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SPACELIKE HYPERSURFACES OF CONSTANT MEAN CURVATURE AND CALABI-BERNSTEIN TYPE PROBLEMS[1]

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Abstract. Spacelike graphs of constant mean curvature over compact Riemannian manifolds in Lorentzian manifolds with constant sectional curvature are studied. The corresponding Calabi-Bernstein type problems are stated. In the case of nonpositive sectional curvature all their solutions are obtained, and for positive sectional curvature well-known results are extended.

1. Introduction. The solutions to the differential equation

$$nH(f^2(u)-g(\nabla u,\nabla u))^{3/2}=(f^2(u)-g(\nabla u,\nabla u))\left(nf'(u)+\frac{1}{f(u)}\Delta u\right)+\frac{1}{f(u)}\nabla^2u(\nabla u,\nabla u)-f'(u)g(\nabla u,\nabla u)\,, \tag{1}$$

with

$$g(\nabla u,\nabla u)<f^2(u)\,,$$

on an n-dimensional Riemannian manifold (F, g), represent the spacelike graphs of constant mean curvature (CMC) H in a Generalized Robertson-Walker (GRW) space with fiber F, base $I\subseteq R$ and warping function f (in our definition in Section 2, the fiber is not assumed to be of constant sectional curvature).

This equation represents a general setup to formulate Calabi-Bernstein type problems, and GRW spaces become the natural ambient Lorentzian manifolds for these problems. In fact, two relevant special cases are the following: First, F is the n-dimensional Euclidean space of curvature $C=0$, $I=R$ and $f\equiv 1$. This corresponds to CMC spacelike graphs in the Minkowski space L^{n+1}. In this case, Calabi [5] (for $n\leq 4$)

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and Cheng and Yau [6] (for arbitrary n) proved that the only (entire) solutions to (1) for $H=0$ are affine functions, whereas for the case $H\neq 0$ many solutions have been found (see, for instance, [11] and [12]). Secondly, F is the round n-sphere with curvature $C=1$, $I=R$ and $f=\cosh$. This corresponds to CMC spacelike graphs in the de Sitter space S_1^{n+1}, and it has been studied by Montiel in [9], where the (entire) solutions to (1) were determined.

In this paper we are interested in studying the solutions to this CMC differential equation and its relation with the associated geometric problem of CMC spacelike hypersurfaces in GRW spaces. Roughly, our main aim is to find all the solutions to (1) when the ambient space has non-positive constant sectional curvature. Moreover, in the case of positive constant sectional curvature we extend Montiel's result (Proposition 1).

On the other hand, the equation (1) has been first considered by Choquet-Bruhat [7] for $H=0$ and by the authors [2] for arbitrary H, in both cases under an additional hypothesis on the ambient space which is natural from the point of view of the general relativity: the so called timelike convergence condition. Now, we will consider this problem under more restricted situations related to classical Calabi-Bernstein assumptions and so, we will assume the ambient Lorentzian manifold to be at least Einstein.

As the first approach to the problem, we show that every compact spacelike hypersurface of CMC in an Einstein GRW space must be totally umbilical (Proposition 1). This result can be seen as a positive answer to Goddard's conjecture [8] in the case of Einstein GRW spaces. On the other hand, it is not difficult to see that the *spacelike slices* of a GRW space (those with constant universal time) are totally umbilical spacelike hypersurfaces of CMC. In this respect, we are able to characterize the spacelike slices as the only compact spacelike hypersurfaces in an Einstein GRW space with CMC and non-positive definite Ricci tensor (Theorem 2 and Corollary 3). This allows us to give a very simple uniqueness result for compact spacelike hypersurfaces of CMC in a classical Robertson-Walker spacetime (Corollaries 4 and 5), as well as the corresponding uniqueness results for the associated Calabi-Bernstein type problem described by the CMC differential equation (1) (Theorems 6 and 7). These results fulfill our main aims in this paper.

Some remarks and applications of our main results are given in Section 4. In particular we see that the compactness of the hypersurface can be often derived just by imposing completeness (Remarks 2 and 3). We also discuss how our uniqueness results can be used to obtain some information on the equation (1) for certain non-compact complete Riemannian manifolds (Remark 4).

2. Preliminaries. Let (F, g) be an n-dimensional, $n\geq 2$, (connected) Riemannian manifold and let $I\subseteq R$ be an open interval in R endowed with the metric $-dt^2$. Throughout this paper we will denote by $(\bar{M}, \langle\ ,\ \rangle)$ the $(n+1)$-dimensional product manifold $I \times F$ with the Lorentzian metric

$$\langle\ ,\ \rangle = \pi_I^*(-dt^2) + f^2(\pi_I)\pi_F^*(g)\,, \tag{2}$$

where $f > 0$ is a smooth function on I, and π_I and π_F denote the projections onto I and F, respectively. Namely, $(\bar{M}, \langle\ ,\ \rangle)$ is a Lorentzian warped product with base $(I, -dt^2)$, fiber (F, g) and warping function f. We will refer to $\bar{M}$ as a *Generalized Robertson-Walker* (GRW) space.

Let $x: M \to \bar{M}$ be a connected, immersed spacelike hypersurface in $\bar{M}$. As usual, we will denote by $\langle\ ,\ \rangle$ both the Lorentzian metric on $\bar{M}$ given by (2) and the Riemannian metric induced on M via the corresponding spacelike immersion x. Set $\partial_t = \partial/\partial t \in \mathscr{X}(\bar{M})$, which is a unit timelike vector field globally defined on $\bar{M}$ and determines a time-orientation on $\bar{M}$. Thus the time-orientability of $\bar{M}$ allows us to define $N \in \mathscr{X}^{\perp}(M)$ as the globally defined unit timelike vector field normal to M whose time-orientation coincides with that of ∂_t.

Following the usual terminology, a GRW space is said to be *spatially closed* when the fiber F is compact. The existence of compact spacelike hypersurfaces in a GRW space implies that the fiber must be compact. In fact, if M is such a hypersurface, then its projection on the fiber F is a covering map and, in particular, F is compact. Therefore, the GRW space which must be considered are necessarily spatially closed (see [2] for details).

Finally, it is not difficult to see that $(\bar{M}, \langle\ ,\ \rangle)$ is Einstein with $\overline{\mathrm{Ric}} = \bar{c}\langle\ ,\ \rangle$, $\bar{c}$ being a real constant, if and only if 1) (F, g) has constant Ricci curvature c; and 2) f satisfies the differential equations

$$\frac{f''}{f} = \frac{\bar{c}}{n} \quad \text{and} \quad \frac{\bar{c}(n-1)}{n} = \frac{c + (n-1)(f')^2}{f^2} \tag{3}$$

(see, for example, [4, Corollary 9.107]). Moreover, $\bar{M}$ has constant sectional curvature $\bar{C}$ if and only if F has constant sectional curvature C (that is, $\bar{M}$ is a *classical* Robertson-Walker (RW) spacetime) and f satisfies (3) with $c = (n-1)C$ and $\bar{c} = n\bar{C}$.

In Table, the positive solutions to (3) are collected (in each case, the interval of definition I of f is the maximal one where f is positive).

3. Main results. Let us consider on $\bar{M}$ the timelike vector field $\xi \in \mathscr{X}(\bar{M})$ given by $\xi = f(\pi_I)\partial_t$ and put along x

$$\xi = \xi^T - \langle \xi, N\rangle N\,, \tag{4}$$

where $\xi^T \in \mathscr{X}(M)$ is tangent to M. If we denote by $\bar{\nabla}$ the Levi-Civita connection on $\bar{M}$ of the Lorentzian metric $\langle\ ,\ \rangle$, it follows that

$$\bar{\nabla}_Z \xi = f'(\pi_I)Z\,, \tag{5}$$

for all vector field Z on $\bar{M}$. Let us denote by ∇ the Levi-Civita connection on M. By

TABLE.

1	$\bar{c}>0$	$c>0$	$f(t)=a\exp(bt)+\dfrac{cn}{4a\bar{c}(n-1)}\exp(-bt)$, $a>0$, $b=\sqrt{\bar{c}/n}$
2	$\bar{c}>0$	$c=0$	$f(t)=a\exp(\varepsilon bt)$, $a>0$, $\varepsilon=\pm1$, $b=\sqrt{\bar{c}/n}$
3	$\bar{c}>0$	$c<0$	$f(t)=a\exp(bt)+\dfrac{cn}{4a\bar{c}(n-1)}\exp(-bt)$, $a\neq0$, $b=\sqrt{\bar{c}/n}$
4	$\bar{c}=0$	$c=0$	$f(t)=a$, $a>0$
5	$\bar{c}=0$	$c<0$	$f(t)=\varepsilon\sqrt{-c/(n-1)}t+a$, $\varepsilon=\pm1$
6	$\bar{c}<0$	$c<0$	$f(t)=a_1\cos(bt)+a_2\sin(bt)$, $a_1^2+a_2^2=cn/\bar{c}(n-1)$, $b=\sqrt{-\bar{c}/n}$

taking covariant derivative in (4) and using the Gauss and Weingarten formulas for the spacelike hypersurface, it is not difficult to get from (5) that

$$\nabla_X\xi^T=f'(\pi)X-\langle\xi, N\rangle AX, \tag{6}$$

for all vector field X tangent to M, where $\pi=\pi_I\circ x$, and A stands for the Weingarten endomorphism associated to N. Therefore, directly from (6) we have the equation,

$$\operatorname{div}(\xi^T)=nf'(\pi)+nH\langle\xi, N\rangle, \tag{7}$$

where div denotes the divergence on M and H stands for the mean curvature function corresponding to N (note that we are taking $H=-\operatorname{tr}(A)/n$).

From a reasoning as above and using now the Codazzi equation we have

$$\operatorname{div}(A\xi^T)=-n\langle\nabla H, \xi\rangle-\overline{\operatorname{Ric}}(\xi^T, N)-nf'(\pi)H-\langle\xi, N\rangle\operatorname{tr}(A^2), \tag{8}$$

where ∇ denotes the gradient on M (see [3] for details). Now, if $\bar{M}$ is assumed to be Einstein and M is compact, then (7) and (8) allow us to obtain the integral formula

$$\int_M((n-1)\langle\nabla H, \xi\rangle+\langle\xi, N\rangle(\operatorname{tr}(A^2)-nH^2))dV=0.$$

Note that, from the Schwarz inequality, the function $U=\operatorname{tr}(A^2)-nH^2$ is non-negative everywhere and $U\equiv0$ if and only if x is a totally umbilical immersion. Thus, taking into account the fact that $\langle\xi, N\rangle\leq-f(\pi)<0$ everywhere, we have:

PROPOSITION 1. *Every compact spacelike hypersurface of constant mean curvature in a spatially closed Einstein* GRW *space must be totally umbilical.*

In orther to go further, observe that $\xi^T = -f(\pi)\nabla\pi$, which implies that π is constant if and only if $\xi^T \equiv 0$, and also since M is compact there exists a point $p_0 \in M$ such that $\xi^T(p_0) = 0$. Therefore, π is constant if and only if ξ^T is a parallel vector field on M. Moreover, when x is totally umbilical it follows from (6) that

$$\nabla_X \xi^T = (f'(\pi) + H\langle N, \xi\rangle)X\,,$$

and so ξ^T is a *conformal Killing* vector field on M. Using now the fact that every conformal Killing vector field on a compact Riemannian manifold with non-positive definite Ricci tensor must be parallel [13], we obtain from Proposition 1 the following result.

THEOREM 2. *Let $\bar{M}$ be a spatially closed Einstein* GRW *space. Then, every compact spacelike hypersurface in $\bar{M}$ with constant mean curvature and non-positive definite Ricci tensor must be a spacelike slice.*

Let us recall that each spacelike slice $F(t_0) = \{t_0\} \times F$ is a totally umbilical hypersurface in $\bar{M}$ with constant mean curvature $H = f'(t_0)/f(t_0)$, and it is homothetic to (F, g) with scale factor $1/f(t_0)$ (see [3] for details). Therefore, Theorem 2 implies that *in the case* 1 of Table

> *there exists no compact spacelike hypersurface in $\bar{M}$ with constant mean curvature and non-positive definite Ricci tensor.*

On the other hand, *in the remaining cases* in Table, Theorem 2 yields the following characterization of spacelike slices.

COROLLARY 3. *Spacelike slices are the only compact spacelike hypersurfaces in $\bar{M}$ with constant mean curvature and non-positive definite Ricci tensor.*

On the other hand, writing Ric for the Ricci tensor of M, it follows from the Gauss equation that

$$\text{(9)} \qquad \mathrm{Ric}(X, X) = \bar{c} - \bar{K}(X \wedge N) - \mathrm{tr}(A)\langle AX, X\rangle + \langle AX, AX\rangle$$

for every unit vector X tangent to M, where $\bar{K}(X \wedge N)$ stands for the sectional curvature in $\bar{M}$ of the timelike plane $X \wedge N$. In order to compute $\bar{K}(X \wedge N)$, observe that

$$Y = \cosh\theta X + \sinh\theta N\,,$$

$$T = \sinh\theta X + \cosh\theta N\,,$$

where $\theta \in R$ is given by

$$\tanh\theta = -\frac{\langle X, \partial_t\rangle}{\langle N, \partial_t\rangle}\,,$$

form an orthonormal basis of $X \wedge N$ which satisfies $\langle Y, \partial_t\rangle = 0$. Writing now $T = -\langle T, \partial_t\rangle\partial_t + T^F$, where the superscript F denotes the projection on the fiber, and using the expression for the curvature tensor $\bar{R}$ of $\bar{M}$ in terms of its warping function and the curvature of the fiber [10, Chapter 7], it is not difficult to obtain

$$\langle \bar{R}(Y, T)T, Y\rangle = \langle T, \partial_t\rangle^2 \langle \bar{R}(Y, \partial_t)\partial_t, Y\rangle + \langle \bar{R}(Y, T^F)T^F, Y\rangle .$$

Moreover,

$$\langle \bar{R}(Y, \partial_t)\partial_t, Y\rangle = \frac{-f''}{f} ,$$

and

$$\langle \bar{R}(Y, T^F)T^F, Y\rangle = \langle T^F, T^F\rangle \frac{(f')^2}{f^2} + \frac{1}{f^2}\langle R^F(Y, T^F)T^F, Y\rangle ,$$

where R^F stands for the curvature tensor of the fiber. Using now (3), we conclude that

$$\bar{K}(X\wedge N) = \bar{K}(Y\wedge T) = \frac{\bar{c}}{n} - \left(g(R^F(Y, T^F)T^F, Y) - \frac{c}{n-1} g(T^F, T^F) \right). \tag{10}$$

Now, if M is a compact spacelike hypersurface in $\bar{M}$ with constant mean curvature, we know from Proposition 1 that it is totally umbilical, and (9) and (10) yield

$$\mathrm{Ric}(X, X) = (n-1)\left(\frac{\bar{c}}{n} - H^2 \right) + g(R^F(Y, T^F)T^F, Y) - \frac{c}{n-1} g(T^F, T^F) , \tag{11}$$

for every unit vector X tangent to M. This expression is specially meaningful when (F, g) has constant sectional curvature C, that is, when $\bar{M}$ is a classical RW spacetime. In that case $C = c/(n-1)$ and Corollary 3 can be stated, using (11), as follows:

COROLLARY 4. *Let $\bar{M}$ be a spatially closed classical* RW *spacetime with constant sectional curvature $\bar{C}$. Then, the only compact spacelike hypersurfaces in $\bar{M}$ with constant mean curvature H such that $H^2 \geq \bar{C}$ are the spacelike slices.*

The following special case of Corollary 4 yields a full geometric answer to our main question in this paper.

COROLLARY 5. *Let $\bar{M}$ be a spatially closed classical* RW *spacetime with non-positive constant sectional curvature. Then, the only compact spacelike hypersurfaces in $\bar{M}$ with constant mean curvature are the spacelike slices.*

Note that the CMC spacelike hypersurfaces in de Sitter space S_1^{n+1} show that Corollary 5 cannot be extended to RW spaces with positive constant curvature.

Finally, taking into account the constant mean curvature differential equation (1) for spacelike graphs in $\bar{M}$ we can give the following uniqueness results.

THEOREM 6. *Let (F, g) be a flat compact Riemannian manifold. Let H be a real constant and let $f: R \to (0, \infty)$ be one of the functions*

$$f(t)=a\,, \qquad a>0\,,$$

and

$$f(t)=ae^{bt}\,, \quad a>0\,,\ b\neq 0 \qquad \text{with} \quad b^2\leq H^2\,.$$

The only solutions $u\colon F\to R$ to the constant mean curvature differential equation on (F, g)

$$nH(f^2(u)-g(\nabla u, \nabla u))^{3/2}=(f^2(u)-g(\nabla u, \nabla u))\left(nf'(u)+\frac{1}{f(u)}\,\Delta u\right)$$
$$+\frac{1}{f(u)}\,\nabla^2 u(\nabla u, \nabla u)-f'(u)g(\nabla u, \nabla u)\,,$$

with

$$g(\nabla u, \nabla u)<f^2(u)\,,$$

are the constant functions.

THEOREM 7. *Let (F, g) be a hyperbolic compact Riemannian manifold with sectional curvature -1. Let H be a real constant and let $f\colon I\subset R\to(0, \infty)$ be one of the functions*

$$f(t)=ae^{bt}-\frac{1}{4ab^2}\,e^{-bt}\,, \quad a\neq 0\,,\ b>0 \qquad \text{with} \quad b^2\leq H^2\,,$$

$$f(t)=t+a\,,$$

and

$$f(t)=a_1\cos bt+a_2\sin bt\,, \quad b>0\,,\ a_1^2+a_2^2=\frac{1}{b^2}\,.$$

The only solutions $u\colon F\to R$ to the constant mean curvature differential equation on (F, g)

$$nH(f^2(u)-g(\nabla u, \nabla u))^{3/2}=(f^2(u)-g(\nabla u, \nabla u))\left(nf'(u)+\frac{1}{f(u)}\,\Delta u\right)$$
$$+\frac{1}{f(u)}\,\nabla^2 u(\nabla u, \nabla u)-f'(u)g(\nabla u, \nabla u)\,,$$

with

$$g(\nabla u, \nabla u)<f^2(u)\,,$$

are the constant functions.

4. Some remarks and applications.

REMARK 1. It should be noted that, even without any assumption on the curvature of $\bar{M}$ or F, if the warping function of a GRW space is constant then the only compact spacelike hypersurfaces in $\bar{M}$ with signed mean curvature function $H \leq 0$ or $H \geq 0$ are the spacelike slices, which are totally geodesic. This follows from (7), by observing that in that case $\operatorname{div}(\xi^T)$ is, up to a non-zero constant, the Laplacian of π. As a consequence, on every compact Riemannian manifold (F, g) and for every signed function H on F, the only solutions to

$$(1-g(\nabla u, \nabla u))\Delta u + \nabla^2 u(\nabla u, \nabla u) \geq 0 \quad \text{or} \ \leq 0 ,$$

with

$$g(\nabla u, \nabla u) < 1 ,$$

are the constant functions (cf. [2]).

REMARK 2. Recall that if M is a compact spacelike hypersurface in $\bar{M}$, then its projection $X = \pi_F \circ x$ on the fiber is a covering map. When M is assumed to be just complete, this result still holds if $f(\pi_I)$ is bounded on M. Thus if the fiber is also simply connected, then M is compact (see [2] for details). In particular, in the cases 4 and 6 of Table, *every complete spacelike hypersurface of constant mean curvature is totally umbilical, when the fiber is simply connected.*

We point out that there are simply connected compact Riemannian manifolds which can be taken as fibers in these cases. For example, let F be a complex hypersurface of degree d in an m-dimensional complex projective space, $m > 2$ and $d \geq m+1$, (F must be simply connected by the Lefschetz hyperplane theorem). If $d = m+1$ then the first Chern class of F is zero and, from a well known result by Yau (see, for instance, [14, Theorems 1 and 2]), F admits a non-flat Riemannian metric which is Ricci flat; in particular, when $m = 3$, M is a $K3$ surface. When $d > m+1$, a similar result by Yau [14, Theorems 1 and 3] says that there is a negatively Ricci curved Einstein metric on F.

REMARK 3. It follows from the Gauss equation (9) that

$$\operatorname{Ric}(X, X) \geq \bar{c} - \frac{n^2 H^2}{4} - \bar{K}(X \wedge N) ,$$

for every unit vector X tangent to M. So, if there exists an upper bound $c_1 < \bar{c}$ for the sectional curvature of timelike planes, we know from Bonnet-Myers' theorem that every complete spacelike hypersurface of constant mean curvature H satisfying

$$H^2 < \frac{4(\bar{c} - c_1)}{n^2}$$

is compact and Proposition 1 implies that it is totally umbilical. For instance, suppose that $\bar{M}$ has constant sectional curvature $\bar{C}$, that is, F has also constant sectional curvature C. If $\bar{C} > 0$ (the first three cases in Table) then $\bar{c} = n\bar{C}$ and we can take $c_1 = \bar{C}$.

Therefore, every complete spacelike hypersurface of constant mean curvature H such that $H^2 < 4(n-1)\bar{C}/n^2$ is totally umbilical (compare with the Theorem in [1]).

REMARK 4. It is worth pointing out that Theorems 6 and 7 yield also some consequences about solutions to the CMC differential equation (1) when the Riemannian manifold (F, g) is either the Euclidean space R^n or the hyperbolic space H^n. In fact, if $u: R^n \to R$ (resp. $u: H^n \to R$) is a solution to (1) with f as in Theorem 6 (resp. Theorem 7), and u is invariant under a discrete subgroup Γ of rigid motions of R^n (resp. isometries of H^n) with compact quotient R^n/Γ (resp. H^n/Γ), then u must be constant.

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