TOKYO J. MATH.
VOL. 25, NO. 2, 2002

Spacelike Maximal Surfaces in 4-dimensional Space Forms of Index 2

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(Communicated by R. Miyaoka)

Abstract. We give necessary and sufficient conditions for the existence of spacelike maximal surfaces in 4-dimensional space forms of index 2. We also discuss spacelike maximal surfaces with constant Gaussian curvature or constant normal curvature, and a rigidity type problem.

1. Introduction.

Let $N_p^n(c)$ denote the n-dimensional simply connected semi-Riemannian space form of constant curvature c and index p, where we write $N^n(c)$ if $p = 0$. We are interested in comparing the geometry of minimal surfaces in $N^4(c)$, spacelike minimal surfaces in $N_1^4(c)$, and spacelike maximal surfaces in $N_2^4(c)$.

In [2], Guadalupe and Tribuzy gave necessary and sufficient conditions for the existence of minimal surfaces in $N^4(c)$, which are generalizations of the Ricci condition for minimal surfaces in $N^3(c)$ (cf. [4]). In the previous paper [7], we obtained a Lorentzian version of their result for spacelike minimal surfaces in $N_1^4(c)$. In this paper, we will discuss the case of spacelike maximal surfaces in $N_2^4(c)$.

Let M be a spacelike maximal surface in $N_2^4(c)$ with Gaussian curvature K and normal curvature K_ν. Then $K \geq c$, where the equality holds at p if and only if p is a geodesic point. Also we have $(K-c)^2 - K_\nu^2 \geq 0$, or $K - c \geq |K_\nu|$, where the equality holds at p if and only if p is an isotropic point.

THEOREM 1. (i) *Let M be a spacelike maximal surface in $N_2^4(c)$. We denote by K, K_ν and Δ the Gaussian curvature, the normal curvature and the Laplacian of M, respectively. Then*

$$\Delta \log(K - c + K_\nu) = 2(2K + K_\nu)\,, \tag{1.1}$$

$$\Delta \log(K - c - K_\nu) = 2(2K - K_\nu) \tag{1.2}$$

at non-isotropic points.

(ii) *Conversely, let M be a 2-dimensional simply connected Riemannian manifold with Gaussian curvature $K(> c)$ and Laplacian Δ. If K_ν is a function on M satisfying $(K-c)^2 -$*

Received November 13, 2001

$K_\nu^2 > 0$ and (1.1), (1.2), *then there exists an isometric maximal immersion of M into $N_2^4(c)$ with normal curvature K_ν.*

THEOREM 2. *Let $f : M \to N_2^4(c)$ be a non-isotropic isometric maximal immersion of a 2-dimensional simply connected Riemannian manifold M into $N_2^4(c)$ with normal curvature K_ν. Then there exists a π-periodic family of isometric maximal immersions $f_\theta : M \to N_2^4(c)$ with the same normal curvature K_ν. Moreover, if $\tilde{f} : M \to N_2^4(c)$ is another isometric maximal immersion with the same normal curvature K_ν, then there exists $\theta \in [0, \pi]$ such that $\tilde{f}$ and f_θ coincide up to congruence.*

THEOREM 3. (i) *Let M be an isotropic spacelike maximal surface in $N_2^4(c)$ with Gaussian curvature K and Laplacian Δ. Then*

$$\Delta \log(K - c) = 2(3K - c) \tag{1.3}$$

at non-geodesic points.

(ii) *Conversely, let M be a 2-dimensional simply connected Riemannian manifold with Gaussian curvature $K (> c)$ and Laplacian Δ. If M satisfies* (1.3), *then there exists an isotropic isometric maximal immersion f of M into $N_2^4(c)$. Moreover, if $\tilde{f} : M \to N_2^4(c)$ is another isotropic isometric maximal immersion, then $\tilde{f}$ and f coincide up to congruence.*

Next we discuss spacelike maximal surfaces with constant Gaussian curvature in $N_2^4(c)$. By Theorem 3 (ii), we can see that for $c < 0$, there exists an isotropic isometric maximal immersion of the hyperbolic plane of constant curvature $c/3$ into $N_2^4(c)$.

We note that $N_1^3(c)$ is naturally included in $N_2^4(c)$. Let $R_2^4 = N_2^4(0)$ be the 4-dimensional semi-Euclidean space with coordinate system (x_1, x_2, x_3, x_4) and metric

$$ds^2 = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2 .$$

For $c < 0$, set

$$H_1^3(c) = \{(x_1, x_2, x_3, x_4) \in R_2^4 | x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1/c\} ,$$

whose universal covering space is $N_1^3(c)$. We define a map $F : R^2 \to H_1^3(c)$ by

$$F(u, v) = \frac{1}{\sqrt{-2c}}(\sinh(\sqrt{-2c} \cdot u), \sinh(\sqrt{-2c} \cdot v), \cosh(\sqrt{-2c} \cdot u), \cosh(\sqrt{-2c} \cdot v)) .$$

Then the surface given by F is a unique flat spacelike maximal surface in $H_1^3(c)$. Let $\tilde{F} : R^2 \to N_1^3(c)$ be the lift of F.

THEOREM 4. *Let M be a spacelike maximal surface with constant Gaussian curvature K in $N_2^4(c)$. Then either* (i) *$K = c$ and M is totally geodesic,* (ii) *$c < 0$, $K = c/3$ and M is isotropic, or* (iii) *$c < 0$, $K = 0$ and M is congruent to the surface given by $\tilde{F}$ in a totally geodesic $N_1^3(c)$.*

REMARK 1. (i) Theorem 4 should be compared with the Riemannian case in [3].

(ii) The author does not know the explicit representation of the surface in the case (ii) of Theorem 4.

We also discuss spacelike maximal surfaces with constant normal curvature in $N_2^4(c)$.

THEOREM 5. *Let M be a spacelike maximal surface with constant normal curvature K_ν in $N_2^4(c)$. Then either* (i) *M lies in a totally geodesic $N_1^3(c)$, or* (ii) *$c < 0$ and M has constant Gaussian curvature $c/3$.*

Finally we give the following rigidity type theorem.

THEOREM 6. *Let M be a spacelike maximal surface in $N_2^4(c)$. If M is locally isometric to a spacelike maximal surface in $N_1^3(c)$, then M lies in a totally geodesic $N_1^3(c)$.*

REMARK 2. Theorem 6 should be compared with the Riemannian case in [6].

Our results suggest that the geometry of spacelike maximal surfaces in $N_2^4(c)$ is somewhat similar to that of minimal surfaces in $N^4(c)$. But it seems that the Lorentzian case is different from these two cases (cf. [7]).

The author wishes to thank the referee for useful comments.

2. Preliminaries.

In this section, we recall the method of moving frames for spacelike surfaces in $N_2^4(c)$. Unless otherwise stated, we shall use the following convention on the ranges of indices:

$$1 \le A, B, \cdots \le 4, \quad 1 \le i, j, \cdots \le 2, \quad 3 \le \alpha, \beta, \cdots \le 4.$$

Let $\{e_A\}$ be a local orthonormal frame field in $N_2^4(c)$, and $\{\omega^A\}$ be the dual coframe. Here the metric of $N_2^4(c)$ is given by

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 - (\omega^3)^2 - (\omega^4)^2 .$$

We can define the connection forms $\{\omega_B^A\}$ by

$$de_B = \sum_A \omega_B^A e_A .$$

Then

$$\omega_j^i + \omega_i^j = 0, \quad \omega_\beta^\alpha + \omega_\alpha^\beta = 0, \quad \omega_\alpha^i = \omega_i^\alpha . \tag{2.1}$$

The structure equations are given by

$$d\omega^A = -\sum_B \omega_B^A \wedge \omega^B , \tag{2.2}$$

$$d\omega_B^A = -\sum_C \omega_C^A \wedge \omega_B^C + \frac{1}{2}\sum_{C,D} R^A{}_{BCD}\omega^C \wedge \omega^D , \tag{2.3}$$

$$R^A{}_{BCD} = c\varepsilon_B(\delta^A{}_C\delta_{BD} - \delta^A{}_D\delta_{BC}) , \tag{2.4}$$

where $\varepsilon_i = 1$ and $\varepsilon_\alpha = -1$.

Let M be a spacelike surface in $N_2^4(c)$, that is, the induced metric on M is Riemannian. We choose the frame $\{e_A\}$ so that $\{e_i\}$ are tangent to M. Then $\omega^\alpha = 0$ on M. In the following, our argument will be restricted to M. By (2.2),

$$0 = -\sum_i \omega_i^\alpha \wedge \omega^i \,.$$

So there is a symmetric tensor h_{ij}^α such that

$$\omega_i^\alpha = \sum_j h_{ij}^\alpha \omega^j \,, \tag{2.5}$$

where h_{ij}^α are the components of the second fundamental form h of M. A point p on M is called isotropic if $\langle h(X,X), h(X,X)\rangle$ is constant for any unit tangent vector X at p. We say that M is isotropic if every point on M is isotropic.

The Gaussian curvature K and the normal curvature K_ν of M are given by

$$d\omega_2^1 = K\omega^1 \wedge \omega^2 \,, \quad d\omega_4^3 = K_\nu \omega^1 \wedge \omega^2 \,. \tag{2.6}$$

Then by (2.1), (2.3), (2.4) and (2.5) we have

$$K = c - h_{11}^3 h_{22}^3 + (h_{12}^3)^2 - h_{11}^4 h_{22}^4 + (h_{12}^4)^2 \,, \tag{2.7}$$

$$K_\nu = -(h_{11}^3 h_{12}^4 - h_{12}^3 h_{11}^4 + h_{12}^3 h_{22}^4 - h_{22}^3 h_{12}^4) \,. \tag{2.8}$$

The mean curvature vector H of M is given by

$$H = \frac{1}{2}\sum_{i,\alpha} h_{ii}^\alpha e_\alpha \,.$$

The surface M is called maximal if $H = 0$ on M.

In the following we assume that M is maximal. Then by (2.7) and (2.8),

$$K = c + (h_{11}^3)^2 + (h_{12}^3)^2 + (h_{11}^4)^2 + (h_{12}^4)^2 \,, \quad K_\nu = -2(h_{11}^3 h_{12}^4 - h_{12}^3 h_{11}^4) \,.$$

Thus we have $K \geq c$, where the equality holds at p if and only if p is a geodesic point. By the computation we can show that

$$\begin{aligned} (K-c)^2 - K_\nu^2 &= \{(h_{11}^3)^2 + (h_{11}^4)^2 - (h_{12}^3)^2 - (h_{12}^4)^2\}^2 + 4(h_{11}^3 h_{12}^3 + h_{11}^4 h_{12}^4)^2 \\ &= \{(h_{11}^3)^2 + (h_{12}^3)^2 - (h_{11}^4)^2 - (h_{12}^4)^2\}^2 + 4(h_{11}^3 h_{11}^4 + h_{12}^3 h_{12}^4)^2 \geq 0 \,, \end{aligned} \tag{2.9}$$

where the equality holds at p if and only if p is an isotropic point.

Around a non-isotropic point where $(K-c)^2 - K_\nu^2 > 0$, by (2.9), we may choose a smooth function θ so that

$$\{(h_{11}^3)^2 + (h_{12}^3)^2 - (h_{11}^4)^2 - (h_{12}^4)^2\}\sin 2\theta + 2(h_{11}^3 h_{11}^4 + h_{12}^3 h_{12}^4)\cos 2\theta = 0 \,.$$

Set

$$\tilde{e}_3 = e_3 \cos\theta - e_4 \sin\theta \,, \quad \tilde{e}_4 = e_3 \sin\theta + e_4 \cos\theta \,,$$

and let $\tilde{h}_{ij}^\alpha$ be the components of h with respect to the frame $\{e_i, \tilde{e}_\alpha\}$. Then we have

$$\tilde{h}_{11}^3 \tilde{h}_{11}^4 + \tilde{h}_{12}^3 \tilde{h}_{12}^4 = 0 \,.$$

By (2.9) we may assume that $(\tilde{h}^3_{11})^2 + (\tilde{h}^3_{12})^2 > (\tilde{h}^4_{11})^2 + (\tilde{h}^4_{12})^2$. Then we may choose the frame $\{e_i\}$ so that $\tilde{h}^3_{12} = 0$, and we have also $\tilde{h}^4_{11} = 0$. Therefore,

LEMMA 1. *Around a non-isotropic point on a spacelike maximal surface M in $N^4_2(c)$, we may choose the frame $\{e_A\}$ so that*

$$\omega^3_1 = a\omega^1 , \quad \omega^3_2 = -a\omega^2 , \quad \omega^4_1 = b\omega^2 , \quad \omega^4_2 = b\omega^1 , \quad a^2 > b^2 . \tag{2.10}$$

Here a and b are determined by K and K_ν through the equations:

$$a^2 + b^2 = K - c , \quad ab = -\frac{1}{2}K_\nu .$$

We assume that M is isotropic maximal and $K > c$. Then by (2.9) we have

$$(h^3_{11})^2 + (h^3_{12})^2 = (h^4_{11})^2 + (h^4_{12})^2 > 0 , \quad h^3_{11}h^4_{11} + h^3_{12}h^4_{12} = 0 .$$

So $h^3_{12} \neq 0$ or $h^4_{12} \neq 0$. Then we may choose the frame $\{e_\alpha\}$ such that $h^3_{12} = 0$, and we have also $h^4_{11} = 0$. Therefore,

LEMMA 2. *On an isotropic spacelike maximal surface M with $K > c$ in $N^4_2(c)$, we may choose the frame $\{e_\alpha\}$ so that*

$$\omega^3_1 = a\omega^1 , \quad \omega^3_2 = -a\omega^2 , \quad \omega^4_1 = a\omega^2 , \quad \omega^4_2 = a\omega^1 . \tag{2.11}$$

Here a satisfies $2a^2 = K - c$.

3. Proof of Theorems 1 and 2.

PROOF OF THEOREM 1. (i) Around a non-isotropic point, using (2.2), (2.3), (2.4) and (2.10), we have

$$\begin{aligned} d\omega^3_1 &= da \wedge \omega^1 - a\omega^1_2 \wedge \omega^2 \\ &= -\omega^3_2 \wedge \omega^2_1 - \omega^3_4 \wedge \omega^4_1 \\ &= a\omega^2 \wedge \omega^2_1 - \omega^3_4 \wedge b\omega^2 . \end{aligned}$$

So, using the notation like

$$da = a_1\omega^1 + a_2\omega^2 , \quad db = b_1\omega^1 + b_2\omega^2 ,$$

$$\omega^1_2 = (\omega^1_2)_1\omega^1 + (\omega^1_2)_2\omega^2 = -\omega^2_1 , \quad \omega^3_4 = (\omega^3_4)_1\omega^1 + (\omega^3_4)_2\omega^2 = -\omega^4_3 ,$$

we get

$$2a(\omega^1_2)_1 - b(\omega^3_4)_1 = -a_2 .$$

Similarly, from the exterior derivative of ω^3_2, ω^4_1 and ω^4_2,

$$\begin{aligned} 2a(\omega^1_2)_2 - b(\omega^3_4)_2 &= a_1 , \\ 2b(\omega^1_2)_2 - a(\omega^3_4)_2 &= b_1 , \\ 2b(\omega^1_2)_1 - a(\omega^3_4)_1 &= -b_2 . \end{aligned}$$

Thus we have

$$2a\omega_2^1 - b\omega_4^3 = *da\,, \quad 2b\omega_2^1 - a\omega_4^3 = *db\,,$$

where $*$ denotes the Hodge star operator on M. Noting that

$$K = c + a^2 + b^2\,, \quad K_\nu = -2ab\,,$$

$$(K-c)^2 - K_\nu^2 = (a^2-b^2)^2\,,$$

we get

$$\omega_2^1 = \frac{1}{4} * d\log|a^2-b^2| = \frac{1}{8} * d\log\{(K-c)^2 - K_\nu^2\}\,, \tag{3.1}$$

$$\omega_4^3 = \frac{b*da - a*db}{a^2-b^2} = \frac{1}{4} * d\log\left(\frac{K-c+K_\nu}{K-c-K_\nu}\right)\,. \tag{3.2}$$

Taking the exterior derivative of these equations, together with (2.6), we have

$$\Delta\log\{(K-c)^2 - K_\nu^2\} = 8K\,, \tag{3.3}$$

$$\Delta\log\left(\frac{K-c+K_\nu}{K-c-K_\nu}\right) = 4K_\nu\,. \tag{3.4}$$

By (3.3) $\pm$ (3.4), we obtain the equations (1.1) and (1.2).

(ii) We may assume that M is a small neighborhood. Let ds^2 be the metric on M. By (1.1) + (1.2)

$$\Delta\log\{(K-c)^2 - K_\nu^2\} = 8K\,,$$

which implies that the metric

$$d\hat{s}^2 = \{(K-c)^2 - K_\nu^2\}^{1/4} ds^2$$

is flat. So there exists a coordinate system (x^1, x^2) such that

$$ds^2 = \{(K-c)^2 - K_\nu^2\}^{-1/4}\{(dx^1)^2 + (dx^2)^2\}\,.$$

Set

$$\omega^i = \{(K-c)^2 - K_\nu^2\}^{-1/8} dx^i\,, \tag{3.5}$$

so that $\{\omega^i\}$ is an orthonormal coframe field with dual frame $\{e_i\}$. By

$$d\omega^1 = -\omega_2^1 \wedge \omega^2\,, \quad d\omega^2 = -\omega_1^2 \wedge \omega^1\,,$$

we can find that the connection form $\omega_2^1 = -\omega_1^2$ is given by

$$\omega_2^1 = -\omega_1^2 = \frac{1}{8} * d\log\{(K-c)^2 - K_\nu^2\}\,.$$

As $(K-c)^2 - K_\nu^2 > 0$, we may choose smooth functions a and b so that

$$a^2 + b^2 = K - c\,, \quad ab = -\frac{1}{2}K_\nu\,, \quad a^2 > b^2\,.$$

Let E be a 2-plane bundle over M with metric $\langle\ ,\ \rangle$ and orthonormal sections $\{e_\alpha\}$ such that $\langle e_\alpha, e_\beta\rangle = -\delta_{\alpha\beta}$. Let h be a symmetric section of $\mathrm{Hom}(TM\times TM, E)$ such that

$$(h^3_{ij}) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad (h^4_{ij}) = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix},$$

and set

$$\omega^3_1 = \omega^1_3 = a\omega^1, \quad \omega^3_2 = \omega^2_3 = -a\omega^2,$$
$$\omega^4_1 = \omega^1_4 = b\omega^2, \quad \omega^4_2 = \omega^2_4 = b\omega^1.$$

We define a compatible connection ${}^\perp\nabla$ of E so that

$${}^\perp\nabla e_3 = \omega^4_3 e_4, \quad {}^\perp\nabla e_4 = \omega^3_4 e_3,$$

where

$$\omega^3_4 = -\omega^4_3 = \frac{1}{4} * d\log\left(\frac{K-c+K_\nu}{K-c-K_\nu}\right).$$

Now, almost reversing the argument in (i), we can find that $\{\omega^A_B\}$ satisfy the structure equations:

$$d\omega^1_2 = -\omega^1_3\wedge\omega^3_2 - \omega^1_4\wedge\omega^4_2 + c\omega^1\wedge\omega^2,$$
$$d\omega^3_1 = -\omega^3_2\wedge\omega^2_1 - \omega^3_4\wedge\omega^4_1, \quad d\omega^3_2 = -\omega^3_1\wedge\omega^1_2 - \omega^3_4\wedge\omega^4_2,$$
$$d\omega^4_1 = -\omega^4_2\wedge\omega^2_1 - \omega^4_3\wedge\omega^3_1, \quad d\omega^4_2 = -\omega^4_1\wedge\omega^1_2 - \omega^4_3\wedge\omega^3_2,$$
$$d\omega^3_4 = -\omega^3_1\wedge\omega^1_4 - \omega^3_2\wedge\omega^2_4,$$

which are the integrability conditions. Therefore, by the fundamental theorem, there exists an isometric immersion of M into $N^4_2(c)$, which is maximal and has normal curvature K_ν.

Let us note the following fact.

PROPOSITION. *Let M be a spacelike maximal surface in $N^4_2(c)$. If the normal curvature K_ν of M is identically zero, then M lies in a totally geodesic $N^3_1(c)$.*

PROOF. When M is isotropic, by (2.9), $K = c$ and M is totally geodesic. When M is non-isotropic, from the argument in the proof of Theorem 1, we have $\omega^1_4 = \omega^2_4 = \omega^3_4 = 0$, and we get the conclusion.

PROOF OF THEOREM 2. For $f : M \to N^4_2(c)$, let a, b and ω^A_B be as in the proof of Theorem 1. For each $\theta \in [0, \pi]$, let $h(\theta)$ be a symmetric section of $\mathrm{Hom}(TM\times TM, T^\perp M)$ such that

$$(h^3_{ij}(\theta)) = \begin{pmatrix} a\cos 2\theta & a\sin 2\theta \\ a\sin 2\theta & -a\cos 2\theta \end{pmatrix}, \quad (h^4_{ij}(\theta)) = \begin{pmatrix} -b\sin 2\theta & b\cos 2\theta \\ b\cos 2\theta & b\sin 2\theta \end{pmatrix},$$

and set

$$\omega_1^3(\theta) = \omega_3^1(\theta) = (a\cos 2\theta)\omega^1 + (a\sin 2\theta)\omega^2 = \omega_1^3\cos 2\theta - \omega_2^3\sin 2\theta\,,$$
$$\omega_2^3(\theta) = \omega_3^2(\theta) = (a\sin 2\theta)\omega^1 - (a\cos 2\theta)\omega^2 = \omega_1^3\sin 2\theta + \omega_2^3\cos 2\theta\,,$$
$$\omega_1^4(\theta) = \omega_4^1(\theta) = -(b\sin 2\theta)\omega^1 + (b\cos 2\theta)\omega^2 = \omega_1^4\cos 2\theta - \omega_2^4\sin 2\theta\,,$$
$$\omega_2^4(\theta) = \omega_4^2(\theta) = (b\cos 2\theta)\omega^1 + (b\sin 2\theta)\omega^2 = \omega_1^4\sin 2\theta + \omega_2^4\cos 2\theta\,.$$

Let $\omega_2^1(\theta) = -\omega_1^2(\theta) = \omega_2^1$ and $\omega_4^3(\theta) = -\omega_3^4(\theta) = \omega_4^3$, for convenience. Then by the computation, we can see that $\{\omega_B^A(\theta)\}$ satisfy the structure equations. Hence, for each $\theta \in [0,\pi]$, there exists an isometric maximal immersion $f_\theta : M \to N_2^4(c)$ with the same normal curvature K_ν.

Let $\tilde{f} : M \to N_2^4(c)$ be another isometric maximal immersion with the same normal curvature K_ν. By Lemma 1, we may choose the frame $\{\tilde{e}_A\}$ so that

$$\tilde{\omega}_1^3 = a\tilde{\omega}^1\,, \quad \tilde{\omega}_2^3 = -a\tilde{\omega}^2\,, \quad \tilde{\omega}_1^4 = b\tilde{\omega}^2\,, \quad \tilde{\omega}_2^4 = b\tilde{\omega}^1\,.$$

Then as in (3.1) and (3.2), we have $\tilde{\omega}_2^1 = \omega_2^1$ and $\tilde{\omega}_4^3 = \omega_4^3$. Also as in (3.5), there exists a coordinate system $\{\tilde{x}^1, \tilde{x}^2\}$ such that

$$\tilde{\omega}^i = \{(K-c)^2 - K_\nu^2\}^{-1/8} d\tilde{x}^i\,.$$

Let θ be the angle between $\partial/\partial x^1$ and $\partial/\partial\tilde{x}^1$. Then using

$$\frac{\partial}{\partial\tilde{x}^1} = \cos\theta\frac{\partial}{\partial x^1} + \sin\theta\frac{\partial}{\partial x^2}\,, \qquad \frac{\partial}{\partial\tilde{x}^2} = -\sin\theta\frac{\partial}{\partial x^1} + \cos\theta\frac{\partial}{\partial x^2}\,,$$

together with $[\partial/\partial\tilde{x}^1, \partial/\partial\tilde{x}^2] = 0$, we find that θ is constant. We note that

$$e_1 = (\cos\theta)\tilde{e}_1 - (\sin\theta)\tilde{e}_2\,, \qquad e_2 = (\sin\theta)\tilde{e}_1 + (\cos\theta)\tilde{e}_2\,.$$

By the computation, we can see that the connection forms along $\tilde{f}$ with respect to the frame $\{e_i, \tilde{e}_\alpha\}$ are the same as those along f_θ with respect to $\{e_i, e_\alpha\}$. That is, with respect to those frames, $\tilde{f}$ and f_θ have the same second fundamental forms and normal connections. Therefore $\tilde{f}$ and f_θ coincide up to congruence.

4. Proof of Theorem 3.

(i) As in Section 3, from the exterior derivative of (2.11), we can get

$$a(2\omega_2^1 - \omega_4^3) = *da\,.$$

Noting that

(4.1) $$K - c = -K_\nu = 2a^2\,,$$

we have

$$2\omega_2^1 - \omega_4^3 = \frac{1}{2} * d\log(K-c)$$

at points where $K > c$. Taking the exterior derivative of this equation, together with (2.6) and (4.1), we obtain the equation (1.3).

(ii) We may assume that M is a small neighborhood. Let $\{\omega^i\}$ be an orthonormal coframe field with dual frame $\{e_i\}$ and connection form $\omega_2^1 = -\omega_1^2$. Let E be a 2-plane bundle over M with metric $\langle\ ,\ \rangle$ and orthonormal sections $\{e_\alpha\}$ such that $\langle e_\alpha, e_\beta\rangle = -\delta_{\alpha\beta}$. Set $a = \sqrt{(K-c)/2}$. Let h be a symmetric section of $\mathrm{Hom}(TM \times TM, E)$ such that

$$(h_{ij}^3) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix},$$

and set

$$\begin{aligned} \omega_1^3 &= \omega_3^1 = a\omega^1, \quad \omega_2^3 = \omega_3^2 = -a\omega^2, \\ \omega_1^4 &= \omega_4^1 = a\omega^2, \quad \omega_2^4 = \omega_4^2 = a\omega^1. \end{aligned}$$

We define a compatible connection ${}^\perp\nabla$ of E so that

$${}^\perp\nabla e_3 = \omega_3^4 e_4, \quad {}^\perp\nabla e_4 = \omega_4^3 e_3,$$

where

$$\omega_4^3 = -\omega_3^4 = 2\omega_2^1 - \frac{1}{2} * d\log(K-c). \tag{4.2}$$

By the computation, we can show that $\{\omega_B^A\}$ satisfy the structure equations. Therefore, there exists an isometric immersion f of M into $N_2^4(c)$, which is maximal and isotropic.

Let $\tilde{f} : M \to N_2^4(c)$ be another isotropic isometric maximal immersion. By Lemma 2, we may choose the frame $\{\tilde{e}_\alpha\}$ so that, with respect to the frame $\{e_i, \tilde{e}_\alpha\}$,

$$\tilde{\omega}_1^3 = a\omega^1, \quad \tilde{\omega}_2^3 = -a\omega^2, \quad \tilde{\omega}_1^4 = a\omega^2, \quad \tilde{\omega}_2^4 = a\omega^1.$$

Then as in (4.2), we have $\tilde{\omega}_4^3 = \omega_4^3$. With respect to the frames $\{e_i, \tilde{e}_\alpha\}$ and $\{e_i, e_\alpha\}$, $\tilde{f}$ and f have the same second fundamental forms and normal connections. Hence $\tilde{f}$ and f coincide up to congruence.

5. Proof of Theorem 4.

When M is isotropic, from the equation (1.3), we have either $K = c$, or $K = c/3$ $(c < 0)$. In the following we consider the case that M is non-isotropic.

As K is constant, using the equations (1.1) and (1.2), we get

$$\Delta K_\nu = 2(5K - c)K_\nu + \frac{2K_\nu^3}{K-c} =: P(K_\nu),$$

$$|\nabla K_\nu|^2 = -4K(K-c)^2 + 2(K+c)K_\nu^2 + \frac{2K_\nu^4}{K-c} =: Q(K_\nu),$$

where ∇ is the Riemannian connection of M. By Lemma 3.3 of [1], on $M_1 = \{p \in M | \nabla K_\nu \neq 0\}$ we have

$$KQ + (P - Q')\left(P - \frac{1}{2}Q'\right) + Q\left(P' - \frac{1}{2}Q''\right) = 0,$$

where the prime denotes the differentiation with respect to K_ν. By the computation, this equation turns to

$$-4K(9K-4c)(K-c)^2 + (90K^2 - 86cK + 16c^2)K_\nu^2 - \frac{2(27K-8c)}{K-c}K_\nu^4 = 0\,,$$

which is a nontrivial equation of K_ν. Thus K_ν must be constant on M_1, and we have a contradiction if M_1 is nonempty. So M_1 is empty and K_ν is constant. Then by (1.1) and (1.2) we have $K = K_\nu = 0$ ($c < 0$). By the Proposition, M lies in a totally geodesic $N_1^3(c)$, and M is congruent to the surface given by $\tilde{F}$ in the introduction. Thus the proof is complete.

6. Proof of Theorem 5.

Assume that M does not lie in any totally geodesic $N_1^3(c)$. Then by the Proposition, K_ν is a non-zero constant. When M is isotropic, K is also constant by (2.9). So by Theorem 4, we have $c < 0$ and $K = c/3$. In the following we consider the case that M is non-isotropic.

As K_ν is a non-zero constant, using the equations (1.1) and (1.2), we get

$$\Delta K = 10K^2 - 12cK + 2c^2 + 2K_\nu^2 =: P(K)\,,$$
$$|\nabla K|^2 = 2(3K-c)\{(K-c)^2 - K_\nu^2\} =: Q(K)\,.$$

By Lemma 3.3 of [1], on $M_1 = \{p \in M | \nabla K \neq 0\}$ we have

$$KQ + (P - Q')\left(P - \frac{1}{2}Q'\right) + Q\left(P' - \frac{1}{2}Q''\right) = 0\,,$$

where the prime denotes the differentiation with respect to K. By the computation, this equation turns to

$$10(K^2 - cK + 2c^2 - 4K_\nu^2)\{(K-c)^2 - K_\nu^2\} = 0\,,$$

which is a nontrivial equation of K. Thus K must be constant on M_1, and we have a contradiction if M_1 is nonempty. So M_1 is empty and K is constant. But by Theorem 4, there are no non-isotropic spacelike maximal surfaces with constant Gaussian curvature and non-zero constant normal curvature in $N_2^4(c)$. So we have a contradiction. Thus we have proved the theorem.

7. Proof of Theorem 6.

Assume that M does not lie in any totally geodesic $N_1^3(c)$. Set

$$M_1 = \{p \in M | K > c,\ K_\nu \neq 0\}\ (\neq \emptyset)\,.$$

We note that every spacelike maximal surface in $N_1^3(c)$ may be seen as a spacelike maximal surface with vanishing normal curvature in $N_2^4(c)$. As M is locally isometric to a spacelike maximal surface in $N_1^3(c)$, from the above note and Theorem 1, we have

$$\Delta \log(K-c) = 4K \tag{7.1}$$

on M_1.

If M is isotropic, then the equation (1.3) is valid on M_1. From (7.1) and (1.3) we have a contradiction. So M is not isotropic.

Set

$$M_2 = \{p \in M | K > c,\ K_\nu \neq 0,\ p \text{ is non-isotropic}\}\,.$$

Let $F = K_\nu/(K - c)$. Then by (1.1), (1.2) and (7.1) we get

$$\Delta F = 2(K - c)F(F^2 + 1)\,, \tag{7.2}$$

$$|\nabla F|^2 = 2(K - c)F^2(F^2 - 1) \tag{7.3}$$

on M_2. Let $\tilde{K}, \tilde{\nabla}, \tilde{\Delta}$ denote the Gaussian curvature, the Riemannian connection and the Laplacian of M_2 with respect to the metric $d\tilde{s}^2 = (K - c)ds^2$, respectively. Then

$$\tilde{K} = \frac{K}{K - c} - \frac{1}{2(K - c)}\Delta \log(K - c) = \frac{K}{c - K} \tag{7.4}$$

on M_2, where we use (7.1) for the second equality. The equations (7.2) and (7.3) can be rewritten as

$$\tilde{\Delta} F = 2F(F^2 + 1) =: P(F)\,, \tag{7.5}$$

$$|\tilde{\nabla} F|^2 = 2F^2(F^2 - 1) =: Q(F) \tag{7.6}$$

on M_2. As $0 < |F| < 1$ on M_2, $|\tilde{\nabla} F|^2 \neq 0$ on M_2 by (7.6). Hence by Lemma 3.3 of [1], we have

$$\tilde{K} Q + (P - Q')\left(P - \frac{1}{2}Q'\right) + Q\left(P' - \frac{1}{2}Q''\right) = 0 \tag{7.7}$$

on M_2, where the prime denotes the differentiation with respect to F. Noting that $0 < |F| < 1$ on M_2, we have by (7.4)–(7.7), $K = 8c/9$ on M_2. As $K > c$ on M_2, we find that $c < 0$. But by Theorem 4, there are no spacelike maximal surfaces with constant Gaussian curvature $8c/9$ in $N_2^4(c)$ where $c < 0$. So we have a contradiction.

Therefore, M lies in a totally geodesic $N_1^3(c)$.

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