i-1} ||e_i^k(\beta_{i-1}L, \cdot)||_\infty + s_i s_{i-1} ||e_{i-2}^k(\alpha_{i-1}L, \cdot)||_\infty,
\end{aligned}
\tag{3.5}
$$

for $i = 1, \ldots, N-1$, where the ratios of the overlaps are given by

$$
r_i = \frac{\beta_{i-1} - \alpha_i}{\beta_i - \alpha_i},\ p_i = \frac{\beta_i - \beta_{i-1}}{\beta_i - \alpha_i},\ q_i = \frac{\alpha_{i+1} - \alpha_i}{\beta_i - \alpha_i},\ s_i = \frac{\beta_i - \alpha_{i+1}}{\beta_i - \alpha_i}.
\tag{3.6}
$$

Proof. By Corollary 2.2 we have

$$
||e_i^{k+2}(x, \cdot)||_\infty \leq \frac{x - \alpha_i L}{(\beta_i - \alpha_i)L} ||e_{i+1}^{k+1}(\beta_i L, \cdot)||_\infty + \frac{\beta_i L - x}{(\beta_i - \alpha_i)L} ||e_{i-1}^{k+1}(\alpha_i L, \cdot)||_\infty.
\tag{3.7}
$$

Since this result holds on all the subdomains Ω_i, we can recursively apply it to the errors on the right in (3.7), namely

$$
\begin{aligned}
||e_{i+1}^{k+1}(\beta_i L, \cdot)||_\infty &\leq \frac{\beta_i - \alpha_{i+1}}{\beta_{i+1} - \alpha_{i+1}} ||e_{i+2}^k(\beta_{i+1}L, \cdot)||_\infty + \frac{\beta_{i+1} - \beta_i}{\beta_{i+1} - \alpha_{i+1}} ||e_i^k(\alpha_{i+1}L, \cdot)||_\infty, \\
||e_{i-1}^{k+1}(\alpha_i L, \cdot)||_\infty &\leq \frac{\alpha_i - \alpha_{i-1}}{\beta_{i-1} - \alpha_{i-1}} ||e_i^k(\beta_{i-1}L, \cdot)||_\infty + \frac{\beta_{i-1} - \alpha_i}{\beta_{i-1} - \alpha_{i-1}} ||e_{i-2}^k(\alpha_{i-1}L, \cdot)||_\infty.
\end{aligned}
$$

Substituting these equations back into the right hand side of (3.7) and evaluating (3.7) at $x = \beta_{i-1}L$ leads to inequality (3.4). Evaluation at $x = \alpha_{i+1}$ leads to inequality (3.5). $\square$

This result is different from the result in the two subdomain case (Lemma 2.3), because we cannot get the error directly as a function of the error at the same location two steps before. The error at a given location depends on the errors at different locations also. This leads to the two independent linear systems of inequalities,

$$\boldsymbol{\xi}^{k+2} \leq D\boldsymbol{\xi}^k \quad \text{and} \quad \boldsymbol{\eta}^{k+2} \leq E\boldsymbol{\eta}^k, \tag{3.8}$$

where the inequality sign here means less than or equal for each component of the vectors $\boldsymbol{\xi}^{k+2}$ and $\boldsymbol{\eta}^{k+2}$. These vectors and the matrices D and E are slightly different if the number of subdomains N is even or odd. We assume in the sequel that N is even. The case where N is odd can be treated in a similar way. For N even we have

$$\boldsymbol{\xi}^k = \begin{pmatrix} ||e_1^k(\alpha_2 L,\cdot)||_\infty \\ ||e_3^k(\beta_2 L,\cdot)||_\infty \\ ||e_3^k(\alpha_4 L,\cdot)||_\infty \\ ||e_5^k(\beta_4 L,\cdot)||_\infty \\ \vdots \\ ||e_{N-1}^k(\beta_{N-2} L,\cdot)||_\infty \\ ||e_{N-1}^k(\alpha_N L,\cdot)||_\infty \end{pmatrix} \quad \text{and} \quad \boldsymbol{\eta}^k = \begin{pmatrix} ||e_2^k(\beta_1 L,\cdot)||_\infty \\ ||e_2^k(\alpha_3 L,\cdot)||_\infty \\ ||e_4^k(\beta_3 L,\cdot)||_\infty \\ ||e_4^k(\alpha_5 L,\cdot)||_\infty \\ \vdots \\ ||e_{N-2}^k(\alpha_{N-1} L,\cdot)||_\infty \\ ||e_N^k(\beta_{N-1} L,\cdot)||_\infty \end{pmatrix}$$

and the banded $(N-1)\times(N-1)$ matrices

$$D = \begin{bmatrix} q_1p_2 & q_1r_2 & & & & & \\ p_3s_2 & p_3q_2 & r_3p_4 & r_3r_4 & & & \\ s_3s_2 & s_3q_2 & q_3p_4 & q_3r_4 & & & \\ & & p_5s_4 & p_5q_4 & r_5p_6 & r_5r_6 & \\ & & s_5s_4 & s_5q_4 & q_5p_6 & q_5r_6 & \\ & & & \ddots & & \ddots & \\ & & & & p_{N-1}s_{N-2} & p_{N-1}q_{N-2} & r_{N-1}p_N \\ & & & & s_{N-1}s_{N-2} & s_{N-1}q_{N-2} & q_{N-1}p_N \end{bmatrix}. \tag{3.9}$$

and

$$E = \begin{bmatrix} p_2q_1 & r_2p_3 & r_2r_3 & & & & \\ s_2q_1 & q_2p_3 & q_2r_3 & & & & \\ & p_4s_3 & p_4q_3 & r_4p_5 & r_4r_5 & & \\ & s_4s_3 & s_4q_3 & q_4p_5 & q_4r_5 & & \\ & & \ddots & & \ddots & & \\ & & & p_{N-2}s_{N-3} & p_{N-2}q_{N-3} & r_{N-2}p_{N-1} & r_{N-2}r_{N-1} \\ & & & s_{N-2}S_{N-3} & s_{N-2}q_{N-3} & q_{N-2}p_{N-1} & q_{N-2}r_{N-1} \\ & & & & & p_Ns_{N-1} & p_Nq_{N-1} \end{bmatrix}. \tag{3.10}$$

Note that the infinity norm of D and E equals one. This can be seen for example for D by looking at the row sum of interior rows,

$$\begin{aligned} p_is_{i-1}+p_iq_{i-1}+r_ip_{i+1}+r_ir_{i+1} &= p_i(s_{i-1}+q_{i-1})+r_i(p_{i+1}+r_{i+1})=p_i+r_i=1, \\ s_is_{i-1}+s_iq_{i-1}+q_ip_{i+1}+q_ir_{i+1} &= s_i(s_{i-1}+q_{i-1})+q_i(p_{i+1}+r_{i+1})=s_i+q_i=1. \end{aligned} \tag{3.11}$$

The boundary rows however sum up to a value less than one, namely

$$\begin{aligned} q_1p_2+q_1r_2 &= q_1(p_2+r_2)=q_1<1 \\ p_{N-1}s_{N-2}+p_{N-1}q_{N-2}+r_{N-1}p_N &= p_{N-1}(s_{N-2}+q_{N-2})+r_{N-1}p_N \\ &= p_{N-1}+r_{N-1}p_N<1 \\ s_{N-1}s_{N-2}+s_{N-1}q_{N-2}+q_{N-1}p_N &= s_{N-1}(s_{N-2}+q_{N-2})+q_{N-1}p_N \\ &= s_{N-1}+q_{N-1}p_N<1. \end{aligned} \tag{3.12}$$

A similar result holds for the matrix E. Since the infinity norm of both D and E equals one, convergence is not obvious at first glance. In the special case with two subdomains treated in section 2 the matrices E and D degenerate to the scalar $q_1 p_2$, which is strictly less than one and convergence follows. In the case of N subdomains the information from the boundary needs to propagate inward to the interior subdomains, before the algorithm exhibits convergence. Hence we expect that the infinity norm of E and D raised to a certain power becomes strictly less than one. We need the following Lemmas to prove convergence.

LEMMA 3.2. *Let $\boldsymbol{r}(A) \in \mathbb{R}^p$ denote the vector containing the row sums of the $p \times p$ square matrix A. Then $\boldsymbol{r}(A^{n+1}) = A^n \boldsymbol{r}(A)$.*

Proof. Let $\mathbb{1} = (1, 1, \ldots, 1)^T$. Then we have $\boldsymbol{r}(A) = A\mathbb{1}$ and hence $\boldsymbol{r}(A^{n+1}) = A^{n+1}\mathbb{1} = A^n A \mathbb{1} = A^n \boldsymbol{r}(A)$. $\square$

LEMMA 3.3. *Let A be a real $p \times q$ matrix with $a_{ij} \geq 0$ and B be a real $q \times r$ matrix with $b_{ij} \geq 0$. Define the sets $I_i(A) := \{k : a_{ik} > 0\}$ and $J_j(A) := \{k : b_{kj} > 0\}$. Then for the product $C := AB$ we have*

$$I_i(C) = \{k : I_i(A) \cap J_k(B) \neq \emptyset\}$$

Proof. We have, since $a_{ik}, b_{kj} \geq 0$

$$c_{ij} > 0 \Longleftrightarrow \sum_{k=1}^{q} a_{ik} b_{kj} > 0 \Longleftrightarrow \exists k \text{ s.t. } a_{ik} > 0 \text{ and } b_{kj} > 0 \Longleftrightarrow I_i(A) \cap J_j(B) \neq \emptyset.$$

Hence for fixed i, $c_{ij} > 0$ if and only if $I_i(A) \cap J_j(B) \neq \emptyset$. $\square$

LEMMA 3.4. *D^k and E^k have strictly positive entries for all integer $k \geq \frac{N-1}{2}$.*

Proof. We show the proof for the matrix D, the proof for E is similar. The row index sets $I_i(D)$ are given by

$$I_i(D) = \begin{cases} \{1, \ldots, i+2\} & i \text{ even} \\ \{1, \ldots, i+1\} & i \text{ odd} \end{cases} \Big\} \quad 1 \leq i < 4$$
$$\qquad\quad \begin{cases} \{i-1, \ldots, i+2\} & i \text{ even} \\ \{i-2, \ldots, i+1\} & i \text{ odd} \end{cases} \Big\} \quad 4 \leq i \leq N-3$$
$$\qquad\quad \begin{cases} \{i-1, \ldots, N-1\} & i \text{ even} \\ \{i-2, \ldots, N-1\} & i \text{ odd} \end{cases} \Big\} \quad N-3 < i \leq N-1$$

The column index sets are given by

$$J_j(D) = \begin{cases} \{1, \ldots, 3\} & & 1 \leq j < 3 \\ \{j-1, \ldots, j+2\} & j \text{ odd} & \multirow{2}{*}{$\Big\}\ 3 \leq j \leq N-2$} \\ \{j-2, \ldots, j+1\} & j \text{ even} & \\ \{N-2, N-1\} & & j = N-1 \end{cases}$$

We are interested in the growth of the index sets $I_i(D^k)$ as a function of k. Once every index set contains all the numbers $1 \leq j \leq N-1$, the matrix D^k has strictly positive entries. We show that every multiplication with D enlarges the index sets $I_i(D^k)$ on both sides by two elements, as long as the elements 1 and $N-1$ are not yet reached. The proof is done by induction: For D^2 we have using Lemma 3.3

$$I_i(D^2) = \begin{cases} \{1,\dots,i+4\} & i \text{ even} \\ \{1,\dots,i+3\} & i \text{ odd} \end{cases} \quad 1 \le i < 6$$
$$ \begin{cases} \{i-3,\dots,i+4\} & i \text{ even} \\ \{i-4,\dots,i+3\} & i \text{ odd} \end{cases} \quad 6 \le i \le N-5$$
$$ \begin{cases} \{i-3,\dots,N-1\} & i \text{ even} \\ \{i-4,\dots,N-1\} & i \text{ odd} \end{cases} \quad N-5 < i \le N-1$$

Now suppose that for k we obtained the sets

$$I_i(D^k) = \left\{ \begin{array}{lll} \{1,\dots,i+2k\} & i \text{ even} & \multirow{2}{*}{$1 \le i < 2+2k$} \\ \{1,\dots,i+2k-1\} & i \text{ odd} & \\ \{i-2k+1,\dots,i+2k\} & i \text{ even} & \multirow{2}{*}{$2+2k \le i \le N-2k-1$} \\ \{i-2k,\dots,i+2k-1\} & i \text{ odd} & \\ \{i-2k+1,\dots,N-1\} & i \text{ even} & \multirow{2}{*}{$N-2k-1 < i \le N-1$} \\ \{i-2k,\dots,N-1\} & i \text{ odd} & \end{array} \right.$$

Then for $k+1$ we have applying Lemma 3.3 again

$$I_i(D^{k+1}) = \left\{ \begin{array}{lll} \{1,\dots,i+2(k+1)\} & i \text{ even} & \multirow{2}{*}{$1 \le i < 2+2(k+1)$} \\ \{1,\dots,i+2(k+1)-1\} & i \text{ odd} & \\ \{i-2(k+1)-1,\dots,i+2(k+1)\} & i \text{ even} & \multirow{2}{*}{$2+2(k+1) \le i \le N-2(k+1)-1$} \\ \{i-2(k+1),\dots,i+2(k+1)-1\} & i \text{ odd} & \\ \{i-2(k+1)-1,\dots,N-1\} & i \text{ even} & \multirow{2}{*}{$N-2(k+1)-1 < i \le N-1$} \\ \{i-2(k+1),\dots,N-1\} & i \text{ odd} & \end{array} \right.$$

Hence every row index set $I_i(D^k)$ grows on both sides by 2 when D^k is multiplied by D, as long as the boundary numbers 1 and $N-1$ are not yet reached. Now the index set $I_1(D^k) = \{1,\dots,2k\}$ has to grow most to reach the boundary number $N-1$, so we need for the number of iterations

$$k \ge \frac{N-1}{2}$$

for the matrix D^k to have strictly positive entries. □

The infinity norm of a vector $\boldsymbol{v}$ in $\mathbb{R}^n$ and a matrix A in $\mathbb{R}^{n\times n}$ is defined by

$$||\boldsymbol{v}||_\infty := \max_{1\le j\le n} |\boldsymbol{v}(j)|, \quad ||A||_\infty := \max_{1\le i\le n} \sum_{j=1}^{n} |A_{ij}|.$$

LEMMA 3.5. *For all $k > \frac{N}{2}$ there exists $\gamma = \gamma(k) < 1$ such that*

$$||D^k||_\infty \le \gamma \quad \textit{and} \quad ||E^k||_\infty \le \gamma$$

Proof. We prove the result for D; the proof for E is similar. We have from (3.11) and (3.12) that

$$\boldsymbol{r}(D) = \begin{pmatrix} q_1 \\ 1 \\ \vdots \\ 1 \\ p_{N-1} + r_{N-1}p_N \\ s_{N-1} + q_{N-1}p_N \end{pmatrix}.$$

By Lemma 3.4 D^k has strictly positive entries for any $k \geq \frac{N}{2}$. Note also that $||D^k||_\infty \leq 1$ since $||D||_\infty \leq 1$. Now by Lemma 3.2 we have

$$||D^{k+1}||_\infty = \max_i r_i(D^{k+1}) = \max_i \sum_j D^k_{ij} r_j(D) < 1$$

since $D^k_{ij} > 0$ for all i, j, $\sum_j D^k_{ij} \leq 1$ for all i, $r_j(D) \in [0,1]$ and $r_1(D) < 1$, $r_{N-1}(D) < 1$ and $r_N(D) < 1$. □

Remark: It suffices for each row index set to reach one of the boundaries, either 1 or $N-1$, for the infinity norm to start decaying. Hence it is enough that there are no more index sets $I_i(D^k)$ (compare the proof of Lemma 3.4) such that $2 + 2k \leq i \leq N - 1 - 2k$ so that the requirement $k \geq \frac{N-1}{2}$ can be relaxed to $k > \frac{N-3}{4}$.

We now fix some $k > \frac{N-3}{4}$ and set

$$\gamma := \max(||D^k||_\infty, ||E^k||_\infty) < 1. \tag{3.13}$$

LEMMA 3.6. *The vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy*

$$||\boldsymbol{\xi}^{2km}||_\infty \leq \gamma^m ||\boldsymbol{\xi}^0||_\infty \tag{3.14}$$

$$||\boldsymbol{\eta}^{2km}||_\infty \leq \gamma^m ||\boldsymbol{\eta}^0||_\infty. \tag{3.15}$$

Proof. By induction on (3.8), using that the entries of D, E, $\boldsymbol{\xi}^k$ and $\boldsymbol{\eta}^k$ are non-negative, we get

$$\boldsymbol{\xi}^{2km} \leq D^{km}\boldsymbol{\xi}^0 \quad \text{and} \quad \boldsymbol{\eta}^{2km} \leq E^{km}\boldsymbol{\eta}^0.$$

Taking norms on both sides and applying Lemma 3.5 the result follows. □

THEOREM 3.7. *The Schwarz iteration for the heat equation with N subdomains converges in the infinity norm in time and space. We have*

$$\max_{1 \leq 2i \leq N} ||e^{2km+1}_{2i}(\cdot,\cdot)||_{\infty,\infty} \leq \gamma^m ||\boldsymbol{\xi}^0||_\infty \tag{3.16}$$

$$\max_{1 \leq 2i+1 \leq N} ||e^{2km+1}_{2i+1}(\cdot,\cdot)||_{\infty,\infty} \leq \gamma^m ||\boldsymbol{\eta}^0||_\infty. \tag{3.17}$$

Proof. We use again the maximum principle. Since the error e^k_i is in the kernel of the heat operator, by the maximum principle e^k_i attains its maximum on the initial line or on the boundary. On the initial line e^k_i vanishes, therefore

$$\max_{1 \leq 2i \leq N} ||e^{2km+1}_{2i}(\cdot,\cdot)||_{\infty,\infty} \leq ||\boldsymbol{\xi}^{2km}||_\infty, \quad \max_{1 \leq 2i+1 \leq N} ||e^{2km+1}_{2i+1}(\cdot,\cdot)||_{\infty,\infty} \leq ||\boldsymbol{\eta}^{2km}||_\infty.$$

Using Lemma 3.6 the result follows. □

Note that the bound for the rate of convergence in Theorem 3.7 is not explicit. This is unavoidable for the level of generality employed. But, if we assume for simplicity that the overlaps are all of the same size then we can get more explicit rates of convergence. We set $r_i = s_i = r \in (0,1)$ and $p_i = q_i = p \in (0,1)$ where $p + r = 1$.

The matrices D and E then simplify to

$$\tilde{D} = \begin{bmatrix} p^2 & pr & & & & & & \\ pr & p^2 & pr & r^2 & & & & \\ r^2 & pr & p^2 & pr & & & & \\ & & pr & p^2 & pr & r^2 & & \\ & & r^2 & pr & p^2 & pr & & \\ & & & \ddots & & \ddots & & \\ & & & & pr & p^2 & pr & \\ & & & & r^2 & pr & p^2 & \end{bmatrix}, \quad \tilde{E} = \begin{bmatrix} p^2 & pr & r^2 & & & & & \\ pr & p^2 & pr & & & & & \\ & pr & p^2 & pr & r^2 & & & \\ & r^2 & pr & p^2 & pr & & & \\ & & \ddots & & \ddots & & & \\ & & & pr & p^2 & pr & r^2 & \\ & & & r^2 & pr & p^2 & pr & \\ & & & & & pr & p^2 & \end{bmatrix}.$$

In this case we can bound the spectral norm of $\tilde{D}$ and $\tilde{E}$ by an explicit expression less than one. We use common notation for the spectral norm, namely

$$||\boldsymbol{v}||_2 := \sqrt{\sum_{i=1}^{n} \boldsymbol{v}(i)^2}, \quad ||A||_2 := \sup_{||\boldsymbol{v}||_2=1} ||A\boldsymbol{v}||_2.$$

LEMMA 3.8. *The spectral norms of $\tilde{D}$ and $\tilde{E}$ are bounded by*

$$||\tilde{D}||_2 \leq 1 - 4pr\sin^2\frac{\pi}{2(N+1)}, \quad ||\tilde{E}||_2 \leq 1 - 4pr\sin^2\frac{\pi}{2(N+1)}.$$

Proof. We prove the bound for $\tilde{D}$. The bound for $\tilde{E}$ can be obtained similarly. We can estimate the spectral norm of $\tilde{D}$ by letting $\tilde{D} = J + r^2F$ where J is tridiagonal and F has only $O(N)$ nonzero entries and these are equal to 1. In fact $||F||_2 = 1$. Using that the eigenvalues of J are given by

$$\lambda_j(J) = p^2 + 2pr\cos\frac{\pi j}{N+1},$$

the spectral norm of $\tilde{D}$ can be estimated by

$$\begin{aligned} ||\tilde{D}||_2 \leq ||J||_2 + r^2||F||_2 &= p^2 + 2pr\cos\frac{\pi}{N+1} + r^2 \\ &= p^2 + 2pr + r^2 - 4pr\sin^2\frac{\pi}{2(N+1)} = 1 - 4pr\sin^2\frac{\pi}{2(N+1)}, \end{aligned}$$

since $p + r = 1$. ☐

LEMMA 3.9. *Assume that all the N subdomains overlap at the same ratio $r \in (0, 0.5]$. Then the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy*

$$\begin{aligned} ||\boldsymbol{\xi}^{2k}||_2 &\leq \left(1 - 4r(1-r)\sin^2\frac{\pi}{2(N+1)}\right)^k ||\boldsymbol{\xi}^0||_2 \\ ||\boldsymbol{\eta}^{2k}||_2 &\leq \left(1 - 4r(1-r)\sin^2\frac{\pi}{2(N+1)}\right)^k ||\boldsymbol{\eta}^0||_2. \end{aligned}$$

Proof. The proof follows as in Lemma 3.6. ☐

Note that $r = 0.5$, which minimizes the upper bound in Lemma 3.9, corresponds to the maximum possible overlap in this setting, namely $\beta_{i-1} = \alpha_{i+1}$ in Figure 3.2.

THEOREM 3.10. *The Schwarz iteration for the heat equation with N subdomains that overlap at the same ratio $r \in (0, 0.5]$ converges in the infinity norm in time and space. Specifically we have*

$$\max_{1\le 2i \le N} ||e_{2i}^{2k}(\cdot,\cdot)||_{\infty,\infty} \le \left(1 - 4r(1-r)\sin^2\frac{\pi}{2(N+1)}\right)^k ||\boldsymbol{\xi}^0||_2 \tag{3.18}$$

$$\max_{1\le 2i+1 \le N} ||e_{2i+1}^{2k}(\cdot,\cdot)||_{\infty,\infty} \le \left(1 - 4r(1-r)\sin^2\frac{\pi}{2(N+1)}\right)^k ||\boldsymbol{\eta}^0||_2. \tag{3.19}$$

Proof. From the proof of Theorem 3.7 we have

$$\max_{1\le 2i \le N} ||e_{2i}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} \le ||\boldsymbol{\xi}^{2k}||_\infty, \quad \max_{1\le 2i+1 \le N} ||e_{2i+1}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} \le ||\boldsymbol{\eta}^{2k}||_\infty.$$

Since the infinity norm is bounded by the spectral norm we get

$$\max_{1\le 2i \le N} ||e_{2i}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} \le ||\boldsymbol{\xi}^{2k}||_2, \quad \max_{1\le 2i+1 \le N} ||e_{2i+1}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} \le ||\boldsymbol{\eta}^{2k}||_2.$$

Using Lemma 3.9 the result follows. □

The results derived above for the continuous heat equation remain valid as in the two subdomain case, when the heat equation is discretized. Details of this analysis can be found in [7].

4. Numerical Experiments. We perform numerical experiments to measure the actual convergence rate of the algorithm for the example problem

$$\begin{array}{rcll} \dfrac{\partial u}{\partial t} & = & \dfrac{\partial^2 u}{\partial x^2} - e^{-(t-1)^2-(x-\frac{1}{4})^2} & 0 < x < 1,\ 0 < t < 3 \\ u(0,t) & = & e^{-2t} & 0 < t < 3 \\ u(1,t) & = & e^{-t} & 0 < t < 3 \\ u(x,0) & = & 1 & 0 < x < 1. \end{array} \tag{4.1}$$

To solve the semi-discrete heat equation, we use the Backward Euler method in time. The first experiment is done splitting the domain $\Omega = [0,1] \times [0,3]$ into the two subdomains $\Omega_1 = [0,\alpha] \times [0,3]$ and $\Omega_2 = [\beta,1] \times [0,3]$ for three pairs of values $(\alpha,\beta) \in \{(0.4,0.6),(0.45,0.55),(0.48,0.52)\}$. Figure 4.1 shows the convergence of the algorithm on the grid point b for $\Delta x = 0.01$ and $\Delta t = 0.01$. The solid line is the predicted convergence rate according to Theorem 2.8 and the dashed line is the measured one. The measured error displayed is the difference between the numerical solution on the whole domain and the solution obtained from the domain decomposition algorithm. As initial guess for the iteration we used the initial condition constant in time. We also checked the robustness of the method by refining the time step and obtained similar results.

We solved the same problem (4.1) using eight subdomains which overlap by 35%. Figure 4.2 shows the decay of the infinity norm of $\boldsymbol{\xi}^k$. The dashed line shows the measured decay rate and the solid line the predicted one. Note that in the initial phase of the iteration the error stagnates, since information has to be propagated across domains.

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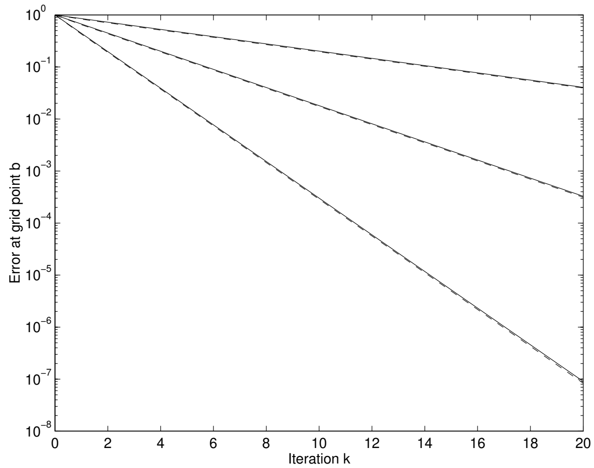

FIG. 4.1. *Theoretical and measured decay rate of the error for two subdomains and three different sizes of the overlap*

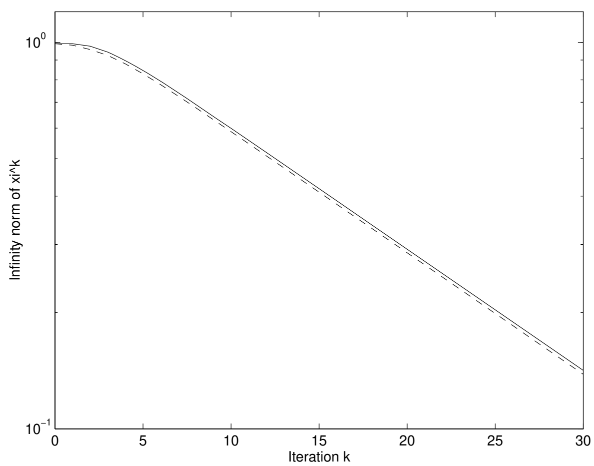

FIG. 4.2. *Theoretical and measured decay rate of the error in the case of eight subdomains*

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