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Spacelike Maximal Surfaces with Constant Scalar Normal Curvature in a Normal Contact Lorentzian Manifold

IKAWA TOSHIHIKO

Nihon University, Dept. of Math., School of Medicine, Itabashi, Tokyo, Japan, 173
e-mail: tikawa@med.nihon-u.ac.jp

Abstract. If the scalar normal curvature of a spacelike maximal surface in a 5-dimensional normal contact Lorentzian manifold with constant ϕ-sectional curvature is constant, then the surface is totally geodesic or nonpositively curved.

1. Introduction

On some odd dimensional manifolds, the normal contact Riemannian metric structure (or Sasakian structure) can be defined. The study of manifolds with this structure has a long history.

If we change the Riemannian metric of the Sasakian structure to a Lorentzian one, we can define the normal contact Lorentzian structure. This definition was given at the starting time of the study of the Sasakian structure. But practical study of it has not been given sufficiently yet (*cf.* [5], [7]). In [3], [4], we study the fundamental properties of manifolds with the normal contact Lorentzian structure.

In this paper, we shall study the scalar normal curvature for spacelike maximal surfaces in a 5-dimensional normal contact Lorentzian manifold of constant ϕ-sectional curvature and prove.

Theorem. *Let $\overline{M}^5$ be a 5-dimensional normal contact Lorentzian manifold with constant ϕ-sectional curvature k and M^2 a spacelike maximal surface with vector field ξ normal to M^2. Assume that the scalar normal curvature K_N of M^2 in $\overline{M}^5$ is constant. Then M^2 is totally geodesic with Gauss curvature $K = \frac{k-3}{4}$ or a nonpositive curved surface.*

2. Spacelike Submanifold

Let $\overline{M}$ be a $(2n+1)$-dimensional $(n \geq 2)$ manifold. The *normal contact Lorentzian structure* (ϕ, ξ, η, g) of $\overline{M}$ is given by a $(1,1)$-type skew-symmetric tensor field ϕ, a vector field ξ, a 1-form η and a Lorentzian metric g as

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \\ &g(\xi,\xi) = -1, \quad \eta(X) = -g(X,\xi), \\ &\left(\overline{\nabla}_X \eta\right) Y = g(\phi X, Y), \quad \overline{\nabla}_X \xi = \phi X, \\ &\left(\overline{\nabla}_X \phi\right) Y = -\eta(Y) X - g(X,Y)\xi, \end{aligned} \tag{2.1}$$

where X is a vector field of $\overline{M}$ and $\overline{\nabla}$ is the covariant derivative with respect to g ([3], [4]). When the curvature tensor field of $K(X,Y)Z$ of $\overline{M}$ has the following form

$$\begin{aligned} 4K(X,Y)Z &= (k-3)(g(Y,Z)X - g(X,Z)Y) \\ &+ (k+1)\big(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi + g(\phi Y,Z)\phi X + g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\big), \end{aligned} \tag{2.2}$$

$\overline{M}$ is called a *space of constant* ϕ*-sectional curvature k*.

Let M be an n-dimensional submanifold of $\overline{M}$. By ∇ we denote the covariant derivative of M determined by the induced metric on M. Let $\mathfrak{X}(\overline{M})$ (resp. $\mathfrak{X}(M)$) be the Lie algebra of vector fields on $\overline{M}$ (resp. M) and $\mathfrak{X}^{\perp}(M)$ the set of all vector fields normal to M.

The Gauss-Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y), \quad \overline{\nabla}_X N = -A^N(X) + D_X N, \tag{2.3}$$
$$X, Y \in \mathfrak{X}(M), \quad N \in \mathfrak{X}^{\perp}(M),$$

where D is the normal connection [6]. B is called the second fundamental form tensor and A the shape operator, and they satisfy

$$g(A^N(X), Y) = g(B(X,Y), N). \tag{2.4}$$

If the induced metric on M is positive-definite, then M is called a *spacelike submanifold*.

Let M be a spacelike submanifold of $\overline{M}$ with the vector field ξ normal to M, then from (2.1), (2.3) and (2.4), M satisfies following properties.

Proposition. *Let M be an m-dimensional spacelike submanifold in a normal contact Lorentzian manifold* $\overline{M}^{2n+1}$ *with structure* (ϕ, ξ, η, g). *Then*

(i) *The dimension m of M satisfies* $m \le n$.
(ii) *The shape operator of* ξ *direction is identically zero.*
(iii) *If* $X \in \mathfrak{X}(M)$ *then* $\phi X \in \mathfrak{X}^{\perp}(M)$.
(iv) *If* $m=n$, *then* $A^{\phi X}(Y) = A^{\phi Y}(X)$, for $X, Y \in \mathfrak{X}(M)$.

3. Local Formulas

We consider a spacelike surface M^2 in a 5-dimensional normal contact Lorentzian manifold $\overline{M}$. Let $\{e_1, e_2, e_3, e_4, \xi\}$ be an orthonormal frame field on $\overline{M}^5$ so that

$$e_1,\ e_2 \in \mathfrak{X}(M), \quad e_{1^*} := \phi e_1 = e_3, \quad e_{2^*} := \phi e_2 = e_4.$$

We shall make use of the following convention on the ranges of indices:

$$1 \le A, B, \cdots \le 5, \qquad 1 \le i, j, \cdots \le 2,$$
$$3 \le i^* \ j^*, \cdots \le 4, \qquad 3 \le \alpha, \beta, \cdots \le 5.$$

Let $\{w^1, w^2, w^{1^*}, w^{2^*}, w^5\}$ be the field of dual frames. Then the structure equations of $\overline{M}$ are given by

$$\begin{aligned}
dw^A &= -\sum \varepsilon_B\, w^A_B \wedge w^B, \quad w^A_B + w^B_A = 0,\\
dw^A_B &= -\sum \varepsilon_C\, w^A_C \wedge w^C_B + \Phi^A_B,\\
\Phi^A_B &= \frac{1}{2}\varepsilon_C\,\varepsilon_D \sum K^A_{BCD}\, w^C \wedge w^D,\\
K^A_{BCD} &+ K^A_{BDC} = 0.
\end{aligned}$$

Restricting these forms to M^2, we have $w^\alpha = 0$. Since $0 = dw^\alpha = -\sum w^\alpha_i \wedge w^i$, by Cartan's lemma we may write

$$w^\alpha_i = \sum h^\alpha_{ij} w^j, \quad h^\alpha_{ij} = h^\alpha_{ji}.$$

From these formulas we obtain

$$\begin{aligned}
dw^i &= -\sum w^i_j \wedge w^j, \quad w^i_j + w^j_i = 0,\\
dw^i_j &= \sum w^i_k \wedge w^k_j + \Omega^i_j,\\
\Omega^i_j &= \frac{1}{2}\sum R^i_{jk\ell}\, w^k \wedge w^\ell,\\
R^i_{jk\ell} &= K^i_{jk\ell} + \sum \varepsilon_\alpha \left(h^\alpha_{ik} h^\alpha_{j\ell} - h^\alpha_{i\ell} h^\alpha_{jk}\right).
\end{aligned}$$

$$\begin{aligned}
dw^\alpha_\beta &= -\sum \varepsilon_\gamma\, w^\alpha_\gamma \wedge w^\gamma_\beta + \Omega^\alpha_\beta,\\
\Omega^\alpha_\beta &= \frac{1}{2}\sum R^\alpha_{\beta k\ell} w^k \wedge w^\ell,\\
R^\alpha_{\beta k\ell} &= K^\alpha_{\beta k\ell} + \sum \left(h^\alpha_{ik} h^\beta_{i\ell} - h^\alpha_{i\ell} h^\beta_{ik}\right).
\end{aligned}$$

An immersion is said to be *maximal* if $\sum_i h^\alpha_{ii} = 0$ for all α.

We define h^{α}_{ijk} and $h^{\alpha}_{ijk\ell}$ by

$$\sum h^{\alpha}_{ijk}\, w^{k} = dh^{\alpha}_{ij} - \sum h^{\alpha}_{i\ell}\, w^{\ell}_{j} - \sum h^{\alpha}_{\ell j}\, w^{\ell}_{i} + \sum \varepsilon_{\beta}\, h^{\beta}_{ij}\, w^{\alpha}_{\beta}\,, \tag{3.1}$$

$$\sum h^{\alpha}_{ijk\ell}\, w^{\ell} = dh^{\alpha}_{ijk} - \sum h^{\alpha}_{\ell jk}\, w^{\ell}_{i} - \sum h^{\alpha}_{i\ell k}\, w^{k}_{j} - \sum h^{\alpha}_{ij\ell}\, w^{\ell}_{k} + \sum \varepsilon_{\beta}\, h^{\beta}_{ijk}\, w^{\alpha}_{\beta}\,.$$

The Laplacian Δh^{α}_{ij} is given by

$$\Delta h^{\alpha}_{ij} = \sum h^{\alpha}_{ijkk}\,.$$

When M^2 is maximal in $\overline{M}^5$, that is $\sum h^{\alpha}_{kk} = 0$ for all α, Δh^{α}_{ij} can be written as

$$\Delta h^{\alpha}_{ij} = \sum \left(2K^{\alpha}_{\beta ki}\, h^{\beta}_{jk} - K^{\alpha}_{k\beta k}\, h^{\beta}_{ij} + 2K^{\alpha}_{\beta kj}\, h^{\beta}_{ki}\right) + \sum \left(K^{m}_{mik}\, h^{\alpha}_{mj} + K^{m}_{kjk}\, h^{\alpha}_{mi} + 2K^{m}_{ijk}\, h^{\alpha}_{mk}\right)$$
$$+ \sum \left(2h^{\alpha}_{km}\, h^{\beta}_{ki}\, h^{\beta}_{mj} - h^{\alpha}_{mk}\, h^{\beta}_{km}\, h^{\beta}_{ij} - h^{\alpha}_{mi}\, h^{\beta}_{mk}\, h^{\beta}_{kj} - h^{\alpha}_{mj}\, h^{\beta}_{ki}\, h^{\beta}_{mk}\right). \tag{3.2}$$

The *scalar normal curvature* K_N of M^2 is defined by

$$K_N = \sum \varepsilon_{\alpha}\, \varepsilon_{\beta}\, S^{\alpha}_{\beta ij}\, S^{\alpha}_{\beta ij}\,, \qquad S^{\alpha}_{\beta ij} = \sum \left(h^{\alpha}_{ik}\, h^{\beta}_{jk} - h^{\alpha}_{jk}\, h^{\beta}_{ik}\right).$$

The covariant derivative of $S^{\alpha}_{\beta ij}$ is defined by

$$S^{\alpha}_{\beta ij\ell} = \sum \left(h^{\alpha}_{ik\ell}\, h^{\beta}_{jk} + h^{\alpha}_{ik}\, h^{\beta}_{jk\ell} - h^{\alpha}_{jk\ell}\, h^{\beta}_{ik} - h^{\alpha}_{jk}\, h^{\beta}_{ik\ell}\right).$$

Then for the Laplacian of K_N, we have the following formula

$$\frac{1}{2}\Delta K_N = \sum \varepsilon_{\alpha}\, \varepsilon_{\beta}\, (S^{\alpha}_{\beta ijk})^2 + \sum S^{\alpha}_{\beta ij}\left(h^{\alpha}_{ik\ell}\, h^{\beta}_{ji\ell} - h^{\beta}_{ik\ell}\, h^{\alpha}_{jk\ell}\right)$$
$$+ 4\sum \varepsilon_{\alpha}\, \varepsilon_{\beta}\, S^{\alpha}_{\beta ij}\, (\Delta h^{\alpha}_{ik})\, h^{\beta}_{jk}\,.$$

4. Proof of Theorem

Let M^2 be a spacelike maximal surface in a normal contact Lorentzian manifold $\overline{M}^5$ of constant ϕ-sectional curvature k. Then, from Proposition, we obtain

$$(h^{1^*}_{ij}) = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \quad (h^{2^*}_{ij}) = \begin{bmatrix} 0 & -a \\ -a & 0 \end{bmatrix}, \quad (h^{5}_{ij}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.1}$$

From (3.2), it follows that

$$\begin{aligned}
\Delta h^{1^*}_{11} &= \frac{3k-5}{4}a - 6a^3, \quad \Delta h^{1^*}_{12} = \Delta h^{1^*}_{21} = 0,\\
\Delta h^{1^*}_{22} &= -\frac{3k-5}{4}a + 6a^3, \quad \Delta h^{2^*}_{11} = \Delta h^{2^*}_{22} = 0,\\
\Delta h^{2^*}_{12} &= \Delta h^{2^*}_{21} = -\frac{3k-5}{4}a + 6a^3,\\
\Delta h^{5}_{11} &= \Delta h^{5}_{12} = \Delta h^{5}_{21} = \Delta h^{5}_{22} = 0,
\end{aligned}$$

by virtue of (2.2).

From (3.1), we have

$$\begin{aligned}
h^{1^*}_{111} &= -h^{1^*}_{221} = -h^{2^*}_{121} = -h^{2^*}_{221}\\
&= -h^{1^*}_{122} = -h^{1^*}_{212} = -h^{2^*}_{112} = h^{2^*}_{222} = a,_1,\\
h^{1^*}_{112} &= -h^{1^*}_{222} = -h^{2^*}_{122} = -h^{2^*}_{212}\\
&= h^{1^*}_{121} = h^{1^*}_{211} = h^{2^*}_{111} = -h^{2^*}_{221} = a,_2,\\
h^{5}_{112} &= h^{0}_{121} = h^{0}_{221} = h^{0}_{222} = 0,\\
h^{0}_{122} &= h^{0}_{221} = -a.
\end{aligned}$$

Since

$$S^{2^*}_{1^*11} = S^{2^*}_{1^*22} = 0, \quad S^{2^*}_{1^*12} = -S^{2^*}_{1^*21} = 2a^2,$$

by virtue of (4.1), we obtain

$$\begin{aligned}
\sum (S^{\alpha}_{\beta ijk})^2 &= \sum (S^{\ell^*}_{m^* ijk})^2 + 2\varepsilon_0 \sum (S^{0}_{\beta ijk})^2\\
&= 4\Big(((2a^2),_1)^2 + ((2a^2),_2)^2 \Big) - 32a^4,
\end{aligned}$$

$$\sum S^{\alpha}_{\beta ij}\Big(h^{\alpha}_{ik\ell} h^{\beta}_{jk\ell} - h^{\beta}_{ik\ell} h^{\alpha}_{jk\ell} \Big) = 2\Big(((2a^2),_1)^2 + ((2a^2),_2)^2 \Big),$$

$$\sum S^{\alpha}_{\beta ij} (\Delta h^{\alpha}_{jk}) h^{\beta}_{jk} = 8a^4 \Big(\frac{3k-5}{4} - 6a^2 \Big),$$

so that

$$\frac{1}{2}\Delta K_N = 8\Big(((2a^2),_1)^2 + ((2a^2),_2)^2 \Big) + 32a^4 \Big(\frac{3k-9}{4} - 6a^2 \Big).$$

If we assume $K_N \equiv constant$, then since a is continuous, this equation reduces to $a \equiv 0$ or $a^2 \geq \frac{k-3}{8}$ everywhere. On the other hand, the Gauss curvature K of M^2 is given by $K = \frac{k-3}{4} - 2a^2$. Hence if $a^2 \geq \frac{k-3}{8}$ then $K \leq 0$. This completes the proof.

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