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MATHEMATISCHE NACHRICHTEN

Space-time fractional Dirichlet problems

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Abstract
This paper establishes explicit solutions for fractional diffusion problems on bounded domains. It also gives stochastic solutions, in terms of Markov processes time-changed by an inverse stable subordinator whose index equals the order of the fractional time derivative. Some applications are given, to demonstrate how to specify a well-posed Dirichlet problem for space-time fractional diffusions in one or several variables. This solves an open problem in numerical analysis.

KEYWORDS
bounded domain, fractional Cauchy problem, infinitesimal generator, killed Feller process

MSC (2010)
26A33, 60J35, 60K99, 35S11

1 | INTRODUCTION

Fractional derivatives were invented by Leibnitz in 1695 [37]. Recently they have found new applications in many areas of science and engineering, see for example these books [27,33,36,37,43,45,46,48]. In particular, partial differential equations that employ fractional derivatives in time are used to model sticking and trapping, a kind of memory effect [12,39,40,44,56]. For practical applications, it is often necessary to employ numerical methods to solve these time-fractional partial differential equations. A variety of effective numerical schemes have been developed to solve fractional partial differential equations on a bounded domain, along with proofs of stability and convergence, see for example [18,21–23,34,35,57]. An important open problem in this area is to show that these problems are well-posed, see discussion in Defterli et al. [20].

In this paper, we take a step in that direction, by establishing explicit solutions to a broad class of time-fractional Cauchy problems [3] $\partial_t^\beta u(x,t) = Lu(x,t); u(x,0) = f(x)$ on a regular bounded domain Ω in d-dimensional Euclidean space, where ∂_t^β is the Caputo fractional derivative of order $0 < \beta < 1$ [37,43], and L is the semigroup generator of some Markov process on $\mathbb{R}^d$ [2,13,47]. In particular, we allow the operator L to be nonlocal in space. This includes the cases where L is a space-fractional derivative in one dimension [10], a tempered fractional derivative [6], the fractional Laplacian in $d \geq 1$ dimensions [15], or a multiscaling fractional derivative in $d > 1$ dimensions [55]. One important outcome of this research is to describe the appropriate version of these nonlocal operators on a bounded domain.

Our method of proof uses a fundamental result [3, Theorem 3.1] from the theory of semigroups, along with some ideas from the theory of Markov processes. This probabilistic method also establishes stochastic solutions for these equations, i.e., we describe a stochastic process whose probability density functions solve the time-fractional and space-nonlocal diffusion problem on the bounded domain. This extends the recent work of Chen et al. [16] where L is the (nonlocal) fractional Laplacian, and

Meerschaert et al. [42] where L is a (local) diffusion operator. However, since we do not assume that L is self-adjoint in this paper, standard spectral theory does not apply, and hence our approach is quite different.

2 | THE GENERATOR OF A KILLED FELLER PROCESS

We denote by $C_0(\mathbb{R}^d)$ the Banach space of continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ vanishing at infinity and endowed with the supremum norm $\|f\| := \sup\{|f(x)| : x \in \mathbb{R}^d\}$. Let $X := \{X_t\}_{t\geq 0}$ be a Feller process on $\mathbb{R}^d$. That is, for any $x \in \mathbb{R}^d$, we assume that the linear operators defined by $P_t f(x) := \mathbb{E}^x[f(X_t)]$ for all $t \geq 0$ form a strongly continuous, contraction semigroup on $C_0(\mathbb{R}^d)$, such that $P_t f \in C_0(\mathbb{R}^d)$ for all $f \in C_0(\mathbb{R}^d)$. By strongly continuous we mean that $\|P_t f - f\| \to 0$ as $t \searrow 0$ for all $f \in C_0(\mathbb{R}^d)$, and by contraction we mean that $\|f\| \leq 1$ implies $\|P_t f\| \leq 1$ for all $f \in C_0(\mathbb{R}^d)$. Then the infinitesimal generator of X is defined by

$$Lf := \lim_{t\searrow 0} \frac{P_t f - f}{t} \quad \text{in } C_0(\mathbb{R}^d). \tag{2.1}$$

We denote by $\mathcal{D}(L)$ the domain of L in $C_0(\mathbb{R}^d)$. Since f is a function of $x \in \mathbb{R}^d$, we can also write the *pointwise formula*

$$L^\sharp f(x) := \lim_{t\searrow 0} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t} \quad \text{in } \mathbb{R}^d. \tag{2.2}$$

Since convergence in $C_0(\mathbb{R}^d)$ implies pointwise convergence in $\mathbb{R}^d$, we have $Lf(x) = L^\sharp f(x)$ for all $f \in \mathcal{D}(L)$ and $x \in \mathbb{R}^d$. Conversely, an application of the Maximum Principle [13, Lemma 1.28] shows that, for any Feller semigroup, if (2.2) holds for each $x \in \mathbb{R}^d$, and if the limit $L^\sharp f \in C_0(\mathbb{R}^d)$, then (2.1) also holds [13, Theorem 1.33].

This leads to an explicit pointwise formula for the generator: Let $C_0^k(\mathbb{R}^d)$ denote the set of $f \in C_0(\mathbb{R}^d)$ whose derivatives up to order k also belong to $C_0(\mathbb{R}^d)$, and write $C_c^\infty(\mathbb{R}^d)$ for the functions in $C_0^\infty(\mathbb{R}^d)$ that vanish off a compact set. If $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(L)$, then [13, Theorem 2.37] shows that for any $f \in C_0^2(\mathbb{R}^d)$ we have $Lf(x) = L^\sharp f(x) = L^p f(x)$ for every $x \in \mathbb{R}^d$, where the pseudodifferential operator L^p is given by

$$\begin{aligned} L^p f(x) :=& -c(x)f(x) + l(x)\cdot\nabla f(x) + \nabla\cdot Q(x)\nabla f(x) \\ &+ \int_{\mathbb{R}^d\setminus\{0\}} \Big(f(x+y) - f(x) - \nabla f(x)\cdot y I_{B_1}(y)\Big) N(x,dy) \end{aligned} \tag{2.3}$$

for some $c(x) \geq 0$, $l(x) \in \mathbb{R}^d$, $Q(x) \in \mathbb{R}^{d\times d}$ symmetric and positive definite, $N(x,\cdot)$ a positive measure satisfying $\int_{\mathbb{R}^d\setminus\{0\}} \min(|y|^2, 1) N(x,dy) < \infty$, and B_1 the unit ball. The goal of this section is to apply this same procedure to killed Feller processes on a bounded domain.

Remark 2.1. In applications, there are no generally useful sufficient conditions that guarantee $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(L)$, so one has to check this on a case-by-case basis, see for example [13, Chapter 3]. In the special case of a Lévy process X_t, where $c = 0$ and l, Q, N do not depend on $x \in \mathbb{R}^d$, it follows from Sato [52, Theorem 31.5] that $C_c^\infty(\mathbb{R}^d) \subset C_0^2(\mathbb{R}^d) \subset \mathcal{D}(L)$. Hence we always have $Lf(x) = L^\sharp f(x) = L^p f(x)$ for all $f \in C_0^2(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$ in this case.

From now on we let $\Omega \subset \mathbb{R}^d$ be a bounded domain (connected open set) and let $C_0(\Omega)$ denote the set of continuous real-valued functions on Ω that tend to zero as $x \in \Omega$ approaches the boundary. Then $C_0(\Omega)$ is a Banach space with the supremum norm, as it can be identified with the closed subspace of $C_0(\mathbb{R}^d)$ consisting of zero extensions of functions in $C_0(\Omega)$. For a Feller process X_t on $\mathbb{R}^d$ we define the first exit time from Ω for X_t by

$$\tau_\Omega = \inf\{t > 0 : X_t \notin \Omega\}. \tag{2.4}$$

Let X_t^Ω denote the killed process on Ω, i.e.,

$$X_t^\Omega = \begin{cases} X_t, & t < \tau_\Omega, \\ \partial, & t \geq \tau_\Omega, \end{cases} \tag{2.5}$$

where ∂ denotes a cemetery point. We naturally extend any $f \in C_0(\Omega)$ to $\Omega \cup \{\partial\}$ by setting $f(\partial) = 0$. A boundary point x of Ω is said to be regular for Ω if $\mathbb{P}^x(\tau_\Omega = 0) = 1$. We say that Ω is regular if every boundary point of Ω is regular for Ω. We say that a Markov process X_t on $\mathbb{R}^d$ or its semigroup $P_t f(x) = \mathbb{E}_x[f(X_t)]$ is *strong Feller* if for any bounded measurable real-valued function f with compact support on $\mathbb{R}^d$, $P_t f(x)$ is bounded and continuous on $\mathbb{R}^d$. We say that a Feller process (resp, semigroup) is *doubly Feller* if it also has the strong Feller property (e.g., see [53]).

Lemma 2.2. *Suppose that X_t is a doubly Feller process on $\mathbb{R}^d$ and that Ω is regular. Then*

$$P_t^\Omega f(x) := \mathbb{E}^x\left[f\left(X_t^\Omega\right)\right], \quad x \in \Omega,\ t \ge 0, \tag{2.6}$$

defines a Feller semigroup on $C_0(\Omega)$.

Proof. Since X_t is doubly Feller and Ω is regular, the theorem on page 68 of Chung [19] implies that X_t^Ω is also doubly Feller. In particular, we have that P_t^Ω is a Feller semigroup on $C_0(\Omega)$. □

We will say that an open set U is compactly contained in Ω, and write $U \subset\subset \Omega$, if $\bar{U}$, the closure of U, defined as the intersection of all closed sets containing U, satisfies $\bar{U} \subset \Omega$. Since Ω is bounded, $\bar{U}$ is compact for any $U \subset\subset \Omega$. If $f_n(x) \to f(x)$ for all $x \in \Omega$, and uniformly on $x \in U$ for any $U \subset\subset \Omega$, we say that $f_n \to f$ uniformly on compacta in Ω. If P_t^Ω is a Feller semigroup on $C_0(\Omega)$, then it has a generator

$$L_\Omega f := \lim_{t\searrow 0} \frac{P_t^\Omega f - f}{t} \quad \text{in } C_0(\Omega), \tag{2.7}$$

with domain $\mathcal{D}(L_\Omega) \subset C_0(\Omega)$. The next result shows that this generator L_Ω can be computed using the pointwise formula (2.2) for the *original* Feller generator on $C_0\left(\mathbb{R}^d\right)$. Given a function $f \in C_0(\Omega)$, we apply the formula (2.2) to the zero extension of f, i.e., we set $f(x) = 0$ for all $x \notin \Omega$, to get an element of $C_0\left(\mathbb{R}^d\right)$. Then we will write $L^\sharp f \in C_0(\Omega)$ to mean that the function defined by (2.2) exists for all $x \in \Omega$, is continuous on Ω, and tends to zero as $x \in \Omega$ approaches the boundary. This *does not* require the limit in (2.2) to exist for any $x \notin \Omega$.

Theorem 2.3. *Assume that X_t is a doubly Feller process on $\mathbb{R}^d$ with $C_c^\infty\left(\mathbb{R}^d\right)$ contained in the domain of its generator, and let $\Omega \subset \mathbb{R}^d$ be a regular bounded domain. Then the domain of the killed generator* (2.7) *is given by*

$$\mathcal{D}(L_\Omega) = \left\{f \in C_0(\Omega) : L^\sharp f \in C_0(\Omega)\right\}. \tag{2.8}$$

Also $L_\Omega f(x) = L^\sharp f(x)$ for all $x \in \Omega$, and (2.2) *holds uniformly on compacta in Ω.*

Proof. Since X_t is a doubly Feller, it follows from Lemma 2.2 that X_t^Ω is a Feller process, whose semigroup (2.6) has a generator (2.7) on $C_0(\Omega)$. Let $f \in \mathcal{D}(L_\Omega)$. Then there exists $g \in C_0(\Omega)$ such that

$$g(x) = \lim_{t\to 0} \frac{P_t^\Omega f(x) - f(x)}{t}$$

for all $x \in \Omega$. Set $f(x) = 0$ for $x \notin \Omega$, and recall that $f(\partial) = 0$. We have

$$\begin{aligned} &P_t^\Omega f(x) - P_t f(x) \\ &\quad = \mathbb{E}^x f\left(X_t^\Omega\right) - \mathbb{E}^x f(X_t) \\ &\quad = \mathbb{E}^x\left[f\left(X_t^\Omega\right) I\left\{\tau_\Omega > t\right\}\right] + \mathbb{E}^x\left[f\left(X_t^\Omega\right) I\left\{\tau_\Omega \le t\right\}\right] - \mathbb{E}^x[f(X_t) I\{\tau_\Omega > t\}] - \mathbb{E}^x[f(X_t) I\{\tau_\Omega \le t\}] \\ &\quad = -\mathbb{E}^x[f(X_t) I\{\tau_\Omega \le t\}]. \end{aligned} \tag{2.9}$$

Indeed, the first and third terms cancel because $X_t^\Omega = X_t$ for $t < \tau_\Omega$, and the second term vanishes because $X_t^\Omega = \partial$ for $t \ge \tau_\Omega$. Furthermore, since X_t has a.s. right-continuous sample paths, we have $f(X_{\tau_\Omega}) = 0$ a.s. Therefore

$$\frac{P_t^\Omega f(x) - f(x)}{t} - \frac{P_t f(x) - f(x)}{t} = \frac{\mathbb{E}^x\left[\left(f\left(X_{\tau_\Omega}\right) - f(X_t)\right) I\{\tau_\Omega \le t\}\right]}{t}. \tag{2.10}$$

By the Strong Markov Property [31, Proposition 7.9] we have

$$\mathbb{E}^x[f(X_t) I\{\tau_\Omega \le t\}] = \mathbb{E}^x\left[P_{t-\tau_\Omega} f\left(X_{\tau_\Omega}\right) I\{\tau_\Omega \le t\}\right], \tag{2.11}$$

since $I\{\tau_\Omega \le t\}$ is measurable with respect to $\mathcal{F}_{\tau_\Omega}$. Recall that $P_t f \to f$ in the sup norm as $t \to 0$. Hence for any $\varepsilon > 0$, for some $\delta > 0$ we have $|P_t f(x) - f(x)| < \varepsilon$ for all $0 < t < \delta$ and all $x \in \mathbb{R}^d$. Note that for $0 < t < \delta$, if $\tau_\Omega \le t$, then $0 < t - \tau_\Omega < \delta$ as well, and otherwise $I\{\tau_\Omega \le t\} = 0$. Then using (2.11), it follows that in (2.10) we have

$$\left|\mathbb{E}^x\left[\left(f\left(X_{\tau_\Omega}\right) - f(X_t)\right) I\{\tau_\Omega \le t\}\right]\right| = \left|\mathbb{E}^x\left[\left(f\left(X_{\tau_\Omega}\right) - P_{t-\tau_\Omega} f(X_{\tau_\Omega})\right) I\{\tau_\Omega \le t\}\right]\right| \le \varepsilon \mathbb{P}^x[\tau_\Omega \le t] \tag{2.12}$$

for $0 < t < \delta$. Given $U \subset\subset \Omega$, choose $r > 0$ so that $B(x,r) := \left\{y \in \mathbb{R}^d : |x - y| < r\right\} \subset \Omega$ for all $x \in U$. Let $\tau_r^x := \inf\left\{t \ge 0 : |X_t - x| \ge r\right\}$ for the process started at $X_0 = x \in U$. Then $\mathbb{P}^x[\tau_\Omega \le t] \le \mathbb{P}^x[\tau_r^x \le t]$, and by [13, Theorem 5.1 and Proposition 2.27(d)] there exists some $M > 0$ such that

$$\frac{\mathbb{P}^x[\tau_r^x \le t]}{t} < M, \quad \text{for all } x \in U \text{ and } t > 0. \tag{2.13}$$

Then

$$\frac{\left|P_t^\Omega f(x) - P_t f(x)\right|}{t} \le \varepsilon \frac{\mathbb{P}^x[\tau_r^x \le t]}{t} < \varepsilon M$$

for all $x \in U$ and $0 < t < \delta$. Hence we have

$$\frac{P_t^\Omega f(x) - P_t f(x)}{t} \to 0 \quad \text{uniformly on compacta in } \Omega, \tag{2.14}$$

as $t \to 0$. Therefore any $f \in \mathcal{D}(L_\Omega)$ is also contained in the set on the right-hand side of Equation (2.8), and in addition, (2.14) holds.

Conversely, suppose $f \in C_0(\Omega)$ and that $(P_t f(x) - f(x))/t \to g(x)$ as $t \to 0$ for some $g \in C_0(\Omega)$, for all $x \in \Omega$. As L_Ω is the generator of a contraction semigroup on $C_0(\Omega)$, its resolvent $(\lambda I - L_\Omega)^{-1}$ exists for all $\lambda > 0$, and maps $C_0(\Omega)$ onto $\mathcal{D}(L_\Omega)$ [49, Chapter VII, Proposition (1.4)]. Then there exists some $h \in \mathcal{D}(L_\Omega)$ such that $(I - L_\Omega)h = f - g$. By (2.14) applied to h,

$$L_\Omega h(x) - g(x) = \lim_{t\to 0} \frac{P_t h(x) - h(x) - (P_t f(x) - f(x))}{t}, \quad x \in \Omega.$$

Hence, for $u = h - f$ we get

$$u(x) = \lim_{t\to 0} \frac{P_t u(x) - u(x)}{t}, \quad x \in \Omega.$$

Without loss of generality let $x_0 \in \Omega$ be such that $\|u\| = \sup_{x\in\Omega} |u(x)| = u(x_0) > 0$ (otherwise consider $-u$). Since P_t is a contraction, $P_t u(x_0) \le \|P_t u\| \le \|u\| = u(x_0)$ and therefore

$$0 \ge \left(P_t u(x_0) - u(x_0)\right)/t \to u(x_0) > 0$$

as $t \to 0$, which is a contradiction. Hence $\sup_{x\in\Omega} |u(x)| = 0$ and therefore $h = f$. Thus any f in the set on the right-hand side of Equation (2.8) is also an element of $\mathcal{D}(L_\Omega)$. □

Remark 2.4. An important consequence of Theorem 2.3 is that $C_c^\infty(\Omega)$ is not contained in $\mathcal{D}(L_\Omega)$ for a large class of pure-jump doubly Feller processes, including all the examples in Section 4 of this paper. Let X_t be a doubly Feller process on $\mathbb{R}^d$ with $C_c^\infty\left(\mathbb{R}^d\right) \subset \mathcal{D}(L)$, so that L is given by (2.3) for $f \in C_0^2\left(\mathbb{R}^d\right)$. Suppose that the local parts c, l, Q are zero and that $N(x, dy) = n(x, y)dy$, where $n(x, y)$ is strictly positive for all x and y. Let Ω be regular and choose $f \in C_c^\infty(\Omega)$, $f \ge 0$ and not identically zero. Set $f(x) = 0$ for $x \notin \Omega$. Then $f \in C_0^2\left(\mathbb{R}^d\right)$ and for any $x \in \partial\Omega$ we have

$$L^\sharp f(x) = \int_{\mathbb{R}^d\setminus\{0\}} f(x + y)n(x, y)\, dy > 0.$$

Hence $f \notin \mathcal{D}(L_\Omega)$ in view of (2.8).

Next we show that functions in $\mathcal{D}(L_\Omega)$ can be characterized as functions in $C_0(\Omega)$ that are locally in the domain of L. This will be used for explicitly computing the killed generator.

Theorem 2.5. *Assume that X_t is a doubly Feller process on $\mathbb{R}^d$ with $C_c^\infty(\mathbb{R}^d)$ contained in the domain of its generator, and let $\Omega \subset \mathbb{R}^d$ be a regular bounded domain. Then*

$$\mathcal{D}(L_\Omega) = \{ f \in C_0(\Omega) : \exists g \in C_0(\Omega), (f_n) \subset \mathcal{D}(L) \text{ such that } f_n \to f \text{ in } C_0(\mathbb{R}^d) \text{ and } Lf_n \to g \text{ unif. on compacta in } \Omega \}, \tag{2.15}$$

and for f, g as in (2.15) *we have $L_\Omega f = g$.*

Proof. First we show that the limit g in (2.15) is unique for any given f. Assume that for some $f_n \in \mathcal{D}(L)$ we have $f_n \to 0$ uniformly on $\mathbb{R}^d$ and $Lf_n(x) \to g(x)$ for all $x \in \Omega$, uniformly on compacta in Ω. We claim that $g(x) = 0$ for all $x \in \Omega$. Assume $g(x) > \delta$ for all $x \in B(x_0; r) \subset \Omega$ for some $x_0 \in \Omega$ and $\delta, r > 0$. Choose $h \in C_c^\infty$ such that $h(x_0) > 0$ is the only local maximum. Let $\epsilon > 0$ be small enough that $U = \{x : h(x_0) - h(x) < \epsilon\} \subset B(x_0, r)$ and let $y = \sup_{x\in\Omega} |Lh(x)|$. Consider

$$h_n = h + 4\frac{y}{\delta} f_n.$$

Let n be large enough such that $|4\frac{y}{\delta} f_n(x)| < \epsilon/2$ for all $x \in \Omega$ and $Lf_n(x) > \delta/2$ for all $x \in U$. Then

$$4\frac{y}{\delta} Lf_n(x) > 4\frac{y}{\delta}\frac{\delta}{2} = 2y \quad \text{for all } x \in \Omega,$$

and since $Lh(x) \le y$ for all $x \in \Omega$, it follows that $Lh_n(x) > y$ for all $x \in \Omega$. For all $x \notin U$ we have $h(x) \le h(x_0) - \varepsilon$, and hence $h_n(x) \le h(x_0) - \varepsilon/2$ for all $x \notin U$. Since $h_n(x_0) > h(x_0) - \varepsilon/2$, it follows that h_n attains its maximum at some point $x_n \in U$. Then the positive maximum principle [31, Theorem 17.11 (iii)] implies that $Lh_n(x_n) \le 0$, and this contradicts the fact that $Lh_n(x) > 0$ for all $x \in \Omega$. Hence $g \le 0$. Considering the sequence $-f_n$, we obtain that $-g \le 0$ and hence $g = 0$. Given two sequences f_n and f_n' in $\mathcal{D}(L)$ that both converge to f in $C_0(\mathbb{R}^d)$, and such that $Lf_n \to g$ and $Lf_n' \to g'$ in $C_0(\mathbb{R}^d)$, it follows that $f_n - f_n' \to 0$ in $C_0(\mathbb{R}^d)$, and hence $L(f_n - f_n') \to g - g' = 0$, which proves uniqueness.

Next we show that functions $f \in \mathcal{D}(L_\Omega)$ can be approximated locally in the graph norm by functions in the domain of L, namely by the functions

$$f_\lambda = (\lambda - L)^{-1} \lambda f.$$

As $P_t f$ is continuous in t and $\|P_t f\| \le \|f\|$, it is not hard to check that $f_\lambda = \lambda \int_0^\infty e^{-\lambda t} P_t f \, dt$ and

$$\lim_{\lambda\to\infty} f_\lambda = P_0 f = f$$

in $C_0(\mathbb{R}^d)$. Furthermore, $f_\lambda \in \mathcal{D}(L)$ and by definition,

$$Lf_\lambda = \lambda f_\lambda - \lambda f.$$

Theorem 2.3 implies that $\frac{P_t f(x) - f(x)}{t} \to L_\Omega f(x)$ uniformly in $x \in U \subset\subset \Omega$, and then it is not hard to check that, using a substitution $u = \lambda t$,

$$\begin{aligned} \lim_{\lambda\to\infty} Lf_\lambda(x) &= \lim_{\lambda\to\infty} \lambda^2 \int_0^\infty e^{-\lambda t} P_t f(x)\, dt - \lambda f(x) \\ &= \lim_{\lambda\to\infty} \lambda^2 \int_0^\infty e^{-\lambda t} (P_t f(x) - f(x))\, dt \\ &= \lim_{\lambda\to\infty} \lambda^2 \int_0^\infty t e^{-\lambda t} \frac{P_t f(x) - f(x)}{t}\, dt \\ &= \lim_{\lambda\to\infty} \int_0^\infty u e^{-u} \frac{P_{(u/\lambda)} f(x) - f(x)}{(u/\lambda)}\, du \\ &= L_\Omega f(x) \end{aligned} \tag{2.16}$$

uniformly in $x \in U$. Hence $\mathcal{D}(L_\Omega)$ is contained in the set on the right-hand side of (2.15).

To prove the reverse set inclusion, suppose that $f \in C_0(\Omega)$ and for some $f_n \in \mathcal{D}(L)$ we have $f_n \to f$ in $C_0(\mathbb{R}^d)$ and $Lf_n(x) \to g(x)$ uniformly in $x \in U \subset\subset \Omega$ for some $g \in C_0(\Omega)$. Let $h = (I - L_\Omega)^{-1}(f - g)$ so that

$$h - f = L_\Omega h - g.$$

Since the resolvent maps $C_0(\Omega)$ onto $\mathcal{D}(L_\Omega)$, the function h lies in the set on the right-hand side of (2.15) by what we have already proven. Hence there exist $h_n \in \mathcal{D}(L)$ such that $h_n \to h$ in $C_0(\mathbb{R}^d)$ and $Lh_n(x) \to L_\Omega h(x)$ for all $x \in \Omega$, uniformly on compacta. Let $u = h - f$ and assume (without loss of generality) that $u(x_0) = \|u\| > \epsilon$ for some $\epsilon > 0$. Let $u_n = h_n - f_n$ so that $Lu_n(x) \to L_\Omega h(x) - g(x) = u(x)$ uniformly in $x \in U \subset\subset \Omega$. However, as u_n converges uniformly to u there exists $N > 0$ and $U \subset\subset \Omega$ such that $\{x_n : u_n(x_n) = \|u_n\|\} \subset U$ for all $n > N$. As $u_n(x_n) > \epsilon/2$ for large n and $Lu_n(x_n) \le 0$ by the maximum principle [31, Theorem 17.11 (iii)], $u_n(x) - Lu_n(x)$ cannot converge uniformly on U to 0. This is a contradiction, and hence $u \equiv 0$. Then $h = f \in \mathcal{D}(L_\Omega)$, which completes the proof. □

Even if $f \notin C_0^2(\mathbb{R}^d)$, the pointwise limit (2.2) might still exist for some $x \in \mathbb{R}^d$. The next result shows that we still have $L^\sharp f(x) = L^p f(x)$ for functions that are locally twice differentiable.

Lemma 2.6. *Assume that X_t is a Feller process on $\mathbb{R}^d$ with $C_c^\infty(\mathbb{R}^d)$ contained in the domain of its generator. Let $f \in C_0(\mathbb{R}^d)$ with f twice continuously differentiable in an open neighborhood U of x. Then $L^\sharp f(x) = L^p f(x)$, where L^p is given by* (2.3).

Proof. Let r be such that $B(x, 2r) \subset U$ and pick $f_n \in C_0^2(\mathbb{R}^d)$ with the property that $f_n \to f$ uniformly and $f_n(y) = f(y)$ for all $y \in B(x, r)$. Then

$$\begin{aligned} \left|L^\sharp f(x) - L^p f(x)\right| &= \left|L^\sharp f(x) - Lf_n(x) + Lf_n(x) - L^p f(x)\right| \\ &= \left|\lim_{t\searrow 0} \frac{\mathbb{E}^x[f(X_t) - f_n(X_t)]}{t} + \int_{\mathbb{R}^d\setminus\{0\}} \big(f(x+y) - f_n(x+y)\big)\, N(x, dy)\right| \\ &\le \lim_{t\searrow 0} \frac{\mathbb{P}^x\{\tau_r^x < t\}}{t}\|f - f_n\| + \left|\int_{|y|>r} \big(f(x+y) - f_n(x+y)\big)\, N(x, dy)\right| \\ &\le M_r\|f - f_n\| \to 0, \end{aligned} \tag{2.17}$$

where $M_r = C_r + N_r$ with C_r as in [13, Theorem 5.1] given by

$$\mathbb{P}^x\{\tau_r^x < t\} \le tC_r$$

and $N_r = C/r^2$ with C given as in [13, Theorem 2.31b] by

$$\int_{\mathbb{R}^d\setminus\{0\}} \min\big(|y|^2, 1\big) N(x, dy) < C.$$

This concludes the proof. □

The following theorem is the main result of this section. It shows that we can evaluate the generator $L_\Omega f(x)$ of the killed Markov process pointwise for $x \in \Omega$ using the explicit formula (2.3) for $Lf(x)$. In what follows, for a function $f \in C_0(\Omega) \cap C^2(\Omega)$ we mean by $L^p f$ the operator (2.3) applied to the zero extension of f.

Theorem 2.7. *Assume that X_t is a doubly Feller process on $\mathbb{R}^d$, and let $\Omega \subset \mathbb{R}^d$ be a regular bounded domain. Suppose that $C_0^2(\mathbb{R}^d)$ is a core for L, so that $Lf(x) = L^\sharp f(x) = L^p f(x)$ for every $x \in \mathbb{R}^d$ and $f \in C_0^2(\mathbb{R}^d)$. Then:*

(1) *for every $f \in \mathcal{D}(L_\Omega)$ there exists $f_n \in C_0(\Omega) \cap C^2(\Omega)$ such that $f_n \to f$ uniformly and $L^p f_n$ converges uniformly on compact subsets of Ω to $L_\Omega f$;*

(2) *if $f_n \in C_0(\Omega) \cap C^2(\Omega)$ is such that $f_n \to f \in C_0(\Omega)$ uniformly and $L^p f_n \to g \in C_0(\Omega)$ converges uniformly on compact subsets of Ω, then $f \in \mathcal{D}(L_\Omega)$ and $L_\Omega f = g$.*

In particular, if $f \in C_0(\Omega) \cap C^2(\Omega)$ and $L^p f \in C_0(\Omega)$, then $f \in \mathcal{D}(L_\Omega)$ and $L_\Omega f(x) = L^p f(x)$ is given by (2.3) *for every $x \in \Omega$.*

Proof. Consider a sequence of open sets $\Omega_n \subset\subset \Omega_{n+1}$ for $n \ge 1$ with $\bigcup \Omega_n = \Omega$. Take $\psi_n \in C_c^\infty(\mathbb{R}^d)$ with $I_{\Omega_n} \le \psi_n \le I_{\Omega_{n+1}}$.

To prove (1), by Theorem 2.5 and the definition of a core there exists $f_n^\infty \in C_0^2(\mathbb{R}^d)$ such that $f_n^\infty \to f$ uniformly and $Lf_n^\infty(x) = L^p f_n^\infty(x) \to L_\Omega f(x)$ uniformly on compact subsets of Ω. To see this, suppose that $f \in \mathcal{D}(L_\Omega)$, and extend f to an element of $C_0(\mathbb{R}^d)$ by setting $f(x) = 0$ for $x \notin \Omega$. Apply Theorem 2.5 to obtain a sequence $(f_n) \subset \mathcal{D}(L)$ such that $f_n \to f$ in $C_0(\mathbb{R}^d)$, and $Lf_n(x) \to L_\Omega f(x)$ uniformly on compacta in Ω. Then for any compact set $U \subset \Omega$ and any integer $k > 0$, for some n_0, we have $\|f_n - f\| < 1/k$ for all $n \geq n_0$, and $|Lf_n(x) - L_\Omega f(x)| < 1/k$ for all $x \in U$ and all $n \geq n_0$. Since $C_0^2(\mathbb{R}^d)$ is a core, for each f_n there exists a sequence $f_{nm}^\infty \in C_0^2(\mathbb{R}^d)$ such that $\|f_n - f_{nm}^\infty\| + \|Lf_n - Lf_{nm}^\infty\| \to 0$ as $m \to \infty$. Hence for any $n > 0$ there is an m_0 such that $\|f_n - f_{nm}^\infty\| + \|Lf_n - Lf_{nm}^\infty\| < 1/n$ for all $m \geq m_0$. Define $f_n^\infty = f_{nm_0}^\infty$. Then for $n \geq n_1 := \max(n_0, k)$ we have for all $n \geq n_1$ that, by the triangle inequality, $\|f - f_n^\infty\| < 2/k$ for all $n \geq n_1$ and $|L_\Omega f(x) - L^p f_n^\infty(x)| < 2/k$ for all $x \in U$ and all $n \geq n_1$. Let $f_n = \psi_n f_n^\infty$. Then $f_n \in C_0(\Omega) \cap C^2(\Omega)$ and by Lemma 2.6

$$\left|L^p f_n(x) - L^p f_n^\infty(x)\right| = \left|\int_{x+y\notin\Omega_n} \left(f_n(x+y) - f_n^\infty(x+y)\right) N(x, dy)\right|.$$

Let U be a compact subset of Ω. Then there exists n_0 such that $U \subset \Omega_{n_0}$ and since the closure of Ω_{n_0} is compact, it is not hard to check that for any $n > n_0$, there exists some $\epsilon > 0$ such that $z \notin \Omega_n$ implies that $|z - x| > \epsilon$ for all $x \in \Omega_{n_0}$.

To see this, write $B(x, r) = \{w : |w - x| < r\}$ and note that, since $\bar{\Omega}_{n_0} \subset \Omega_n$ open, for each $x \in \bar{\Omega}_{n_0}$ there exists some $r > 0$ such that $B(x, 2r) \subset \Omega_n$. The collection of sets $\{B(x, r) : x \in \bar{\Omega}_{n_0}\}$ covers the compact set $\bar{\Omega}_{n_0}$, hence there exists a finite subcover $B(x_j, r_j)$ for $j = 1, \ldots, J$ such that $\bar{\Omega}_{n_0} \subset \bigcup_{j=1}^J B(x_j, r_j)$. For any $x \in \bar{\Omega}_{n_0}$ we have $|x - x_j| < r_j$ for some $j = 1, \ldots, J$ and $|x_j - z| > 2r_j$ for all $z \notin \Omega_n$, so that $|x - z| \geq |x_j - z| - |x - x_j| > r_j$. Then the claim holds with $\varepsilon = \min\{r_j : 1 \leq j \leq J\}$.

By [13, Proposition 2.27 (d)],

$$\int_{x+y\notin\Omega_n} \left|f_n(x+y) - f_n^\infty(x+y)\right| N(x, dy) \leq \|f_n - f_n^\infty\| \int_{|y|>\epsilon} N(x, dy) \to 0$$

uniformly on U, and hence $L^p f_n$ converges uniformly on U to $L_\Omega f$.

To prove (2), let $f_n^\infty = \psi_n f_n$. Then $f_n^\infty \in C_0^2(\mathbb{R}^d)$ and, with the same argument as above, $L^p f_n^\infty \to g$ uniformly on compact subsets of Ω. By Theorem 2.5, $f \in \mathcal{D}(L_\Omega)$ and $L_\Omega f = g$. □

Remark 2.8. In general, we do not know whether $L_\Omega f(x)$ can be computed by the pointwise formula (2.3) for every $f \in \mathcal{D}(L_\Omega)$. However, Theorem 2.7 shows that we can always write $L_\Omega f(x) = \lim_{n\to\infty} L^p f_n(x)$ for some $f_n \in C_0(\Omega) \cap C^2(\Omega)$, so that the pointwise formula (2.3) applies to $L^p f_n(x)$. Hence L_Ω is the unique closed extension to $\mathcal{D}(L_\Omega)$ of the formula (2.3) on $C_0(\Omega) \cap C^2(\Omega)$, compare [13, Theorem 2.37 (a)]. This is similar to the manner in which the Fourier transform is defined as an isometry on $L_2(\mathbb{R}^d)$: the pointwise definition is valid on a dense subset $L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$, and the isometry is the unique continuous extension to $L_2(\mathbb{R}^d)$.

Remark 2.9. In the case $c \equiv 0$, $l(x) \equiv l$, $Q(x) \equiv Q$, and $N(x, dy) \equiv N(dy)$, (2.3) is the generator of a Lévy process on $\mathbb{R}^d$. Then Hawkes [26, Lemma 2.1] shows that X_t is doubly Feller if and only if X_t has a Lebesgue density for each $t > 0$. It follows from Sato [52, Theorem 31.5] that $C_c^\infty(\mathbb{R}^d) \subset C_0^2(\mathbb{R}^d) \subset \mathcal{D}(L)$ in this case. Hence the conditions of Theorem 2.7 are satisfied for any Lévy process with a density.

Remark 2.10. Note that the equality $L_\Omega f = L^p f$ for suitable functions f is proved in [54, Corollary 3.8] under the assumption $C_c^\infty(\Omega) \subseteq \mathcal{D}(L_\Omega)$, which do not apply in our case, see Remark 2.4.

Remark 2.11. For Markov processes on $\mathbb{R}^d$, it is typical to first write a pointwise formula (2.3) for the generator, and then prove that there exists a Markov process with this generator (e.g., solve the martingale problem). See for example Ethier and Kurtz [25] or Taira [58]. Our problem is the reverse: Given a Markov process on a bounded domain, we want to compute the generator. The existence of a Feller process on $\mathbb{R}^d$ with generator (2.3) can be established by several different methods, see Böttcher, Schilling and Wang [13, Chapter 3] for a nice review. Theorem 2.3 is the analogue of [13, Theorem 1.33] on bounded domains. In the special case where $L = -(-\Delta)^{\alpha/2}$ is the fractional Laplacian, Chen, Meerschaert, and Nane [16, Lemma 4.1] give the pointwise formula for the killed generator. Theorem 2.7 extends that result to a more general Feller process. Theorem 9.4.1 in Taira [58] establishes a pointwise formula similar to (2.3) for Feller processes on $\bar{\Omega}$, assuming that $f \in C^2(\bar{\Omega})$. However, this is insufficient for our purposes, since $\mathcal{D}(L_\Omega)$ typically contains functions that are not C^2 at the boundary, e.g., see Example 4.1. Imkeller and Pavlyukevich [28,29] use probabilistic arguments to study the first exit time of stable-driven stochastic processes, and the analogous processes driven by a Lévy process where the tail of the Lévy measure is regularly varying. They compute the mean and tail bounds of τ_Ω, and show that the first exit time for these jump processes is determined by the large jumps. Dybiec, Gudowska–Nowak, and Hänggi [24] compute the mean and distribution of τ_Ω by solving a fractional boundary value problem derived from the Fokker–Planck equation of the underlying stable-driven process.

3 | FRACTIONAL CAUCHY PROBLEMS

In this section, we recall some results on (fractional) Cauchy problems that will be useful in Section 4. If X_t is a doubly Feller process on $\mathbb{R}^d$ and Ω is a regular bounded domain, then Lemma 2.2 implies that the semigroup P_t^Ω associated with the killed process, defined by (2.6), is a Feller semigroup on $C_0(\Omega)$. The generator L_Ω of this semigroup and its domain $\mathcal{D}(L_\Omega)$ are given in Theorem 2.3. If $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(L)$, then Theorem 2.7 gives an explicit pointwise formula (2.3) for L_Ω, valid for all $x \in \Omega$ and all $f \in C_0(\Omega) \cap C^2(\Omega)$ with $L^p f \in C_0(\Omega)$. Remark 2.8 explains that L_Ω is the unique extension of (2.3) to $\mathcal{D}(L_\Omega)$. Since P_t^Ω is a Feller semigroup, the function $u(t) = P_t^\Omega f$ solves the abstract Cauchy problem

$$\partial_t u(x,t) = L_\Omega u(x,t), \quad u(x,0) = f(x) \tag{3.1}$$

for any $f \in \mathcal{D}(L_\Omega)$, e.g., see [2, Proposition 3.1.9 (h)]. Furthermore $P_t^\Omega f$ is a *mild solution* to the Cauchy problem (3.1) for *any* $f \in C_0(\Omega)$ [2, Proposition 3.1.9 (b)]. That is, $u(x,t) = P_t^\Omega f(x)$ is the unique solution in $C_0(\Omega)$ to the corresponding integral equation

$$u(t) = f + L_\Omega \int_0^t u(s)\,ds \tag{3.2}$$

for all $t \geq 0$.

The function

$$u(x,t) = P_t^\Omega f(x) + \int_0^t P_s^\Omega g(x,t-s)\,ds \tag{3.3}$$

is the unique solution to the inhomogeneous Cauchy problem

$$\partial_t u(x,t) = L_\Omega u(x,t) + g(x,t); \quad u(x,0) = f(x) \tag{3.4}$$

for any $g(x,t) = g_0(x) + \int_0^t \partial_s g(x,s)\,ds \in C_0(\Omega)$ such that $\partial_t g(x,t) \in L^1_{\mathrm{loc}}(\mathbb{R}^+, C_0(\Omega))$ [2, Corollary 3.1.17]. The same formula (3.3) gives the unique mild solution to (3.4) for any $f \in C_0(\Omega)$ and any $g \in L^1([0,T), C_0(\Omega))$, see [2, Theorem 3.1.16]. That is, it solves the integral equation

$$u(t) = f + L_\Omega \int_0^t u(s)\,ds + \int_0^t g(s)\,ds. \tag{3.5}$$

In practice, the condition $f \in \mathcal{D}(L_\Omega)$ can be hard to check. In numerical analysis theory, it is therefore common to prove results like the Lax Equivalence Theorem for mild solutions, which can then be approximated by strong solutions, see for example [30, Chapter 10].

The positive and negative Riemann–Liouville fractional integrals of a suitable function $f : \mathbb{R} \to \mathbb{R}$ are defined by

$$\begin{aligned}\mathbb{I}^\alpha_{[L,x]} f(x) &= \frac{1}{\Gamma(\alpha)} \int_L^x f(y)(x-y)^{\alpha-1}\,dy,\\ \mathbb{I}^\alpha_{[x,R]} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^R f(y)(y-x)^{\alpha-1}\,dy\end{aligned} \tag{3.6}$$

for any $\alpha > 0$ and any $-\infty \leq L < x < R \leq \infty$, see for example [50, Definition 2.1]. The positive and negative Riemann–Liouville fractional derivatives are defined by

$$\begin{aligned}\mathbb{D}^\alpha_{[L,x]} f(x) &= \left(\frac{d}{dx}\right)^n \mathbb{I}^{n-\alpha}_{[L,x]} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_L^x f(y)(x-y)^{n-\alpha-1} dy,\\ \mathbb{D}^\alpha_{[x,R]} f(x) &= \left(-\frac{d}{dx}\right)^n \mathbb{I}^{n-\alpha}_{[x,R]} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^R f(y)(y-x)^{n-\alpha-1} dy\end{aligned} \tag{3.7}$$

for any non-integer $\alpha > 0$ and any $-\infty \leq L < x < R \leq \infty$, where $n-1 < \alpha < n$, see for example [50, p. 31]. The positive and negative Caputo fractional derivatives are defined by

$$\partial^{\alpha}_{[L,x]} f(x) = \mathbb{I}^{n-\alpha}_{[L,x]}\left(\frac{d}{dx}\right)^n f(x) = \frac{1}{\Gamma(n-\alpha)}\int_L^x f^{(n)}(y)(x-y)^{n-\alpha-1}\,dy,$$
$$\partial^{\alpha}_{[x,R]} f(x) = \mathbb{I}^{n-\alpha}_{[x,R]}\left(-\frac{d}{dx}\right)^n f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^R f^{(n)}(y)(y-x)^{n-\alpha-1}\,dy, \tag{3.8}$$

see for example [43, Eq. (2.16)]. If $0 < \beta < 1$, then for a function $f : \mathbb{R}^+ \to \mathbb{R}$ with Laplace transform

$$\tilde{f}(s) := \int_0^\infty e^{-st} f(t)\,dt \tag{3.9}$$

it is not hard to show that $\partial^{\beta}_{[0,t]} f(t)$ has Laplace transform $s^\beta \tilde{f}(s) - s^{\beta-1} f(0)$, extending the well-known formula for integer order derivatives. Since $\mathbb{D}^{\beta}_{[0,t]} f(t)$ has Laplace transform $s^\beta \tilde{f}(s)$, and since $s^{\beta-1}$ is the Laplace transform of the function $t^{-\beta}/\Gamma(1-\beta)$, it follows by the uniqueness of the Laplace transform that

$$\partial^{\beta}_{[0,t]} f(t) = \mathbb{D}^{\beta}_{[0,t]} f(t) - \frac{t^{-\beta}}{\Gamma(1-\beta)} f(0), \tag{3.10}$$

see [43, p. 39] for more details.

Let $g_\beta(u)$ denote the probability density function of the standard stable subordinator, with Laplace transform

$$\int_0^\infty e^{-su} g_\beta(u)\,du = e^{-s^\beta} \tag{3.11}$$

for some $0 < \beta < 1$. Suppose that D_t is a Lévy process such that $g_\beta(u)$ is the probability density of D_1, and define the *inverse stable subordinator* (first passage time)

$$E_t = \inf\{u > 0 : D_u > t\}. \tag{3.12}$$

A general result from the theory of semigroups [3, Theorem 3.1] (see also Remark 3.1) implies that the function

$$v(x,t) := \int_0^\infty g_\beta(r) P^{\Omega}_{(t/r)^\beta} f(x)\,dr \tag{3.13}$$

is the unique solution to the time-fractional Cauchy problem

$$\mathbb{D}^{\beta}_t v(x,t) = L_\Omega v(x,t) + \frac{t^{-\beta}}{\Gamma(1-\beta)} f(0); \quad v(x,0) = f(x) \tag{3.14}$$

for any $f \in \mathcal{D}(L_\Omega)$. Using (3.10), it follows that the same function also solves

$$\partial^{\beta}_t v = L_\Omega v; \quad v(0) = f \tag{3.15}$$

for any $f \in \mathcal{D}(L_\Omega)$. Since

$$h(w,t) = \frac{t}{\beta} w^{-1-1/\beta} g_\beta\left(t w^{-1/\beta}\right) \tag{3.16}$$

is the probability density function of the inverse stable subordinator E_t [40, Corollary 3.1], it follows by a simple change of variables that

$$v(x,t) = \int_0^\infty u(x,w) h(w,t)\,dw = \mathbb{E}^x\left[f\left(X^{\Omega}_{E_t}\right)\right], \tag{3.17}$$

where $u(x,w) = P^{\Omega}_w f(x)$.

Remark 3.1. The proof in [3, Theorem 3.1] uses Laplace transforms, and although it is not explicitly stated, this also leads to a simple proof of uniqueness: If $v(x,t)$ solves the fractional Cauchy problem (3.14), then its Laplace transform satisfies

$\tilde{v} = \left(s^\beta - L\right)^{-1} s^{\beta-1} f$. As L generates a semigroup, $\left(s^\beta - L\right)^{-1}$ is a bounded operator for all s^β in the right half plane. In particular $\left(s^\beta - L\right)^{-1} 0 = 0$ and hence by the uniqueness of the Laplace transform, we have $v = 0$ for initial data $f = 0$. Then, given two solutions v_1, v_2 to (3.14), their difference $v = v_1 - v_2$ solves (3.14) with $f = 0$, and hence $v_1 = v_2$. Therefore, (3.13) is the unique solution to the fractional Cauchy problem (3.14). The uniqueness of solutions is well known, and was used, for example, in [5].

Baeumer et al. [4] consider the inhomogeneous fractional Cauchy problem

$$\partial_t^\beta v(x,t) = L_\Omega v(x,t) + r(x,t); \quad v(x,0) = f(x) \tag{3.18}$$

with $0 < \beta < 1$. Assuming that $t \mapsto v(x,t)$ is differentiable and $r(x,0) \equiv 0$, they show that (3.18) can also be written in Volterra integral form

$$v(x,t) = L_\Omega \mathbb{I}^\beta_{[0,t]} v(x,t) + f(x) + \int_0^t R(x,s)\, ds \tag{3.19}$$

with $R(x,t) = \partial_t^{1-\beta} r(x,t)$ (and then $R(x,t) = \mathbb{D}_t^{1-\beta} r(x,t)$ as well). Note that the forcing function $R(x,t)$ has the traditional meaning, and the units of x/t, unlike the function $r(x,t)$. Any solution to the integral equation (3.19) will be called a *mild solution* to the inhomogeneous fractional Cauchy problem (3.18). Then the inhomogeneous fractional Cauchy problem (3.18) with $r(x,0) \equiv 0$, and $R(t) \in L^1_{\text{loc}}\left(\mathbb{R}^+; C_0(\Omega)\right)$ has a unique mild solution

$$v(x,t) = \int_0^\infty P_s^\Omega f(x) h(s,t)\, ds + \int_0^t \int_0^\infty P_u^\Omega R(x,s) h(u, t-s)\, du\, ds, \tag{3.20}$$

where h is given by (3.16), see Baeumer et al. [4, Theorem 1].

4 | APPLICATIONS

In many applications, including numerical analysis, it is necessary to consider fractional partial differential equations on a bounded domain with Dirichlet boundary conditions. However, the theoretical foundations have been lacking. Using the results of Section 2 on the generator of the killed process, along with the results from Section 3 on fractional Cauchy problems, we can establish existence and uniqueness of solutions to many fractional partial differential equations on a bounded domain with Dirichlet boundary conditions. The main technical condition is that the underlying Markov process is doubly Feller (defined just before Lemma 2.2). In this section, we provide some example applications to illustrate the power of our method.

Example 4.1. This example clarifies that the solution to the Cauchy problem (3.1) on the bounded domain need not solve the corresponding Cauchy problem $\partial_t u(x,t) = Lu(x,t)$ on $C_0\left(\mathbb{R}^d\right)$. Suppose that $f \geq 0$ is a smooth function with compact support in $\Omega = (0, M) \subset \mathbb{R}$, and that $L = \Delta = \partial_x^2$, the generator of a Brownian motion X_t on $\mathbb{R}$. The Cauchy problem

$$\partial_t U(x,t) = \Delta U(x,t); \quad U(x,0) = f(x) \tag{4.1}$$

has a unique solution

$$U(x,t) = \int_{y \in \mathbb{R}^d} f(y) p(x-y,t)\, dy$$

on $C_0(\mathbb{R})$, where $p(x,t) = (4\pi t)^{-1/2} e^{-x^2/(4t)}$ is the Gaussian density with mean zero and variance $2t$. Then $U(x,t) > 0$ for all $t > 0$ and all $x \in \mathbb{R}$, so $U(x,t)$ does not vanish off Ω, and hence is not a solution to (3.1). In this case, the solution to (3.1) can be written explicitly in the form

$$u(x,t) = \sum_{n=1}^\infty f_n e^{-\lambda_n t} \psi_n(x)$$

where $\lambda_n = (n\pi/M)^2, n = 1,2,3,\ldots$ are the eigenvalues and $\psi_n(x) = \sin(n\pi x/M)$ are the corresponding eigenfunctions of the generator L_Ω of the killed semigroup, and $f_n = (2/M) \int \psi_n(x) f(x)\, dx$, see for example [1, Eq. (8) with $\alpha = 1$]. This solution belongs to $C_0(\Omega) \cap C^2(\Omega)$ for each $t \geq 0$, and hence we have $L_\Omega u(x,t) = \Delta u(x,t)$ for all $x \in \Omega$ and all $t > 0$. Hence the function $u(x,t)$ also solves the differential equation $\partial_t u(x,t) = \Delta u(x,t)$, with the same initial condition $u(x,0) = f(x)$, at every point $(x,t) \in \Omega \times (0,\infty)$. Let $v(x,t) := f_n e^{-\lambda_n t}\psi_n(x)$ for some $n \in \mathbb{N}$ and $(x,t) \in \Omega \times (0,\infty)$, and set $v(x,t) = 0$ for $x \notin \Omega$, so that $v(\cdot,t) \in C_0(\mathbb{R})$ for all $t > 0$. Then $\partial_x v(x,t)$ does not exist at $x = 0$ or $x = M$, and hence $v(\cdot,t) \notin \mathcal{D}(L)$. Furthermore, the zero extension of $u(x,t)$ to $\mathbb{R} \times (0,\infty)$ cannot be twice differentiable in x at the boundary points $x = 0, M$, otherwise $u(x,t)$ would be another solution to (4.1) on $C_0(\mathbb{R})$, which would violate uniqueness. This example shows, in particular, that an element of $\mathcal{D}(L_\Omega)$ need not be extendable to an element of $\mathcal{D}(L)$.

Example 4.2. Here we compute the generator of a killed stable process X_t on $\mathbb{R}$ with index $1 < \alpha < 2$ in terms of fractional derivatives, see Theorem 4.3. Given a suitable function $f : \mathbb{R} \to \mathbb{R}$, the *generator form* of the positive fractional derivative is defined by

$$\mathbf{D}^\alpha_{(-\infty,x]} f(x) := \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_0^\infty \left[f(x-y) - f(x) + yf'(x)\right] y^{-1-\alpha}\, dy \tag{4.2}$$

for $1 < \alpha < 2$ [43, Eq. (2.18)]. The generator form of the negative fractional derivative is defined by

$$\mathbf{D}^\alpha_{[x,\infty)} f(x) := \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_0^\infty \left[f(x+y) - f(x) - yf'(x)\right] y^{-1-\alpha}\, dy \tag{4.3}$$

for $1 < \alpha < 2$ [43, Eq. (3.33)]. After a change of variables $y \mapsto -y$, it is not hard to see that these are special cases of the formula (2.3). The generator of any α-stable semigroup on $\mathbb{R}$ with index $1 < \alpha < 2$ can be written as

$$Lf(x) = -af'(x) + \int_{y\neq 0} \left[f(x-y) - f(x) + yf'(x)\right] \phi(dy) \tag{4.4}$$

with

$$\phi(dy) = \begin{cases} b\dfrac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} y^{-\alpha-1} dy & \text{for } y > 0, \text{ and} \\ c\dfrac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} |y|^{-\alpha-1} dy & \text{for } y < 0. \end{cases} \tag{4.5}$$

and a computation [43, Example 3.24] shows that

$$Lf(x) = -a\partial_x f(x) + b\mathbf{D}^\alpha_{(-\infty,x]} f(x) + c\mathbf{D}^\alpha_{[x,\infty)} f(x). \tag{4.6}$$

The fractional partial differential equation $\partial_t u = Lu$ with generator (4.6) is useful for modeling anomalous diffusion, where a cloud of particles spreads at a faster rate than the Brownian motion (the special case $\alpha = 2$), see Benson et al. [11].

An important open problem for fractional diffusion modeling is to identify the appropriate governing equation and boundary conditions on a bounded domain $\Omega = (L, R)$, see Defterli et al. [20] for additional discussion. The next result solves this problem in the case of zero Dirichlet boundary conditions.

Theorem 4.3. *Assume that X_t is a stable Lévy process on $\mathbb{R}^1$ with generator* (4.4) *and Lévy measure* (4.5) *for some $1 < \alpha < 2$. Let $\Omega = (L, R)$. Then the killed generator* (2.7) *is given by $L_\Omega f(x) = L^p f(x)$ for all $x \in \Omega$, for any $f \in C_0(\Omega) \cap C^2(\Omega)$ such that $L^p f(x) \in C_0(\Omega)$, where*

$$\begin{aligned} L^p f(x) &= -a\partial_x f(x) + \int_{-\infty}^\infty \left[f(x-y) - f(x) + yf'(x)\right] \phi(dy) \\ &= -a\partial_x f(x) + b\,\mathbb{D}^\alpha_{[L,x]} f(x) + c\,\mathbb{D}^\alpha_{[x,R]} f(x), \end{aligned} \tag{4.7}$$

using the Riemann–Liouville fractional derivatives (3.7)*. The integral formula in* (4.7) *is applied to the zero extension of $f \in C_0(\Omega) \cap C^2(\Omega)$, a function $f \in C_0(\mathbb{R})$ defined by setting $f(x) = 0$ for $x \notin \Omega$.*

Proof. In order to apply the results of Section 2, we need to show that Ω is regular. For $x \in \mathbb{R}$, define the first hitting time of x by $T_x = \inf\{t > 0 : X_t = x\}$. Since $1 < \alpha < 2$, we have $\mathbb{P}^x(T_x = 0) = 1$ for all $x \in \mathbb{R}$, see for example Sato [52, Example 43.22, p. 325]. This implies that the boundary points L and R are both regular for Ω. Since X_t has a smooth density for any

$t > 0$ [38, Theorem 7.2.7], Remark 2.9 shows that Theorem 2.7 applies, and hence the killed generator is given by the formula (4.4) applied to the zero extension of a function $f \in C_0(\Omega) \cap C^2(\Omega)$. For any such function, use (4.4) to write

$$L^p f(x) = -af'(x) + bI_1 + cI_2 \tag{4.8}$$

where

$$I_1 = \int_0^\infty [f(x-y) - f(x) + yf'(x)] \frac{y^{-\alpha-1}}{\Gamma(-\alpha)}\, dy$$

and

$$I_2 = \int_{-\infty}^0 [f(x-y) - f(x) + yf'(x)] \frac{|y|^{-\alpha-1}}{\Gamma(-\alpha)}\, dy.$$

Write

$$I_1 = \int_{x-L}^\infty \left[0 - f(x) + yf'(x)\right] \frac{y^{-\alpha-1}}{\Gamma(-\alpha)}\, dy + \int_0^{x-L} \left[f(x-y) - f(x) + yf'(x)\right] \frac{y^{-\alpha-1}}{\Gamma(-\alpha)}\, dy$$

and integrate by parts, noting that $f(x-y) - f(x) + yf'(x) = O\big(y^2\big)$ as $y \to 0$. The remaining boundary terms from the two integrals cancel, and then a change of variable $y \mapsto x - y$ yields

$$I_1 = f'(x)\frac{(x-L)^{1-\alpha}}{\Gamma(2-\alpha)} + \int_L^x \left[f'(y) - f'(x)\right] \frac{(x-y)^{-\alpha}}{\Gamma(1-\alpha)}\, dy. \tag{4.9}$$

Write $D = d/dx$ and use (3.8) to see that

$$\mathbb{D}^\alpha_{[L,x]} f(x) = D^2 \mathbb{I}^{2-\alpha}_{[L,x]} f(x) = D\left[D\mathbb{I}^{1-(\alpha-1)}_{[L,x]} f(x)\right] = D\mathbb{D}^{\alpha-1}_{[L,x]} f(x).$$

The Caputo and Riemann–Liouville fractional derivatives are related by

$$\mathbb{D}^{\alpha-1}_{[L,x]} f(x) = \partial^\alpha_{[L,x]} f(x) + f(L)\frac{(x-L)^{1-\alpha}}{\Gamma(2-\alpha)},$$

see for example [32, Eq. (2.4.6)]. Since $f(L) = 0$, this implies that

$$\begin{aligned}
\mathbb{D}^\alpha_{[L,x]} f(x) &= D\partial^{\alpha-1}_{[L,x]} f(x)\\
&= \frac{d}{dx}\left[\int_L^x f'(y)\frac{(x-y)^{1-\alpha}}{\Gamma(2-\alpha)}\, dy\right]\\
&= \frac{d}{dx}\left[\int_L^x \left[f'(y) - f'(x)\right]\frac{(x-y)^{1-\alpha}}{\Gamma(2-\alpha)}\, dy\right] + \frac{d}{dx}\left[f'(x)\int_L^x \frac{(x-y)^{1-\alpha}}{\Gamma(2-\alpha)}\, dy\right]\\
&= \int_L^x \left[f'(y) - f'(x)\right]\frac{(x-y)^{-\alpha}}{\Gamma(1-\alpha)}\, dy + f'(x)\frac{d}{dx}\int_L^x \frac{(x-y)^{1-\alpha}}{\Gamma(2-\alpha)}\, dy,
\end{aligned}$$

which reduces to (4.9). Similarly, $I_2 = \mathbb{D}^\alpha_{[x,R]} f(x)$. □

Now the results stated in Section 3 can be applied. Suppose that X_t is any stable Lévy process with index $1 < \alpha < 2$, specified by its generator (4.6). Recall from Remark 2.8 that L_Ω is the unique extension of (4.7). In what follows, we will also denote this extension by $L_\Omega f(x) = -a\partial_x + b\,\mathbb{D}^\alpha_{[L,x]} f(x) + c\,\mathbb{D}^\alpha_{[x,R]} f(x)$. Then the function $u(x,t) = \mathbb{E}^x[f(X_t) I\{\tau_\Omega < t\}]$ for $\Omega = (L,R)$ is the unique solution to the space-fractional Dirichlet problem

$$\begin{aligned}
\partial_t u(x,t) &= -a\partial_x u(x,t) + b\,\mathbb{D}^\alpha_{[L,x]} u(x,t) + c\,\mathbb{D}^\alpha_{[x,R]} u(x,t), \quad \text{for all } x \in \Omega,\ t > 0\\
u(x,0) &= f(x), \quad \text{for all } x \in \Omega;\\
u(x,t) &= 0, \quad \text{for all } x \notin \Omega, t \geq 0,
\end{aligned} \tag{4.10}$$

for any $f \in \mathcal{D}(L_\Omega)$, and the unique mild solution to (4.10) for any $f \in C_0(\Omega)$. If u_1, u_2 are the corresponding solutions to (4.10) for initial functions f_1, f_2, then $\|u_2(t) - u_1(t)\| = \left\|P_t^\Omega(f_2 - f_1)\right\| \le \|f_2 - f_1\|$ in the supremum norm, so the solution depends continuously on the initial condition. Hence the Dirichlet problem (4.10) is well posed.

Also, for any $0 < \beta < 1$ the function $v(x,t) = \mathbb{E}^x\left[f\left(X^\Omega_{E_t}\right)\right]$ is the unique solution to the space-time fractional Dirichlet problem

$$
\begin{aligned}
\partial_t^\beta v(x,t) &= -a\partial_x v(x,t) + b\,\mathbb{D}^\alpha_{[l,x]} v(x,t) + c\,\mathbb{D}^\alpha_{[x,r]} v(x,t), \quad \text{for all } x \in \Omega,\ t > 0,\\
v(x,0) &= f(x), \quad \text{for all } x \in \Omega;\\
v(x,t) &= 0, \quad \text{for all } x \notin \Omega, t \ge 0,
\end{aligned} \tag{4.11}
$$

for any $f \in \mathcal{D}(L_\Omega)$, and the unique mild solution to (4.10) for any $f \in C_0(\Omega)$. Note that the process $X^\Omega_{E_t}$ is not Markov, and the family of operators $T_t f(x) = \mathbb{E}^x\left[f\left(X^\Omega_{E_t}\right)\right]$ is not a semigroup. Write $v(x,t)$ in terms of $u(x,t)$ using (3.17), where h is given by (3.16). Since $w \mapsto h(w,t)$ is the probability density function of the nonnegative random variable E_t, we have

$$
\|v(t)\| = \sup_{x\in\Omega}\left|\int_0^\infty P^\Omega_w f(x) h(w,t)\,dw\right| \le \int_0^\infty \left\|P^\Omega_w f\right\| h(w,t)\,dw \le \|f\| \int_0^\infty h(w,t)\,dw = \|f\| \tag{4.12}
$$

using the fact that $\left\|P^\Omega_t f\right\| \le \|f\|$ in the supremum norm on $C_0(\Omega)$. It follows that the space-time fractional diffusion equation (4.11) is also well-posed.

Example 4.4. The following is a typical example from numerical analysis, see for example [41,57]. Consider the inhomogeneous fractional partial differential equation

$$
\partial_t u(x,t) = b\mathbb{D}^\alpha_{[0,x]} u(x,t) + c\mathbb{D}^\alpha_{[x,1]} u(x,t) + g(x,t) \tag{4.13}
$$

on a finite domain $\Omega = (0,1)$ with $1 < \alpha < 2$, positive coefficients $b \ne c$, initial condition $u(x,0) = 0$ for all $x \in \Omega$, Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$ for all $t \ge 0$, and forcing function

$$
g(x,t) = x^2(1-x)^2 - t\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)} g_{2-\alpha}(x) - 2\frac{\Gamma(4)}{\Gamma(4-\alpha)} g_{3-\alpha}(x) + \frac{\Gamma(5)}{\Gamma(5-\alpha)} g_{4-\alpha}(x)\right]
$$

where $g_p(x) = ax^p + b(1-x)^p$. Using the well-known formulae [43, Example 2.7]

$$
\begin{aligned}
\mathbb{D}^\alpha_{[L,x]}(x-L)^p &= \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}(x-L)^{p-\alpha},\\
\mathbb{D}^\alpha_{[x,R]}(R-x)^p &= \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}(R-x)^{p-\alpha}
\end{aligned} \tag{4.14}
$$

for $p > \alpha$, it is easy to check that the exact solution is $u(x,t) = tx^2(1-x)^2$. However, up to now, it was not known whether this solution was well-posed, or even unique, see [20, Section 3] for further details.

Since both $g(x,t) \in C_0(\Omega)$ and $\partial_t g(x,t) \in C_0(\Omega)$ for all $t \ge 0$, it follows from Example 4.2 and (3.4) that this is the *unique* solution to the inhomogeneous Cauchy problem

$$
\begin{aligned}
\partial_t u(x,t) &= b\,\mathbb{D}^\alpha_{[0,x]} u(x,t) + c\,\mathbb{D}^\alpha_{[x,1]} u(x,t) + g(x,t), \quad \text{for all } x \in \Omega,\ t > 0,\\
u(x,0) &= 0, \quad \text{for all } x \in \Omega;\\
u(x,t) &= 0, \quad \text{for all } x \notin \Omega, t \ge 0.
\end{aligned} \tag{4.15}
$$

Furthermore, uniqueness and (3.3) imply that $u(x,t) = \int_0^t \mathbb{E}^x\left[g\left(X^\Omega_t, t-s\right)\right] ds$. Since the initial function $f(x) \equiv 0$, we certainly have $P^\Omega_t f \in C_0(\Omega) \cap C^2(\Omega)$ for all $t \ge 0$. Hence $u(x,t)$ is the unique solution to (4.15) in the classical sense, i.e., the generator can be explicitly computed by the pointwise formulae (3.7) for the Riemann–Liouville fractional derivatives.

Remark 4.5. An important question in the theory of fractional partial differential equations is how to write appropriate boundary conditions. From the point of view of killed Markov processes, it is natural to impose the condition that $u(x,t) = 0$ for all $x \notin \Omega$ and all $t \ge 0$. On the other hand, the problem (4.13) only assumes $u(x,t) = 0$ for x on the boundary of Ω. However, the problem

(4.13) as stated is indeed well-posed, because the definition of the Riemann–Liouville fractional derivative (3.7) implicitly incorporates the zero exterior condition.

Remark 4.6. In some applications, the Caputo fractional derivatives (3.8) in the spatial variable x are used instead of the Riemann–Liouville. For the problem (4.13), these two forms are equivalent, because both $u(x,t)$ and $\partial_x u(x,t)$ vanish at the boundary, see for example Podlubny [48, Eq. (2.165)].

Remark 4.7. The generator of an α-stable Lévy process X_t on $\mathbb{R}$ with index $0 < \alpha < 1$ can be written in the form

$$Lf(x) = -af'(x) + \int_{y\neq 0} [f(x-y) - f(x)]\,\phi(dy) \tag{4.16}$$

where

$$\phi(dy) = \begin{cases} b\,\dfrac{\alpha}{\Gamma(1-\alpha)} y^{-\alpha-1} dy & \text{for } y > 0, \text{ and} \\ c\,\dfrac{\alpha}{\Gamma(1-\alpha)} |y|^{-\alpha-1} dy & \text{for } y < 0. \end{cases} \tag{4.17}$$

Using the generator form of the positive fractional derivative [43, Eq. (2.15)]

$$\mathbf{D}^{\alpha}_{[-\infty,x]} f(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} [f(x) - f(x-y)]\, y^{-1-\alpha} dy \tag{4.18}$$

and the negative fractional derivative [43, Eq. (3.31)]

$$\mathbf{D}^{\alpha}_{[x,\infty]} f(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} [f(x) - f(x+y)]\, y^{-1-\alpha} dy \tag{4.19}$$

for $0 < \alpha < 1$, we can also write

$$Lf(x) = -a\partial_x f(x) - b\mathbf{D}^{\alpha}_{[-\infty,x]} f(x) - c\mathbf{D}^{\alpha}_{[x,\infty]} f(x), \tag{4.20}$$

see [43, Example 3.24] for details. The question whether $\Omega = (L, R) \subset \mathbb{R}$ is regular can be answered in terms of the first passage time of X_t, which is defined by

$$T_{(x,\infty)} = \inf\left\{t > 0 : X_t > x\right\}, \quad x \in \mathbb{R}.$$

Since X_t is continuous in probability, it follows that R is regular for Ω if and only if $\mathbb{P}^R\big(T_{(R,\infty)} = 0\big) = 1$, and the regularity of L can be described analogously in terms of $T_{(-\infty,L)}$. It follows using [52, Theorem 47.6] that Ω is regular if and only if $b > 0$, $c > 0$ and $a = 0$. Then an argument similar to Theorem 4.3 shows that the generator of the killed stable Lévy process is given by

$$L_\Omega f(x) = -b\,\mathbf{D}^{\alpha}_{[L,x]} f(x) - c\,\mathbf{D}^{\alpha}_{[x,R]} f(x) \tag{4.21}$$

for all $x \in \Omega$, for any $f \in C_0(\Omega) \cap C^2(\Omega)$ such that the right-hand side of (4.21) belongs to $C_0(\Omega)$. It also follows from [52, Theorem 47.6] that Ω is always regular for X_t when $\alpha = 1$. One can also compute the generator of the corresponding killed process on Ω, but the formula is more complicated, because the centering term $f'(x) y I_{B_1}(y)$ in (2.3) cannot be simplified.

Remark 4.8. Suppose that $c = 0$ and $1 < \alpha < 2$ in (4.7). Then Theorem 3.4.4 and Theorem 4.3.3 in the recent PhD thesis of Sankaranarayanan [51] show that the domain of the killed generator L_Ω for $\Omega = (0,1)$ can be characterized completely as

$$\mathcal{D}(L_\Omega) = \left\{ f \in C_0(\Omega) : f = \mathbb{I}^{\alpha}_{[0,x]} g - x^{\alpha-1} \mathbb{I}^{\alpha}_{[0,x]} g(1)\ \exists\, g \in C_0(\Omega) \right\}.$$

The second term $x^{\alpha-1}\mathbb{I}^{\alpha}_{[0,x]} g(1)$ ensures that $f(1) = 0$. Then $L_\Omega f = b\mathbb{D}^{\alpha}_{[0,x]} f = bg$, since $\mathbb{D}^{\alpha}_{[0,x]}\mathbb{I}^{\alpha}_{[0,x]} f = f$ for all $f \in C_0(\Omega)$ [48, Eq. (2.106)], and

$$L_\Omega\left[x^{\alpha-1}\right] = bD^2 \mathbb{I}^{2-\alpha}_{[0,x]} x^{\alpha-1} = bD^2\,[\Gamma(\alpha)x] = 0$$

for all $x \in (0,1)$, where $D = d/dx$. Hence the pointwise formula (4.7) for $L_\Omega f(x)$ is valid for all $f \in \mathcal{D}(L_\Omega)$ in this case. Write any $f \in \mathcal{D}(L_\Omega)$ as $f = \mathbb{I}^\alpha_{[0,x]} g - a x^{\alpha-1}$, where $a = \mathbb{I}^\alpha_{[0,x]} g(1)$. Note that $L_\Omega f = \mathbb{D}^\alpha_{[0,x]} f = D\mathbb{D}^{\alpha-1}_{[0,x]} f$ and write $\mathbb{D}^{\alpha-1}_{[0,x]} f = A + B$, where

$$A = \mathbb{D}^{\alpha-1}_{[0,x]}\mathbb{I}^\alpha_{[0,x]} g = D\mathbb{I}^{2-\alpha}_{[0,x]}\mathbb{I}^\alpha_{[0,x]} g = D\mathbb{I}^2_{[0,x]} g = \mathbb{I}_{[0,x]} g = \int_0^x g(y)\,dy$$

tends to zero as $x \to 0$, and

$$B = \mathbb{D}^{\alpha-1}_{[0,x]}\left[a x^{\alpha-1}\right] = D\mathbb{I}^{2-\alpha}_{[0,x]}\left[a x^{\alpha-1}\right] = D[bx] = b,$$

where $b = a\Gamma(\alpha)$. Hence $\mathbb{D}^{\alpha-1}_{[0,x]} f(0+) = b > 0$. Set $f(x) = 0$ for $x \notin (0,1)$. Then we have $\mathbb{D}^{\alpha-1}_{[0,x]} f(x) = 0$ for all $x < 0$. Hence $\mathbb{D}^{\alpha-1}_{[0,x]} f$ is not continuous at $x = 0$, and so $\mathbb{D}^\alpha_{[0,x]} f(x) = D\mathbb{D}^{\alpha-1}_{[0,x]} f(x)$ does not exist at $x = 0$. This shows that the zero extension of f is not in $\mathcal{D}(L)$.

Example 4.9. Meerschaert and Tadjeran [41] consider

$$\partial_t u(x,t) = a(x)\mathbb{D}^{1.8}_{[0,x]} u(x,t) + b(x)\mathbb{D}^{1.8}_{[x,2]} u(x,t) + g(x,t) \tag{4.22}$$

on a finite domain $0 < x < 2$ and $t > 0$ with the coefficient functions

$$a(x) = \Gamma(1.2)x^{1.8} \quad \text{and} \quad b(x) = \Gamma(1.2)(2-x)^{1.8},$$

the forcing function

$$g(x,t) = -32e^{-t}\left[x^2 + (2-x)^2 - 2.5\left(x^3 + (2-x)^3\right) + \frac{25}{22}\left(x^4 + (2-x)^4\right)\right],$$

initial condition $u(x,0) = 4x^2(2-x)^2$, and Dirichlet boundary conditions $u(0,t) = u(2,t) = 0$. Using (4.14), is easy to check that $u(x,t) = 4e^{-t}x^2(2-x)^2$ is the exact solution. This test problem is used in [41] to demonstrate the effectiveness of an implicit Euler solution method. The method is proven to be unconditionally stable and consistent, and hence convergent, but whether the problem is well-posed is an open question, see Defterli et al. [20] for additional discussion. The operator $L = a(x,t)\mathbb{D}^{1.8}_{[-\infty,x]} + b(x,t)\mathbb{D}^{1.8}_{[x,\infty]}$ can be computed from (2.3) with c, l, Q equal to zero and

$$N(x,dy) = c(x,y)\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}|y|^{-\alpha-1}dy, \quad c(x,y) = b(x)I(y>0) + a(x)I(y<0). \tag{4.23}$$

However, it is not known whether this stable-like operator generates a Markov process on $\mathbb{R}$. In particular, the coefficients do not satisfy the usual growth conditions for a stochastic differential equation, see [14, Theorem A.1]. We can, however, prove uniqueness using the following well-known result.

Proposition 4.10. *Suppose that Ω is a bounded domain in $\mathbb{R}^d$, and $F(r) \geq F(s)$ for $r \leq s$. Define the operator $If(x) = F(Lf(x))$ where $Lf(x)$ is given by (2.3). If u, v are two solutions to*

$$\begin{aligned} \partial_t u(x,t) + Iu(x,t) &= 0; \qquad && x \in \Omega,\ 0 < t < T, \\ u(x,t) &= h(t,x), && x \notin \Omega,\ 0 < t < T, \\ u(x,0) &= f(x), && x \in \Omega, \end{aligned} \tag{4.24}$$

for some $T > 0$, then $u(x,t) = v(x,t)$ for all $x \in \mathbb{R}^d$ and all $t \geq 0$.

Proof. [Thanks to Andrzej Swiech] Suppose that $u(y,s) > v(y,s)$ at some point $y \in \Omega$ and $0 < s < T$. For $\delta > 0$, define

$$u^\delta(x,t) := u(x,t) - \frac{\delta}{T-t}.$$

If $\delta > 0$ is sufficiently small, then $u^\delta(y,s) - v(y,s) > 0$, and hence the function $u^\delta(x,t) - v(x,t)$ attains its positive maximum at some point $(x,t) \in \Omega \times (0,T)$. Then at this point we have $\partial_t u^\delta(x,t) = \partial_t v(x,t)$, and $\nabla u^\delta(x,t) = \nabla v(x,t)$. Since $u^\delta(x+z,t) - v(x+z,t) \leq u^\delta(x,t) - v(x,t)$, we also have $u^\delta(x+z,t) - u^\delta(x,t) \leq v(x+z,t) - v(x,t)$, and it follows that $Lu^\delta(x,t) \leq Lv(x,t)$.

Hence $Iu^\delta(x,t) \geq Iv(x,t)$. Thus we obtain

$$0 = \partial_t v(x,t) + Iv(x,t) \leq \partial_t u^\delta(x,t) + Iu^\delta(x,t) = \frac{-\delta}{(T-t)^2}$$

which is a contradiction. □

Since we know that $u(x,t) = 4e^{-t}x^2(2-x)^2$ solves the Dirichlet problem (4.22), we can apply Proposition 4.10 with $F(u) = -u$ to show that this solution is unique. Hence the numerical method in [41] indeed converges to the unique solution, which resolves an open question in that paper.

Example 4.11. The generator of any α-stable semigroup on $\mathbb{R}^d$ with index $1 < \alpha < 2$ can be written in the form

$$Lf(x) = -a\nabla f(x) + \int_{y\neq 0} [f(x-y) - f(x) + y\cdot\nabla f(x)]\,\phi(dy) \tag{4.25}$$

where

$$\phi(dy) = b\,\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} r^{-\alpha-1} dr\, M(d\theta) \tag{4.26}$$

in polar coordinates $r = |y|$ and $\theta = y/|y|$, where the spectral measure $M(d\theta)$ is any probability measure on the unit sphere. A calculation shows that

$$Lf(x) = -a\nabla f(x) + b\nabla_M^\alpha f(x), \tag{4.27}$$

where the vector fractional derivative is defined by

$$\nabla_M^\alpha f(x) = \int_{|\theta|=1} \mathbf{D}_\theta^\alpha f(x) M(d\theta)$$

and $\mathbf{D}_\theta^\alpha$ the fractional directional derivative, i.e., the one dimensional fractional derivative $\mathbf{D}_r^\alpha g(r)$ (in generator form) of the projection $g(r) = f(x + r\theta)$ for $r \in \mathbb{R}$. See [43, Example 6.29] for complete details.

If M is uniform over the sphere, it follows that $\nabla_M^\alpha f(x) = -c_{d,\alpha}(-\Delta)^{\alpha/2} f(x)$, where the fractional Laplacian $-(-\Delta)^{\alpha/2} f(x)$ has Fourier transform $-\|k\|^\alpha \hat{f}(k)$, and

$$c_{d,\alpha} = |\cos(\pi\alpha/2)| \int_{\|\theta\|=1} |\theta_1|^\alpha M(d\theta)$$

where $\theta = (\theta_1, \ldots, \theta_d)$, see [43, Example 6.24].

For any stable Lévy process with index $1 < \alpha < 2$, Remark 2.9 shows that Theorem 2.7 applies for any regular bounded domain $\Omega \subset \mathbb{R}^d$, and hence the killed generator is given by the same formula (4.25) applied to the zero extension a function $f \in C_0(\Omega) \cap C^2(\Omega)$. Now suppose that Ω is a convex domain, so that for every $x \in \Omega$ and $|\theta| = 1$ there exists a unique $R = R(x,\theta) > 0$ such that $x - r\theta \in \Omega$ for $0 < r < R$, and $x - r\theta \notin \Omega$ for $r > R$. Let $C = b\alpha(\alpha-1)/\Gamma(2-\alpha)$. A change of variable $y = r\theta$ in polar coordinates yields

$$L_\Omega f(x) = -a\nabla f(x) + \int_{|\theta|=1}\int_0^\infty [f(x-r\theta) - f(x) + r\theta\cdot\nabla f(x)]\, Cr^{-\alpha-1} dr\, M(d\theta)$$

for any $f \in C_0(\Omega) \cap C^2(\Omega)$ such that the right-hand side belongs to $C_0(\Omega)$. Then the same one dimensional calculation on the inner integral as in Example 4.2 leads to

$$L_\Omega f(x) = -a\nabla f(x) + b\nabla_{M,\Omega}^\alpha f(x) \tag{4.28}$$

where

$$\nabla_{M,\Omega}^\alpha f(x) = \int_{|\theta|=1} \mathbb{D}^\alpha_{[x-R(x,\theta),x],\theta} f(x)\, M(d\theta), \tag{4.29}$$

and $\mathbb{D}^\alpha_{[x-R,x],\theta} f(x)$ is the Riemann–Liouville fractional directional derivative, defined as the one dimensional Riemann–Liouville derivative $\partial^\alpha_{[x-R,x]} g(r)$ of the projection $g(r) = f(x + r\theta)$. Note that $g'(r) = \theta \cdot \nabla f(x + r\theta)$.

Then for any $0 < \beta < 1$ the function $v(x,t) = \mathbb{E}^x\left[f\left(X^\Omega_{E_t}\right)\right]$ is the unique solution to the Dirichlet problem

$$\begin{aligned} \partial_t^\beta v(x,t) &= -a\partial_x v(x,t) + b\nabla^\alpha_{M,\Omega} v(x,t), \quad \text{for all } x \in \Omega,\ t > 0, \\ v(x,0) &= f(x), \quad \text{for all } x \in \Omega; \\ v(x,t) &= 0, \quad \text{for all } x \notin \Omega, t \geq 0, \end{aligned} \tag{4.30}$$

for any $f \in \mathcal{D}(L_\Omega)$, and the unique mild solution to (4.10) for any $f \in C_0(\Omega)$. Then the same argument as in Example 4.2 shows that the space-time fractional diffusion equation (4.30) is well-posed. As in the previous examples, we understand that (4.28) represents the unique extension to $\mathcal{D}(L_\Omega)$.

Remark 4.12. Example 4.11 includes the fractional Laplacian as a special case. Chen et al. [16, Theorem 5.1] established strong solutions to the space-time fractional diffusion equation with Dirichlet boundary conditions (4.30) in the special case where $M(d\theta)$ is uniform over the sphere, i.e., the fractional Laplacian. Here the function $u(x,t)$ is said to be a strong solution if for every $t > 0$, $u(x,t) \in C_0(\Omega)$, $(-\Delta)^{\alpha/2}u(x,t)$ exists pointwise for every $x \in \Omega$, the Caputo fractional derivative $\partial_t^\beta u(x,t)$ exists pointwise for every $t > 0$ and $x \in \Omega$, $\partial_t^\beta u(x,t) = -(-\Delta)^{\alpha/2}u(x,t)$ pointwise in $(0,\infty)\times\Omega$, and $\lim_{t\downarrow 0} u(x,t) = f(x)$ for every $x \in \Omega$. The theorem assumes that the initial condition $f \in \mathcal{D}\left(L^k_\Omega\right)$ for some $k > -1 + (3d+4)/(2\alpha)$. The proof of [16, Theorem 5.1] involves symmetric Dirichlet forms, and an eigenfunction expansion of the fractional Laplacian. It seems difficult to extend that argument to the more general setting of Example 4.11, since the generator L of a stable process need not be self-adjoint, so that standard spectral theory does not apply.

Example 4.13. Bass [7] introduced stable-like processes, where the order $\alpha(x)$ of the fractional derivative varies in space. If $\alpha : \Omega \to [\alpha_1, \alpha_2]$ is a smooth bounded function for some $0 < \alpha_1 < \alpha_2 < 2$, then Schilling and Wang [53, Theorem 3.3] prove that the stable-like process X_t on $\mathbb{R}^d$ with generator $-(-\Delta)^{\alpha(x)/2}$ is doubly Feller. If Ω is a regular bounded domain in $\mathbb{R}^d$, then Bass [8, Theorem 2.1 and Remark 7.1] shows that X_t solves the martingale problem, i.e.,

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds$$

is a $\sigma\{X_s : 0 \leq s \leq t\}$-martingale for any $f \in C^2_b\left(\mathbb{R}^d\right)$, the family of real-valued functions on $\mathbb{R}^d$ such that f and all its derivatives of order 1 or 2 are continuous and bounded. Then it is easy to check, using the definition of the generator, that any function $f \in C^2_0\left(\mathbb{R}^d\right)$ is in $\mathcal{D}(L)$, where L is given by (2.3) with $c = l = Q = 0$ and $N(x,dy) = c_{d,\alpha(x)}|y|^{-d-\alpha(x)}dy$ for any $f \in C^2_0\left(\mathbb{R}^d\right)$. Then Theorem 2.7 shows that the generator of the killed process is given by the same pointwise formula applied to the zero extension of a function $f \in C_0(\Omega) \cap C^2(\Omega)$. Then for any $0 < \beta < 1$ the function $v(x,t) = \mathbb{E}^x\left[f\left(X^\Omega_{E_t}\right)\right]$ is the unique solution to the Dirichlet problem

$$\begin{aligned} \partial_t^\beta v(x,t) &= -(-\Delta)^{\alpha(x)/2} v(x,t), \quad \text{for all } x \in \Omega,\ t > 0, \\ v(x,0) &= f(x), \quad \text{for all } x \in \Omega; \\ v(x,t) &= 0, \quad \text{for all } x \notin \Omega, t \geq 0, \end{aligned} \tag{4.31}$$

for any $f \in \mathcal{D}(L_\Omega)$, and the unique mild solution to (4.31) for any $f \in C_0(\Omega)$. The same argument as in Example 4.2 shows that the Dirichlet problem (4.31) is well-posed. Here again, we define $-(-\Delta)^{\alpha(x)/2}f(x)$ using the zero extension of a function $f \in C_0(\Omega)$, and we have $f \in \mathcal{D}(L_\Omega)$ if the pointwise formula for $-(-\Delta)^{\alpha(x)/2}f(x)$ belongs to $C_0(\Omega)$.

Example 4.14. Bass and Levin [9] consider a different class of stable-like processes on $\mathbb{R}^d$ with generator (2.3) where $c = l = Q = 0$ and $N(x,dy) = \kappa(x,y)|y|^{-d-\alpha}dy$, $0 < \alpha < 2$, $\kappa(x,y) = \kappa(x,-y)$, and $0 < \kappa_1 < \kappa(x,y) < \kappa_2 < \infty$. Here we assume that $\kappa(x,y) = a(x)c_{d,\alpha}$ where $|a(x) - a(y)| \leq a_0|x-y|^\lambda$ for some $0 < \lambda < 1$ and $a_0 > 0$. Theorem 3.19 in Böttcher et al. [13] establishes the existence of a time-homogeneous Feller process X_t with this generator $L = -a(x)(-\Delta)^{\alpha/2}$. Chen and Zhang [17, Eq. (1.18)] observe that X_t solves the stochastic differential equation $dX_t = a\left(X_{t-}\right)^{1/\alpha}dY_t$ where Y_t is the standard symmetric stable Lévy process with generator $L_Y = -(-\Delta)^{\alpha/2}$ for some $0 < \alpha < 2$. It follows from [17, Corollary 1.3] that the transition density $p_t(x,y)$ of X_t (i.e., the Lebesgue probability density of $y = X_{t+s}$ given $X_s = x$) is locally bounded in $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ for any $t > 0$. It is easy to check that T_t is a $C_b\left(\mathbb{R}^d\right)$ semigroup (e.g., see discussion after [53, Theorem 2.1]) and then it follows from Schilling and Wang [53, Corollary 2.2] that X_t is doubly Feller. Then for any regular bounded domain $\Omega \subset \mathbb{R}^d$, Theorem 2.7 shows that the generator of the killed process X^Ω_t is given by the same formula: $L_\Omega f(x) = -a(x)(-\Delta)^{\alpha/2}f(x)$ for all

$f \in C_0(\Omega) \cap C^2(\Omega)$ such that $-a(x)(-\Delta)^{\alpha/2} f(x) \in C_0(\Omega)$, where we define $f(x) = 0$ for $x \notin \Omega$. Hence for any $0 < \beta < 1$ the function $v(x,t) = \mathbb{E}^x\left[f\left(X^{\Omega}_{E_t}\right)\right]$ is the unique solution to the Dirichlet problem

$$\begin{aligned} \partial_t^\beta v(x,t) &= -a(x)(-\Delta)^{\alpha/2} v(x,t), \quad \text{for all } x \in \Omega,\ t > 0, \\ v(x,0) &= f(x), \quad \text{for all } x \in \Omega; \\ v(x,t) &= 0, \quad \text{for all } x \notin \Omega, t \geq 0, \end{aligned} \tag{4.32}$$

for any $f \in \mathcal{D}(L_\Omega)$, and the unique mild solution to (4.32) for any $f \in C_0(\Omega)$. The same argument as in Example 4.2 shows that the Dirichlet problem (4.32) is well-posed. Again, $-a(x)(-\Delta)^{\alpha/2}$ represents the unique extension to $\mathcal{D}(L_\Omega)$, and $f \in \mathcal{D}(L_\Omega)$ if the pointwise formula for $-a(x)(-\Delta)^{\alpha/2} f(x)$ belongs to $C_0(\Omega)$.

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