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Space-Time Structures from IIB Matrix Model

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We derive a long distance effective action for space-time coordinates from a IIB matrix model. It provides us an effective tool to study the structures of space-time. We prove the finiteness of the theory for finite N to all orders of the perturbation theory. Space-time is shown to be inseparable, and its dimensionality is dynamically determined. The IIB matrix model contains a mechanism to ensure a vanishing cosmological constant which does not rely on the manifest supersymmetry. We discuss possible mechanisms to obtain realistic dimensionality and gauge groups from the IIB matrix model.

§1. Introduction

A large N reduced model has been proposed as a nonperturbative formulation of type IIB superstring theory.[1),2)] It is defined by the following action:

$$S = -\frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right), \tag{1·1}$$

where ψ is a ten dimensional Majorana-Weyl spinor field, and A_μ and ψ are $N \times N$ Hermitian matrices. It is formulated in a manifestly covariant way which we believe has a definite advantage over the light-cone formulation[3)] to study nonperturbative aspects of superstring theory. In fact we can in principle predict the dimensionality of space-time, the gauge group, and the matter contents by solving this model. In this paper we report the results of the first such efforts toward this goal.

This action can be related to the Green-Schwarz action of superstring[4)] by using the semiclassical correspondence in the large N limit,

$$\begin{aligned} -i[\ ,\ ] &\to \frac{1}{N}\{\ ,\ \}, \\ \mathrm{Tr} &\to N\int d^2\sigma\sqrt{\hat{g}}. \end{aligned} \tag{1·2}$$

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In fact Eq. (1·1) reduces to the Green-Schwarz action in the Schild gauge:[5]

$$S_{\text{Schild}} = \int d^2\sigma \left[\sqrt{\hat{g}}\alpha \left(\frac{1}{4}\{X^\mu, X^\nu\}^2 - \frac{i}{2}\bar{\psi}\Gamma^\mu\{X^\mu, \psi\} \right) + \beta\sqrt{\hat{g}} \right]. \tag{1·3}$$

Through this correspondence, the eigenvalues of the matrices A_μ are identified with the space-time coordinates $X_\mu(\sigma)$. The $\mathcal{N} = 2$ supersymmetry manifests itself in S_{Schild} as

$$\begin{cases} \delta^{(1)} X_\mu &= i\bar{\epsilon_1}\Gamma_\mu\psi \\ \delta^{(1)}\psi &= -\frac{1}{2}\sigma_{\mu\nu}\Gamma^{\mu\nu}\epsilon_1 \end{cases}, \tag{1·4}$$

and

$$\begin{cases} \delta^{(2)} X_\mu &= 0 \\ \delta^{(2)}\psi &= \epsilon_2 \end{cases}. \tag{1·5}$$

The $\mathcal{N} = 2$ supersymmetry (1·4) and (1·5) is directly translated into the symmetry of S as

$$\begin{cases} \delta^{(1)} A_\mu &= i\bar{\epsilon_1}\Gamma_\mu\psi \\ \delta^{(1)}\psi &= \frac{i}{2}\Gamma^{\mu\nu}[A^\mu, A^\nu]\epsilon_1 \end{cases}, \tag{1·6}$$

and

$$\begin{cases} \delta^{(2)} A_\mu &= 0 \\ \delta^{(2)}\psi &= \epsilon_2 \end{cases}. \tag{1·7}$$

The bosonic part of the action vanishes for commuting matrices $A_\mu^{ij} = x_\mu^i \delta^{ij}$, where i and j are color indices. These are the generic classical vacuum configurations. Namely, the distributions of the eigenvalues determine the extent and the dimensionality of space-time. Hence, the structure of space-time is dynamically determined by the theory. As we show in this paper, space-time exists as a single bunch, and no single eigenvalue can escape from the rest. We may consider $n \times n$ submatrices of our original $N \times N$ matrices, where n is assumed to be still very large. More precisely, we consider a block diagonal background where the first blocks are n dimensional and the rest is a diagonal vacuum configuration. Such a background may represent a D-string (or D-objects) which occupies a certain region of space-time.

Then, the effective action which describes the $n \times n$ submatrices can be obtained by integrating the rest of the degrees of freedom. It has been shown that a term proportional to β in Eq. (1·3) arises if a vacuum energy of order N exists in this model.[1] In fact we will show in this paper that this is indeed the case. The effective action may include all possible terms which are consistent with the symmetry. The term proportional to β is the lowest dimensional term which is consistent with the symmetry, and we expect that the higher dimensional terms are irrelevant for large n. In this way we can understand why this model resembles the effective action for D-objects.[6] However, we emphasize that we regard the action (1·1) as fundamental, and it should be distinguished from the effective action for N D-instantons.

It is straightforward to construct configurations which represent arbitrary numbers of D-objects in space-time in an analogous way. Although they are constrained to be inside the extension of space-time, as we see in this paper, they can be located anywhere in this extension. As is shown in Ref. 1), the long distance interactions

of D-objects are in accord with string theory. Thus it must be clear that the IIB matrix model is definitely not the first quantized theory of a D-string, but the full second quantized theory. Arbitrary numbers of D-objects exist as local excitations of space-time in this formulation, which is certainly not the case if we deal with the effective theory of a fixed number of D-objects.

It has also been proposed that the Wilson loops are the creation and annihilation operators for strings. We consider the following regularized Wilson loops: [2]

$$\begin{aligned} w(C) &= \mathrm{Tr}[v(C)], \\ v(C) &= \prod_{n=1}^{M} U_n, \\ U_n &= \exp\{i\epsilon(k_n^\mu A_\mu + \bar{\lambda}_n \psi)\}. \end{aligned} \tag{1·8}$$

Here k_n^μ are momentum densities distributed along a loop C, and we have also introduced the fermionic sources λ_n. The parameter ϵ in the argument of the exponential is a cutoff factor. As has been shown, ϵ can be regarded as a lattice spacing of the worldsheet. We therefore call $\epsilon \to 0$ limit the continuum limit. This model has been further investigated through the loop equations which govern the Wilson loops. Such investigations have shown that the theory has no infrared divergences, and the string perturbation theory can be reproduced in the double scaling limit.

It is possible to see the infrared finiteness of the theory in a more direct way. We may expand the fields around the vacuum configurations $A_\mu^{ij} = x_\mu^i \delta^{ij}$. While the off-diagonal elements of the matrices possess nonvanishing propagators, the diagonal components do not possess the propagators and may be called the "zeromodes". We denote the bosonic zeromodes as x_μ^i and the fermionic zeromodes as ξ_α^i, where i denotes the color indices.

The perturbative expansion around this background at first sight appears to resemble that of the original large N reduced models. [7] If this is the case, the perturbative expansion around the classical vacuum is identical to the ten dimensional super Yang-Mills theory in the large N limit. The ultraviolet divergences of ten dimensional super Yang-Mills theory appear as the infrared divergences in this model, and hence the theory appears to be ill-defined. However, the existence of the fermionic zeromodes turns out to modify such a picture completely, as is shown in this paper.

The possibly dangerous configurations are such that the eigenvalues x_μ^i are widely separated compared to the scale set by g^2. It is clear that the mass scale of the off-diagonal elements and that of the zeromodes are also widely separated in such a region. Therefore we can adopt a Born-Oppenheimer type approximation, and we can integrate the off-diagonal elements first. Here the expansion parameter is g^2 over the fourth power of the average distance between the eigenvalues. Therefore the one-loop approximation gives an accurate result for widely separated eigenvalues. In this way we can obtain the effective Lagrangian for the zeromodes.

We can subsequently perform the integration over the fermionic zeromodes. The resulting effective action for the bosonic zeromodes turns out to be completely infrared finite for finite N. Since the theory is manifestly ultraviolet finite, we have established the remarkable fact that this model is a finite theory which is free from

both short distance and long distance divergences.

The effective action provides us a useful tool to study the structures of space-time. In §3, we study it to determine the distributions of space-time coordinates. We believe that we have entered a new paradigm in string theory where we are beginning to determine the structure of space-time dynamically. The readily recognizable structure which emerges from our investigation is in fact a four dimensional object, albeit a fractal. We find that space-time points form branched polymers in ten dimensions within the simplest approximation. Obviously, we need some mechanism to flatten this into four dimensions to describe our space-time. Although our investigation to find such a mechanism in the IIB matrix model is still in progress, we explicitly propose possible mechanisms.

We remark that models related to ours have been proposed and studied in Refs. 8)~21). We also note a deep connection between our approach and the noncommutative geometry.[22] This paper consists of four sections. In the first introductory section, we review the conceptual framework of our approach. In §2, we derive the long distance effective action for the zeromodes (super coordinates). In §3, we study the effective action to understand the possible structure of space-time. Section 4 is devoted to conclusions and the discussion. One of the crucial issues in our approach is how to take the large N limit. We address this question in the concluding section.

§2. Effective theory for diagonal elements

In this section, we discuss an effective theory for the diagonal elements of the IIB matrix model. As explained in the Introduction, the theory classically possesses huge moduli, and the diagonal elements of A_μ may assume any values. However, this classical degeneracy is lifted quantum mechanically, and there remains no moduli in the theory, as we show in this section. Let us consider the expansion around the most generic classical moduli, where the gauge group $SU(N)$ is completely broken down to $U(1)^{N-1}$. Then the diagonal elements of A_μ and ψ appear as the zeromodes, while the off-diagonal elements become massive. So we may integrate out the massive modes first and obtain the effective action for the diagonal elements.

We thus decompose A_μ into a diagonal part X_μ and an off-diagonal part $\tilde{A}_\mu$. We also decompose ψ into ξ and $\tilde{\psi}$:

$$A_\mu = X_\mu + \tilde{A}_\mu; \quad X_\mu = \begin{pmatrix} x_\mu^1 & & & \\ & x_\mu^2 & & \\ & & \ddots & \\ & & & x_\mu^N \end{pmatrix},$$

$$\psi = \xi + \tilde{\psi}; \quad \xi = \begin{pmatrix} \xi^1 & & & \\ & \xi^2 & & \\ & & \ddots & \\ & & & \xi^N \end{pmatrix}, \tag{2·1}$$

where x_μ^i and ξ_α^i satisfy the constraints $\sum_{i=1}^N x_\mu^i = 0$ and $\sum_{i=1}^N \xi_\alpha^i = 0$, respectively,

since the gauge group is $SU(N)$. We then integrate out the off-diagonal parts $\tilde{A}_\mu$ and $\tilde{\psi}$ and obtain the effective action for supercoordinates of space-time $S_{\text{eff}}[X, \xi]$. The effective action for space-time coordinates $S_{\text{eff}}[X]$ can be obtained by further integrating out ξ:

$$\begin{aligned} \int dA d\psi e^{-S[A,\psi]} &= \int dX d\xi e^{-S_{\text{eff}}[X,\xi]} \\ &= \int dX e^{-S_{\text{eff}}[X]}, \end{aligned} \tag{2·2}$$

where dX and $d\xi$ represent $\prod_{i=1}^{N-1}\prod_{\mu=0}^{9} dx_\mu^i$ and $\prod_{i=1}^{N-1}\prod_{\alpha=1}^{16} d\xi_\alpha^i$, respectively.

2.1. *Perturbative evaluation of $S_{\text{eff}}[X, \xi]$*

In this subsection, we integrate out the off-diagonal elements $\tilde{A}_\mu$ and $\tilde{\psi}$ by perturbative expansion in g^2. As we see below, this expansion is valid when all of the diagonal elements are widely separated from one another. Since X_μ and ξ do not have propagators, we treat them as collective coordinates.

The original action (1·1) can be expanded as

$$S = S_2 + S_{\text{int}}, \tag{2·3}$$

$$\begin{aligned} S_2 = \frac{1}{2g^2}\text{Tr}\Big(&-[X_\mu, \tilde{A}_\nu][X^\mu, \tilde{A}^\nu] + [X_\mu, \tilde{A}_\nu][X^\nu, \tilde{A}^\mu] \\ &-\bar{\tilde{\psi}}\Gamma^\mu[X_\mu, \tilde{\psi}] - [\bar{\xi}, \tilde{A}_\mu]\Gamma^\mu\tilde{\psi} - \bar{\tilde{\psi}}\Gamma^\mu[\tilde{A}_\mu, \xi]\Big), \end{aligned} \tag{2·4}$$

$$S_{\text{int}} = \frac{1}{2g^2}\text{Tr}\Big(-2[X_\mu, \tilde{A}_\nu][\tilde{A}^\mu, \tilde{A}^\nu] - \frac{1}{2}[\tilde{A}_\mu, \tilde{A}_\nu][\tilde{A}^\mu, \tilde{A}^\nu] - \bar{\tilde{\psi}}\Gamma^\mu[\tilde{A}_\mu, \tilde{\psi}]\Big). \tag{2·5}$$

We observe that ξ has no propagator, since the quadratic term of ξ is absent. As for the gauge fixing, we adopt the following covariant gauge:*)

$$\begin{aligned} S_{\text{gf}} &= -\frac{1}{2g^2}\text{Tr}([X_\mu, A^\mu]^2), \\ S_{\text{FP}} &= -\frac{1}{g^2}\text{Tr}([X_\mu, b][A^\mu, c]), \end{aligned} \tag{2·6}$$

where b and c are the Faddeev-Popov ghost fields.

In terms of the components, $S_2 + S_{\text{gf}}$ can be written as

$$\begin{aligned} S_2 + S_{\text{gf}} = \frac{1}{2g^2}\sum_{i\neq j}\Big(&(x_\nu^i - x_\nu^j)^2\tilde{A}_\mu^{ji}\tilde{A}^{ij\mu} - \bar{\tilde{\psi}}^{ji}\Gamma^\mu(x_\mu^i - x_\mu^j)\tilde{\psi}^{ij} \\ &+(\bar{\xi}^i - \bar{\xi}^j)\Gamma^\mu\tilde{\psi}^{ij}\tilde{A}_\mu^{ji} + \bar{\tilde{\psi}}^{ji}\Gamma^\mu(\xi^i - \xi^j)\tilde{A}_\mu^{ij}\Big). \end{aligned} \tag{2·7}$$

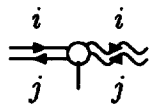
The first and the second terms are the kinetic terms for $\tilde{A}$ and $\tilde{\psi}$ respectively, while the last two terms are $\tilde{A}\tilde{\psi}\xi$ vertices. The basic building blocks of the Feynman rules

*) Actually there is a subtlety for the ghost zeromodes in the covariant gauge. However, the same one-loop effective action can be obtained without such subtlety in the light cone gauge, where A_+ is diagonalized.

are:

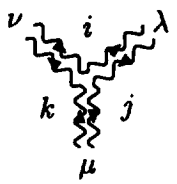
$$\langle \tilde{A}_\mu^{ij} \tilde{A}_\nu^{ji} \rangle \equiv$$  $$= \frac{\eta_{\mu\nu}}{(x^i - x^j)^2}, \tag{2·8}$$

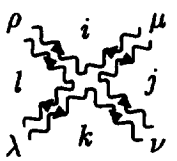
$$\langle \tilde{\psi}^{ij} \bar{\tilde{\psi}}^{ji} \rangle \equiv$$  $$= -\frac{1}{(x^i - x^j)_\mu \Gamma^\mu}, \tag{2·9}$$

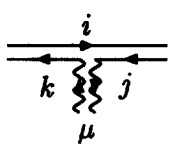
$$\tilde{A}\tilde{\psi}\xi \text{ vertex} \equiv$$  $$= \Gamma^\mu (\xi^i - \xi^j). \tag{2·10}$$

As $\tilde{A}$ and $\tilde{\psi}$ are matrices, the propagators are denoted by double lines with indices i and j.

The interaction terms in Eq. (2·5) are denoted by the following vertices:

boson three-point vertex $\equiv$  , (2·11)

boson four-point vertex $\equiv$  , (2·12)

boson fermion vertex $\equiv$  . (2·13)

One can see from the above Feynman rules that the expansion parameter after the ξ integration is g^2 over the fourth power of the average distances between the space-time coordinates. Therefore, the perturbative expansion is valid when all of the diagonal elements are widely separated from one another. In particular, one-loop approximation is valid in these regions.

A novel feature of the above Feynman rules compared with those of the gauge theory in the large N limit is the presence of ξ insertion vertices (2·10). If we were to set $\xi = 0$, the contributions from $\tilde{A}$ and $\tilde{\psi}$ along from the ghosts would cancel at the one-loop level:

$$\int d\tilde{A} d\tilde{\psi} dbdc \, e^{-(S_2 + S_{\text{gf}} + S_{\text{FP}})} = \prod_{i<j} (x^i - x^j)^{2(-10+2+8)} = 1. \tag{2·14}$$

Thus the diagonal elements would take any value and the classical moduli would remain at least perturbatively. The theory is identical to the large N limit of the super Yang-Mills theory if we identify X with momenta.[7] Therefore the major

difference between the IIB matrix model and super Yang-Mills theory is the presence of dynamical zeromodes X and ξ in the former. Although the zeromodes are order N quantities, we cannot ignore them even in the large N limit, as we show below.

Equation (2·7) shows that the one-loop effective action can be expressed by a super determinant. It turns out to be most useful to integrate out $\tilde{\psi}$ first. Such an integration gives rise to the following quadratic term of $\tilde{A}_\mu$:

$$\frac{1}{g^2}\sum_{i<j}(\bar{\xi}^i-\bar{\xi}^j)\Gamma^{\mu\alpha\nu}(\xi^i-\xi^j)\frac{(x^i_\alpha-x^j_\alpha)}{(x^i-x^j)^2}\tilde{A}^{ji}_\mu\tilde{A}^{ij}_\nu, \tag{2·15}$$

where $\Gamma^{\mu\alpha\nu}=\frac{1}{3!}\Gamma^{[\mu}\Gamma^{\alpha}\Gamma^{\nu]}$. Here the indices within the bracket [ ] are totally anti-symmetrized. In this way we obtain the following one-loop effective action for the zeromodes:

$$\begin{aligned}\int d\tilde{A}d\tilde{\psi}dbdc\, e^{-(S_2+S_{\rm gf}+S_{\rm FP})} &= \prod_{i<j}\det{}_{\mu\nu}\Big(\eta^{\mu\nu}+S^{\mu\nu}_{(ij)}\Big)^{-1}\\ &\equiv e^{-S^{\text{1-loop}}_{\rm eff}[X,\xi]},\end{aligned} \tag{2·16}$$

where

$$S^{\mu\nu}_{(ij)}=(\bar{\xi}^i-\bar{\xi}^j)\Gamma^{\mu\alpha\nu}(\xi^i-\xi^j)\frac{(x^i_\alpha-x^j_\alpha)}{(x^i-x^j)^4}. \tag{2·17}$$

The effective action can be expanded as

$$\begin{aligned}S^{\text{1-loop}}_{\rm eff}[X,\xi] &= \sum_{i<j}\mathrm{tr}\ln(\eta^{\mu\nu}+S^{\mu\nu}_{(ij)})\\ &= -\sum_{i<j}\mathrm{tr}\Big(\frac{S^2_{(ij)}}{2}+\frac{S^4_{(ij)}}{4}+\frac{S^6_{(ij)}}{6}+\frac{S^8_{(ij)}}{8}\Big).\end{aligned} \tag{2·18}$$

Here the symbol tr in the lower case stands for the trace for Lorentz indices. Note that the expansion terminates with the eighth power of $S^{\mu\nu}_{(ij)}$, since ξ has 16 spinor components and $S^{\mu\nu}_{(ij)}$ is bilinear in ξ. Note also that only even powered terms remain, since $S^{\mu\nu}_{(ij)}$ is anti-symmetric under the interchange of μ and ν. Furthermore, the quadratic and sextic terms vanish identically, as shown in Appendix A. Therefore we obtain

$$S^{\text{1-loop}}_{\rm eff}[X,\xi]=-\sum_{i<j}\mathrm{tr}\Big(\frac{S^4_{(ij)}}{4}+\frac{S^8_{(ij)}}{8}\Big). \tag{2·19}$$

We remark that Eq. (2·19) is gauge independent. This is because the longitudinal part of the bosonic propagator which is sensitive to the gauge parameter yields no contribution, since

$$(x^i_\mu-x^j_\mu)S^{\mu\nu}_{(ij)}=(\bar{\xi}^i-\bar{\xi}^j)\Gamma^{\mu\alpha\nu}(\xi^i-\xi^j)\frac{(x^i_\mu-x^j_\mu)(x^i_\alpha-x^j_\alpha)}{(x^i-x^j)^4}=0. \tag{2·20}$$

It is also possible to construct similar reduced models for super Yang-Mills theory in D=3, 4 and 6 by the large N reduction procedure. Then the calculation we have

explained in this subsection is also applicable to these cases, and we find

$$S_{\text{eff}}^{\text{1-loop}}[X,\xi] = \begin{cases} 0 & \text{for } D=3 \\ -\sum_{i<j} \text{tr}\left(\frac{S_{(ij)}^2}{2}\right) & \text{for } D=4 \\ -\sum_{i<j} \text{tr}\left(\frac{S_{(ij)}^2}{2} + \frac{S_{(ij)}^4}{4}\right) & \text{for } D=6 \end{cases} . \tag{2·21}$$

Before we proceed to perform ξ integration and to obtain the effective action for X, we discuss $\mathcal{N}=2$ supersymmetry of the effective action in the next subsection.

2.2. $\mathcal{N}=2$ *SUSY*

In this subsection, we show that the effective action (2·19) has the following $\mathcal{N}=2$ supersymmetry:

$$\begin{cases} \delta^{(1)} x_\mu^i &= i\bar{\epsilon}_1 \Gamma_\mu \xi^i \\ \delta^{(1)} \xi^i &= 0 \end{cases} , \tag{2·22}$$

$$\begin{cases} \delta^{(2)} x_\mu^i &= 0 \\ \delta^{(2)} \xi^i &= \epsilon_2 \end{cases} . \tag{2·23}$$

These symmetries are the remnants of the symmetry in the original theory:

$$\begin{cases} \delta^{(1)} A_\mu &= i\bar{\epsilon}_1 \Gamma_\mu \psi \\ \delta^{(1)} \psi &= \frac{i}{2} \Gamma^{\mu\nu} [A^\mu, A^\nu] \epsilon_1 \end{cases} , \tag{2·24}$$

$$\begin{cases} \delta^{(2)} A_\mu &= 0 \\ \delta^{(2)} \psi &= \epsilon_2 \end{cases} . \tag{2·25}$$

First let us decompose the variables into diagonal and off-diagonal parts in the original transformations (2·24) and (2·25). We may consider $\tilde{A}_\mu$ and $\tilde{\psi}$ as quantities of order $\hbar$. Then one can expand these transformations in terms of $\hbar$. However, the action is invariant under these transformations at each order of $\hbar$. Thus S_2 (2·4) is invariant under the transformations which are linear in $\tilde{A}_\mu$ and $\tilde{\psi}$,

$$\begin{cases} \delta^{(1)} x_\mu^i &= i\bar{\epsilon}_1 \Gamma_\mu \xi^i \\ \delta^{(1)} \tilde{A}_\mu^{ij} &= i\bar{\epsilon}_1 \Gamma_\mu \tilde{\psi}^{ij} \\ \delta^{(1)} \xi^i &= 0 \\ \delta^{(1)} \tilde{\psi}^{ij} &= i(x^i - x^j)_\mu \tilde{A}_\nu^{ij} \Gamma^{\mu\nu} \epsilon_1 \end{cases} , \quad \begin{cases} \delta^{(2)} x_\mu^i &= 0 \\ \delta^{(2)} \tilde{A}_\mu^{ij} &= 0 \\ \delta^{(2)} \xi^i &= \epsilon_2 \\ \delta^{(2)} \tilde{\psi}^{ij} &= 0 \end{cases} . \tag{2·26}$$

After integration over $\tilde{A}$ and $\tilde{\psi}$, the remaining effective action (2·19) shall have the symmetries (2·22) and (2·23).

One can also show the invariance of (2·19) under the transformations (2·22) and (2·23) through explicit calculations. Since $S_{(ij)}^{\mu\nu}$ contains ξ^i only through the combination $\xi^i - \xi^j$, one can see that invariance under (2·23) is satisfied rather trivially. As for (2·22), it introduces an additional ξ^i. The quantity $\text{tr}(S_{(ij)})^8$ is

invariant under (2·22), since it already contains 16 ξ. Some calculations are required to exhibit the invariance of the term $\mathrm{tr}(S_{(ij)})^4$:

$$\mathrm{tr}\delta^{(1)}S_{(ij)}S_{(ij)}^3 = i\frac{1}{(x^4)^4}x_\lambda x_\rho x_\sigma U_\alpha{}^{\alpha\lambda\rho\sigma} - 4i\frac{1}{(x^4)^4}\frac{1}{x^2}x_\mu x_\nu x_\lambda x_\rho x_\sigma U^{\mu\nu\lambda\rho\sigma}, \quad (2\cdot27)$$

where $U^{\mu\nu\lambda\rho\sigma} = (\bar{\epsilon}_1\Gamma^\mu\xi)(\bar{\xi}\Gamma_\alpha{}^{\nu\beta}\xi)(\bar{\xi}\Gamma_\beta{}^{\lambda\gamma}\xi)(\bar{\xi}\Gamma_\gamma{}^{\rho\delta}\xi)(\bar{\xi}\Gamma_\delta{}^{\sigma\alpha}\xi)$ is a tensor which is totally symmetric and traceless with respect to the indices $\nu\lambda\rho\sigma$, as is explained in Appendix A. Here x_μ and ξ denote $x_\mu^i - x_\mu^j$ and $\xi^i - \xi^j$, respectively. We may further decompose $U^{\mu\nu\lambda\rho\sigma}$ into irreducible components as

$$U^{\mu\nu\lambda\rho\sigma}_{\text{sym traceless}} = \text{symmetrization of } U^{\mu\nu\lambda\rho\sigma} - \frac{1}{40}(g^{\mu\nu}U_\alpha{}^{\alpha\lambda\rho\sigma} + 9 \text{ terms}). \quad (2\cdot28)$$

And we can show that $U^{\mu\nu\lambda\rho\sigma}_{\text{sym traceless}} = 0$ by decomposing $SO(10) \supset SO(2) \times SO(8)$ where $\xi = (s_a, c_{\dot{a}}), \epsilon = (b_a, t_{\dot{a}})$. It is then sufficient to demonstrate that $U^{+++++}_{\text{sym traceless}} = s^9 b = 0$. Therefore the following cancellation implies the invariance of $\mathrm{tr}(S_{(ij)})^4$:

$$\mathrm{tr}\delta^{(1)}S_{(ij)}S_{(ij)}^3 = i\frac{1}{(x^4)^4}\left(1 - 4\frac{10}{40}\right)x_\lambda x_\rho x_\sigma U_\alpha{}^{\alpha\lambda\rho\sigma} = 0. \quad (2\cdot29)$$

This completes the proof of the invariance of (2·19) under (2·22) and (2·23).

2.3. *Dynamics of X at long distances*

In this subsection, we study the integration procedure of ξ to obtain an effective theory for the space-time coordinates X:

$$\int dX d\xi\, e^{-S_{\text{eff}}[X,\xi]} = \int dX\, e^{-S_{\text{eff}}[X]}. \quad (2\cdot30)$$

We perform ξ integrations explicitly at the one-loop level in this subsection and explain that they can be represented graphically in terms of trees which connect the space-time points. Although the vacuum energy of order N^2 has been canceled between the bosonic and fermionic degrees of freedom, it does not vanish at order N in the IIB matrix model, since $S_{\text{eff}}[X]$ is shown to be nontrivial. Through the perturbative evaluation of $S_{\text{eff}}[X]$ to all orders, we prove the finiteness of the X integration in Eq. (2·30) for finite N.

2.3.1. One-loop evaluation of $S_{\text{eff}}[X]$

We substitute the one-loop effective action $S_{\text{eff}}^{\text{1-loop}}[X,\xi]$ of Eq. (2·19) into Eq. (2·30) and consider the type of terms that survive after ξ integrations:

$$\int dX d\xi e^{-S_{\text{eff}}^{\text{1-loop}}[x,\xi]} = \int dX d\xi \prod_{i<j}\left[1 + \frac{\mathrm{tr}(S_{(ij)}^4)}{4} + \left(\frac{1}{2}\left(\frac{\mathrm{tr}(S_{(ij)}^4)}{4}\right)^2 + \frac{\mathrm{tr}(S_{(ij)}^8)}{8}\right)\right]. \quad (2\cdot31)$$

Here the products are taken over all possible different pairs of color indices (ij). When we expand the multi-products, we select one of the three different factors, 1,

$\mathrm{tr}(S^4_{(ij)})/4$ or $(\mathrm{tr}(S^8_{(ij)})/8 + (\mathrm{tr}(S^4_{(ij)}))^2/32)$ for each pair of (ij). Since the last two factors are functions of $(x^i_\mu - x^j_\mu)$, they can be visualized by bonds that connect the "space-time points" x^i_μ and x^j_μ. More precisely, in order to remind us that the factors $\mathrm{tr}(S^4_{(ij)})/4$ and $(\mathrm{tr}(S^8_{(ij)})/8 + (\mathrm{tr}(S^4_{(ij)}))^2/32)$ contain 8 and 16 spinor components of $\xi^i_\alpha - \xi^j_\alpha$, we draw 8 or 16 bonds between x^i_μ and x^j_μ, depending on whether we take $\mathrm{tr}(S^4_{(ij)})/4$ or $(\mathrm{tr}(S^8_{(ij)})/8 + (\mathrm{tr}(S^4_{(ij)}))^2/32)$. That is, each bond corresponds to each component of a spinor $\xi^i_\alpha - \xi^j_\alpha$. We call these sets of 8 and 16 bonds "8-fold bond" and "16-fold bond", respectively. In this way we can associate each term in the expansion of multi-products in Eq. (2·31) with a graph connecting the space-time points by 8-fold bonds or 16-fold bonds. We do not assign any bond to the factor 1. Therefore the multi-products in Eq. (2·31) can be replaced by a summation over all possible graphs.

Let us consider the spinor ξ^i_α, component by component, in order to discuss what kind of graphs survive after ξ integrations. Out of the 16 spinor components of ξ^i_α, we may focus on a particular spinor component, such as the first component ξ^i_1. We rewrite Eq. (2·31) as

$$\int dX \int \prod_{i=1}^{N-1} d\xi^i_2 \cdots d\xi^i_{16} \prod_{i=1}^{N-1} d\xi^i_1 \prod_{i<j} (C_0 + C_1 \cdot (\xi^i_1 - \xi^j_1)). \tag{2·32}$$

Here C_0 and C_1 are functions of $x^i_\mu - x^j_\mu$ and the other spinor components, $\xi^i_2 - \xi^j_2, \cdots, \xi^i_{16} - \xi^j_{16}$. We can then expand the multi-products, regarding the factor $C_1 \cdot (\xi^i_1 - \xi^j_1)$ as a single bond. Then it is clear from the following considerations that the integrations of ξ^i_1 for all color indices i generate subgraphs associated with the first component of spinors, which connect all the N points without a loop. Such graphs are called "maximal trees".

i) Since ξ^i_1 has $N-1$ independent color components, the subgraphs having $N-1$ bonds remain after ξ^i_1 integration.

ii) If there is a loop in the subgraph, the contribution vanishes, since a product of delta functions of grassmann variables on the loop vanishes:

$$\delta(\xi^{i_1}_1 - \xi^{i_2}_1)\delta(\xi^{i_2}_1 - \xi^{i_3}_1) \cdots \delta(\xi^{i_{(k-1)}}_1 - \xi^{i_k}_1)\delta(\xi^{i_k}_1 - \xi^{i_1}_1) = 0. \tag{2·33}$$

We also note that all maximal trees contribute equally as we can demonstrate by performing ξ^i_1 integrations from the end points of the maximal trees.

Therefore the integrations of the fermionic degrees of freedom generate graphs which are superpositions of 16 maximal trees. However, these 16 maximal trees are not independent. At each bond they are constrained to be bunched into a set of 8 or 16. From these considerations, we find a graphical representation of the one-loop effective action:

$$\int dX d\xi e^{-S^{\text{1-loop}}_{\text{eff}}[x,\xi]} = \int dX d\xi \sum_{G:\text{graph}} \prod_{(ij):\text{bond of } G} \times \left[\left(\frac{\mathrm{tr}(S^4_{(ij)})}{4} \right) \text{ or } \left(\frac{1}{2} \left(\frac{\mathrm{tr}(S^4_{(ij)})}{4} \right)^2 + \frac{\mathrm{tr}(S^8_{(ij)})}{8} \right) \right]. \tag{2·34}$$

Here we sum over all possible graphs consisting of 8-fold and 16-fold bonds which can be expressed as superpositions of 16 maximal trees. For each bond (ij) of G, we assign the first or the second factor, depending on whether it is an 8-fold or 16-fold bond.

Since each 16-fold bond contains 16 spinors, $(\xi^i - \xi^j)$ and $(x^i - x^j)$ must form Lorentz singlets by themselves:

$$\left(\frac{1}{2}\left(\frac{\mathrm{tr}(S^4_{(ij)})}{4}\right)^2 + \frac{\mathrm{tr}(S^8_{(ij)})}{8}\right) \sim \delta^{(16)}(\xi^i - \xi^j)\,\frac{1}{(x^i - x^j)^{24}}. \tag{2·35}$$

In an 8-fold bond, however, $(\xi^i - \xi^j)$ and $(x^i - x^j)$ couple as

$$\mathrm{tr}(S^4_{(ij)}) = T^{\mu\nu\lambda\rho}(x^i - x^j)_\mu (x^i - x^j)_\nu (x^i - x^j)_\lambda (x^i - x^j)_\rho / (x^i - x^j)^{16}. \tag{2·36}$$

Here

$$\begin{aligned} T^{\mu\nu\lambda\rho} = {} & [(\bar{\xi}^i - \bar{\xi}^j)\Gamma^{\mu\alpha}{}_\beta(\xi^i - \xi^j)][(\bar{\xi}^i - \bar{\xi}^j)\Gamma^{\nu\beta}{}_\gamma(\xi^i - \xi^j)] \\ & \times[(\bar{\xi}^i - \bar{\xi}^j)\Gamma^{\lambda\gamma}{}_\delta(\xi^i - \xi^j)][(\bar{\xi}^i - \bar{\xi}^j)\Gamma^{\rho\delta}{}_\alpha(\xi^i - \xi^j)] \end{aligned} \tag{2·37}$$

is a totally symmetric traceless tensor, as shown in Appendix A.

If there were only 16-fold bonds, considerable simplifications take place, since the graphs are reduced to maximal trees with 16-fold degeneracy, and the interactions between the points only depend on their distances. We may cite the four dimensional model in Eq. (2·21) as a simple model which shares such a property. The effective action of the four dimensional model can be represented as

$$\begin{aligned} \int dX d\xi e^{-S^{\text{1-loop}}_{\text{eff}}[x,\xi]} &= \int dX d\xi \prod_{i<j}\left[1 + \frac{\mathrm{tr}(S^2_{(ij)})}{2}\right] \\ &= \int dX d\xi \sum_{G:\text{graph}} \prod_{(ij):\text{bond of } G} \frac{\mathrm{tr}(S^2_{(ij)})}{2}, \end{aligned} \tag{2·38}$$

where we sum over all graphs whose bonds form maximal trees. Since ξ has four spinor components in four dimensions, we have

$$\mathrm{tr}(S^2_{(ij)}) \sim \delta^{(4)}(\xi^i - \xi^j)\,\frac{1}{(x^i - x^j)^6}. \tag{2·39}$$

Therefore, in the four-dimensional model, the distribution of X becomes of a "branched polymer" type:

$$\int dX e^{-S^{\text{1-loop}}_{\text{eff}}[X]} = \int dX \sum_{G:\text{maximal tree}} \prod_{(ij):\text{bond of } G} \frac{1}{(x^i - x^j)^6}. \tag{2·40}$$

Note that all points are connected by the bonds, and each x^{ij} integration can be performed independently and converges for large x^{ij} on each bond, where $x^{ij} = x^i - x^j$. Thus this system is infrared convergent. Although the integrations seem

to be divergent for short distances, this is due to the failure of the one-loop approximation, since the theory itself is manifestly finite at short distances. We consider the short distance behavior of the theory in subsection 2.4. As we see in Appendix C, the dynamics of branched polymers are well-known, and its Hausdorff dimension is four. Therefore this model constitutes an example of models for dynamical generation of space-time which predicts four-dimensional space-time. As shown in the subsequent discussion, the IIB matrix model is much more complicated. Nevertheless, we expect that the structure of space-time is also determined dynamically in the IIB matrix model.

2.3.2. Infrared convergence

In what follows we show that the X integral is convergent in the infrared region for finite N to all orders of perturbation theory:

$$\int dX e^{-S_{\text{eff}}[X]} < \infty. \qquad \text{(infrared)} \tag{2·41}$$

This shows that all points are gathered as a single bunch, and hence space-time is inseparable. Indeed we can show that *the multiple integral, Eq. (2·41), is absolutely convergent.*

We use the following theorem to prove the above statement: *An M-dimensional multiple integral is absolutely convergent when the superficial degrees of divergences of sub-integrals on any m-dimensional hyperplanes within $\boldsymbol{R}^M$ are negative.* Let us apply the identical linear transformations T to the variables X and ξ as

$$\begin{pmatrix} y^1 \\ \vdots \\ y^{N-1} \end{pmatrix} = T \begin{pmatrix} x^1 \\ \vdots \\ x^{N-1} \end{pmatrix}, \qquad \begin{pmatrix} \eta^1 \\ \vdots \\ \eta^{N-1} \end{pmatrix} = T \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^{N-1} \end{pmatrix}, \tag{2·42}$$

and consider the case where the values of $y^1 \cdots y^m$ become large, while $y^{m+1} \cdots y^{N-1}$ are fixed. We observe from the Feynman rules in subsection 2.1 that we can associate the factor $(x^i - x^j)^{-\frac{3}{2}}$ with every $\xi^i - \xi^j$ by absorbing half of the powers of the bosonic and the fermionic propagators which are connected to $\xi^i - \xi^j$ in Eq. (2·10). Although the X dependences of the diagrams are not entirely absorbed by this procedure at higher orders, the remaining factors make the infrared convergence properties even better. We explain this point using concrete examples at the end of this subsection.

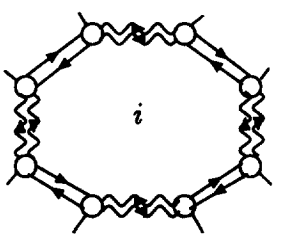
We can reexpress $(x^i - x^j)^{-\frac{3}{2}}$ and $\xi^i - \xi^j$ in terms of y and η through identical linear transformations as $(\sum_k C_k^{ij} y^k)^{-\frac{3}{2}}$ and $\sum_k C_k^{ij} \eta^k$, respectively. The perturbative expansion of the effective action consists of many terms which contain fermionic variables through the combinations of $\xi^i - \xi^j$. Let us consider the contribution from a term which contains η_1^1. If it is associated with the bond (ij), it means that $C_1^{ij} \neq 0$, and we obtain the factor $(C_1^{ij} y^1 + \cdots)^{-\frac{3}{2}}$ after the integration of η_1^1. We obtain analogous factors from integrations over all spinor components of fermionic variables

up to η^m. Thus we can conclude that

$$| e^{-S_{\text{eff}}[y]} | < \frac{1}{\prod_{i=1}^{m} \prod_{\alpha=1}^{16} P_\alpha^i}, \tag{2·43}$$

where the P_α^i are homogeneous functions of degree 3/2 of $y^1 \cdots y^m$. Hence the superficial degree of divergence is negative on this $10m$-dimensional hyperplane. Therefore the superficial degrees of divergences are negative on any hyperplanes in $\boldsymbol{R}^{10(N-1)}$. This completes the proof that the theory is infrared finite for finite N to all orders. Since the theory is manifestly finite at short distances, it also establishes that the IIB matrix model is a finite theory for finite N.

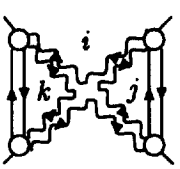
Finally, we give concrete examples which illustrate the fact that the factor $(x^i - x^j)^{-3/2}$ can be associated with every $\xi^i - \xi^j$. At the one-loop level, the diagrams are made of the bosonic propagators (2·8), the fermionic propagators (2·9) and the ξ insertion vertex (2·10). The Feynman diagrams contain them consecutively along the loop as depicted in the following figure:



$$= \text{tr} S_{(ij)}^4. \tag{2·44}$$

Therefore we can diagrammatically understand that we can precisely assign the factor $(x^i - x^j)^{-3/2}$ to $(\xi^i - \xi^j)$ in the one-loop effective action.

Since the expansion parameter is g^2 over the fourth power of the average distances between the points, the infrared convergence is expected to become better at higher orders. In fact, at higher loop level, there remain some vertices with nontrivial X dependences which make the infrared convergence even better after absorbing half of the powers of the bosonic and the fermionic propagators connected to every ξ. For example,



(2·45)

is proportional to $(\xi^{ij})^2/(x^{ij})^3 \cdot (\xi^{ik})^2/(x^{ik})^3 \cdot g^2/((x^{ij})^2(x^{ik})^2)$, where the third factor is associated with the four gluon vertex at the center of this diagram and indeed improves the infrared convergence. Here we have introduced the notation $x_\mu^{ij} = x_\mu^i - x_\mu^j$ and $\xi_\alpha^{ij} = \xi_\alpha^i - \xi_\alpha^j$.

2.4. *Short distances*

Until now we have considered the effective action for the diagonal elements which is valid when all of the diagonal elements are well separated. This is analogous to

the most generic moduli space in the super Yang-Mills theory, although precisely speaking there is no moduli in the IIB matrix model. It is natural to next consider the second most generic moduli space. Let us suppose that a pair of bosonic coordinates are degenerate, but the rest of the coordinates are well separated from one another and from the center of mass coordinates of the pair.

We then decompose the $N \times N$ matrix valued variables into 2×2 submatrices and the rest. The rest consist of Hermitian $(N-2) \times (N-2)$ matrices and complex $2 \times (N-2)$ matrices. Let us further decompose the 2×2 submatrices into the center of mass part and the remaining $SU(2)$ degrees of freedom. Since the off-diagonal elements of $(N-2) \times (N-2)$ matrices and the complex $2 \times (N-2)$ matrices are massive, we can integrate them out as before. We write the remaining degrees of freedom as

$$A_\mu \ni \begin{pmatrix} A_\mu^{U(2)} & & & \\ & x_\mu^3 & & \\ & & \ddots & \\ & & & x_\mu^N \end{pmatrix}, \qquad A_\mu^{U(2)} = x_\mu \mathbf{1} + A_\mu^{SU(2)}, \tag{2·46}$$

$$\psi \ni \begin{pmatrix} \psi^{U(2)} & & & \\ & \xi^3 & & \\ & & \ddots & \\ & & & \xi^N \end{pmatrix}, \qquad \psi^{U(2)} = \xi \mathbf{1} + \psi^{SU(2)}. \tag{2·47}$$

The resulting effective action can be written as

$$\begin{aligned} S_{\text{eff}}[A_\mu^{SU(2)}, \psi^{SU(2)}; x_\mu, \xi; x_\mu^3, \cdots, x_\mu^N, \xi^3, \cdots, \xi^N] \\ = S[A_\mu^{SU(2)}, \psi^{SU(2)}] + S'_{\text{eff}}[x_\mu, \xi; x_\mu^3, \cdots, x_\mu^N, \xi^3, \cdots, \xi^N] \\ + S_{\text{int}}[A_\mu^{SU(2)}, \psi^{SU(2)}; x_\mu, \xi; x_\mu^3, \cdots, x_\mu^N, \xi^3, \cdots, \xi^N]. \end{aligned} \tag{2·48}$$

The first term is the $SU(2)$ part of the original action. The second term is essentially identical to the $SU(N-1)$ case we have discussed in this section. The center of mass coordinates of the degenerate pair play the role of $(N-1)$-th coordinates. The only difference is that the strength of the potential between the center of mass coordinates and the others has doubled. The third term denotes the interaction between $SU(2)$ part and the diagonal elements of the $(N-2) \times (N-2)$ matrix. We neglect this since it is small compared to the first term in many cases, as is discussed in the concluding section.

We still need to consider the $SU(2)$ part to determine the dynamics of the relative coordinates of the pair of points. Here we can cite the exact solution for the $SU(2)$ case. As we show in Appendix B, the distribution for the relative coordinates r_μ is

$$\begin{aligned} &\int d^{10} r f(r), \\ &f(r) \sim \begin{cases} 1/r^{24} & r^2 \gg g \\ r^8 & r^2 \ll g \end{cases}. \end{aligned} \tag{2·49}$$

We conclude that there is a pairwise repulsive potential of $-8 \ln r$ type when two coordinates are close to each other. It is clear that these considerations are valid for arbitrary numbers of degenerate pairs, although the center of mass coordinates should be well separated. Although it is possible to repeat these considerations for cases with higher degeneracy, the analysis becomes more complicated. Therefore we choose to adopt a phenomenological approach and assume the existence of a hardcore repulsive potential of the form

$$S_{\text{core}}[X] = \sum_{i<j} g(x^i - x^j), \tag{2·50}$$

where

$$g(x^i - x^j) = \begin{cases} -4\ln((x^i - x^j)^2/g) & \text{for } (x^i - x^j)^2 \ll g \\ 0 & \text{for } (x^i - x^j)^2 \gg g \end{cases} . \tag{2·51}$$

In the following section, we investigate the structure of space-time by using the one-loop effective action for the diagonal elements plus the phenomenological hardcore potential S_{core}. Although our effective action is not exact at short distances, it presumably captures the essential feature of the IIB matrix model, namely, the incompressibility. If so, our effective action may be in the same universality class as the full IIB matrix model.

§3. Possible scenarios of dynamical generation of space-time

So far we have derived an effective action for the diagonal elements of A_μ and ψ. We have identified the eigenvalues of A_μ with the space-time coordinates X_μ through the semiclassical correspondence, as explained in the Introduction. Let us suppose that the eigenvalue distribution of A_μ is extensive in d dimensions but supported only by finite intervals in the remaining $10 - d$ dimensions. Such a distribution may be interpreted in such a manner that space-time is extensive in d dimensions but has shrunk in the remaining directions. The dimensionality of such space-time is obviously d. If such a distribution is derived from our effective action, we may conclude that the dimensionality of space-time is d.

In the perturbative formulation of string theory, we need to assume a particular background on which strings propagate. Although certain consistency conditions are required for the backgrounds, we have no principle to select a particular background. However, we can in principle determine the possible structure of space-time from the IIB matrix model uniquely. Our effective action provides us an exciting possibility to realize this.

In this section we study the effective action in order to see whether the IIB matrix model can explain the structure of space-time, and its dimensionality in particular. As explained shortly in more detail, our effective action is still rather complicated. One possible way to avoid this is to consider further simplified models which are thought to contain the essential features of our effective action. Such models may

contain several parameters which may not be easily determined. We therefore draw the phase diagrams of these models for all possible parameter regions and identify the structures of space-time in the respective phases.

The aim here is to show the existence of models which can realize four dimensional space-time. We then need to show that the IIB matrix model indeed belongs to the same universality class. Although these investigations are still in progress, we find it very plausible that the IIB matrix model realizes four dimensional flat space-time. We explicitly propose some of possible mechanisms to realize realistic space-time.

3.1. *Models*

We consider the following ensemble in ten dimensions which is controlled by the one-loop effective action (2·19) and the core potential (2·50):

$$\begin{aligned}&\int dX e^{-S_{\text{core}}} d\xi e^{-S_{\text{eff}}^{\text{1-loop}}[X,\xi]} \\ &= \int dX e^{-S_{\text{core}}} d\xi \sum_{G:\text{graph}} \prod_{(ij):\text{bond of } G} \\ &\quad \times \left[\left(\frac{\text{tr}(S_{(ij)}^4)}{4} \right) \text{ or } \left(\frac{1}{2} \left(\frac{\text{tr}(S_{(ij)}^4)}{4} \right)^2 + \frac{\text{tr}(S_{(ij)}^8)}{8} \right) \right]. \end{aligned} \tag{3·1}$$

As explained in §2, we call the bond in association with $\text{tr}(S_{(ij)})^4$ and the bond with $(\text{tr}(S_{(ij)})^4)^2/32 + \text{tr}(S_{(ij)})^8/8$ a 8-fold bond and a 16-fold bond, respectively. The former gives 8 fermionic variables per bond, while the latter gives a product of all 16 components and is proportional to $(x^i - x^j)^{-24}\delta^{(16)}(\xi^i - \xi^j)$.

If only 16-fold bonds appear in a graph, ξ integration can be easily performed. This gives a branched polymer, as we saw in the previous section. The integrations can be performed bond by bond, and the resultant interactions between the x^i are functions of distances only, that is, $(x^i - x^j)^{-24}$ for each bond. Thus we can estimate the partition function as

$$\int dX e^{-S_{\text{core}}} d\xi e^{-S_{\text{eff}}^{\text{1-loop}}[X,\xi]} \longrightarrow \int dX e^{-S_{\text{core}}} \sum_{G:\text{maximal tree}} \prod_{(ij):\text{bond of } G} \frac{1}{(x^i - x^j)^{24}}, \tag{3·2}$$

where the summation of G is over all maximal tree graphs.

When 8-fold bonds appear in the graph, the simple branched polymer picture breaks down. As is seen from Eqs. (2·36) and (2·37), $\text{tr}(S_{(ij)})^4$ can be rewritten as

$$\text{tr}(S_{(ij)})^4 \propto C^{\mu\nu\lambda\rho\alpha_1\ldots\alpha_8} \xi_{\alpha_1}^{ij} \cdots \xi_{\alpha_8}^{ij} V_{\mu\nu\lambda\rho}^{ij} / (x^{ij})^{16}, \tag{3·3}$$

where $\xi_\alpha^{ij} = \xi_\alpha^i - \xi_\alpha^j$, $x_\mu^{ij} = x_\mu^i - x_\mu^j$, and $C^{\mu\nu\lambda\rho\alpha_1\ldots\alpha_8}$ is an invariant tensor. Here $V_{\mu\nu\lambda\rho}^{ij}$ is a fourth rank symmetric traceless tensor constructed from x^{ij}:

$$\begin{aligned} V_{\mu\nu\lambda\rho}^{ij} &= x_\mu^{ij} x_\nu^{ij} x_\lambda^{ij} x_\rho^{ij} - \frac{(x^{ij})^2}{D+4} (x_\mu^{ij} x_\nu^{ij} \delta_{\lambda\rho} + x_\mu^{ij} x_\lambda^{ij} \delta_{\nu\rho} + \cdots) \\ &\quad + \frac{(x^{ij})^4}{(D+2)(D+4)} (\delta_{\mu\nu}\delta_{\lambda\rho} + \delta_{\mu\lambda}\delta_{\nu\rho} + \cdots), \end{aligned} \tag{3·4}$$

where $D = 10$. Therefore, at each 8-fold bond there are many choices to select 8 fermionic components out of 16, and each choice gives a different x_μ^{ij} dependence which is specified by $C^{\mu\nu\lambda\rho\alpha_1\cdots\alpha_8}$. Hence, integration over ξ generally gives complicated interactions, which involve many bonds and depend on relative directions of relevant bonds. An important point here is that the tensor $V_{\mu\nu\lambda\rho}^{ij}$ is traceless and vanishes if we naively take the orientation average for x_μ^{ij}. Therefore it is natural to expect that the appearance of an 8-fold bond is suppressed to some extent if the eigenvalues of A_μ are distributed uniformly in ten dimensions.

As a simpler model which may capture the essential feature of our effective action, we may consider a model where we average the orientation dependence of the 8-fold bonds. Instead of considering the graphs consisting of 8-fold and 16-fold bonds, we introduce two independent maximal trees, each of which maximally connects the N points with coordinate x_μ^i. When the two trees share a bond (ij), we regard it as a 16-fold bond and give the corresponding factor $((x^{ij})^2 + g)^{-12}$. For a bond which is contained in one tree only, we regard it as an 8-fold bond and assign a factor $((x^{ij})^2 + g)^{-6}$, which represents the residual effect of the 8-fold bond after orientation average. Here we have introduced a short distance cutoff of order g in order to avoid the meaningless singularity at $x^{ij} = 0$. We also assume the hardcore repulsive potentials as given by S_{core} at short distances:

$$\int dX e^{-S_{\text{core}}} d\xi e^{-S_{\text{eff}}^{1\text{-loop}}[X,\xi]} \longrightarrow \int dX e^{-S_{\text{core}}} \sum_{G:\text{two maximal trees}} \prod_{(ij):\text{bond of } G} \times [((x^{ij})^2 + g)^{-6} e^{-\lambda/2} \text{ or } ((x^{ij})^2 + g)^{-12}]. \tag{3·5}$$

Here the meaning of the factor $e^{-\lambda/2}$ is as follows. As we have discussed in the previous paragraph, the orientation average suppresses the appearance of 8-fold bonds. Therefore we can account for the effect of angle-dependence at least partially by introducing a suppression factor $e^{-\lambda/2}$ for each 8-fold bond. We call this model a "double tree model".

If the IIB matrix model is really approximated by this model, λ should be in principle fixed, because the IIB matrix model has no free parameter. However, to determine λ precisely may be as difficult as solving the IIB matrix model itself. Therefore we treat λ as a parameter, and examine the phase structure as λ is varied from $-\infty$ to ∞. We show in the next subsection that two apparent phases exist in the two opposite limiting cases, $\lambda \to \infty$ and $\lambda \to -\infty$.

An important factor which is not taken into account in the double tree model is the fact that λ is a sort of mean field and it may strongly depend on the state of the system, that is, the distribution of eigenvalues. Actually, we see in subsection 3.3 that λ decreases when the eigenvalues distribute in lower dimensional space-time, and this effect favors the generation of lower dimensional space-time. By introducing a rough approximation we can show that the free energy is minimized if the space-time is compactified to four dimensions. It appears that more detailed analysis of this effect is indispensable to arrive at our goal.

3.2. *Branched polymer phase and droplet phase*

In this subsection we analyze the double tree model defined by Eq. (3·5). It contains a parameter λ, and we consider two limiting situations. First we take λ sufficiently large. Then most of the bonds on one tree are bound to bonds on the other tree, and the system behaves as an ordinary branched polymer made of 16-fold bonds only. We call this state a branched polymer (BP) state. As reviewed in Appendix C, its partition function is given by

$$Z_{\mathrm{BP}}(N) = N!(\hat{f}_1(0)\alpha_c)^N \sim N!\, e^{-NF_{\mathrm{BP}}},$$
$$\hat{f}_1(0) = \int_{x_c} d^D x \frac{1}{x^{24}}, \tag{3·6}$$

where we have introduced free energy per bond defined by $F_{\mathrm{BP}} = -\ln(\alpha_c \hat{f}_1(0))$. If we neglect the effect of intersections due to the core potential, we can set $\alpha_c = e$. We have also multiplied this by $N!$, since we distinguish N points x^i. The quantity x_c is a short distance cutoff determined by the core potential. Since the Hausdorff dimension of BP is four, we cannot neglect the effect of intersections in dimensions less than eight, and Eq. (3·6) gives an overestimation of the partition function. In other words, α_c is reduced from e. Furthermore, in dimensions less than four, most BPs cannot be packed, and the number of possible graphs drastically decreases. Without any other interactions, other than 16-fold bond interactions, BP favorably expands in ten dimensions, and we cannot obtain a lower dimensional distribution of eigenvalues.

On the other hand if we take λ small, the number of 8-fold bonds increases, and 16-fold bonds totally disappear in the $\lambda \to -\infty$ limit. Actually, even for some positive values of λ, most of the 16-fold bonds disappear, since the entropy increase caused by resolving 16-fold bonds into 8-fold bonds overcompensates the effect of positive values of λ which favors 16-fold bonds. Our numerical simulation supports this fact, as we see below. Then entropy is gained for such configurations for which a maximal number of points gather around one another, so that rearrangements of bonds occur easily. Indeed, without a core potential, all points are condensed into a finite area whose size does not depend on N, as we show at the end of this subsection. If we take the core potential into account, they cannot condense into an infinitely dense state. Instead, all of the cores are packed in an area whose size grows with N. We call this state a droplet state. When a droplet state is realized in d dimensions, the size of the droplet is given by $R = l(N)^{1/d}$, where l is the core size. Since bonds flip locally in a droplet, we regard the droplet as a collection of clusters, in each of which, bonds can freely flip. The cluster is assumed to have n points inside. Then the volume of the cluster is $v = l^d n$, and the typical length of bonds in the cluster can be identified with the radius of the cluster $r = ln^{1/d}$. Therefore each cluster is expected to contribute to the partition function an amount

$$z(n) = \left(n!\left(\frac{\alpha_c}{r^{12}}\right)^n\right)^2 v^n e^{-\lambda n_8/2}, \tag{3·7}$$

where n_8 is the number of 8-fold bonds in the cluster. The first factor is the partition function of BP consisting of n points, and it is squared since we sum over two

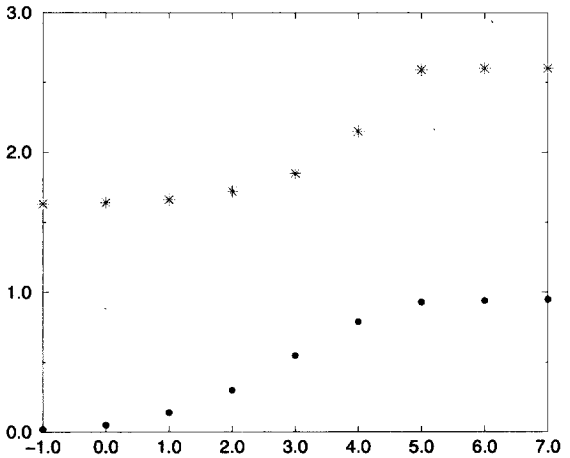


Fig. 1. Numerical simulation for $N = 1000$. The horizontal axis is λ. Stars indicate root mean squares of the eigenvalues, and dots indicate the ratio of 16-fold bonds. For large values of λ, the system is in a BP phase. For small values of λ, it is in a droplet phase.

independent maximal trees. We can then roughly estimate the partition function of the double tree model in the droplet phase by

$$\begin{aligned} Z_{\text{droplet}} &\sim (z(n))^{N/n} \frac{N!}{(n!)^{N/n}} \\ &\sim N! n^{2N(1-12/d)} (\alpha_c^2 \, l^{d-24})^N e^{-\lambda N_8/2}, \end{aligned} \tag{3·8}$$

where N_8 is the total number of the 8-fold bonds. Since $d \leq 10$, Z_{droplet} is maximized by a small value of n, which should be independent of N. Therefore Z_{droplet} can be written as

$$Z_{\text{droplet}} \sim N! \; e^{-NF_{\text{droplet}} - \lambda N_8/2}. \tag{3·9}$$

Comparing the partition function of BP Eq. (3·6) and that of the droplet Eq. (3·9) indicates that there is a phase transition or a cross-over between the two phases at some value of λ. For larger values of λ, the system is in a BP phase and for smaller values of λ, it is in a droplet phase. Our numerical simulation supports this transition (see Fig. 1). The nature of the transition is not clear at the present stage, although it is similar to a liquid-gas transition, since we have no order parameter associated with a symmetry. The points distribute in full ten dimensions both in these two phases. Other possible interactions, such as angle dependent interactions, will modify these phases and we may obtain a phase with lower dimensionality.

Finally in this subsection, we investigate what happens if we do not have the core potential. In this case we can use a mean field approximation to determine the one-particle density in the following way. Let us denote p_n as the probability that a point is connected to n other points. Then the self-consistent equation for the

one-particle density $\rho(x)$ is given by

$$\rho(x) = \frac{1}{Z} \exp\left(\sum_{n=1}^{\infty} p_n n \int d^d y \ln(f(x-y))\rho(y)\right),$$
$$\int d^d x \rho(x) = 1, \tag{3·10}$$

where $f(x)$ is the Boltzmann factor for each bond and is given by $(x^2+g)^{-6}$ in our case. Since N does not appear in these equations, the one-particle density does not depend on N. Hence all points are condensed into a finite area V which does not depend on N. We call this state a mean-field state. We can find a lower bound of the partition function by replacing the length of each bond with the size of the whole system:

$$Z_{\mathrm{MF}} > (N!\alpha_c^N)^2 (V^{-12N/d})^2 V^N e^{-\lambda N_8/2}. \tag{3·11}$$

Since Eq. (3·11) contains extra $N!$ compared with Z_{BP} in Eq. (3·6), we have to increase λ as $\log N \to \infty$ in order to keep the state in the BP phase. Thus we find that any state falls into a mean-field phase if we do not have the core potential.

3.3. *A mechanism favoring lower dimensions*

Although the double tree model has taught us a lot about the IIB matrix model, we may need to extend it in order to be more faithful to our effective action. As we saw in subsection 3.1, the interaction of 8-fold bonds generally depends on the relative angles of the vectors x_μ^{ij}. If eigenvalues extend in ten dimensions, averaging over directions gives a suppression factor to each 8-fold bond, because the tensor $V_{\mu\nu\lambda\rho}$ is traceless. This suppression becomes weak if eigenvalues collapse into lower dimensions. Therefore if there are a considerable number of 8-fold bonds, the partition function is suppressed in higher dimensions. However, the entropy which is associated with possible orientations of bonds favors higher dimensions. Therefore there is a competition between the two, and it is conceivable that four dimensional space-time is realized as a compromise. In what follows we roughly estimate the partition function assuming that the system is in a droplet phase and show the possibility of this scenario.

Assuming that x_μ^{ij} lie isotropically in a d dimensional subspace, we can estimate the orientation average of $V_{\mu\nu\lambda\rho}^{ij}$ as

$$\begin{aligned} V_{\mu\nu\lambda\rho}^{ij} \sim & \left(\frac{2(x^{ij})^4}{d(d+4)} - \frac{(x^{ij})^4}{(d+2)(d+4)}\right)(\hat{\delta}_{\mu\nu}\hat{\delta}_{\lambda\rho} + \hat{\delta}_{\mu\lambda}\hat{\delta}_{\nu\rho} + \cdots) \\ & - \frac{(x^{ij})^4}{d(D+4)}(\hat{\delta}_{\mu\nu}\delta_{\lambda\rho} + \hat{\delta}_{\mu\lambda}\delta_{\nu\rho} + \cdots) \\ & + \frac{(x^{ij})^4}{(D+2)(D+4)}(\delta_{\mu\nu}\delta_{\lambda\rho} + \delta_{\mu\lambda}\delta_{\nu\rho} + \cdots). \end{aligned} \tag{3·12}$$

Here $\hat{\delta}$ stands for the projection operator to the d dimensional space-time, while δ is the Kronecker delta in the original $D = 10$ dimensions. The interactions of 8-fold

bonds in general involve many $V_{\mu\nu\lambda\rho}$ whose Lorentz indices are contracted in various ways. It is not easy to estimate the combinatorics which emerge from such contractions exactly even after such simplification as Eq. (3·12). It may be qualitatively valid to replace $\hat{\delta}$ by δ. Then the relevant combinatorics become independent of d. Under such an approximation, each 8-fold bond effectively comes to possess a weight which is proportional to

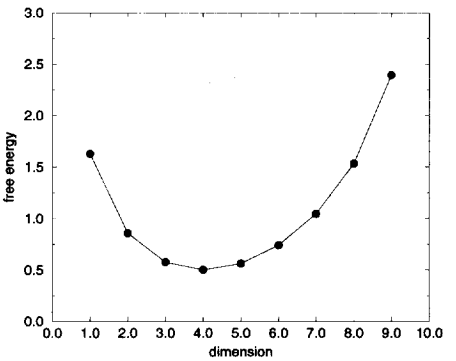


Fig. 2. Free energy as a function of dimension. We assume $\gamma = 17/6$. Free energy is minimized at four dimensions.

$$e^{-\lambda/2} \sim f_{\text{eff}}(d) = \left(\frac{1}{(D+2)(D+4)} - \frac{1}{(d+2)(d+4)} + \frac{2}{d}\left(\frac{1}{d+4} - \frac{1}{D+4} \right) \right). \tag{3·13}$$

The weight $f_{\text{eff}}(d)$ is a decreasing function and favors lower dimensions. Competition between this weight and entropy, which favors higher dimensions, determines space-time dimensions in this scenario. If we assume that all bonds are 8-fold and the entropy per bond increases as d^{γ}, the partition function for a droplet can be expressed as

$$Z_{\text{droplet}} = N! e^{-2NF(d)} \sim N! (f_{\text{eff}}(d))^{2N} d^{2\gamma N}. \tag{3·14}$$

The free energy F has a minimum at $d = 4$ when γ is between 2.6 and 3.1. Figure 2 shows F as a function of d. If this is the mechanism to generate our space-time, we need to determine detailed numerical factors to prove it. We stress here that the model is already fixed by nature and we do not have freedom to tune parameters.

3.4. *Other mechanisms for understanding space-time structure*

There are two other possible mechanisms to generate lower dimensional structure of space-time.

One way is to approach from a BP phase in which we regard 8-fold bonds in Eq. (3·1) as perturbations to BP consisting of 16-fold bonds only. Since the Hausdorff dimension of BP is four, there is a possibility that small perturbations might compress the system into four dimensions. If we neglect the effect of self-intersections and naively confine BP into d-dimensional space, the mean density becomes zero for $d > 4$ and diverges for $d < 4$ in the large N limit. Then for $d < 4$, the core potential extends the bond length, and the energy drastically increases. In other words most of the topologies of BP are prohibited to avoid the increase of energy, and the entropy is decreased very much. For $d > 4$, large distances between the points prevent 16-fold bonds from resolving into pairs of 8-fold bonds, and we cannot gain the entropy of having 8-fold bonds at various places on the polymer. On the other hand, at $d = 4$, 8-fold bonds can move around on the polymer, and

the system behaves as a gas of 8-fold bonds. Therefore, only at $d = 4$ can we gain the entropy of an 8-fold bond gas without too much excess energy due to the core potential. The mechanism discussed here is based on the picture of BP and seems totally different from the one discussed in subsection 3.3, which is based on the picture of droplet. In four dimensions, however, these two pictures can be complementary, since the size of space-time predicted in each of these pictures is both $N^{1/4}l$.

Finally, we point out a possibility that four-dimensional space-time is realized in the intermediate region of Fig. 1 even for the naive double tree model, where dimensional dependence of λ is not taken into account. Here we consider the dimensional dependence of the ratio of 8-fold bonds, r_8. In subsection 3.2, we saw the transition between two limiting phases, namely a BP phase for large λ and a droplet phase for small λ. For intermediate values of λ, the partition function can be expressed as

$$Z(\lambda) \sim N!\, e^{-N(F(\lambda)+\lambda r_8)}, \tag{3·15}$$

where $F(\lambda = \infty) = F_{\mathrm{BP}}$ and $F(\lambda = -\infty) = F_{\mathrm{droplet}}$. Obviously, the free energy $F(\lambda)$ defined above is always smaller in higher dimensions. Therefore the system always expands in full dimensions if r_8 is independent of dimension. Our numerical simulations, however, showed that r_8 is smaller in lower dimensions for fixed values of λ. This is consistent with the intuitive idea that each point has a smaller number of neighbors in lower dimensions, and the probability that two 8-fold bonds coincide to form a 16-fold bond is larger. If the dimensional dependence of λr_8 exceeds that of $F(\lambda)$, there is a possibility that a lower dimensional phase becomes more stable than the ten dimensional phase for some value of λ.

§4. Conclusions and discussion

In this concluding section, we first estimate the magnitude of the higher loop contributions to the effective action. Since our expansion parameter is g^2 over the fourth power of the average distances between the points, long distance behavior is described well by the one-loop effective action $S_{\mathrm{eff}}^{\text{1-loop}}[X]$. However, there is a possibility that small contributions may pile up at each loop, since N color indices circulate around the loop. Our strategy in this section is to assume that the one-loop level effective action determines the distribution $\rho(x)$ of the eigenvalues. We then use $\rho(x)$ to estimate the magnitude of the higher loop effects. This procedure is self-consistent if we find that the higher loop effects are small. Of course, our perturbative expansion breaks down if the expansion parameter grows as a positive power of N. What we show here is that the higher loop corrections are indeed finite in the large N limit for finite coupling constant, and our approximation based on the loop expansion can be fully justified.

Although we have argued that higher loop effects improve the infrared convergence properties in general, there are diagrams which possess infrared convergence properties which are identical to those with the one-loop diagrams in the estimation procedures just explained. For example, we may consider the one-loop self energy corrections to the propagators of the off-diagonal components. We then consider

their insertions into the propagators of the one-loop diagrams which contribute to the effective action. This is a part of the two-loop level contributions.

Recall that the matrix model (1·1) has been constructed by the large N reduction procedure from the $SU(N)$ gauge theory. Thus, to leading order in the $1/N$ expansion, the perturbative expansions in g^2 in the matrix model would be identical to those in the gauge theory were it not for the fermionic zeromodes. They are precisely the part of the quantum corrections we quoted above. Then, the ratio of $(l+1)$-loop contributions to l-loop contributions may be estimated by dimensional analysis as

$$\begin{aligned} R_{g^2} &= g^2 \sum_k \frac{1}{(x^i - x^k)^4} \\ &= g^2 N \frac{\sum_k (x^i - x^k)^{-4}}{\sum_k 1} \\ &= g^2 N \frac{\int d^{10}x\, \rho(x) x^{-4}}{\int d^{10}x\, \rho(x)}, \end{aligned} \tag{4·1}$$

where $\rho(x) = \sum_k \delta(x - x^k)$ is a single-particle distribution of the eigenvalues.

We first assume that x is distributed uniformly in a d dimensional manifold of size R. From the correspondence with gauge theories, we can say that there is no divergences from small x regions for $d > 4$, and from large x regions for $d < 4$. Thus,

$$\int^R d^d x\, x^{-4} = \begin{cases} R^{d-4} & d > 4 \\ l^{d-4} & d < 4 \end{cases}, \tag{4·2}$$

where l is a typical size of a short distance cutoff. Therefore,

$$R_{g^2} = \begin{cases} \dfrac{g^2 N}{R^4} = N \left(\dfrac{l}{R}\right)^4 & d > 4 \\ \dfrac{g^2 N}{R^d l^{4-d}} = N \left(\dfrac{l}{R}\right)^d & d < 4 \end{cases}, \tag{4·3}$$

where we used the fact that the core size is $l \sim g^{1/2}$. Since $N \sim (R/l)^d$, we may conclude that the quantum corrections are divergent for $d > 4$, but finite for $d < 4$ in the large N limit. These arguments seem to imply a logarithmic divergence for $d = 4$. However, here we need to take supersymmetry into account. It is well-known that dimensional reduction from ten dimensional super Yang-Mills theory down to four dimensions implies $\mathcal{N} = 4$ supersymmetry in four dimensions. Furthermore, four dimensional super Yang-Mills theory with $\mathcal{N} = 4$ supersymmetry is also known to be finite. And therefore we expect that the quantum corrections for $d = 4$ case are finite.

We have found a divergence in the above argument for $d > 4$. However, the situation is totally different if we assume that space-time is branched polymer like for

$d > 4$. We recall that $R \sim N^{1/4} l$ for such space-time, since its Hausdorff dimension is four. If so, the estimation of Eq. (4·1) is essentially the same as the four-dimensional case, and we may conclude that quantum corrections are finite also for $d > 4$. Thus we can show that the corrections arising from planar diagrams are finite in the large N limit if the coupling g^2 is kept fixed.

Next, let us consider higher order effects of the $1/N$ expansion. It is clear that the leading $1/N^2$ corrections contain two fewer summations over color indices compared with the planar contributions. However, they do contribute since they are at least finite. In fact, they can be of the same orders of magnitude with the planar contributions, since they are more singular at small x regions. For example, let us consider a three-loop correction to the gluon self-energy. Such a nonplanar contribution should have the form

$$g^6 \sum_k \frac{1}{(x^i - x^j)^6} \frac{1}{(x^i - x^k)^2} \frac{1}{(x^j - x^k)^4}. \tag{4·4}$$

Here, the important point is that the superficial degree of divergence for the integration variable x^k is reduced compared with that of the corresponding planar diagram. For dimensions up to four, the above expression may be evaluated as follows:

$$g^6 N \frac{\int d^{10}x \rho(x) \frac{1}{(x^i - x^j)^6} \frac{1}{(x^i - x)^2} \frac{1}{(x^j - x)^4}}{\int d^{10}x \rho(x)} = g^6 N \frac{1}{l^{12-d} R^d}, \tag{4·5}$$

where we set $(x^i - x^j) = l \sim g^{1/2}$ to estimate the magnitude of the correction at short distances. Since $N \sim (R/l)^d$, the contribution is again nonvanishing and finite in the large N limit.

Therefore, infinite numbers of diagrams contribute in dimensions up to four. This means that all orders of the $1/N$ expansion contribute equally. On the other hand, the nonplanar contributions diverge in general for $d > 4$, although they are much smaller compared with the planar contributions, unless space-time is branched polymer like. If space-time is branched polymer like, the situation is essentially identical to the four-dimensional case, since its fractal dimension is also four. So we may conclude that all orders of $1/N$ expansion contribute in all dimensions.

Our findings may be interpreted to imply that $g_{\rm st} \sim 1$ since we have argued that the string coupling constant is inversely proportional to N.[2] Apparently, the double scaling limit is naturally taken in the IIB matrix model. However, our investigations here also imply that the string scale satisfies $(\alpha')^2 \sim g^2$. This is in disagreement with our previous estimate $(\alpha')^2 \sim g^2 N$ based on the loop equations. We believe that a reexamination of the loop equations should be able to reconcile this discrepancy.

We also would like to comment on the universality of IIB matrix models. It is clear from the investigation in this section that the renormalizability and finiteness of four dimensional super Yang-Mills theory is crucial to control the quantum corrections in the IIB matrix model. If we were to add higher dimensional terms to the IIB matrix model, we may need to renormalize the theory nonperturbatively. We expect that such a renormalized theory belongs to the same universality class

as ours. Therefore we may argue that the IIB matrix model is universal, and our Lagrangian may correspond to the fixed-point Lagrangian.

In this paper, we have derived an effective action for the IIB matrix model which is valid when the eigenvalues of A_μ are widely separated. This is the effective action in terms of the super coordinates of space-time. It has clearly been shown that the theory has no infrared divergences, and the universe never disintegrates. The exciting possibility is that we can determine the dimensionality of space-time by studying such an effective action.

There are order N pairwise attractive potentials between the space-time coordinates. Such interactions may be classified into the 16-fold bond and the 8-fold bond types. If there were only 16-fold bond interactions, the space-time coordinates would form branched polymer-type configurations with fractal dimension four which stretch in ten dimensional space-time. Such a distribution of the eigenvalues cannot be considered as a four-dimensional continuous space-time. We have found that the closely packed distributions of the eigenvalues can be realized when we take the 8-fold bond interactions into account. Such distributions can be interpreted as continuous space-time. Here, the existence of the hard core repulsive potentials between the eigenvalues is also very important. Although more detailed investigations are required to show that four dimensional space-time is realized in this model, we have proposed possible mechanisms to realize the realistic dimensionality.

We believe that what we have already learned from the IIB matrix model is very significant. It is not defined on a fixed background metric, but space-time is dynamically determined as the vacuum of this model. The extent and the dimensionality of space-time can be obtained from the eigenvalue distributions of A_μ. In this model, the vacuum is not empty and may be classically represented by the diagonal matrices. Matter (for example D-objects) corresponds to the local fluctuations which float on the vacuum. In this way space-time and matter are inseparable, and they determine each other. Note that such a picture never emerges as long as we deal with effective Lagrangians for a finite number of D-objects.

If four-dimensional space-time is generated dynamically, it is plausible that space-time has four dimensional Lorentz invariance. Also it would be intrinsically flat. Here, the core potential plays an important role to protect space-time from shrinking. On the other hand, long-distance dynamics protects it from expanding infinitely. This implies the vanishing of the cosmological constant. In our scenario, the stability of generated space-time guarantees absence of a cosmological constant in an effective theory of gravity, that is, Einstein gravity. If vacuum energy is generated, eigenvalues are rearranged so as to restore stability of space-time. This energy is dynamically tuned to be zero. The existence of the hardcore repulsive potential at short distances and the infrared finiteness are responsible for this remarkable phenomenon. It does not explicitly depend on the supersymmetry, and hence it should be independent of whether supersymmetry is spontaneously broken or not. We believe that this fact alone constitutes a major achievement and hence underscores the validity of the IIB matrix model.

We recall the $\mathcal{N} = 2$ SUSY transformations for the zeromodes:

$$\begin{cases} \delta^{(1)} x_\mu &= i\bar{\epsilon}_1 \Gamma_\mu \xi \\ \delta^{(1)} \xi &= 0 \end{cases}, \tag{4·6}$$

and

$$\begin{cases} \delta^{(2)} x_\mu &= 0 \\ \delta^{(2)} \xi &= \epsilon_2 \end{cases}. \tag{4·7}$$

The linear combinations of the above translations $\tilde{\delta}^{\pm} = \delta^1 \pm \delta^2$ form the $\mathcal{N} = 2$ supersymmetry algebra. The vacuum expectation value of the supersymmetry transformation of ξ must vanish if supersymmetry is to remain unbroken by the vacuum. It is clear from Eq. (4·7) that the $\mathcal{N} = 2$ supersymmetry of $\tilde{\delta}^{\pm}$ are broken completely. This is in contrast to the D-string case, where half of the supersymmetry can be preserved. If so, this model realizes a vanishing cosmological constant without supersymmetry, as we have argued.

It is also possible to write a scenario to obtain realistic gauge groups in this model. Although we have considered the most generic case in which all eigenvalues of A_μ are distinct, we may assume that some of the eigenvalues remain degenerate. Then the vacuum configurations would be represented by block diagonal matrices where each block is m dimensional. Then the gauge group $SU(m)$ is realized (note that the $U(1)$ part is used to make the space-time coordinates). It is also possible to realize the standard model gauge group in an analogous way. We may again integrate out the off-diagonal elements and obtain the effective action for the $U(m)$ submatrices. The resultant effective action must closely resemble the gauge theory with the Planck scale cutoff, since we have local gauge invariance as the manifest symmetry of the effective action.

Since we have argued that it is possible to obtain realistic gauge groups in four dimensions in this model, it is natural to ask to what kind of string compactification it corresponds. It presumably corresponds to the compactification into the interior of S^5, that is B^6. It may be possible to play analogous games with Calabi-Yau compactifications of a heterotic string. It is possible to obtain chiral spinors in the fundamental representations of the gauge group in four dimensions from the chiral spinors in the adjoint representations in ten dimensions. For this purpose we need to assume that there is nontrivial gauge field configuration in B^6 which produces a nontrivial index for the Dirac operator. We immediately think of a two dimensional example such as magnetic vortices. Then the tensor products of three of these are the possible candidates for such nontrivial gauge configurations.

We also note the similarity between our effective Lagrangian and the dynamical triangulation approach for quantum gravity. Due to the existence of the hardcore potential of the Planck scale, the unit cell of the effective action is also the Planck scale. The pairing interaction between the eigenvalues may be identified with the bonds in the dynamical triangulation approach. The integration over the fermionic zeromodes ensures that the model sums contributions with all possible connectivities of the bonds, just like the dynamical triangulation approach. This analogy may be useful to reconstruct the metric out of the IIB matrix model. We also conjecture that general coordinate invariance is present in this model due to the following reasoning.

There is a permutation symmetry S_N which permutes the color indices. This is a subgroup of the full gauge group $SU(N)$ and is an exact symmetry of our effective action. Since it does not change the density of the eigenvalues, it should be part of the volume-preserving diffeomorphism group in the continuum limit. Similar argument has been made in the dynamical triangulation approach. However, it suffers from the infamous conformal mode instability, while the IIB matrix model is free from such a problem, which is again a remarkable merit of this model. So the gauge and the diffeomorphism invariances may be unified into the $SU(N)$ symmetry of the IIB matrix model.

Although the investigation of this model is still in an initial stage, this model turns out to be a finite theory which is free from both short distance and long distance divergences. The existence of such a theory itself is very impressive. It is a manifestly covariant formulation of superstring which enables us to determine the structure of the vacuum, namely space-time. We have argued that it can solve problems such as the cosmological constant problem which appeared to be insurmountable before its advent. Therefore we believe that it will reveal further truths concerning the structure of space-time and matter.

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Appendix A

—— Fierz Transformation ——

In this appendix, we investigate the totally symmetric tensors which can be constructed out of a ten dimensional Majorana-Weyl spinor ξ. The effective action contains such terms as $\mathrm{tr}S^n$, where $S^\mu = \bar{\xi}\Gamma^{\mu\alpha\nu}\xi\ x_\alpha/x^4$, and n is an even integer up to 8. Therefore we are interested in totally symmetric tensors.

We first point out that the only nonvanishing tensors which are quadratic in ξ are $\bar{\xi}\Gamma^{\mu\nu\lambda}\xi$. Thus the Fierz transformation is performed quite easily, and we obtain the following identities:

$$(\bar{\xi}\Gamma^{\mu\alpha\beta}\xi)(\bar{\xi}\Gamma^{\nu}{}_{\alpha\beta}\xi) = 0, \tag{A·1}$$

$$(\bar{\xi}\Gamma^{\mu\nu\alpha}\xi)(\bar{\xi}\Gamma^{\lambda\rho}{}_{\alpha}\xi) = (\bar{\xi}\Gamma^{\mu\lambda\alpha}\xi)(\bar{\xi}\Gamma^{\nu\rho}{}_{\alpha}\xi) - (\bar{\xi}\Gamma^{\mu\rho\alpha}\xi)(\bar{\xi}\Gamma^{\nu\lambda}{}_{\alpha}\xi). \tag{A·2}$$

From Eq. (A·1) and Eq. (A·2), we can prove the following identity:

$$(\bar{\xi}\Gamma^{\mu\alpha}{}_{\beta}\xi)(\bar{\xi}\Gamma^{\nu\beta}{}_{\gamma}\xi)(\bar{\xi}\Gamma^{\lambda\gamma}{}_{\alpha}\xi) = 0. \tag{A·3}$$

Proof: Since the first factor on the left-hand side is antisymmetric in α and β, the other factors can be antisymmetrized as the right-hand side of Eq. (A·2). Therefore using Eq. (A·2) we have

$$\text{L.H.S.} = -1/2(\bar{\xi}\Gamma^{\mu\alpha}{}_{\beta}\xi)(\bar{\xi}\Gamma^{\nu\lambda}{}_{\gamma}\xi)(\bar{\xi}\Gamma^{\beta}{}_{\alpha}{}^{\gamma}\xi), \tag{A·4}$$

and this vanishes after the contractions of α and β due to Eq. (A·1). (q.e.d.)

Equation (A·1) shows that any totally symmetric tensor made of four ξ vanishes. Since sixteen ξ form a Lorentz singlet, we can see from the "interchange of particles and holes" that any totally symmetric tensor made of twelve ξ must vanish. From Eq. (A·3), we can also conclude the absence of the totally symmetric tensors made of six and ten spinors in an analogous way.

Thus we have shown that the totally symmetric tensors made of ξ are exhausted by the following list: $\{1,\ T^{\mu\nu\lambda\rho} = (\bar{\xi}\Gamma^{\mu\alpha}{}_{\beta}\xi)(\bar{\xi}\Gamma^{\nu\beta}{}_{\gamma}\xi)(\bar{\xi}\Gamma^{\lambda\gamma}{}_{\delta}\xi)(\bar{\xi}\Gamma^{\rho\delta}{}_{\alpha}\xi),\ \xi^{16}\}$. It is easily checked that $T^{\mu\nu\lambda\rho}$ is a totally symmetric traceless tensor in the following way. Similar use of Eqs. (A·2) and (A·3) to the above gives the antisymmetric part of $T^{\mu\nu\lambda\rho}$ as

$$\begin{aligned} T^{\mu\nu[\lambda\rho]} &= -1/2(\bar{\xi}\Gamma^{\mu\alpha}{}_{\beta}\xi)(\bar{\xi}\Gamma^{\nu\beta}{}_{\gamma}\xi)(\bar{\xi}\Gamma^{\lambda\rho}{}_{\delta}\xi)(\bar{\xi}\Gamma^{\gamma}{}_{\alpha}{}^{\delta}\xi) \\ &= 0, \end{aligned} \tag{A·5}$$

and from Eq. (A·1) we immediately see that $T^{\mu\nu\lambda}{}_{\lambda} = 0$. Therefore $T^{\mu\nu\lambda\rho}$ is totally symmetric and traceless.

Appendix B
—— *SU(2) Matrix Model* ——

In this appendix, we solve supersymmetric $SU(2)$ matrix models in various dimensions, $D = 3, 4, 6, 10$. *) We have

$$S = -\frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2}{}^t\psi\alpha^\mu[A_\mu, \psi]\right), \tag{B·1}$$

where $A_\mu(\mu = 0, 1, \cdots, D-1)$ and ψ are 2×2 Hermite matrices, and ψ is a spinor in D dimensional super Yang Mills theory. Each spinor consists of p_D real components, where p_D=2, 4, 8 and 16 for D=3, 4, 6 and 10, respectively. We denote D-dimensional alpha matrices by the α^μ ($\alpha^\mu = \Gamma^0\Gamma^\mu$), and the first three of these can be represented as follows:

$$\begin{aligned} \alpha^i &= \rho^i \otimes 1_{p_D/2}, \\ \rho^i &= \begin{cases} 1_2 & \text{for } i{=}0 \\ \sigma^1 & \text{for } i{=}1 \\ \sigma^3 & \text{for } i{=}2 \end{cases}. \end{aligned} \tag{B·2}$$

Performing the integral over fermionic variables gives the pfaffian **)

$$\mathrm{Pf}(\alpha^\mu A_\mu) = \mathrm{Pf}(\alpha^\mu T^a A^a_\mu), \tag{B·3}$$

where T^a is the generator of $SU(2)$ in the adjoint representation, namely $(T^a)_{bc} = \epsilon_{bac}$. Now we "rotate" A_μ by a Lorentz transformation so that only $A_0, A_1$and A_2

*) Some results presented here have already been reported in Ref. 23). Similar calculations have also been done in a different context in Ref. 24).

**) Although the integration does not lead to a pfaffian in $D = 6$, the result (B·4) holds in this case as well.

are nonvanishing. Equation (B·3) then reduces to the three-dimensional calculation,

$$\left\{\mathrm{Pf}(\rho^i T^a A_i^a)\right\}^{p_D/2} \propto \left\{\epsilon_{ijk}\epsilon_{abc}A_i^a A_j^b A_k^c\right\}^{p_D/2}, \tag{B·4}$$

which corresponds to $(\det A)^{p_D/2}$ when we regard A_i^a as a 3×3 matrix.

The next step is the integration over three ten-dimensional vectors A_μ^a, which we reduce to the integration over three-dimensional vectors. The Jacobian for this reduction is the volume of the parallelepiped spanned by the three vectors A^1, A^2 and A^3, to the $(D-3)$-th, which is nothing but $|\det A|^{D-3}$.

To estimate the behavior of the integral, we take the parametrization

$$\begin{pmatrix} r \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} a \\ \boldsymbol{y} \end{pmatrix}, \begin{pmatrix} b \\ \boldsymbol{z} \end{pmatrix}, \tag{B·5}$$

for three vectors A_i^a, where $\boldsymbol{y}$ and $\boldsymbol{z}$ are two dimensional vectors, while $\mathbf{0}$ is the two dimensional zero vector. After these considerations, we obtain

$$\begin{aligned}\int dAd\psi e^{-S} = \int drr^2 dadbd^2\boldsymbol{y}d^2\boldsymbol{z}\left\{r(\boldsymbol{y}\times\boldsymbol{z})\right\}^{p_D/2}|r(\boldsymbol{y}\times\boldsymbol{z})|^{D-3} \\ \times\exp\Big(-\frac{1}{g^2}\left[r^2(\boldsymbol{y}^2+\boldsymbol{z}^2)+(a\boldsymbol{z}-b\boldsymbol{y})^2+(\boldsymbol{y}\times\boldsymbol{z})^2\right]\Big).\end{aligned} \tag{B·6}$$

Integrating over a and b yields

$$\int dadbe^{-(a\boldsymbol{z}-b\boldsymbol{y})^2} = \left(\boldsymbol{z}^2\boldsymbol{y}^2-(\boldsymbol{z}\cdot\boldsymbol{y})^2\right)^{-1/2} = |\boldsymbol{y}\times\boldsymbol{z}|^{-1}. \tag{B·7}$$

Next, let us integrate over $\boldsymbol{y}$ and $\boldsymbol{z}$. For the D=3 case, the integral becomes identically zero, since $\{r(\boldsymbol{y}\times\boldsymbol{z})\}$ takes both signs. For the remaining dimensions, we can estimate the integration for the cases in which r is either large or small. These suffice for the present purpose. When r is large, the first term of the argument of the exponential in Eq. (B·6) becomes dominant. We can perform the integrations over $\boldsymbol{y}$ and $\boldsymbol{z}$ by rescaling them as $\boldsymbol{y}/r$ and $\boldsymbol{z}/r$. In this way Eq. (B·6) is estimated to be

$$\begin{aligned}\int drr^{-(p_D/2+D-3)} &= \int r^{D-1}drr^{-(p_D/2+2D-4)} \\ &= \begin{cases} \int r^3 dr\, r^{-6} & \text{for } D{=}4 \\ \int r^5 dr\, r^{-12} & \text{for } D{=}6 \\ \int r^9 dr\, r^{-24} & \text{for } D{=}10 \end{cases}. \qquad (r^2 \gg g)\end{aligned} \tag{B·8}$$

When r is small, the first term of the argument of the exponential becomes negligible. We can perform the integrations over $\boldsymbol{y}$ and $\boldsymbol{z}$ independently from r. Equation (B·6)

is estimated in this case to be

$$\int dr r^{2+(p_D/2+D-3)} = \int r^{D-1} dr r^{p_D/2}$$
$$= \begin{cases} \int r^3 dr\, r^2 & \text{for } D=4 \\ \int r^5 dr\, r^4 & \text{for } D=6 \\ \int r^9 dr\, r^8 & \text{for } D=10 \end{cases} . \quad (r^2 \ll g) \qquad \text{(B·9)}$$

Finally, we confirm that the long-distance behavior of the r integral (B·8) agrees with the result of the perturbative calculation in the text. Recall that the one-loop effective action for r and ξ is written as

$$S_{\text{eff}}^{\text{1-loop}}[r, \xi] = \text{tr} \ln(1 + S^{\mu\nu}), \qquad \text{(B·10)}$$

where

$$S^{\mu\nu} = \bar{\xi} \Gamma^{\mu\alpha\nu} \xi \frac{r_\alpha}{r^4} \sim \xi^2 / r^3. \qquad \text{(B·11)}$$

Thus the integration over ξ,

$$\int d^D r d^{p_D} \xi e^{-S_{\text{eff}}[r,\xi]}, \qquad \text{(B·12)}$$

leads to Eq. (B·8).

Appendix C
—— *Branched Polymer* ——

In this appendix we give a brief introduction to the thermodynamics of branched polymers (BP) without a self-avoiding effect in D dimensions (see for example Ref. 25)). The partition function of BP with N points and $(N-1)$ bonds is given by

$$Z_N = \sum_{\text{BP}} \int \prod_{i=1}^{N-1} d^D x^i \prod_{(ij):\text{bond}} f(x^i - x^j) \prod_{n=1}^{\infty} t_n^{\#n}, \qquad \text{(C·1)}$$

where the summation is taken over all possible topologies of branched polymers. The quantity t_n denotes a positive weight assigned for the points to which n bonds are connected, and $\#n$ denotes the number of such points in a given configuration. The weight function for bonds $f(x)$ can be Fourier transformed as

$$\hat{f}(p) = \int d^D x f(x) e^{-ipx} = \hat{f}(0)(1 - c_2 p^2 + c_4 p^4 + \cdots), \qquad \text{(C·2)}$$

with a positive coefficient c_2. Since we are interested in the thermodynamic limit in which N goes to infinity, we consider a grand canonical partition function:

$$Z = \sum_{N=1}^{\infty} N Z_N k_0^{(N-1)}. \qquad \text{(C·3)}$$

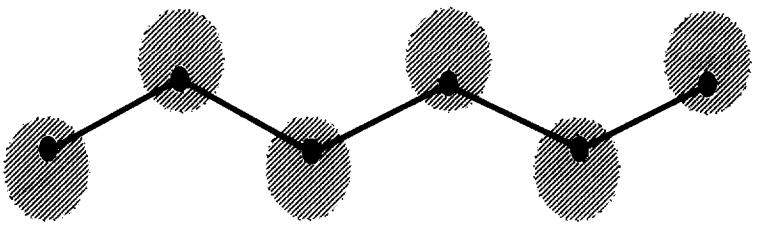

Fig. 3. Graphical representation of a two-point function. A pair of points in a given configuration can be connected by bonds in a unique way. In this figure they are connected by five bonds. Such bonds form a random-walk type object. The remaining points in a branched polymer are connected to the points in this object. They are represented by the blobs in this figure.

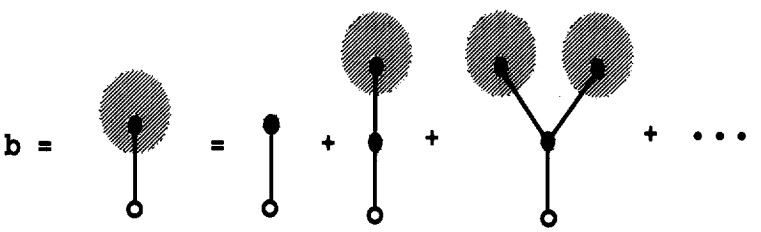

Fig. 4. Graphical representation of the Schwinger Dyson equation for b. The white circles are not associated with any weight.

In order to demonstrate the scaling behavior, we consider a two-point correlation function. If we pick up a pair of points in a particular configuration, they are uniquely connected by n bonds. If we assume that they are separated by a fixed distance, n varies from one configuration to another. Therefore we can express a two point function in momentum space as follows:

$$\hat{G}(p,k_0) = \int d^D(x-y)G(x-y,k_0)e^{ip(x-y)}$$
$$\propto \sum_{n=1}^{\infty} k_0^{n+1}\left(t_1 + t_2 b + \frac{t_3}{2!}b^2 + \cdots\right)^2 (\hat{f}(p))^n B^{n-1}. \qquad \text{(C·4)}$$

A graphical illustration of this equation is given in Fig. 3.The factor $(t_1 + t_2 b + \cdots)$ is the contribution from each end point in Fig. 3. B represents each blob in Fig. 3 (except those at the endpoints). b is graphically represented in Fig. 4. It satisfies the Schwinger Dyson equation

$$b = k_0\hat{f}(0)\left(t_1 + t_2 b + \frac{t_3}{2!}b^2 + \cdots\right) = kh(b), \qquad \text{(C·5)}$$

where $k = k_0\hat{f}(0)$. The blob B can be related to $h(b)$ as

$$B = t_2 + t_3 b + \frac{t_4}{2!}b^2 + \cdots = h'(b). \qquad \text{(C·6)}$$

From these considerations, we can see that $\hat{G}(p, k_0)$ can be written as

$$\hat{G}(p, k_0) \propto \frac{k_0^2 h(b)^2 \hat{f}(p)}{1 - k_0 \hat{f}(p) h'(b)}. \quad \text{(C·7)}$$

We also note that $Z'(b) = h(b)$, since the partition function (C·3) is given by

$$Z(b) = 1 + t_1 b + t_2 b^2/2! + t_3 b^2/3! + \cdots. \quad \text{(C·8)}$$

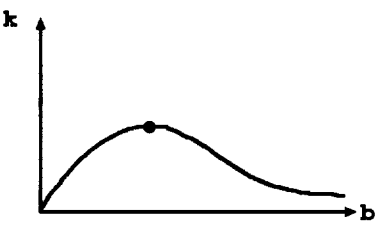

Fig. 5. Typical relation between k and b. The black dot in the figure indicates the critical point. The large N limit is taken by approaching this point from the left.

Since the averaged number of the points in the branched polymer is given by

$$\bar{N} \sim \frac{k}{Z}\frac{\partial Z}{\partial k} = \frac{k h(b)}{Z(b)\left(\frac{\partial k}{\partial b}\right)}, \quad \text{(C·9)}$$

we must tune the fugacity k_0 so that $\partial k/\partial b$ approaches zero from a positive value. Equation (C·5) solves k as a function of b: $k = b/h(b)$. A typical solution is illustrated in Fig. 5. Generally $t_1 \neq 0$, and some of the t_n with n larger than 2 are non-zero. We observe that $k(b)$ vanishes at $b = 0$ and $b = \infty$. Since it is positive definite, there is a critical fugacity k_c, where an averaged number becomes infinite.

Near the critical point, $k(b)$ can be approximated by

$$k \sim k_c - \frac{c}{2}(b - b_c)^2, \quad \text{(C·10)}$$

where c is a positive constant determined by the set of t_n. The partition function is given by the integral of $h(b)$ over b, and it behaves near the critical point as

$$Z = \text{const} - \frac{b_c}{k_c}\sqrt{\frac{2(k_c - k)}{c}}. \quad \text{(C·11)}$$

The universal part (the second term) determines the power $-3/2$ in the following expansion:

$$Z = \sum_{N=1}^{\infty} N^{-3/2}\left(\frac{k}{k_c}\right)^N \sim \sum_{N=1}^{\infty} N^{-3/2} e^{-\frac{k_c - k}{k_c}N}. \quad \text{(C·12)}$$

This is because the universal part of the infinite sum of the r.h.s. in Eq. (C·12) can be estimated by the integral $\int dN N^{-3/2} \exp(-(k_c - k)N/k_c)$. Therefore the partition function for a fixed N is $Z_N = N^{-5/2}(\hat{f}(0)/k_c)^N$. The leading non-universal behavior is given by

$$Z_N \sim (\hat{f}(0)\alpha_c)^N, \quad \text{(C·13)}$$

where $\alpha_c = 1/k_c$ takes a value of order one. In the special case that $t_n = 1$ for all n, we find that $k = be^{-b}$, and the critical values are $k_c = 1/e$ and $b_c = 1$. This is the case relevant to our analysis.

Inserting $kh'(b) = 1 - h(b)k'(b)$ (which is derived from Eq. (C·5)) into Eq. (C·7), we obtain the scaling behavior of a two-point function near the critical point. For small p, it behaves as

$$\begin{aligned}\hat{G}(p, k_0) &\propto \frac{k_{0,c}^2 h^2(b_c)\hat{f}(p)}{1 - (\hat{f}(p)/\hat{f}(0))(1 - \sqrt{2c(k_c - k)}h(b_c))} \\ &\propto \frac{1}{p^2 + M^2\sqrt{(k_c - k)}} \propto \frac{1}{(p_{phy}^2 + M^2)\sqrt{(k_c - k)}},\end{aligned} \tag{C·14}$$

where $M^2 = \sqrt{2c}h(b_c)/c_2$. In order to take the scaling limit, we have introduced the physical momentum $p_{phy} = p(k_c - k)^{-1/4}$. In this way, we have obtained the scaling relation

$$(k_c - k)^{-1/2} G(x(k_c - k)^{-1/4}, k_0) \overset{k \to k_c}{\longrightarrow} g(x), \tag{C·15}$$

where $g(x)$ is a certain function. Since the dominant contribution comes from $N \sim k_c/(k_c - k)$, we can observe from Eq. (C·15) that the Hausdorff dimension of a branched polymer is four.

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