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Spacelike Singularities and Hidden Symmetries of Gravity

Marc Henneaux
Physique Théorique et Mathématique
Université Libre de Bruxelles & International Solvay Institutes
Boulevard du Triomphe, ULB – C.P. 231
B-1050 Bruxelles, Belgium
email: henneaux@ulb.ac.be
http://www.ulb.ac.be/sciences/ptm/pmif/membres/henneaux.html

Daniel Persson
Physique Théorique et Mathématique
Université Libre de Bruxelles & International Solvay Institutes
Boulevard du Triomphe, ULB – C.P. 231
B-1050 Bruxelles, Belgium
email: dpersson@ulb.ac.be

Philippe Spindel
Service de Mécanique et Gravitation
Université de Mons-Hainaut, Académie Wallonie-Bruxelles
Avenue du Champ de Mars 6, B-7000 Mons, Belgium
email: philippe.spindel@umh.ac.be

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Abstract

We review the intimate connection between (super-)gravity close to a spacelike singularity (the "BKL-limit") and the theory of Lorentzian Kac–Moody algebras. We show that in this limit the gravitational theory can be reformulated in terms of billiard motion in a region of hyperbolic space, revealing that the dynamics is completely determined by a (possibly infinite) sequence of reflections, which are elements of a Lorentzian Coxeter group. Such Coxeter groups are the Weyl groups of infinite-dimensional Kac–Moody algebras, suggesting that these algebras yield symmetries of gravitational theories. Our presentation is aimed to be a self-contained and comprehensive treatment of the subject, with all the relevant mathematical background material introduced and explained in detail. We also review attempts at making the infinite-dimensional symmetries manifest, through the construction of a geodesic sigma model based on a Lorentzian Kac–Moody algebra. An explicit example is provided for the case of the hyperbolic algebra E_{10}, which is conjectured to be an underlying symmetry of M-theory. Illustrations of this conjecture are also discussed in the context of cosmological solutions to eleven-dimensional supergravity.

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1 Introduction

It has been realized long ago that spacetime singularities are generic in classical general relativity [91]. However, their exact nature is still far from being well understood. Although it is expected that spacetime singularities will ultimately be resolved in a complete quantum theory of gravity, understanding their classical structure is likely to shed interesting light and insight into the nature of the mechanisms at play in the singularity resolution. Furthermore, analyzing general relativity close to such singularities also provides important information on the dynamics of gravity within the regime where it breaks down. Indeed, careful investigations of the field equations in this extreme regime has revealed interesting and unexpected symmetry properties of gravity.

In the late 1960's, Belinskii, Khalatnikov and Lifshitz ("BKL") [16] gave a general description of spacelike singularities in the context of the four-dimensional vacuum Einstein theory. They provided convincing evidence that the generic solution of the dynamical Einstein equations, in the vicinity of a spacelike singularity, exhibits the following remarkable properties:

- The spatial points dynamically decouple, i.e., the partial differential equations governing the dynamics of the spatial metric asymptotically reduce, as one goes to the singularity, to ordinary differential equations with respect to time (one set of ordinary differential equations per spatial point).
- The solution exhibits strong chaotic properties of the type investigated independently by Misner [137] and called "mixmaster behavior". This chaotic behavior is best seen in the hyperbolic billiard reformulation of the dynamics due to Chitre [31] and Misner [138] (for pure gravity in four spacetime dimensions).

1.1 Cosmological billiards and hidden symmetries of gravity

This important work has opened the way to many further fruitful investigations in theoretical cosmology. Recently, a new – and somewhat unanticipated – development has occurred in the field, with the realisation that for the gravitational theories that have been studied most (pure gravity and supergravities in various spacetime dimensions) the dynamics of the gravitational field exhibits strong connections with Lorentzian Kac–Moody algebras, as discovered by Damour and Henneaux [45], suggesting that these might be "hidden" symmetries of the theory.

These connections appear for the cases at hand because in the BKL-limit, not only can the equations of motion be reformulated as dynamical equations for billiard motion in a region of hyperbolic space, but also this region possesses unique features: It is the fundamental Weyl chamber of some Kac–Moody algebra. The dynamical motion in the BKL-limit is then a succession of reflections in the walls bounding the fundamental Weyl chamber and defines "words" in the Weyl group of the Kac–Moody algebra.

Which billiard region of hyperbolic space actually emerges – and hence which Kac–Moody algebra is relevant – depends on the theory at hand, i.e., on the spacetime dimension, the menu of matter fields, and the dilaton couplings. The most celebrated case is eleven-dimensional supergravity, for which the billiard region is the fundamental region of $E_{10} \equiv E_8^{++}$, one of the four hyperbolic Kac–Moody algebras of highest rank 10. The root lattice of E_{10} is furthermore one of the few even, Lorentzian, self-dual lattices – actually the only one in 10 dimensions – a fact that could play a key role in our ultimate understanding of M-theory.

Other gravitational theories lead to other billiards characterized by different algebras. These algebras are closely connected to the hidden duality groups that emerge upon dimensional reduction to three dimensions [41, 95].

That one can associate a regular billiard and an infinite discrete reflection group (Coxeter group) to spacelike singularities of a given gravitational theory in the BKL-limit is a robust fact (even though the BKL-limit itself is yet to be fully understood), which, in our opinion, will survive future developments. The mathematics necessary to appreciate the billiard structure and its connection to the duality groups in three dimensions involve hyperbolic Coxeter groups, Kac–Moody algebras and real forms of Lie algebras.

The appearance of infinite Coxeter groups related to Lorentzian Kac–Moody algebras has triggered fascinating conjectures on the existence of huge symmetry structures underlying gravity [47]. Similar conjectures based on different considerations had been made earlier in the pioneering works [113, 167]. The status of these conjectures, however, is still somewhat unclear since, in particular, it is not known how exactly the symmetry would act.

The main purpose of this article is to explain the emergence of infinite discrete reflection groups in gravity in a self-contained manner, including giving the detailed mathematical background needed to follow the discussion. We shall avoid, however, duplicating already existing reviews on BKL billiards.

Contrary to the main core of the review, devoted to an explanation of the billiard Weyl groups, which is indeed rather complete, we shall also discuss some paths that have been taken towards revealing the conjectured infinite-dimensional Kac–Moody symmetry. Our goal here will only be to give a flavor of some of the work that has been done along these lines, emphasizing its dynamical relevance. Because we feel that it would be premature to fully review this second subject, which is still in its infancy, we shall neither try to be exhaustive nor give detailed treatments.

1.2 Outline of the paper

Our article is organized as follows. In Section 2, we outline the key features of the BKL phenomenon, valid in any number of dimensions, and describe the billiard formulation which clearly displays these features. Since the derivation of these aspects have been already reviewed in [48], we give here only the results without proof. Next, for completeness, we briefly discuss the status of the BKL conjecture – assumed to be valid throughout our review.

In Sections 3 and 4, we develop the mathematical tools necessary for apprehending those aspects of Coxeter groups and Kac–Moody algebras that are needed in the BKL analysis. First, in Section 3, we provide a primer on Coxeter groups (which are the mathematical structures that make direct contact with the BKL billiards). We then move on to Kac–Moody algebras in Section 4, and we discuss, in particular, some prominent features of *hyperbolic* Kac–Moody algebras.

In Section 5 we then make use of these mathematical concepts to relate the BKL billiards to Lorentzian Kac–Moody algebras. We show that there is a simple connection between the relevant Kac–Moody algebra and the U-duality algebras that appear upon toroidal dimensional reduction to three dimensions, when these U-duality algebras are split real forms. The Kac–Moody algebra is then just the standard overextension of the U-duality algebra in question.

To understand the non-split case requires an understanding of real forms of finite-dimensional semi-simple Lie algebras. This mathematical material is developed in Section 6. Here, again, we have tried to be both rather complete and explicit through the use of many examples. We have followed a pedagogical approach privileging illustrative examples over complete proofs (these can be found in any case in the references given in the text). We explain the complementary Vogan and Tits–Satake approaches, where maximal compact and maximal noncompact Cartan subalgebras play the central roles, respectively. The concepts of restricted root systems and of the Iwasawa decomposition, central for understanding the emergence of the billiard, have been given particular attention. For completeness we provide tables listing all real forms of finite Lie algebras, both in terms of Vogan diagrams and in terms of Tits–Satake diagrams. In Section 7 we use these mathematical developments to relate the Kac–Moody billiards in the non-split case to the U-duality algebras appearing in three dimensions.

Up to (and including) Section 7, the developments present well-established results. With Section 8 we initiate a journey into more speculative territory. The presence of hyperbolic Weyl groups suggests that the corresponding infinite-dimensional Kac–Moody algebras might, in fact, be true underlying symmetries of the theory. How this conjectured symmetry should actually act on the physical fields is still unclear, however. We explore one approach in which the symmetry is realized nonlinearly on a $(1+0)$-dimensional sigma model based on $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$, which is the case relevant to eleven-dimensional supergravity. To this end, in Section 8 we introduce the concept of a level decomposition of some of the relevant Kac–Moody algebras in terms of finite regular subalgebras. This is necessary for studying the sigma model approach to the conjectured infinite-dimensional symmetries, a task undertaken in Section 9. We show that the sigma model for $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$ spectacularly reproduces important features of eleven-dimensional supergravity. However, we also point out important limitations of the approach, which probably does not constitute the final word on the subject.

In Section 10 we show that the interpretation of eleven-dimensional supergravity in terms of a manifestly $\mathcal{E}_{10}$-invariant sigma model sheds interesting and useful light on certain cosmological solutions of the theory. These solutions were derived previously but without the Kac–Moody algebraic understanding. The sigma model approach also suggests a new method of uncovering novel solutions. Finally, in Section 11 we present a concluding discussion and some suggestions for future research.

2 The BKL Phenomenon

In this section, we explain the main ideas of the billiard description of the BKL behavior. Our approach is based on the billiard review [48], from which we adopt notations and conventions. We shall here only outline the logic and provide the final results. No attempt will be made to reproduce the (sometimes heuristic) arguments underlying the derivation.

2.1 The general action

We are interested in general theories describing Einstein gravity coupled to bosonic "matter" fields. The only known bosonic matter fields that consistently couple to gravity are p-form fields, so our collection of fields will contain, besides the metric, p-form fields, including scalar fields ($p = 0$). The action reads

$$S\left[g_{\mu\nu}, \phi, A^{(p)}\right] = \int d^D x \sqrt{-{}^{(D)}\mathrm{g}}\left[R - \partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\sum_p \frac{e^{\lambda^{(p)}\phi}}{(p+1)!} F^{(p)}_{\mu_1\cdots\mu_{p+1}} F^{(p)\,\mu_1\cdots\mu_{p+1}}\right] + \text{“more”}, \tag{2.1}$$

where we have chosen units such that $16\pi G = 1$. The spacetime dimension is left unspecified. The Einstein metric $\mathrm{g}_{\mu\nu}$ has Lorentzian signature $(-, +, \cdots, +)$ and is used to lower or raise the indices. Its determinant is ${}^{(D)}\mathrm{g}$, where the index D is used to avoid any confusion with the determinant of the spatial metric introduced below. We assume that among the scalars, there is only one dilaton[1], denoted ϕ, whose kinetic term is normalized with weight 1 with respect to the Ricci scalar. The real parameter $\lambda^{(p)}$ measures the strength of the coupling to the dilaton. The other scalar fields, sometimes called axions, are denoted $A^{(0)}$ and have dilaton coupling $\lambda^{(0)} \neq 0$. The integer $p \geq 0$ labels the various p-forms $A^{(p)}$ present in the theory, with field strengths $F^{(p)} = dA^{(p)}$,

$$F^{(p)}_{\mu_1\cdots\mu_{p+1}} = \partial_{\mu_1} A^{(p)}_{\mu_2\cdots\mu_{p+1}} \pm p \text{ permutations.} \tag{2.2}$$

We assume the form degree p to be strictly smaller than $D-1$, since a $(D-1)$-form in D dimensions carries no local degree of freedom. Furthermore, if $p = D - 2$ the p-form is dual to a scalar and we impose also $\lambda^{(D-2)} \neq 0$.

The field strength, Equation (2.2), could be modified by additional coupling terms of Yang–Mills or Chapline–Manton type [20, 29] (e.g., $F_C = dC^{(2)} - C^{(0)}dB^{(2)}$ for two 2-forms $C^{(2)}$ and $B^{(2)}$ and a 0-form $C^{(0)}$, as it occurs in ten-dimensional type IIB supergravity), but we include these additional contributions to the action in "more". Similarly, "more" might contain Chern–Simons terms, as in the action for eleven-dimensional supergravity [38].

We shall at this stage consider arbitrary dilaton couplings and menus of p-forms. The billiard derivation given below remains valid no matter what these are; all theories described by the general action Equation (2.1) lead to the billiard picture. However, it is only for particular p-form menus, spacetime dimensions and dilaton couplings that the billiard region is regular and associated with a Kac–Moody algebra. This will be discussed in Section 5. Note that the action, Equation (2.1), contains as particular cases the bosonic sectors of all known supergravity theories.

2.2 Hamiltonian description

We assume that there is a spacelike singularity at a finite distance in proper time. We adopt a spacetime slicing adapted to the singularity, which "occurs" on a slice of constant time. We build

[1] This is done mostly for notational convenience. If there were other dilatons among the 0-forms, these should be separated off from the p-forms because they play a distinct role. They would appear as additional scale factors and would increase the dimensions of the relevant hyperbolic billiard (they define additional spacelike directions in the space of scale factors).

the slicing from the singularity by taking pseudo-Gaussian coordinates defined by $N = \sqrt{\mathrm{g}}$ and $N^i = 0$, where N is the lapse and N^i is the shift [48]. Here, $\mathrm{g} \equiv \det(\mathrm{g}_{ij})$. Thus, in some spacetime patch, the metric reads[2]

$$ds^2 = -\mathrm{g}(dx^0)^2 + \mathrm{g}_{ij}(x^0, x^i)\, dx^i\, dx^j, \tag{2.3}$$

where the local volume g collapses at each spatial point as $x^0 \to +\infty$, in such a way that the proper time $dT = -\sqrt{\mathrm{g}}\, dx^0$ remains finite (and tends conventionally to 0^+). Here we have assumed the singularity to occur in the past, as in the original BKL analysis, but a similar discussion holds for future spacelike singularities.

2.2.1 Action in canonical form

In the Hamiltonian description of the dynamics, the canonical variables are the spatial metric components g_{ij}, the dilaton ϕ, the spatial p-form components $A^{(p)}_{m_1\cdots m_p}$ and their respective conjugate momenta π^{ij}, π_ϕ and $\pi^{m_1\cdots m_p}_{(p)}$. The Hamiltonian action in the pseudo-Gaussian gauge is given by

$$S\left[\mathrm{g}_{ij}, \pi^{ij}, \phi, \pi_\phi, A^{(p)}_{m_1\cdots m_p}, \pi^{m_1\cdots m_p}_{(p)}\right] = \int dx^0 \left[\int d^d x \left(\pi^{ij}\dot{\mathrm{g}}_{ij} + \pi_\phi \dot{\phi} + \sum_p \pi^{m_1\cdots m_p}_{(p)} \dot{A}^{(p)}_{m_1\cdots m_p}\right) - H\right], \tag{2.4}$$

where the Hamiltonian is

$$\begin{aligned}
H &= \int d^d x\, \mathcal{H}, \\
\mathcal{H} &= K' + V', \\
K' &= \pi^{ij}\pi_{ij} - \frac{1}{d-1}(\pi^i_i)^2 + \frac{1}{4}(\pi_\phi)^2 + \sum_p \frac{(p!)e^{-\lambda^{(p)}\phi}}{2}\, \pi^{m_1\cdots m_p}_{(p)} \pi_{(p)\, m_1\cdots m_p}, \\
V' &= -R\mathrm{g} + \mathrm{g}^{ij}\mathrm{g}\, \partial_i\phi\, \partial_j\phi + \sum_p \frac{e^{\lambda^{(p)}\phi}}{2\,(p+1)!}\, \mathrm{g}\, F^{(p)}_{m_1\cdots m_{p+1}} F^{(p)\, m_1\cdots m_{p+1}}.
\end{aligned} \tag{2.5}$$

In addition to imposing the coordinate conditions $N = \sqrt{\mathrm{g}}$ and $N^i = 0$, we have also set the temporal components of the p-forms equal to zero ("temporal gauge").

The dynamical equations of motion are obtained by varying the above action w.r.t. the canonical variables. Moreover, there are constraints on the dynamical variables, which are

$$\begin{aligned}
\mathcal{H} &= 0 \qquad &&(\text{"Hamiltonian constraint"}), \\
\mathcal{H}_i &= 0 \qquad &&(\text{"momentum constraint"}), \\
\varphi^{m_1\cdots m_{p-1}}_{(p)} &= 0 \qquad &&(\text{"Gauss law" for each } p\text{-form}, p > 0).
\end{aligned} \tag{2.6}$$

Here we have set

$$\begin{aligned}
\mathcal{H}_i &= -2\pi^j_{i\ \,|j} + \pi_\phi \partial_i \phi + \sum_p \pi^{m_1\cdots m_p}_{(p)} F^{(p)}_{i m_1\cdots m_p}, \\
\varphi^{m_1\cdots m_{p-1}}_{(p)} &= -p\, \pi^{m_1\cdots m_{p-1} m_p}_{(p)}{}_{|m_p},
\end{aligned} \tag{2.7}$$

where the subscript $|m_p$ denotes the spatially covariant derivative. These constraints are preserved by the dynamical evolution and need to be imposed only at one "initial" time, say at $x^0 = 0$.

[2] Note that we have for convenience chosen to work with a coordinate coframe dx^i, with the imposed constraint $N = \sqrt{\mathrm{g}}$. In general, one may of course use an arbitrary spatial coframe, say $\theta^i(x)$, for which the associated gauge choice reads $N = w(x)\sqrt{\mathrm{g}}$, with $w(x)$ being a density of weight -1. Such a frame will be used in Section 2.3.1. This general kind of spatial coframe was also used extensively in the recent work [40].

2.2.2 Iwasawa change of variables

In order to study the dynamical behavior of the fields as $x^0 \to \infty$ ($\mathrm{g} \to 0$) and to exhibit the billiard picture, it is particularly convenient to perform the Iwasawa decomposition of the spatial metric. Let $\mathrm{g}(x^0, x^i)$ be the matrix with entries $\mathrm{g}_{ij}(x^0, x^i)$. We set

$$\mathrm{g} = \mathcal{N}^T \mathcal{A}^2 \mathcal{N}, \tag{2.8}$$

where $\mathcal{N}$ is an upper triangular matrix with 1's on the diagonal ($\mathcal{N}_{ii} = 1$, $\mathcal{N}_{ij} = 0$ for $i > j$) and $\mathcal{A}$ is a diagonal matrix with positive elements, which we parametrize as

$$\mathcal{A} = \exp(-\beta), \qquad \beta = \mathrm{diag}(\beta^1, \beta^2, \cdots, \beta^d). \tag{2.9}$$

Both $\mathcal{N}$ and $\mathcal{A}$ depend on the spacetime coordinates. The spatial metric $d\sigma^2$ becomes

$$d\sigma^2 = \mathrm{g}_{ij}\, dx^i\, dx^j = \sum_{k=1}^{d} e^{(-2\beta^k)} (\omega^k)^2 \tag{2.10}$$

with

$$\omega^k = \sum_i \mathcal{N}_{k\,i}\, dx^i. \tag{2.11}$$

The variables β^i of the Iwasawa decomposition give the (logarithmic) scale factors in the new, orthogonal, basis. The variables $\mathcal{N}_{ij}$ characterize the change of basis that diagonalizes the metric and hence they parametrize the off-diagonal components of the original g_{ij}.

We extend the transformation Equation (2.8) in configuration space to a canonical transformation in phase space through the formula

$$\pi^{ij} d\mathrm{g}_{ij} = \pi^i d\beta_i + \sum_{i<j} P_{ij}\, d\mathcal{N}_{ij}. \tag{2.12}$$

Since the scale factors and the off-diagonal variables play very distinct roles in the asymptotic behavior, we split off the Hamiltonian as a sum of a kinetic term for the scale factors (including the dilaton),

$$K = \frac{1}{4}\left[\sum_{i=1}^{d} \pi_i^2 - \frac{1}{d-1}\left(\sum_{i=1}^{d} \pi_i\right)^2 + \pi_\phi^2\right], \tag{2.13}$$

plus the rest, denoted by V, which will act as a potential for the scale factors. The Hamiltonian then becomes

$$\begin{aligned}
\mathcal{H} &= K + V, \\
V &= V_S + V_G + \sum_p V_p + V_\phi, \\
V_S &= \frac{1}{2}\sum_{i<j} e^{-2(\beta^j - \beta^i)}\Big(\sum_m P_{im}\mathcal{N}_{jm}\Big)^2, \\
V_G &= -R\mathrm{g}, \\
V_{(p)} &= V_{(p)}^{\mathrm{el}} + V_{(p)}^{\mathrm{magn}}, \\
V_{(p)}^{\mathrm{el}} &= \frac{p! e^{-\lambda^{(p)}\phi}}{2}\, \pi_{(p)}^{m_1\cdots m_p} \pi_{(p)\, m_1\cdots m_p}, \\
V_{(p)}^{\mathrm{magn}} &= \frac{e^{\lambda^{(p)}\phi}}{2\,(p+1)!}\, \mathrm{g}\, F^{(p)}_{m_1\cdots m_{p+1}} F^{(p)\, m_1\cdots m_{p+1}}, \\
V_\phi &= \mathrm{g}^{ij} \mathrm{g}\, \partial_i\phi\, \partial_j\phi.
\end{aligned} \tag{2.14}$$

The kinetic term K is quadratic in the momenta conjugate to the scale factors and defines the inverse of a metric in the space of the scale factors. Explicitly, this metric reads

$$\sum_i (d\beta^i)^2 - \left(\sum d\beta^i\right)^2 + (d\phi)^2. \tag{2.15}$$

Since the metric coefficients do not depend on the scale factors, that metric in the space of scale factors is flat, and, moreover, it is of Lorentzian signature. A conformal transformation where all scale factors are scaled by the same number ($\beta^i \to \beta^i + \epsilon$) defines a timelike direction. It will be convenient in the following to collectively denote all the scale factors (the β^i's and the dilaton ϕ) as β^μ, i.e., $(\beta^\mu) = (\beta^i, \phi)$.

The analysis is further simplified if we take for new p-form variables the components of the p-forms in the Iwasawa basis of the ω^k's,

$$\mathcal{A}^{(p)}_{i_1\cdots i_p} = \sum_{m_1,\cdots,m_p} (\mathcal{N}^{-1})_{m_1 i_1} \cdots (\mathcal{N}^{-1})_{m_p i_p} A_{(p)m_1\cdots m_p}, \tag{2.16}$$

and again extend this configuration space transformation to a point canonical transformation in phase space,

$$\left(\mathcal{N}_{ij}, P_{ij}, A^{(p)}_{m_1\cdots m_p}, \pi^{m_1\cdots m_p}_{(p)}\right) \quad \rightarrow \quad \left(\mathcal{N}_{ij}, P'_{ij}, \mathcal{A}^{(p)}_{m_1\cdots m_p}, \mathcal{E}^{i_1\cdots i_p}_{(p)}\right), \tag{2.17}$$

using the formula $\sum p\, dq = \sum p'\, dq'$, which reads

$$\sum_{i<j} P_{ij}\dot{\mathcal{N}}_{ij} + \sum_p \pi^{m_1\cdots m_p}_{(p)} \dot{A}^{(p)}_{m_1\cdots m_p} = \sum_{i<j} P'_{ij}\dot{\mathcal{N}}_{ij} + \sum_p \mathcal{E}^{i_1\cdots i_p}_{(p)} \dot{\mathcal{A}}^{(p)}_{m_1\cdots m_p}. \tag{2.18}$$

Note that the scale factor variables are unaffected, while the momenta P_{ij} conjugate to $\mathcal{N}_{ij}$ get redefined by terms involving $\mathcal{E}$, $\mathcal{N}$ and $\mathcal{A}$ since the components $\mathcal{A}^{(p)}_{m_1\cdots m_p}$ of the p-forms in the Iwasawa basis involve the $\mathcal{N}$'s. On the other hand, the new p-form momenta, i.e., the components of the electric field $\pi^{m_1\cdots m_p}_{(p)}$ in the basis $\{\omega^k\}$ are simply given by

$$\mathcal{E}^{i_1\cdots i_p}_{(p)} = \sum_{m_1,\cdots,m_p} \mathcal{N}_{i_1 m_1}\mathcal{N}_{i_2 m_2} \cdots \mathcal{N}_{i_p m_p} \pi^{m_1\cdots m_p}_{(p)}. \tag{2.19}$$

In terms of the new variables, the electromagnetic potentials become

$$\begin{aligned} V^{\text{el}}_{(p)} &= \frac{p!}{2} \sum_{i_1,i_2,\cdots,i_p} e^{-2e_{i_1\cdots i_p}(\beta)} (\mathcal{E}^{i_1\cdots i_p}_{(p)})^2, \\ V^{\text{magn}}_{(p)} &= \frac{1}{2\,(p+1)!} \sum_{i_1,i_2,\cdots,i_{p+1}} e^{-2m_{i_1\cdots i_{p+1}}(\beta)} (\mathcal{F}_{(p)\,i_1\cdots i_{p+1}})^2. \end{aligned} \tag{2.20}$$

Here, $e_{i_1\cdots i_p}(\beta)$ are the electric linear forms

$$e_{i_1\cdots i_p}(\beta) = \beta^{i_1} + \cdots + \beta^{i_p} + \frac{\lambda^{(p)}}{2}\phi \tag{2.21}$$

(the indices i_j are all distinct because $\mathcal{E}^{i_1\cdots i_p}_{(p)}$ is completely antisymmetric) while $\mathcal{F}_{(p)\,i_1\cdots i_{p+1}}$ are the components of the magnetic field $F_{(p)m_1\cdots m_{p+1}}$ in the basis $\{\omega^k\}$,

$$\mathcal{F}_{(p)\,i_1\cdots i_{p+1}} = \sum_{m_1,\cdots,m_{p+1}} (\mathcal{N}^{-1})_{m_1 i_1} \cdots (\mathcal{N}^{-1})_{m_{p+1} i_{p+1}} F_{(p)m_1\cdots m_{p+1}}, \tag{2.22}$$

and $m_{i_1\cdots i_{p+1}}(\beta)$ are the magnetic linear forms

$$m_{i_1\cdots i_{p+1}}(\beta) = \sum_{j\notin\{i_1,i_2,\cdots i_{p+1}\}} \beta^j - \frac{\lambda^{(p)}}{2}\phi. \tag{2.23}$$

One sometimes rewrites $m_{i_1\cdots i_{p+1}}(\beta)$ as $b_{i_{p+2}\cdots i_d}(\beta)$, where $\{i_{p+2}, i_{p+3}, \cdots, i_d\}$ is the set complementary to $\{i_1, i_2, \cdots i_{p+1}\}$, e.g.,

$$b_{1\,2\,\cdots\,d-p-1}(\beta) = \beta^1 + \cdots + \beta^{d-p-1} - \frac{\lambda^{(p)}}{2}\phi = m_{d-p\,\cdots\,d}. \tag{2.24}$$

The exterior derivative $\mathcal{F}$ of $\mathcal{A}$ in the non-holonomic frame $\{\omega^k\}$ involves of course the structure coefficients $C^i{}_{jk}$ in that frame, i.e.,

$$\mathcal{F}_{(p)\,i_1\cdots i_{p+1}} = \partial_{[i_1}\mathcal{A}_{i_2\cdots i_{p+1}]} + \text{“}C\mathcal{A}\text{”-terms}, \tag{2.25}$$

where

$$\partial_{i_1} \equiv \sum_{m_1} (\mathcal{N}^{-1})_{m_1 i_1}(\partial/\partial x^{m_1}) \tag{2.26}$$

is here the frame derivative. Similarly, the potential V_ϕ reads

$$V_\phi = \sum_i e^{-2\bar{m}_i(\beta)}(\mathcal{F}_i)^2, \tag{2.27}$$

where $\mathcal{F}_i$ is

$$\mathcal{F}_i = (\mathcal{N}^{-1})_{ji}\partial_j\phi \tag{2.28}$$

and

$$\bar{m}_i(\beta) = \sum_{j\neq i} \beta^j. \tag{2.29}$$

2.3 Decoupling of spatial points close to a spacelike singularity

So far we have only redefined the variables without making any approximation. We now start the discussion of the BKL-limit, which investigates the leading behavior of the fields as $x^0 \to \infty$ (g $\to$ 0). Although the more recent “derivations” of the BKL-limit treat both elements at once [43, 44, 45, 48], it appears useful – especially for rigorous justifications – to separate two aspects of the BKL conjecture[3].

The first aspect is that the spatial points decouple in the limit $x^0 \to \infty$, in the sense that one can replace the Hamiltonian by an effective “ultralocal” Hamiltonian H^{UL} involving no spatial gradients and hence leading at each point to a set of dynamical equations that are ordinary differential equations with respect to time. The ultralocal effective Hamiltonian has a form similar to that of the Hamiltonian governing certain spatially homogeneous cosmological models, as we shall explain in this section.

The second aspect of the BKL-limit is to take the sharp wall limit of the ultralocal Hamiltonian. This leads directly to the billiard description, as will be discussed in Section 2.4.

[3] The Hamiltonian heuristic derivation of [48] shares many features in common with the work of [122, 109, 112, 123], extended to some higher-dimensional models in [110, 111]. The central feature of [48] is the Iwasawa decomposition which enables one to clearly see the role of off-diagonal variables.

2.3.1 Spatially homogeneous models

In spatially homogeneous models, the fields depend only on time in invariant frames, e.g., for the metric

$$ds^2 = \mathrm{g}_{ij}(x^0)\psi^i\psi^j, \tag{2.30}$$

where the invariant forms fulfill

$$d\psi^i = -\frac{1}{2}f^i{}_{jk}\psi^j \wedge \psi^k.$$

Here, the $f^i{}_{jk}$ are the structure constants of the spatial homogeneity group. Similarly, for a 1-form and a 2-form,

$$A^{(1)} = A_i(x^0)\psi^i, \qquad A^{(2)} = \frac{1}{2}A_{ij}(x^0)\psi^i \wedge \psi^j, \qquad \text{etc.} \tag{2.31}$$

The Hamiltonian constraint yielding the field equations in the spatially homogeneous context[4] is obtained by substituting the form of the fields in the general Hamiltonian constraint and contains, of course, no explicit spatial gradients since the fields are homogeneous. Note, however, that the structure constants $f^i{}_{jk}$ contain implicit spatial gradients. The Hamiltonian can now be decomposed as before and reads

$$\begin{aligned}\mathcal{H}^{\mathrm{UL}} &= K + V^{\mathrm{UL}},\\ V^{\mathrm{UL}} &= V_S + V_G^{\mathrm{UL}} + \sum_p \left(V^{\mathrm{el}}_{(p)} + V^{\mathrm{UL,magn}}_{(p)}\right),\end{aligned} \tag{2.32}$$

where K, V_S and $V^{\mathrm{el}}_{(p)}$, which do not involve spatial gradients, are unchanged and where V_ϕ disappears since $\partial_i\phi = 0$. The potential V_G is given by [61]

$$V_G \equiv -\mathrm{g}R = \frac{1}{4}\sum_{i\neq j, i\neq k, j\neq k} e^{-2\alpha_{ijk}(\beta)}(C^i{}_{jk})^2 + \frac{1}{2}\sum_j e^{-2\bar{m}_j(\beta)}\left(C^i{}_{jk}\,C^k{}_{ji} + \text{“more”}\right), \tag{2.33}$$

where the linear forms $\alpha_{ijk}(\beta)$ (with i, j, k distinct) read

$$\alpha_{ijk}(\beta) = 2\beta^i + \sum_{m\,:\,m\neq i, m\neq j, m\neq k} \beta^m, \tag{2.34}$$

and where “more” stands for the terms in the first sum that arise upon taking $i = j$ or $i = k$. The structure constants in the Iwasawa frame (with respect to the coframe in Equation (2.30)) are related to the structure constants $f^i{}_{jk}$ through

$$C^i{}_{jk} = \sum_{i',j',k'} f^{i'}{}_{j'k'}\mathcal{N}^{-1}_{ii'}\mathcal{N}_{jj'}\mathcal{N}_{kk'} \tag{2.35}$$

and depend therefore on the dynamical variables. Similarly, the potential $V^{\mathrm{magn}}_{(p)}$ becomes

$$V^{\mathrm{magn}}_{(p)} = \frac{1}{2\,(p+1)!}\sum_{i_1,i_2,\cdots,i_{p+1}} e^{-2m_{i_1\cdots i_{p+1}}(\beta)}(\mathcal{F}^h_{(p)\,i_1\cdots i_{p+1}})^2, \tag{2.36}$$

where the field strengths $\mathcal{F}^h_{(p)\,i_1\cdots i_{p+1}}$ reduce to the “AC” terms in dA and depend on the potentials and the off-diagonal Iwasawa variables.

[4] This Hamiltonian exists if $f^i{}_{ik} = 0$, as we shall assume from now on.

2.3.2 The ultralocal Hamiltonian

Let us now come back to the general, inhomogeneous case and express the dynamics in the frame $\{dx^0, \psi^i\}$ where the ψ^i's form a "generic" non-holonomic frame in space,

$$d\psi^i = -\frac{1}{2} f^i{}_{jk}(x^m)\, \psi^j \wedge \psi^k. \tag{2.37}$$

Here the $f^i{}_{jk}$'s are in general space-dependent. In the non-holonomic frame, the exact Hamiltonian takes the form

$$\mathcal{H} = \mathcal{H}^{\mathrm{UL}} + \mathcal{H}^{\mathrm{gradient}}, \tag{2.38}$$

where the ultralocal part $\mathcal{H}^{\mathrm{UL}}$ is given by Equations (2.32) and (2.33) with the relevant $f^i{}_{jk}$'s, and where $\mathcal{H}^{\mathrm{gradient}}$ involves the spatial gradients of $f^i{}_{jk}$, β^m, ϕ and $\mathcal{N}_{ij}$.

The first part of the BKL conjecture states that one can drop $\mathcal{H}^{\mathrm{gradient}}$ asymptotically; namely, the dynamics of a generic solution of the Einstein–p-form-dilaton equations (not necessarily spatially homogeneous) is asymptotically determined, as one goes to the spatial singularity, by the ultralocal Hamiltonian

$$H^{\mathrm{UL}} = \int d^d x\, \mathcal{H}^{\mathrm{UL}}, \tag{2.39}$$

provided that the phase space constants $f^i{}_{jk}(x^m) = -f^i{}_{kj}(x^m)$ are such that all exponentials in the above potentials do appear. In other words, the f's must be chosen such that none of the coefficients of the exponentials, which involve f and the fields, identically vanishes – as would be the case, for example, if $f^i{}_{jk} = 0$ since then the potentials V_G and $V^{\mathrm{magn}}_{(p)}$ are equal to zero. This is always possible because the $f^i{}_{jk}$, even though independent of the dynamical variables, may in fact depend on x and so are not required to fulfill relations "$ff = 0$" analogous to the Bianchi identity since one has instead "$\partial f + ff = 0$".

Comments

1. As we shall see, the conditions on the f's (that all exponentials in the potential should be present) can be considerably weakened. It is necessary that only the relevant exponentials (in the sense defined in Section 2.4) be present. Thus, one can correctly capture the asymptotic BKL behavior of a generic solution with fewer exponentials. In the case of eleven-dimensional supergravity the spatial curvature is asymptotically negligible with respect to the electromagnetic terms and one can in fact take a holonomic frame for which $f^i{}_{jk} = 0$ (and hence also $C^i{}_{jk} = 0$).
2. The actual values of the $f^i{}_{jk}$ (provided they fulfill the criterion given above or rather its weaker form just mentioned) turn out to be irrelevant in the BKL-limit because they can be absorbed through redefinitions. This is for instance why the Bianchi VIII and IX models, even though they correspond to different groups, can both be used to describe the BKL behavior in four spacetime dimensions.

2.4 Dynamics as a billiard in hyperbolic space

The second step in the BKL-limit is to take the sharp wall limit of the potentials.[5] This leads to the billiard picture. It is crucial here that the coefficients in front of the dominant walls are all

[5] In this article we will exclusively restrict ourselves to considerations involving the sharp wall limit. However, in recent work [40] it was argued that in order to have a rigorous treatment of the dynamics close to the singularity also in the chaotic case, it is necessary to go beyond the sharp wall limit. This implies that one should retain the exponential structure of the dominant walls.

positive. Again, just as for the first step, this limit has not been fully justified. Only heuristic, albeit convincing, arguments have been put forward.

The idea is that as one goes to the singularity, the exponential potentials get sharper and sharper and can be replaced in the limit by the corresponding Θ_∞-function, denoted for short Θ and defined by $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = +\infty$ for $x > 0$. Taking into account the facts that $a\Theta(x) = \Theta(x)$ for all $a > 0$, as well as that some walls can be neglected, one finds that the Hamiltonian becomes in the sharp wall limit

$$H = \int d^d x \, \mathcal{H}^{\text{sharp}}, \tag{2.40}$$

with

$$\begin{aligned} \mathcal{H}^{\text{sharp}} = K &+ \sum_{i<j} \Theta\left(-2 s_{ji}(\beta)\right) + \sum_{i \neq j, i \neq k, j \neq k} \Theta(-2\alpha_{ijk}(\beta)) \\ &+ \sum_{i_1 < i_2 < \cdots < i_p} \Theta(-2 e_{i_1 \cdots i_p}(\beta)) + \sum_{i_1 < i_2 < \cdots < i_{p+1}} \Theta(-2 m_{i_1 \cdots i_{p+1}}(\beta)), \end{aligned} \tag{2.41}$$

where $s_{ji}(\beta) = \beta^j - \beta^i$. See [48] for more information.

The description of the motion of the scale factors (at each spatial point) is easy to give in that limit. Because the potential walls are infinite (and positive), the motion is constrained to the region where the arguments of all Θ-functions are negative, i.e., to

$$s_{ji}(\beta) \geq 0 \, (i < j), \qquad \alpha_{ijk}(\beta) \geq 0, \qquad e_{i_1 \cdots i_p}(\beta) \geq 0, \qquad m_{i_1 \cdots i_{p+1}}(\beta) \geq 0. \tag{2.42}$$

In that region, the motion is governed by the kinetic term K, i.e., is a geodesic for the metric in the space of the scale factors. Since that metric is flat, this is a straight line. In addition, the constraint $\mathcal{H} = 0$, which reduces to $K = 0$ away from the potential walls, forces the straight line to be null. We shall assume that the time orientation in the space of the scale factors is such that the straight line is future-oriented ($\mathrm{g} \to 0$ in the future).

It is easy to check that all the walls appearing in Equation (2.41), collectively denoted $F_A(\beta) \equiv F_{A\mu}\beta^\mu = 0$, are timelike hyperplanes. This is because the squared norms of all the F_A's are positive,

$$(F_A|F_A) = \sum_i \left(\frac{\partial F_A}{\partial \beta^i}\right)^2 - \frac{1}{d-1}\left(\sum_i \frac{\partial F_A}{\partial \beta^i}\right)^2 + \left(\frac{\partial F_A}{\partial \phi}\right)^2 > 0. \tag{2.43}$$

Explicitly, one finds

$$\begin{aligned} (s_{ji}|s_{ji}) &= 2, \\ (\alpha_{ijk}|\alpha_{ijk}) &= 2, \\ (e_{i_1 \cdots i_p}|e_{i_1 \cdots i_p}) &= \frac{p(d-p-1)}{d-1} + \frac{\left(\lambda^{(p)}\right)^2}{4}, \\ (m_{i_1 \cdots i_{p+1}}|m_{i_1 \cdots i_{p+1}}) &= \frac{p(d-p-1)}{d-1} + \frac{\left(\lambda^{(p)}\right)^2}{4}. \end{aligned} \tag{2.44}$$

Because the potential walls are timelike, they have a non-empty intersection with the forward light cone in the space of the scale factors. When the null straight line representing the evolution of the scale factors hits one of the walls, it gets reflected according to the rule [43]

$$v^\mu \quad \to \quad v^\mu - 2\frac{v^\nu F_{A\nu}}{(F_A|F_A)} F_A^\mu, \tag{2.45}$$

where v is the velocity vector (tangent to the straight line). This reflection preserves the time orientation since the hyperplanes are timelike and hence belong to the orthochronous Lorentz group $O^{\uparrow}(k,1)$ where $k = d-1$ or d according to whether there is no or one dilaton. The conditions $s_{ji} = 0$ define the "symmetry" or "centrifugal" walls, the conditions $\alpha_{ijk} = 0$ define the "curvature" or "gravitational" walls, the conditions $e_{i_1 \cdots i_p} = 0$ define the "electric" walls, while the conditions $m_{i_1 \cdots i_{p+1}} = 0$ define the "magnetic" walls.

The motion is thus a succession of future-oriented null straight line segments interrupted by reflections against the walls, where the motion undergoes a reflection belonging to $O^{\uparrow}(k,1)$. Whether the collisions eventually stop or continue forever is better visualized by projecting the motion radially on the positive sheet of the unit hyperboloid, as was done first in the pioneering work of Chitre and Misner [31, 138] for pure gravity in four spacetime dimensions. We recall that the positive sheet of the unit hyperboloid $\sum(\beta^i)^2 - (\sum\beta^i)^2 + \phi^2 = -1$, $\sum\beta^i > 0$, provides a model of hyperbolic space (see, e.g., [146]).

The intersection of a timelike hyperplane with the unit hyperboloid defines a hyperplane in hyperbolic space. The region in hyperbolic space on the positive side of all hyperplanes is the allowed dynamical region and is called the "billiard table". It is never compact in the cases relevant to gravity, but it may or may not have finite volume. The projection of the motion of the scale factors on the unit hyperboloid is the same as the motion of a billiard ball in a hyperbolic billiard: geodesic arcs in hyperbolic space within the billiard region, interrupted by collisions against the bounding walls where the motion undergoes a specular reflection.

When the volume of the billiard table is finite, the collisions with the potential walls never end (for generic initial data) and the motion is chaotic. When, on the other hand, the volume is infinite, generic initial data lead to a motion that ultimately freely runs away to infinity. This is non-chaotic. For more information, see [135, 170]. An interesting criterion for chaos (equivalent to finite volume of hyperbolic billiard region) has been given in [111] in terms of illuminations of spheres by point sources.

Comments

1. The task of determining the billiard region is greatly simplified by the observation that some walls are behind others and are thus not relevant. For instance, it is clear that if $\beta^2 - \beta^1 > 0$ and $\beta^3 - \beta^2 > 0$, then $\beta^3 - \beta^1 > 0$. Among the symmetry wall conditions, the only relevant ones are $\beta^{i+1} - \beta^i > 0$, $i = 1, 2, \cdots, d-1$. Similarly, a wall of any given type can be written as a positive combination of the walls of the same type with smallest values of the indices i of the β's and the symmetry walls (e.g., the electric wall condition $\beta^2 > 0$ for a 1-form with zero dilaton coupling can be written as $\beta^1 + (\beta^2 - \beta^1) > 0$ and is thus a consequence of $\beta^1 > 0$ and $\beta^2 - \beta^1 > 0$). Finally, one also verifies that in the presence of true p-forms ($0 < p < d-1$), the gravitational walls are never relevant as they can be written as combinations of p-form walls with positive coefficients [49].

2. It is interesting to determine the spatially homogeneous models that reproduce asymptotically the correct billiard limit. It is clear that in order to do so, homogeneous cosmological models need only contain the relevant walls. It is not necessary that they yield all the walls. Which homogeneity groups are acceptable depends on the system at hand. We list here a few examples. For vacuum gravity in four spacetime dimensions, the appropriate homogeneous models are the so-called Bianchi VIII or IX models. For vacuum gravity in higher dimensions, the structure constants of the homogeneity group must fulfill the conditions of [60] *and the metric must include off-diagonal components* (see also [58]). In the presence of a single p-form and no dilaton ($0 < p < d-1$), the simplest (Abelian) homogeneity group can be taken [44].

2.5 Rules for deriving the wall forms from the Lagrangian – Summary

We have recalled above that the generic behavior near a spacelike singularity of the system with action (2.1) can be described at each spatial point in terms of a billiard in hyperbolic space. The action for the billiard ball reads, in the gauge $N = \sqrt{\mathrm{g}}$,

$$S = \int dx^0 \left[G_{\mu\nu} \frac{d\beta^\mu}{dx^0} \frac{d\beta^\nu}{dx^0} - V(\beta^\mu) \right], \tag{2.46}$$

where we recall that $x^0 \to \infty$ in the BKL-limit (proper time $T \to 0^+$), and $G_{\mu\nu}$ is the metric in the space of the scale factors,

$$G_{\mu\nu}\, d\beta^\mu\, d\beta^\nu = \sum_{i=1}^{d} d\beta^i\, d\beta^i - \left(\sum_{i=1}^{d} d\beta^i \right) \left(\sum_{j=1}^{d} d\beta^j \right) + d\phi\, d\phi \tag{2.47}$$

introduced in Equation (2.15) above. As stressed there, this metric is flat and of Lorentzian signature. Between two collisions, the motion is a free, geodesic motion. The collisions with the walls are controlled by the potential $V(\beta^\mu)$, which is a sum of sharp wall potentials. The walls are hyperplanes and can be inferred from the Lagrangian. They are as follows:

1. Gravity brings in the symmetry walls

$$\beta^{i+1} - \beta^i = 0, \tag{2.48}$$

   with $i = 1, 2, \cdots, d-1$, and the curvature wall

$$2\beta^1 + \beta^2 + \cdots + \beta^{d-2} = 0. \tag{2.49}$$

2. Each p-form brings in an electric wall

$$\beta^1 + \cdots + \beta^p + \frac{\lambda^{(p)}}{2}\phi = 0, \tag{2.50}$$

   and a magnetic wall

$$\beta^1 + \cdots + \beta^{d-p-1} - \frac{\lambda^{(p)}}{2}\phi = 0. \tag{2.51}$$

We have written here only the (potentially) relevant walls. There are other walls present in the potential, but because these are behind the relevant walls, which are infinitely steep in the BKL-limit, they are irrelevant. They are relevant, however, when trying to exhibit the symmetry in a complete treatment where the BKL-limit is the zeroth order term in a gradient expansion yet to be understood [47].

The scalar product dual to the scalar product in the space of the scale factors is

$$(F|G) = \sum_i F_i G_i - \frac{1}{d-1} \left(\sum_i F_i \right) \left(\sum_j G_j \right) + F_\phi G_\phi \tag{2.52}$$

for two linear forms $F = F_i \beta^i + F_\phi \phi$, $G = G_i \beta^i + G_\phi \phi$.

These recipes are all that we shall need for investigating the regularity properties of the billiards associated with the class of actions Equation (2.1).

2.6 More on the free motion: The Kasner solution

The free motion between two bounces is a straight line in the space of the scale factors. In terms of the original metric components, it takes the form of the Kasner solution with dilaton. Indeed, the free motion is given by

$$\beta^\mu = q^\mu x^0 + \beta_0^\mu,$$

where the "velocities" q^μ are subject to

$$\sum_i (q^i)^2 - \left(\sum_i q^i\right)^2 + q_\phi^2 = 0,$$

since the motion is lightlike by the Hamiltonian constraint. The proper time $dT = -\sqrt{\mathrm{g}}\, dx^0$ is then $T = B\exp(-Kx^0)$, with $K = \sum_i q^i$ and for some constant B (we assume, as before, that the singularity is at $T = 0^+$). Redefining then

$$p^\mu = \frac{q^\mu}{\sum_i q^i}$$

yields the celebrated Kasner solution

$$ds^2 = -dT^2 + \sum_i T^{2p^i} \left(dx^i\right)^2, \tag{2.53}$$

$$\phi = -p_\phi \ln T + A, \tag{2.54}$$

subject to the constraints

$$\sum_i p^i = 1, \qquad \sum_i (p^i)^2 + p_\phi^2 = 1, \tag{2.55}$$

where A is a constant of integration and where the coordinates x^i have been suitably rescaled (if necessary).

2.7 Chaos and billiard volume

With our rules for writing down the billiard region, one can determine in which case the volume of the billiard is finite and in which case it is infinite. The finite-volume, chaotic case is also called "mixmaster case", a terminology introduced in four dimensions in [137].

The following results have been obtained:

- Pure gravity in $D \leq 10$ dimensions is chaotic, but ceases to be so for $D \geq 11$ [63, 62].
- The introduction of a dilaton removes chaos [15, 3]. The gravitational four-derivative action in four dimensions, based on R^2, is dynamically equivalent to Einstein gravity coupled to a dilaton [160]. Hence, chaos is removed also for this case.
- p-form gauge fields ($0 < p < d-1$) without scalar fields lead to a finite-volume billiard [44].
- When both p-forms and dilatons are included, the situation is more subtle as there is a competition between two opposing effects. One can show that if the dilaton couplings are in a "subcritical" open region that contains the origin – i.e., "not too big" – the billiard volume is infinite and the system is non chaotic. If the dilaton couplings are outside of that region, the billiard volume is finite and the system is chaotic [49].

2.8 A note on the constraints

We have focused in the above presentation on the dynamical equations of motion. The constraints were only briefly mentioned, with no discussion, except for the Hamiltonian constraint. This is legitimate because the constraints are first class and hence preserved by the Hamiltonian evolution. Thus, they need only be imposed at some "initial" time. Once this is done, one does not need to worry about them any more. Furthermore the momentum constraints and Gauss' law constraints are differential equations relating the initial data at different spatial points. This means that they do not constrain the dynamical variables at a given point but involve also their gradients – contrary to the Hamiltonian constraint which becomes ultralocal. Consequently, at any given point, one can freely choose the initial data on the undifferentiated dynamical variables and then use these data as (part of) the appropriate boundary data necessary to integrate the constraints throughout space. This is why one can assert that all the walls described above are generically present even when the constraints are satisfied.

The situation is different in homogeneous cosmologies where the symmetry relates the values of the fields at all spatial points. The momentum and Gauss' law constraints become then algebraic equations and might remove some relevant walls. But this feature (removal of walls by the momentum and Gauss' law constraints) is specific to some homogeneous cosmologies and does not hold in the generic case where spatial gradients are non-zero.

A final comment: How the spatial diffeomorphism constraints and Gauss' law fit in the conjectured infinite-dimensional symmetry is a point that is still poorly understood. See, however, [52] for recent progress in this direction.

2.9 On the validity of the BKL conjecture – A status report

Providing a complete rigorous justification of the above description of the behavior of the gravitational field in the vicinity of a spacelike singularity is a formidable task that has not been pushed to completion yet. The task is formidable because the Einstein equations form a complicated nonlinear system of partial differential equations. We shall assume throughout our review that the BKL description is correct, based on the original convincing arguments put forward by BKL themselves [16] and the subsequent fruitful investigations that have shed further important light on the validity of the conjecture. The billiard description will thus be taken for granted.

For completeness, we provide in this section a short guide to the work that has been accumulated since the late 1960's to consolidate the BKL phenomenon.

As we have indicated, there are two aspects to the BKL conjecture:

1. The first part of the conjecture states that spatial points decouple as one goes to a spacelike singularity in the sense that the evolution can be described by a collection of systems of ordinary differential equations with respect to time, one such system at each spatial point. (*"A spacelike singularity is local."*)

2. The second part of the conjecture states that the system of ordinary differential equations with respect to time describing the asymptotic dynamics at any given spatial point can be asymptotically replaced by the billiard equations. If the matter content is such that the billiard table has infinite volume, the asymptotic behavior at each point is given by a (generalized) Kasner solution (*"Kasner-like spacelike singularities"*). If, on the other hand, the matter content is such that the billiard table has finite volume, the asymptotic behavior at each point is a chaotic, infinite, oscillatory succession of Kasner epochs. (*"Oscillatory, or mixmaster, spacelike singularities."*)

A third element of the original conjecture was that the matter could be neglected asymptotically. While generically true in four spacetime dimensions (the exception being a massless scalar field, equivalent to a fluid with the stiff equation of state $p = \rho$), this aspect of the conjecture does not remain valid in higher dimensions where the p-form fields might add relevant walls that could change the qualitative asymptotic behavior. We shall thus focus here only on Aspects 1 and 2.

- In the Kasner-like case, the mathematical situation is easier to handle since the conjectured asymptotic behavior of the fields is then monotone and known in closed form. There exist theorems validating (generically) this conjectured asymptotic behavior, starting from the pioneering work of [3] (where the singularities with this behavior are called "quiescent"), which was extended later in [49] to cover more general matter contents. See also [18, 108] for related work.

- The situation is much more complicated in the oscillatory case, where only partial results exist. However, even though as yet incomplete, the mathematical and numerical studies of the BKL analysis has provided overwhelming support for its validity. Most work has been done in four dimensions.

  The first attempts to demonstrate that spacelike singularities are local were done in the simpler context of solutions with isometries. It is only recently that general solutions without symmetries have been treated, but this has been found to be possible only numerically so far [87]. The literature on this subject is vast and we refer to [2, 87, 147] for points of entry into it. Let us note that an important element in the analysis has been a more precise reformulation of what is meant by "local". This has been achieved in [163], where a precise definition involving a judicious choice of scale invariant variables has been proposed and given the illustrative name of *"asymptotic silence"* – the singularities being called *"silent singularities"* since propagation of information is asymptotically eliminated.

  If one accepts that generic spacelike singularities are silent, one can investigate the system of ordinary differential equations that arise in the local limit. In four dimensions, this system is the same as the system of ordinary differential equations describing the dynamics of spatially homogeneous cosmologies of Bianchi type IX. It has been effectively shown analytically in [151] that the Bianchi IX evolution equations can indeed be replaced, in the generic case, by the billiard equations (with only the dominant, sharp walls) that produce the mixmaster behavior. This validates the second element in the BKL conjecture in four dimensions.

The connection between the billiard variables and the scale invariant variables has been investigated recently in the interesting works [92, 162].

Finally, taking for granted the BKL conjecture, one might analyze the chaotic properties of the billiard map (when the volume is finite). Papers exploring this issue are [30, 32, 121, 132] (four dimensions) and [68] (five dimensions).

Let us finally mention the interesting recent paper [40], in which a more precise formulation of the BKL conjecture, aimed towards the chaotic case, is presented. In particular, the main result of this work is an extension of the Fuchsian techniques, employed, e.g., in [49], which are applicable also for systems exhibiting chaotic dynamics. Furthermore, [40] examines the geometric structure which is preserved close to the singularity, and it is shown that this structure has a mathematical description in terms of a so called "partially framed flag".

3 Hyperbolic Coxeter Groups

In this section, we develop the theory of Coxeter groups with a particular emphasis on the hyperbolic case. The importance of Coxeter groups for the BKL analysis stems from the fact that in the case of the gravitational theories that have been studied most (pure gravity, supergravities), the group generated by the reflections in the billiard walls is a Coxeter group. This follows, in turn, from the regularity of the corresponding billiards, whose walls intersect at angles that are integer submultiples of π.

3.1 Preliminary example: The BKL billiard (vacuum $D = 4$ gravity)

To illustrate the regularity of the gravitational billiards and motivate the mathematical developments through an explicit example, we first compute in detail the billiard characterizing vacuum, $D = 4$ gravity. Since this corresponds to the case originally considered by BKL, we call it the "BKL billiard". We show in detail that the billiard reflections in this case are governed by the "extended modular group" $PGL(2,\mathbb{Z})$, which, as we shall see, is isomorphic to the hyperbolic Coxeter group A_1^{++}.

3.1.1 Billiard reflections

There are three scale factors so that after radial projection on the unit hyperboloid, we get a billiard in two-dimensional hyperbolic space. The billiard region is defined by the following relevant wall inequalities,

$$\beta^2 - \beta^1 > 0, \qquad \beta^3 - \beta^2 > 0 \tag{3.1}$$

(symmetry walls) and

$$2\beta^1 > 0 \tag{3.2}$$

(curvature wall). The remarkable properties of this region from our point of view are:

- It is a triangle (i.e., a simplex in two dimensions) because even though we had to begin with 6 walls (3 symmetry walls and 3 curvature walls), only 3 of them are relevant.
- The walls intersect at angles that are integer submultiples of π, i.e., of the form

  $$\frac{\pi}{n}, \tag{3.3}$$

  where n is an integer. The symmetry walls intersect indeed at sixty degrees ($n = 3$) since the scalar product of the corresponding linear forms (of norm squared equal to 2) is -1, while the gravitational wall makes angles of zero ($n = \infty$, scalar product $= -2$) and ninety ($n = 2$, scalar product $= 0$) degrees with the symmetry walls.

These angles are captured in the matrix $A = (A_{ij})_{i,j=1,2,3}$ of scalar products,

$$A_{ij} = (\alpha_i|\alpha_j)\,, \tag{3.4}$$

which reads explicitly

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{3.5}$$

Recall from the previous section that the scalar product of two linear forms $F = F_i\beta^i$ and $G = G_i\beta^i$ is, in a three-dimensional scale factor space,

$$(F|G) = \sum_i F_iG_i - \frac{1}{2}\left(\sum_i F_i\right)\left(\sum_i G_i\right), \tag{3.6}$$

where we have taken $\alpha_1(\beta) \equiv 2\beta^1$, $\alpha_2(\beta) \equiv \beta^2 - \beta^1$ and $\alpha_3(\beta) \equiv \beta^3 - \beta^2$. The corresponding billiard region is drawn in Figure 1.

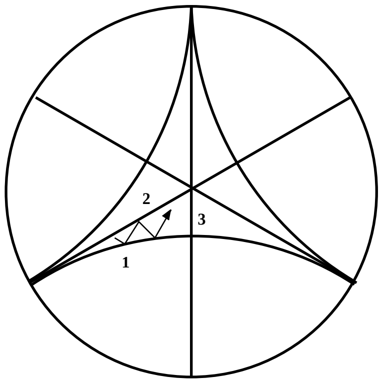


Figure 1: The BKL billiard of pure four-dimensional gravity. The figure represents the billiard region projected onto the hyperbolic plane. The particle geodesic is confined to the fundamental region enclosed by the three walls $\alpha_1(\beta) = 2\beta^1 = 0, \alpha_2(\beta) = \beta^2 - \beta^1 = 0$ and $\alpha_3(\beta) = \beta^3 - \beta^2 = 0$, as indicated by the numbering in the figure. The two symmetry walls $\alpha_2(\beta) = 0$ and $\alpha_3(\beta) = 0$ intersect at an angle of $\pi/3$, while the gravity wall $\alpha_1(\beta) = 0$ intersects, respectively, at angles 0 and $\pi/2$ with the symmetry walls $\alpha_2(\beta) = 0$ and $\alpha_3(\beta) = 0$. The particle has no direction of escape so the dynamics is chaotic.

Because the angles between the reflecting planes are integer submultiples of π, the reflections in the walls bounding the billiard region[6],

$$s_i(\gamma) = \gamma - 2\frac{(\gamma|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i = \gamma - (\gamma|\alpha_i)\alpha_i, \tag{3.7}$$

obey the following relations,

$$s_1 s_3 = s_3 s_1 \quad \leftrightarrow \quad (s_1 s_3)^2 = 1, \qquad (s_2 s_3)^3 = 1. \tag{3.8}$$

The product $s_1 s_3$ is a rotation by $2\pi/2 = \pi$ and hence squares to one; the product $s_2 s_3$ is a rotation by $2\pi/3$ and hence its cube is equal to one. There is no power of the product $s_1 s_2$ that is equal to one, something that one conventionally writes as

$$(s_1 s_2)^\infty = 1. \tag{3.9}$$

The group generated by the reflections s_1, s_2 and s_3 is denoted A_1^{++}, for reasons that will become clear in the following, and coincides with the arithmetic group $PGL(2,\mathbb{Z})$, as we will now show (see also [75, 116, 107]).

[6] s_i is the reflection with respect to the hyperplane defined by $\alpha_i = 0$, because it preserves the scalar product, fixes the plane orthogonal to α_i and maps α_i on $-\alpha_i$. Note that we are here being deliberately careless about notation in order not to obscure the main point, namely that the billiard reflections are elements of a Coxeter group. To be precise, the linear forms $\alpha_i(\beta)$, $i = 1, 2, 3$, really represent the *values* of the linear maps $\alpha_i : \beta \quad \rightarrow \quad \alpha_i(\beta) \in \mathbb{R}$. The billiard ball moves in the space of scale factors, say $\mathcal{M}_\beta$ (β-space), and hence the maps α_i, which define the walls, belong to the dual space $\mathcal{M}_\beta^\star$ of linear forms acting on $\mathcal{M}_\beta$. In order to be compatible with the treatment in Section 2.4 (cf. Equation (2.45)), Equation (3.7) – even though written here as a reflection in the space $\mathcal{M}_\beta^\star$ – really corresponds to a geometric reflection in the space $\mathcal{M}_\beta$, in which the particle moves. This will be carefully explained in Section 5.2 (cf. Equations (5.20) and (5.21)), after the necessary mathematical background has been introduced.

3.1.2 On the group $PGL(2,\mathbb{Z})$

The group $PGL(2,\mathbb{Z})$ is defined as the group of 2×2 matrices C with integer entries and determinant equal to ± 1, with the identification of C and $-C$,

$$PGL(2,\mathbb{Z}) = \frac{GL(2,\mathbb{Z})}{\mathbb{Z}_2}. \tag{3.10}$$

Note that although elements of the real general linear group $GL(2,\mathbb{R})$ have (non-vanishing) unrestricted determinants, the discrete subgroup $GL(2,\mathbb{Z}) \subset GL(2,\mathbb{R})$ only allows for $\det C = \pm 1$ in order for the inverse C^{-1} to also be an element of $GL(2,\mathbb{Z})$.

There are two interesting realisations of $PGL(2,\mathbb{Z})$ in terms of transformations in two dimensions:

- One can view $PGL(2,\mathbb{Z})$ as the group of fractional transformations of the complex plane

  $$C: z \rightarrow z' = \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{Z}, \tag{3.11}$$

  with

  $$ad - cb = \pm 1. \tag{3.12}$$

  Note that one gets the same transformation if C is replaced by $-C$, as one should. It is an easy exercise to verify that the action of $PGL(2,\mathbb{Z})$ when defined in this way maps the complex upper half-plane,

  $$\mathbb{H} = \{z \in \mathbb{C} \,|\, \Im z > 0\}, \tag{3.13}$$

  onto itself whenever the determinant $ad - bc$ of C is equal to $+1$. This is not the case, however, when $\det C = -1$.

- For this reason, it is convenient to consider alternatively the following action of $PGL(2,\mathbb{Z})$,

  $$\begin{aligned} z \rightarrow z' &= \frac{az+b}{cz+d}, \qquad \text{if } ad-cb=1,\\ &\text{or}\\ z \rightarrow z' &= \frac{a\bar{z}+b}{c\bar{z}+d}, \qquad \text{if } ad-cb=-1, \end{aligned} \tag{3.14}$$

  ($a,b,c,d \in \mathbb{Z}$), which does map the complex upper-half plane onto itself, i.e., which is such that $\Im z' > 0$ whenever $\Im z > 0$.

  The transformation (3.14) is the composition of the identity with the transformation (3.11) when $\det C = 1$, and of the complex conjugation transformation, $f: z \rightarrow \bar{z}$ with the transformation (3.11) when $\det C = -1$. Because the coefficients a, b, c, and d are real, f commutes with C and furthermore the map (3.11) $\rightarrow$ (3.14) is a group isomorphism, so that we can indeed either view the group $PGL(2,\mathbb{Z})$ as the group of fractional transformations (3.11), or as the group of transformations (3.14).

An important subgroup of the group $PGL(2,\mathbb{Z})$ is the group $PSL(2,\mathbb{Z})$ for which $ad - cb = 1$, also called the "modular group". The translation $T: z \rightarrow z+1$ and the inversion $S: z \rightarrow -1/z$ are examples of modular transformations,

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.15}$$

It is a classical result that any modular transformation can be written as the product

$$T^{m_1} S T^{m_2} S \cdots S T^{m_k}, \tag{3.16}$$

but the representation is not unique [4].

Let s_1, s_2 and s_3 be the $PGL(2,\mathbb{Z})$-transformations

$$\begin{aligned} s_1 : z &\to -\bar{z}, \\ s_2 : z &\to 1 - \bar{z}, \\ s_3 : z &\to \frac{1}{\bar{z}}, \end{aligned} \tag{3.17}$$

to which there correspond the matrices

$$s_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \qquad s_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.18}$$

The s_i's are reflections in the straight lines $x = 0$, $x = 1/2$ and the unit circle $|z| = 1$, respectively. These are in fact just the transformations of hyperbolic space s_1, s_2 and s_3 described in Section 3.1.1, since the reflection lines intersect at 0, 90 and 60 degrees, respectively.

One easily verifies that $T = s_2 s_1$ and that $S = s_1 s_3 = s_3 s_1$. Since any transformation of $PGL(2,\mathbb{Z})$ not in $PSL(2,\mathbb{Z})$ can be written as a transformation of $PSL(2,\mathbb{Z})$ times, say, s_1 and since any transformation of $PSL(2,\mathbb{Z})$ can be written as a product of S's and T's, it follows that the group generated by the 3 reflections s_1, s_2 and s_3 coincides with $PGL(2,\mathbb{Z})$, as announced above. (Strictly speaking, $PGL(2,\mathbb{Z})$ could be a quotient of that group by some invariant subgroup, but one may verify that the kernel of the homomorphism is trivial (see Section 3.2.5 below).) The fundamental domains for $PGL(2,\mathbb{Z})$ and $PSL(2,\mathbb{Z})$ are drawn in Figure 2. The equivalence between $PGL(2,\mathbb{Z})$ and the Coxeter group A_1^{++} has been discussed previously in [75, 116, 107].

3.2 Coxeter groups – The general theory

We have just shown that the billiard group in the case of pure gravity in four spacetime dimensions is the group $PGL(2,\mathbb{Z})$. This group is generated by reflections and is a particular example of a Coxeter group. Furthermore, as we shall explain below, this Coxeter group turns out to be the Weyl group of the (hyperbolic) Kac–Moody algebra A_1^{++}. Our first encounter with Lorentzian Kac–Moody algebras in more general gravitational theories will also be through their Weyl groups, which are, exactly as in the four-dimensional case just described, particular instances of (non-Euclidean) Coxeter groups, and which arise as the groups of billiard reflections.

For this reason, we start by developing here some aspects of the theory of Coxeter groups. An excellent reference on the subject is [107], to which we refer for more details and information. We consider Kac–Moody algebras in Section 4.

3.2.1 Examples

Coxeter groups generalize the familiar notion of reflection groups in Euclidean space. Before we present the basic definition, let us briefly discuss some more illuminating examples.

The dihedral group $I_2(3) \equiv A_2$

Consider the dihedral group $I_2(3)$ of order 6 of symmetries of the equilateral triangle in the Euclidean plane.

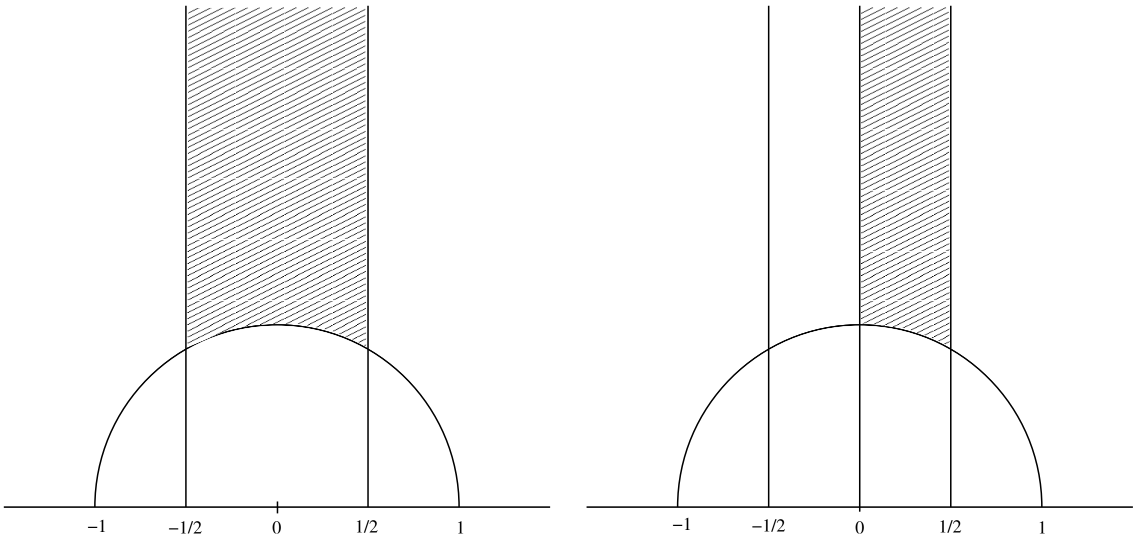


Figure 2: The figure on the left hand side displays the action of the modular group $PSL(2,\mathbb{Z})$ on the complex upper half plane $\mathbb{H} = \{z \in \mathbb{C}\,|\,\Im z > 0\}$. The two generators of $PSL(2,\mathbb{Z})$ are S and T, acting as follows on the coordinate $z \in \mathbb{H} : S(z) = -1/z$; $T(z) = z + 1$, i.e., as an inversion and a translation, respectively. The shaded area indicates the fundamental domain $\mathcal{D}_{PSL(2,\mathbb{Z})} = \{z \in \mathbb{H}\,|\, -1/2 \leq \Re z \leq 1/2;\ |z| \geq 1\}$ for the action of $PSL(2,\mathbb{Z})$ on $\mathbb{H}$. The figure on the right hand side displays the action of the "extended modular group" $PGL(2,\mathbb{Z})$ on $\mathbb{H}$. The generators of $PGL(2,\mathbb{Z})$ are obtained by augmenting the generators of $PSL(2,\mathbb{Z})$ with the generator s_1, acting as $s_1(z) = -\bar{z}$ on $\mathbb{H}$. The additional two generators of $PGL(2,\mathbb{Z})$ then become: $s_2 \equiv s_1 \circ T$; $s_3 \equiv s_1 \circ S$, and their actions on $\mathbb{H}$ are $s_2(z) = 1-\bar{z}$; $s_3(z) = 1/\bar{z}$. The new generator s_1 corresponds to a reflection in the line $\Re z = 0$, the generator s_2 is in turn a reflection in the line $\Re z = 1/2$, while the generator s_3 is a reflection in the unit circle $|z| = 1$. The fundamental domain of $PGL(2,\mathbb{Z})$ is $\mathcal{D}_{PGL(2,\mathbb{Z})} = \{z \in \mathbb{H}\,|\,0 \leq \Re z \leq 1/2;\ |z| \geq 1\}$, corresponding to half the fundamental domain of $PSL(2,\mathbb{Z})$. The "walls" $\Re z = 0, \Re z = 1/2$ and $|z| = 1$ correspond, respectively, to the gravity wall $\alpha_1(\beta) = 0$, the symmetry wall $\alpha_2(\beta) = 0$ and the symmetry wall $\alpha_3(\beta) = 0$ of Figure 1.

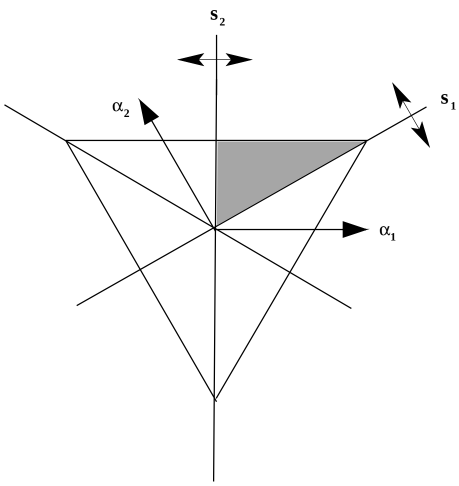


Figure 3: The equilateral triangle with its 3 axes of symmetries. The reflections s_1 and s_2 generate the entire symmetry group. We have pictured the vectors α_1 and α_2 orthogonal to the axes of reflection and chosen to make an obtuse angle. The shaded region $\{w|(w|\alpha_1) \geq 0\} \cap \{w|(w|\alpha_2) \geq 0\}$ is a fundamental domain for the action of the group on the triangle. Note that the fundamental domain for the action of the group on the entire Euclidean plane extends indefinitely beyond the triangle but is, of course, still bounded by the two walls orthogonal to α_1 and α_2.

This group contains the identity, three reflections s_1, s_2 and s_3 about the three medians, the rotation R_1 of $2\pi/3$ about the origin and the rotation R_2 of $4\pi/3$ about the origin (see Figure 3),

$$I_2(3) = \{1, s_1, s_2, s_3, R_1, R_2\}. \tag{3.19}$$

The reflections act as follows[7],

$$s_i(\gamma) = \gamma - 2\frac{(\gamma|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i, \tag{3.20}$$

where $(\ |\ )$ is here the Euclidean scalar product and where α_i is a vector orthogonal to the hyperplane (here, line) of reflection.

Now, all elements of the dihedral group $I_2(3)$ can be written as products of the two reflections s_1 and s_2:

$$1 = s_1^0, \qquad s_1 = s_1, \qquad s_2 = s_2, \qquad R_1 = s_1 s_2, \qquad R_2 = s_2 s_1, \qquad s_3 = s_1 s_2 s_1. \tag{3.21}$$

Hence, the dihedral group $I_2(3)$ is generated by s_1 and s_2. The writing Equation (3.21) is not unique because s_1 and s_2 are subject to the following relations,

$$s_1^2 = 1, \qquad s_2^2 = 1, \qquad (s_1 s_2)^3 = 1. \tag{3.22}$$

The first two relations merely follow from the fact that s_1 and s_2 are reflections, while the third relation is a consequence of the property that the product $s_1 s_2$ is a rotation by an angle of $2\pi/3$. This follows, in turn, from the fact that the hyperplanes (lines) of reflection make an angle of $\pi/3$. There is no other relation between the generators s_1 and s_2 because any product of them can be reduced, using the relations Equation (3.22), to one of the 6 elements in Equation (3.21), and these are independent.

The dihedral group $I_2(3)$ is also denoted A_2 because it is the Weyl group of the simple Lie algebra A_2 (see Section 4). It is isomorphic to the permutation group S_3 of three objects.

[7]Note that the discussion in Footnote 6 applies also here.

The infinite dihedral group $I_2(\infty) \equiv A_1^+$

Consider now the group of isometries of the Euclidean line containing the symmetries about the points with integer or half-integer values of x (x is a coordinate along the line) as well as the translations by an integer. This is clearly an infinite group. It is generated by the two reflections s_1 about the origin and s_2 about the point with coordinate 1/2,

$$s_1(x) = -x, \qquad s_2(x) = -(x-1). \tag{3.23}$$

The product $s_2 s_1$ is a translation by $+1$ while the product $s_1 s_2$ is a translation by -1, so no power of $s_1 s_2$ or $s_2 s_1$ gives the identity. All the powers $(s_2 s_1)^k$ and $(s_1 s_2)^j$ are distinct (translations by $+k$ and $-j$, respectively). The only relations between the generators are

$$s_1^2 = 1 = s_2^2. \tag{3.24}$$

This infinite dihedral group $I_2(\infty)$ is also denoted by A_1^+ because it is the Weyl group of the affine Kac–Moody algebra A_1^+.

3.2.2 Definition

A Coxeter group $\mathfrak{C}$ is a group generated by a finite number of elements s_i ($i = 1, \cdots, n$) subject to relations that take the form

$$s_i^2 = 1 \tag{3.25}$$

and

$$(s_i s_j)^{m_{ij}} = 1, \tag{3.26}$$

where the integers m_{ij} associated with the pairs (i,j) fulfill

$$\begin{aligned} m_{ij} &= m_{ji}, \\ m_{ij} &\geq 2 \qquad (i \neq j). \end{aligned} \tag{3.27}$$

Note that Equation (3.25) is a particular case of Equation (3.26) with $m_{ii} = 1$. If there is no power of $s_i s_j$ that gives the identity, as in our second example, we set, by convention, $m_{ij} = \infty$. The generators s_i are called "reflections" because of Equation (3.25), even though we have not developed yet a geometric realisation of the group. This will be done in Section 3.2.4 below.

The number n of generators is called the rank of the Coxeter group. The Coxeter group is completely specified by the integers m_{ij}. It is useful to draw the set $\{m_{ij}\}$ pictorially in a diagram Γ, called a Coxeter graph. With each reflection s_i, one associates a node. Thus there are n nodes in the diagram. If $m_{ij} > 2$, one draws a line between the node i and the node j and writes m_{ij} over the line, except if m_{ij} is equal to 3, in which case one writes nothing. The default value is thus "3". When there is no line between i and j ($i \neq j$), the exponent m_{ij} is equal to 2. We have drawn the Coxeter graphs for the Coxeter groups $I_2(3)$, $I_2(m)$ and for the Coxeter group H_3 of symmetries of the icosahedron.

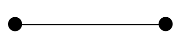


Figure 4: The Coxeter graph of the symmetry group $I_2(3) \equiv A_2$ of the equilateral triangle.

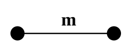


Figure 5: The Coxeter graph of the dihedral group $I_2(m)$.

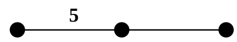


Figure 6: The Coxeter graph of the symmetry group H_3 of the regular icosahedron.

Note that if $m_{ij} = 2$, the generators s_i and s_j commute, $s_i s_j = s_j s_i$. Thus, a Coxeter group $\mathfrak{C}$ is the direct product of the Coxeter subgroups associated with the connected components of its Coxeter graph. For that reason, we can restrict the analysis to Coxeter groups associated with connected (also called irreducible) Coxeter graphs.

The Coxeter group may be finite or infinite as the previous examples show.

Another example: C_2^+

It should be stressed that the Coxeter group can be infinite even if none of the Coxeter exponent is infinite. Consider for instance the group of isometries of the Euclidean plane generated by reflections in the following three straight lines: (i) the x-axis (s_1), (ii) the straight line joining the points $(1, 0)$ and $(0, 1)$ (s_2), and (iii) the y-axis (s_3). The Coxeter exponents are finite and equal to 4 ($m_{12} = m_{21} = m_{23} = m_{32} = 4$) and 2 ($m_{13} = m_{31} = 2$). The Coxeter graph is given in Figure 7. The Coxeter group is the symmetry group of the regular paving of the plane by squares and contains translations. Indeed, the product $s_2 s_1 s_2$ is a reflection in the line parallel to the y-axis going through $(1, 0)$ and thus the product $t = s_2 s_1 s_2 s_3$ is a translation by $+2$ in the x-direction. All powers of t are distinct; the group is infinite. This Coxeter group is of affine type and is called C_2^+ (which coincides with B_2^+).

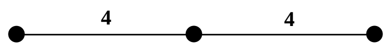


Figure 7: The Coxeter graph of the affine Coxeter group C_2^+ corresponding to the group of isometries of the Euclidean plane.

The isomorphism problem

The Coxeter presentation of a given Coxeter group may not be unique. Consider for instance the group $I_2(6)$ of order 12 of symmetries of the regular hexagon, generated by two reflections s_1 and s_2 with

$$s_1^2 = s_2^2 = 1, \qquad (s_1 s_2)^6 = 1.$$

This group is isomorphic with the rank 3 (reducible) Coxeter group $I_2(3) \times \mathbb{Z}_2$, with presentation

$$r_1^2 = r_2^2 = r_3^2 = 1, \qquad (r_1 r_2)^3 = 1, \qquad (r_1 r_3)^2 = 1, \qquad (r_2 r_3)^2 = 1,$$

the isomorphism being given by $f(r_1) = s_1$, $f(r_2) = s_1 s_2 s_1 s_2 s_1$, $f(r_3) = (s_1 s_2)^3$. The question of determining all such isomorphisms between Coxeter groups is known as the "isomorphism problem of Coxeter groups". This is a difficult problem whose general solution is not yet known [10].

3.2.3 The length function

An important concept in the theory of Coxeter groups is that of the length of an element. The length of $w \in \mathfrak{C}$ is by definition the number of generators that appear in a minimal representation of w as a product of generators. Thus, if $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ and if there is no way to write w as a product of less than l generators, one says that w has length l.

For instance, for the dihedral group $I_2(3)$, the identity has length zero, the generators s_1 and s_2 have length one, the two non-trivial rotations have length two, and the third reflection s_3 has length three. Note that the rotations have representations involving two and four (and even a higher number of) generators since for instance $s_1 s_2 = s_2 s_1 s_2 s_1$, but the length is associated with the representations involving as few generators as possible. There might be more than one such representation as it occurs for $s_3 = s_1 s_2 s_1 = s_2 s_1 s_2$. Both involve three generators and define the length of s_3 to be three.

Let w be an element of length l. The length of ws_i (where s_i is one of the generators) differs from the length of w by an odd (positive or negative) integer since the relations among the generators always involve an even number of reflections. In fact, $l(ws_i)$ is equal to $l+1$ or $l-1$ since $l(ws_i) \leq l(w)+1$ and $l(w \equiv ws_i s_i) \leq l(ws_i)+1$. Thus, in ws_i, there can be at most one simplification (i.e., at most two elements that can be removed using the relations).

3.2.4 Geometric realization

We now construct a geometric realisation for any given Coxeter group. This enables one to view the Coxeter group as a group of linear transformations acting in a vector space of dimension n, equipped with a scalar product preserved by the group.

To each generator s_i, associate a vector α_i of a basis $\{\alpha_1, \cdots, \alpha_n\}$ of an n-dimensional vector space V. Introduce a scalar product defined as follows,

$$B(\alpha_i, \alpha_j) = -\cos\left(\frac{\pi}{m_{ij}}\right), \tag{3.28}$$

on the basis vectors and extend it to V by linearity. Note that for $i = j$, $m_{ii} = 1$ implies $B(\alpha_i, \alpha_i) = 1$ for all i. In the case of the dihedral group A_2, this scalar product is just the Euclidean scalar product in the two-dimensional plane where the equilateral triangle lies, as can be seen by taking the two vectors α_1 and α_2 respectively orthogonal to the first and second lines of reflection in Figure 3 and oriented as indicated. But in general, the scalar product (3.28) might not be of Euclidean signature and might even be degenerate. This is the case for the infinite dihedral group $I_2(\infty)$, for which the matrix B reads

$$B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{3.29}$$

and has zero determinant. We shall occasionally use matrix notations for the scalar product, $B(\alpha, \gamma) \equiv \alpha^T B \gamma$.

However, the basis vectors are always all spacelike since they have norm squared equal to 1. For each i, the vector space V splits then as a direct sum

$$V = \mathbb{R}\alpha_i \oplus H_i, \tag{3.30}$$

where H_i is the hyperplane orthogonal to α_i ($\delta \in H_i$ iff $B(\delta, \alpha_i) = 0$). One defines the geometric reflection σ_i as

$$\sigma_i(\gamma) = \gamma - 2B(\gamma, \alpha_i)\alpha_i. \tag{3.31}$$

It is clear that σ_i fixes H_i pointwise and reverses α_i. It is also clear that $\sigma_i^2 = 1$ and that σ_i preserves B,

$$B\left(\sigma_i(\gamma), \sigma_i(\gamma')\right) = B(\gamma, \gamma'). \tag{3.32}$$

Note that in the particular case of A_2, we recover in this way the reflections s_1 and s_2.

We now verify that the σ_i's also fulfill the relations $(\sigma_i \sigma_j)^{m_{ij}} = 1$. To that end we consider the plane Π spanned by α_i and α_j. This plane is left invariant under σ_i and σ_j. Two possibilities may occur:

1. The induced scalar product on Π is nondegenerate and in fact positive definite, or
2. the induced scalar product is positive semi-definite, i.e., there is a null direction orthogonal to any other direction.

The second case occurs only when $m_{ij} = \infty$. The null direction is given by $\lambda = \alpha_i + \alpha_j$.

- In Case 1, V splits as $\Pi \oplus \Pi^\perp$ and $(\sigma_i\sigma_j)^{m_{ij}}$ is clearly the identity on $\Pi^\perp$ since both σ_i and σ_j leave $\Pi^\perp$ pointwise invariant. One needs only to investigate $(\sigma_i\sigma_j)^{m_{ij}}$ on Π, where the metric is positive definite. To that end we note that the reflections σ_i and σ_j are, on Π, standard Euclidean reflections in the lines orthogonal to α_i and α_j, respectively. These lines make an angle of π/m_{ij} and hence the product $\sigma_i\sigma_j$ is a rotation by an angle of $2\pi/m_{ij}$. It follows that $(\sigma_i\sigma_j)^{m_{ij}} = 1$ also on Π.
- In Case 2, m_{ij} is infinite and we must show that no power of the product $\sigma_i\sigma_j$ gives the identity. This is done by exhibiting a vector γ for which $(\sigma_i\sigma_j)^k(\gamma) \neq \gamma$ for all integers k different from zero. Take for instance α_i. Since one has $(\sigma_i\sigma_j)(\alpha_i) = \alpha_i + 2\lambda$ and $(\sigma_i\sigma_j)(\lambda) = \lambda$, it follows that $(\sigma_i\sigma_j)^k(\alpha_i) = \alpha_i + 2k\lambda \neq \alpha_i$ unless $k = 0$.

As the defining relations are preserved, we can conclude that the map f from the Coxeter group generated by the s_i's to the geometric group generated by the σ_i's defined on the generators by $f(s_i) = \sigma_i$ is a group homomorphism. We will show below that its kernel is the identity so that it is in fact an isomorphism.

Finally, we note that if the Coxeter graph is irreducible, as we assume, then the matrix B_{ij} is *indecomposable*. A matrix A_{ij} is called *decomposable* if after reordering of its indices, it decomposes as a non-trivial direct sum, i.e., if one can slit the indices i, j in two sets J and Λ such that $A_{ij} = 0$ whenever $i \in J, j \in \Lambda$ or $i \in \Lambda, j \in J$. The indecomposability of B follows from the fact that if it were decomposable, the corresponding Coxeter graph would be disconnected as no line would join a point in the set Λ to a point in the set J.

3.2.5 Positive and negative roots

A *root* is any vector in the space V of the geometric realisation that can be obtained from one of the basis vectors α_i by acting with an element w of the Coxeter group (more precisely, with its image $f(w)$ under the above homomorphism, but we shall drop "f" for notational simplicity). Any root α can be expanded in terms of the α_i's,

$$\alpha = \sum_i c_i \alpha_i. \tag{3.33}$$

If the coefficients c_i are all non-negative, we say that the root α is positive and we write $\alpha > 0$. If the coefficients c_i are all non-positive, we say that the root α is negative and we write $\alpha < 0$. Note that we use strict inequalities here because if $c_i = 0$ for all i, then α is not a root. In particular, the α_i's themselves are positive roots, called also "simple" roots. (Note that the simple roots considered here differ by normalization factors from the simple roots of Kac–Moody algebras, as we shall discuss below.) We claim that roots are either positive or negative (there is no root with some c_i's in Equation (3.33) > 0 and some other c_i's < 0). The claim follows from the fact that the image of a simple root by an arbitrary element w of the Coxeter group is necessarily either positive or negative.

This, in turn, is the result of the following theorem, which provides a useful criterion to tell whether the length $l(ws_i)$ of ws_i is equal to $l(w) + 1$ or $l(w) - 1$.

Theorem: $l(ws_i) = l(w) + 1$ if and only if $w(\alpha_i) > 0$.

The proof is given in [107], page 111.

It easily follows from this theorem that $l(ws_i) = l(w) - 1$ if and only if $w(\alpha_i) < 0$. Indeed, $l(ws_i) = l(w) - 1$ is equivalent to $l(w) = l(ws_i) + 1$, i.e., $l((ws_i)s_i) = l(ws_i) + 1$ and thus, by the theorem, $ws_i(\alpha_i) > 0$. But since $s_i(\alpha_i) = -\alpha_i$, this is equivalent to $w(\alpha_i) < 0$.

We have seen in Section 3.2.3 that there are only two possibilities for the length $l(ws_i)$. It is either equal to $l(w) + 1$ or to $l(w) - 1$. From the theorem just seen, the root $w(\alpha_i)$ is positive in the first case and negative in the second. Since any root is the Coxeter image of one of the simple roots α_i, i.e., can be written as $w(\alpha_i)$ for some w and α_i, we can conclude that the roots are either positive or negative; there is no alternative.

The theorem can be used to provide a geometric interpretation of the length function. One can show [107] that $l(w)$ is equal to the number of positive roots sent by w to negative roots. In particular, the fundamental reflection s associated with the simple root α_s maps α_s to its negative and permutes the remaining positive roots.

Note that the theorem implies also that the kernel of the homomorphism that appears in the geometric realisation of the Coxeter group is trivial. Indeed, assume $f(w) = 1$ where w is an element of the Coxeter group that is not the identity. It is clear that there exists one group generator s_i such that $l(ws_i) = l(w) - 1$. Take for instance the last generator occurring in a reduced expression of w. For this generator, one has $w(\alpha_i) < 0$, which is in contradiction with the assumption $f(w) = 1$.

Because f is an isomorphism, we shall from now on identify the Coxeter group with its geometric realisation and make no distinction between s_i and σ_i.

3.2.6 Fundamental domain

In order to describe the action of the Coxeter group, it is useful to introduce the concept of fundamental domain. Consider first the case of the symmetry group A_2 of the equilateral triangle. The shaded region $\mathcal{F}$ in Figure 4 contains the vectors γ such that $B(\alpha_1, \gamma) \geq 0$ and $B(\alpha_2, \gamma) \geq 0$. It has the following important property: Any orbit of the group A_2 intersects $\mathcal{F}$ once and only once. It is called for this reason a "fundamental domain". We shall extend this concept to all Coxeter groups. However, when the scalar product B is not positive definite, there are inequivalent types of vectors and the concept of fundamental domain can be generalized a priori in different ways, depending on which region one wants to cover. (The entire space? Only the timelike vectors? Another region?) The useful generalization turns out not to lead to a fundamental domain of the action of the Coxeter group on the entire vector space V, but rather to a fundamental domain of the action of the Coxeter group on the so-called Tits cone $\mathcal{X}$, which is such that the inequalities $B(\alpha_i, \gamma) \geq 0$ continue to play the central role.

We assume that the scalar product is nondegenerate. Define for each simple root α_i the open half-space

$$A_i = \{\gamma \in V \mid B(\alpha_i, \gamma) > 0\}. \tag{3.34}$$

We define $\mathcal{E}$ to be the intersection of all A_i,

$$\mathcal{E} = \bigcap_i A_i. \tag{3.35}$$

This is a convex open cone, which is non-empty because the metric is nondegenerate. Indeed, as B is nondegenerate, one can, by a change of basis, assume for simplicity that the bounding hyperplanes $B(\alpha_i, \gamma) = 0$ are the coordinate hyperplanes $x_i = 0$. $\mathcal{E}$ is then the region $x_i > 0$ (with appropriate orientation of the coordinates) and $\mathcal{F}$ is $x_i \geq 0$. The closure

$$\begin{aligned} \mathcal{F} = \bar{\mathcal{E}} &= \textstyle\bigcap_i \bar{A}_i, \\ \bar{A}_i &= \{\gamma \in V | B(\alpha_i, \gamma) \geq 0\} \end{aligned} \tag{3.36}$$

is then a closed convex cone[8].

We next consider the union of the images of $\mathcal{F}$ under the Coxeter group,

$$\mathcal{X} = \bigcup_{w \in \mathfrak{C}} w(\mathcal{F}). \tag{3.37}$$

One can show [107] that this is also a convex cone, called the Tits cone. Furthermore, $\mathcal{F}$ is a fundamental domain for the action of the Coxeter group on the Tits cone; the orbit of any point in $\mathcal{X}$ intersects $\mathcal{F}$ once and only once [107]. The Tits cone does not coincide in general with the full space V and is discussed below in particular cases.

3.3 Finite Coxeter groups

An important class of Coxeter groups are the finite ones, like $I_2(3)$ above. One can show that a Coxeter group is finite if and only if the scalar product defined by Equation (3.28) on V is Euclidean [107]. Finite Coxeter groups coincide with finite reflection groups in Euclidean space (through hyperplanes that all contain the origin) and are discrete subgroups of $O(n)$. The classification of finite Coxeter groups is known and is given in Table 1 for completeness. For finite Coxeter groups, one has the important result that the Tits cone coincides with the entire space V [107].

3.4 Affine Coxeter groups

Affine Coxeter groups are by definition such that the bilinear form B is positive semi-definite but not positive definite. The radical $V^\perp$ (defined as the subspace of vectors x for which $B(x,y) \equiv x^T B y = 0$ for all y) is then one-dimensional (in the irreducible case). Indeed, since B is positive semi-definite, its radical coincides with the set N of vectors such that $\lambda^T B \lambda = 0$ as can easily be seen by going to a basis in which B is diagonal (the eigenvalues of B are non-negative). Furthermore, N is at least one-dimensional since B is not positive definite (one of the eigenvalues is zero). Let μ be a vector in $V^\perp \equiv N$. Let ν be the vector whose components are the absolute values of those of μ, $\nu_i = |\mu_i|$. Because $B_{ij} \leq 0$ for $i \neq j$ (see definition of B in Equation (3.28)), one has

$$0 \leq \nu^T B \nu \leq \mu^T B \mu = 0$$

and thus the vector ν belongs also to $V^\perp$. All the components of ν are strictly positive, $\nu_i > 0$. Indeed, let J be the set of indices for which $\nu_j > 0$ and I the set of indices for which $\nu_i = 0$. From $\sum_j B_{kj}\nu_j = 0$ ($\nu \in V^\perp$) one gets, by taking k in I, that $B_{ij} = 0$ for all $i \in I$, $j \in J$, contrary to the assumption that the Coxeter system is irreducible (B is indecomposable). Hence, none of the components of any zero eigenvector μ can be zero. If $V^\perp$ were more than one-dimensional, one could easily construct a zero eigenvector of B with at least one component equal to zero. Hence, the eigenspace $V^\perp$ of zero eigenvectors is one-dimensional.

Affine Coxeter groups can be identified with the groups generated by affine reflections in Euclidean space (i.e., reflections through hyperplanes that may not contain the origin, so that the group contains translations) and have also been completely classified [107]. The translation subgroup of an affine Coxeter group $\mathfrak{C}$ is an invariant subgroup and the quotient $\mathfrak{C}_0$ is finite; the affine Coxeter group $\mathfrak{C}$ is equal to the semi-direct product of its translation subgroup by $\mathfrak{C}_0$. We list all the affine Coxeter groups in Table 2.

[8]Note that in the case of the infinite dihedral group $I_2(\infty)$, for which B is degenerate, the definition does not give anything of interest since $\mathcal{E} = \emptyset$. When B is degenerate, the formalism developed here can nevertheless be carried through but one must go to the dual space V^* [107].

Table 1: Finite Coxeter groups.

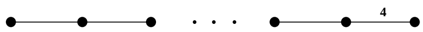
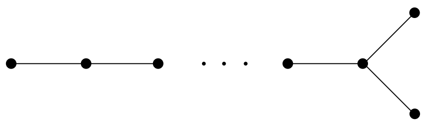
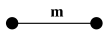
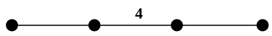
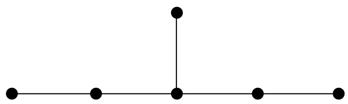
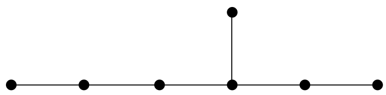
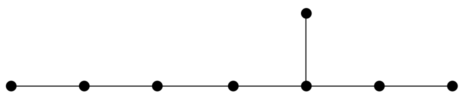
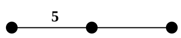
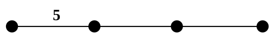
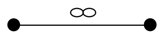
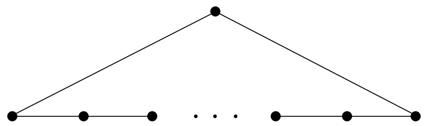
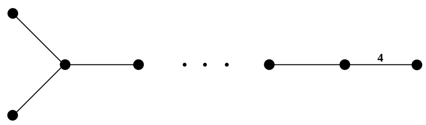
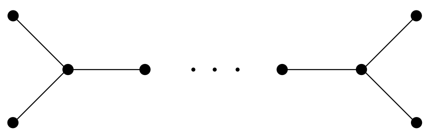
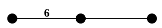
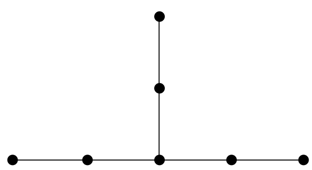
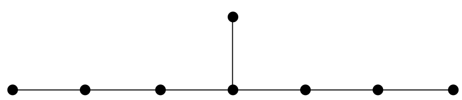
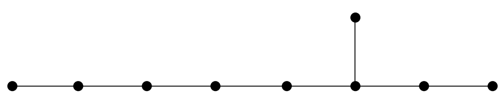
Name	Coxeter graph
A_n	
$B_n \equiv C_n$	
D_n	
$I_2(m)$	
F_4	
E_6	
E_7	
E_8	
H_3	
H_4	

Table 2: Affine Coxeter groups.

Name	Coxeter graph
A_1^+	
A_n^+ $(n > 1)$	
B_n^+ $(n > 2)$	
C_n^+	
D_n^+	
G_2^+	
F_4^+	
E_6^+	
E_7^+	
E_8^+	

3.5 Lorentzian and hyperbolic Coxeter groups

Coxeter groups that are neither of finite nor of affine type are said to be of indefinite type. An important property of Coxeter groups of indefinite type is the following. There exists a positive vector (c_i) such that $\sum_j B_{ij} c_j$ is negative [116]. A vector is said to be positive (respectively, negative) if all its components are strictly positive (respectively, strictly negative). This is denoted $c_i > 0$ (respectively, $c_i < 0$). Note that a vector may be neither positive nor negative, if some of its components are positive while some others are negative. Note also that these concepts refer to a specific basis. This property is demonstrated in Appendix A.

We assume, as already stated, that the scalar product B is nondegenerate. Let $\{\omega_i\}$ be the basis dual to the basis $\{\alpha_i\}$ in the scalar product B,

$$B(\alpha_i, \omega_j) = \delta_{ij}. \tag{3.38}$$

The ω_i's are called "fundamental weights". (The fundamental weights are really defined by Equation (3.38) up to normalization, as we will see in Section 3.6 on crystallographic Coxeter groups. They thus differ from the solutions of Equation (3.38) only by a positive multiplicative factor, irrelevant for the present discussion.)

Consider the vector $v = \sum_i c_i \alpha_i$, where the vector c_i is such that $c_i > 0$ and $\sum_j B_{ij} c_j < 0$. This vector exists since we assume the Coxeter group to be of indefinite type. Let Σ be the hyperplane orthogonal to v. Because $c_i > 0$, the vectors ω_i's all lie on the positive side of Σ, $B(v, \omega_i) = c_i > 0$. By contrast, the vectors α_i's all lie on the negative side of Σ since $B(\alpha_i, v) = \sum_j B_{ij} c_j < 0$. Furthermore, v has negative norm squared, $B(v, v) = \sum_i c_i (\sum_j B_{ij} c_j) < 0$. Thus, in the case of Coxeter groups of indefinite type (with a nondegenerate metric), one can choose a hyperplane such that the positive roots lie on one side of it and the fundamental weights on the other side. The converse is true for Coxeter group of finite type: In that case, there exists $c_i > 0$ such that $\sum_j B_{ij} c_j$ is positive, implying that the positive roots and the fundamental weights are on the same side of the hyperplane Σ.

We now consider a particular subclass of Coxeter groups of indefinite type, called Lorentzian Coxeter groups. These are Coxeter groups such that the scalar product B is of Lorentzian signature $(n-1, 1)$. They are discrete subgroups of the orthochronous Lorentz group $O^+(n-1, 1)$ preserving the time orientation. Since the α_i are spacelike, the reflection hyperplanes are timelike and thus the generating reflections s_i preserve the time orientation. The hyperplane Σ from the previous paragraph is spacelike. In this section, we shall adopt Lorentzian coordinates so that Σ has equation $x^0 = 0$ and we shall choose the time orientation so that the positive roots have a negative time component. The fundamental weights have then a positive time component. This choice is purely conventional and is made here for convenience. Depending on the circumstances, the other time orientation might be more useful and will sometimes be adopted later (see for instance Section 4.8).

Turn now to the cone $\mathcal{E}$ defined by Equation (3.35). This cone is clearly given by

$$\mathcal{E} = \{\lambda \in V | \forall \alpha_i \quad B(\lambda, \alpha_i) > 0\} = \left\{ \sum d_i \omega_i | d_i > 0 \right\}. \tag{3.39}$$

Similarly, its closure $\mathcal{F}$ is given by

$$\mathcal{F} = \{\lambda \in V | \forall \alpha_i \quad B(\lambda, \alpha_i) \geq 0\} = \left\{ \sum d_i \omega_i | d_i \geq 0 \right\}. \tag{3.40}$$

The cone $\mathcal{F}$ is thus the convex hull of the vectors ω_i, which are on the boundary of $\mathcal{F}$.

By definition, a hyperbolic Coxeter group is a Lorentzian Coxeter group such that the vectors in $\mathcal{E}$ are all timelike, $B(\lambda, \lambda) < 0$ for all $\lambda \in \mathcal{E}$. Hyperbolic Coxeter groups are precisely the groups that emerge in the gravitational billiards of physical interest. The hyperbolicity condition forces $B(\lambda, \lambda) \leq 0$ for all $\lambda \in \mathcal{F}$, and in particular, $B(\omega_i, \omega_i) \leq 0$: The fundamental weights are timelike or null. The cone $\mathcal{F}$ then lies within the light cone. This does not occur for generic (non-hyperbolic) Lorentzian algebras.

The following theorem enables one to decide whether a Coxeter group is hyperbolic by mere inspection of its Coxeter graph.

Theorem: Let $\mathfrak{C}$ be a Coxeter group with irreducible Coxeter graph Γ. The Coxeter group is hyperbolic if and only if the following two conditions hold:

- The bilinear form B is nondegenerate but not positive definite.
- For each i, the Coxeter graph obtained by removing the node i from Γ is of finite or affine type.

(Note: By removing a node, one might get a non-irreducible diagram even if the original diagram is connected. A reducible diagram defines a Coxeter group of finite type if and only if each irreducible component is of finite type, and a Coxeter group of affine type if and only if each irreducible component is of finite or affine type with at least one component of affine type.)

Proof:

- It is clear that if a Coxeter group is hyperbolic, then its bilinear form fulfills the first condition. Let ω_i be one of the vectors of the dual basis. The vectors α_j with $j \neq i$ form a basis of the hyperplane Π_i orthogonal to ω_i. Because ω_i is non-spacelike (the group is hyperbolic), the hyperplane Π_i is spacelike or null. The Coxeter graph defined by the α_j with $j \neq i$ (i.e., by removing the node α_i) is thus of finite or affine type.
- Conversely, assume that the two conditions of the theorem hold. From the first condition, it follows that the set $N = \{\lambda \in V \,|\, B(\lambda, \lambda) < 0\}$ is non-empty. Let Π_i be the hyperplane spanned by the α_j with $j \neq i$, i.e., orthogonal to ω_i. From the second condition, it follows that the intersection of N with each Π_i is empty. Accordingly, each connected component of N lies in one of the connected components of the complement of $\bigcup_i \Pi_i$, namely, is on a definite (positive or negative) side of each of the hyperplanes Π_i. These sets are of the form $\sum_i c_i \alpha_i$ with $c_i > 0$ for some i's (fixed throughout the set) and $c_i < 0$ for the others. This forces the signature of B to be Lorentzian since otherwise there would be at least a two-dimensional subspace Z of V such that $Z \setminus \{0\} \subset N$. Because $Z \setminus \{0\}$ is connected, it must lie in one of the subsets just described. But this is impossible since if $\lambda \in Z \setminus \{0\}$, then $-\lambda \in Z \setminus \{0\}$.

We now show that $\mathcal{E} \subset N$. Because the signature of B is Lorentzian, N is the inside of the standard light cone and has two components, the "future" component and the "past" component. From the second condition of the theorem, each ω_i lies on or inside the light cone since the orthogonal hyperplane is non-timelike. Furthermore, all the ω_i's are future pointing, which implies that the cone $\mathcal{E}$ lies in N, as had to be shown (a positive sum of future pointing non spacelike vectors is non-spacelike). This concludes the proof of the theorem.

In particular, this theorem is useful for determining all hyperbolic Coxeter groups once one knows the list of all finite and affine ones. To illustrate its power, consider the Coxeter diagram of Figure 8, with 8 nodes on the loop and one extra node attached to it (we shall see later that it is called A_7^{++}).

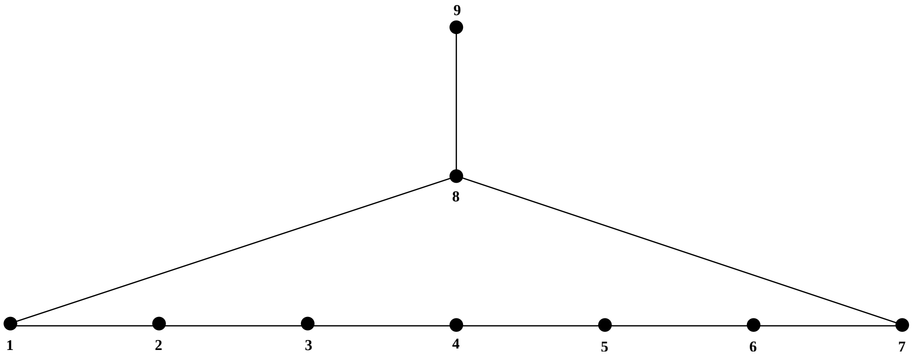


Figure 8: The Coxeter graph of the group A_7^{++}.

The bilinear form is given by

$$\frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{3.41}$$

and is of Lorentzian signature. If one removes the node labelled 9, one gets the affine diagram A_7^+ (see Figure 9). If one removes the node labelled 8, one gets the finite diagram of the direct product group $A_1 \times A_7$ (see Figure 10). Deleting the nodes labelled 1 or 7 yields the finite diagram of A_8 (see Figure 11). Removing the nodes labelled 2 or 6 gives the finite diagram of D_8 (see Figure 12). If one removes the nodes labelled 3 or 5, one obtains the finite diagram of E_8 (see Figure 13). Finally, deleting the node labelled 4 yields the affine diagram of E_7^+ (see Figure 14). Hence, the Coxeter group is hyperbolic.

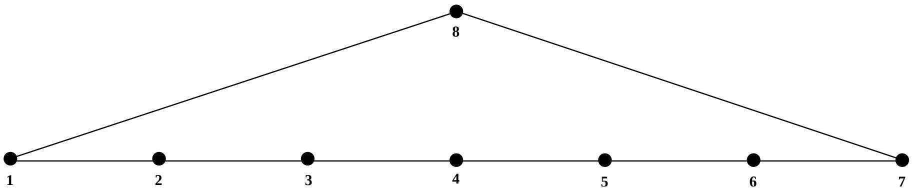


Figure 9: The Coxeter graph of A_7^+.

Consider now the same diagram, with one more node in the loop (A_8^{++}). In that case, if one removes one of the middle nodes 4 or 5, one gets the Coxeter group E_7^{++}, which is neither finite nor affine. Hence, A_8^{++} is not hyperbolic.

Using the two conditions in the theorem, one can in fact provide the list of all irreducible hyperbolic Coxeter groups. The striking fact about this classification is that hyperbolic Coxeter groups exist only in ranks $3 \leq n \leq 10$, and, moreover, for $4 \leq n \leq 10$ there is only a finite number. In the $n = 3$ case, on the other hand, there exists an infinite class of hyperbolic Coxeter groups.

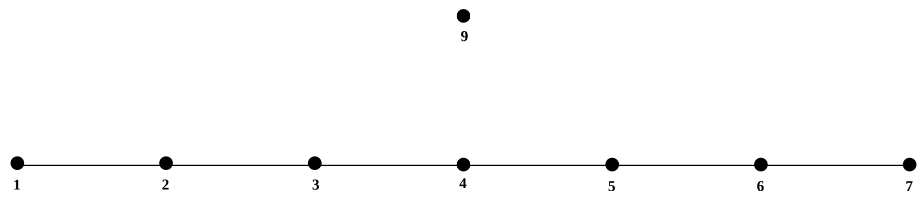


Figure 10: The Coxeter graph of $A_7 \times A_1$.

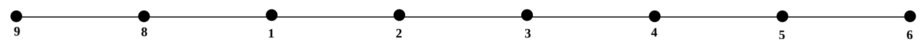


Figure 11: The Coxeter graph of A_8.

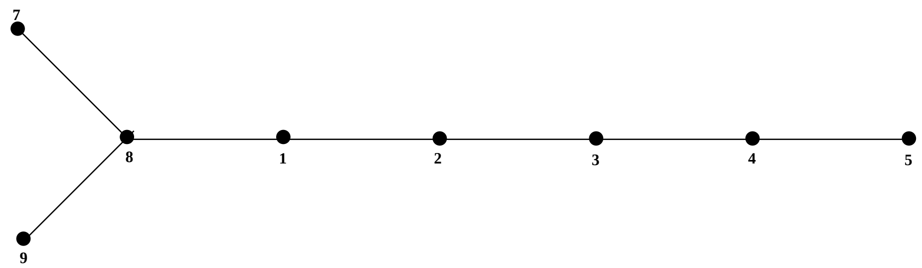


Figure 12: The Coxeter graph of D_8.

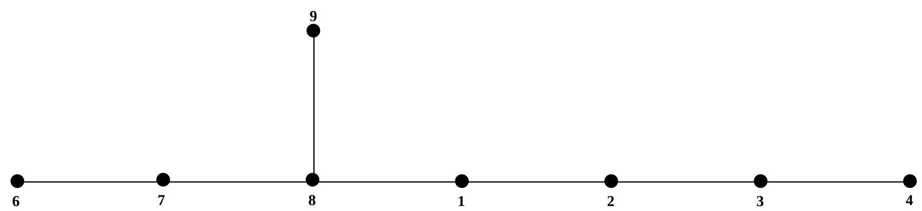


Figure 13: The Coxeter graph of E_8.

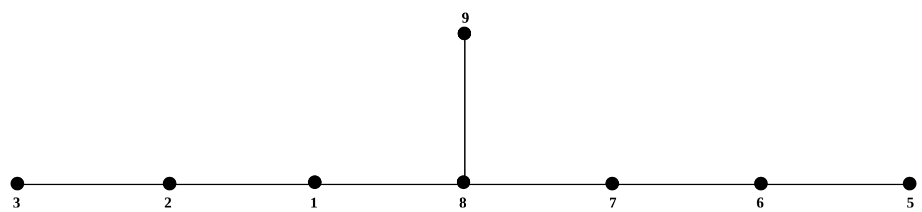


Figure 14: The Coxeter graph of E_7^+.

In Figure 15 we give a general form of the Coxeter graphs corresponding to all rank 3 hyperbolic Coxeter groups, and in Tables 3 – 9 we give the complete classification for $4 \leq n \leq 10$.

Note that the inverse metric $(B^{-1})_{ij}$, which gives the scalar products of the fundamental weights, has only negative entries in the hyperbolic case since the scalar product of two future-pointing non-spacelike vectors is strictly negative (it is zero only when the vectors are both null and parallel, which does not occur here).

One can also show [116, 107] that in the hyperbolic case, the Tits cone $\mathcal{X}$ coincides with the future light cone. (In fact, it coincides with either the future light cone or the past light cone. We assume that the time orientation in V has been chosen as in the proof of the theorem, so that the Tits cone coincides with the future light cone.) This is at the origin of an interesting connection with discrete reflection groups in hyperbolic space (which justifies the terminology). One may realize hyperbolic space $\mathcal{H}_{n-1}$ as the upper sheet of the hyperboloid $B(\lambda, \lambda) = -1$ in V. Since the Coxeter group is a subgroup of $O^+(n-1, 1)$, it leaves this sheet invariant and defines a group of reflections in $\mathcal{H}_{n-1}$. The fundamental reflections are reflections through the hyperplanes in hyperbolic space obtained by taking the intersection of the Minkowskian hyperplanes $B(\alpha_i, \lambda) = 0$ with hyperbolic space. These hyperplanes bound the fundamental region, which is the domain to the positive side of each of these hyperplanes. The fundamental region is a simplex with vertices $\bar{\omega}_i$, where $\bar{\omega}_i$ are the intersection points of the lines $\mathbb{R}\omega_i$ with hyperbolic space. This intersection is at infinity in hyperbolic space if ω_i is lightlike. The fundamental region has finite volume but is compact only if the ω_i are timelike.

Thus, we see that the hyperbolic Coxeter groups are the reflection groups in hyperbolic space with a fundamental domain which (i) is a simplex, and which (ii) has finite volume. The fact that the fundamental domain is a simplex (n vectors in $\mathcal{H}_{n-1}$) follows from our geometric construction where it is assumed that the n vectors α_i form a basis of V.

The group $PGL(2, \mathbb{Z})$ relevant to pure gravity in four dimensions is easily verified to be hyperbolic.

For general information, we point out the following facts:

- Compact hyperbolic Coxeter groups (i.e., hyperbolic Coxeter groups with a compact fundamental region) exist only for ranks 3, 4 and 5, i.e., in two, three and four-dimensional hyperbolic space. All hyperbolic Coxeter groups of rank > 5 have a fundamental region with at least one vertex at infinity. The hyperbolic Coxeter groups appearing in gravitational theories are always of the noncompact type.

- There exist reflection groups in hyperbolic space whose fundamental domains are not simplices. Amazingly enough, these exist only in hyperbolic spaces of dimension ≤ 995. If one imposes that the fundamental domain be compact, these exist only in hyperbolic spaces of dimension ≤ 29. The bound can probably be improved [164].

- Non-hyperbolic Lorentzian Coxeter groups are associated through the above construction with infinite-volume fundamental regions since some of the vectors ω_i are spacelike, which imply that the corresponding reflection hyperplanes intersect beyond hyperbolic infinity.

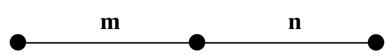


Figure 15: This Coxeter graph corresponds to hyperbolic Coxeter groups for all values of m and n for which the associated bilinear form B is not of positive definite or positive semidefinite type. This therefore gives rise to an infinite class of rank 3 hyperbolic Coxeter groups.

Table 3: Hyperbolic Coxeter groups of rank 4.

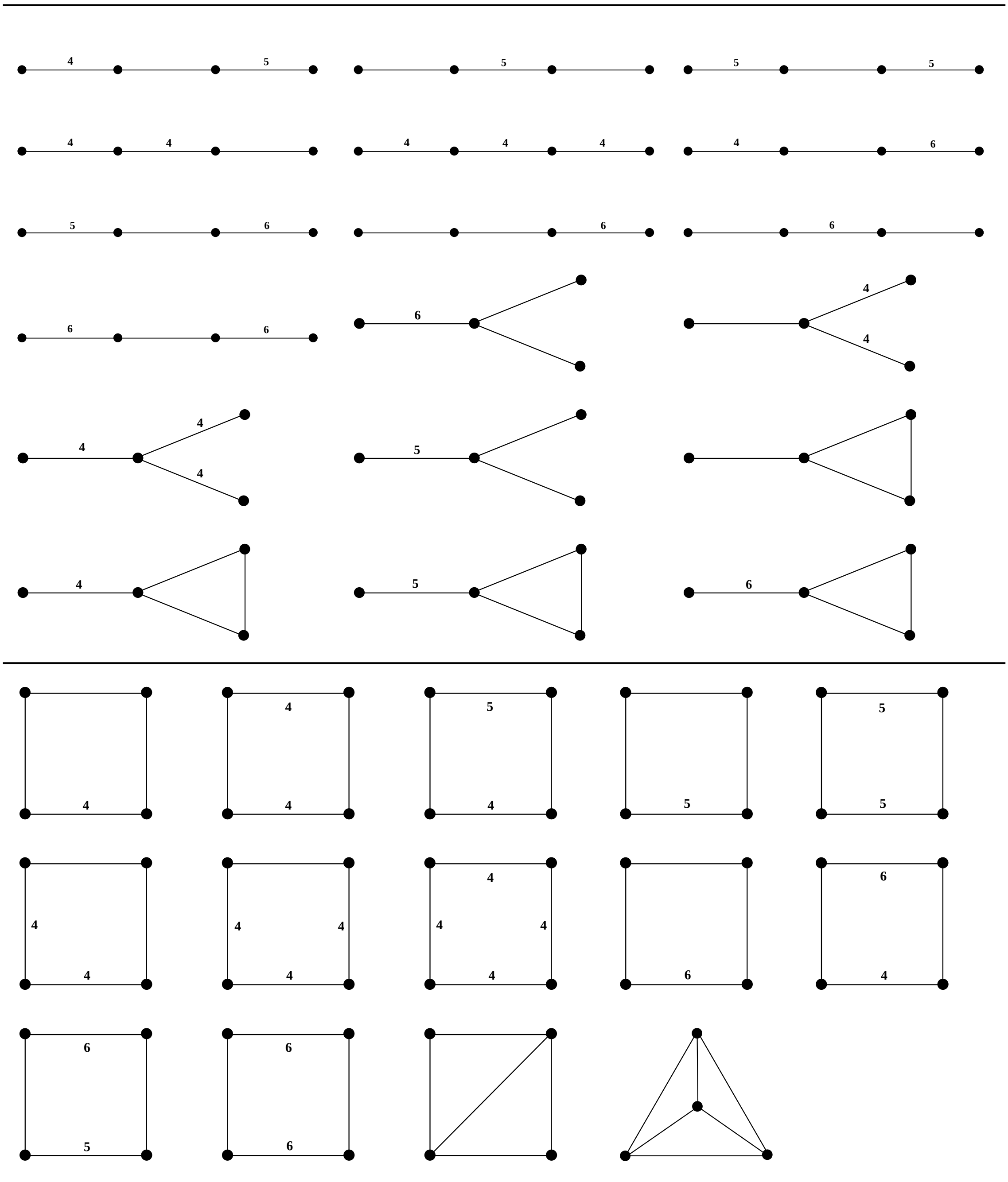


Table 4: Hyperbolic Coxeter groups of rank 5.

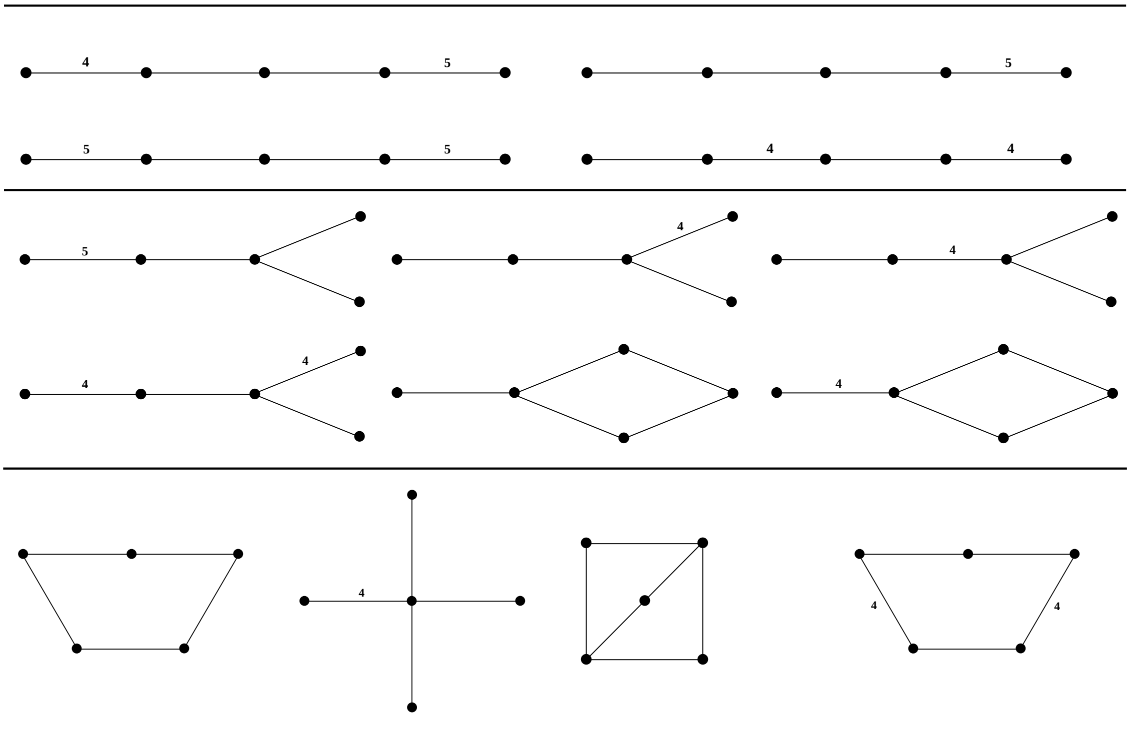


Table 5: Hyperbolic Coxeter groups of rank 6.

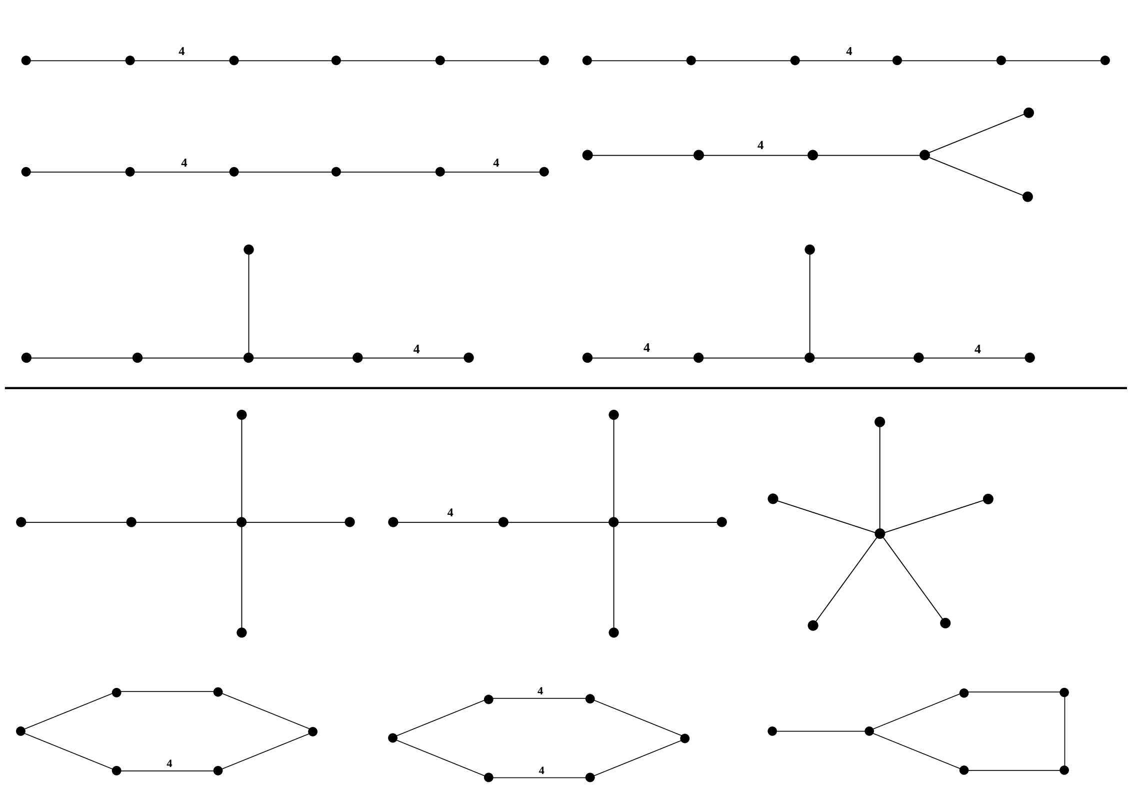


Table 6: Hyperbolic Coxeter groups of rank 7.

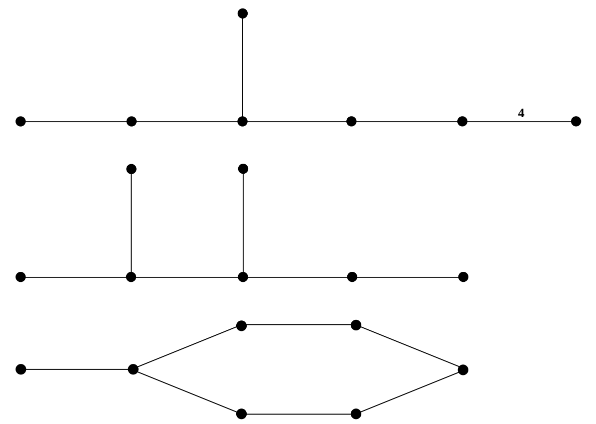


Table 7: Hyperbolic Coxeter groups of rank 8.

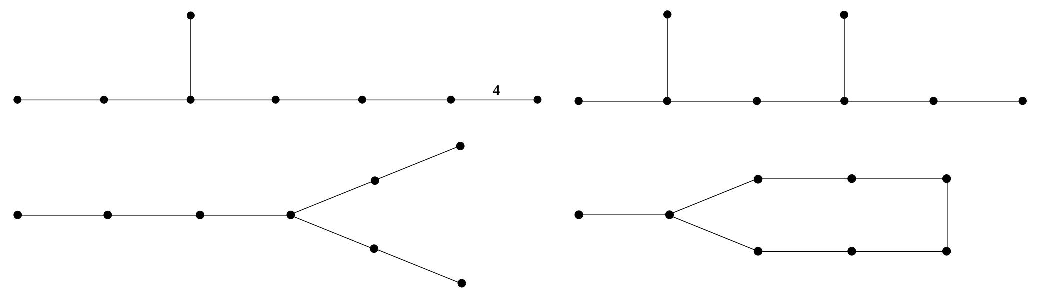


Table 8: Hyperbolic Coxeter groups of rank 9.

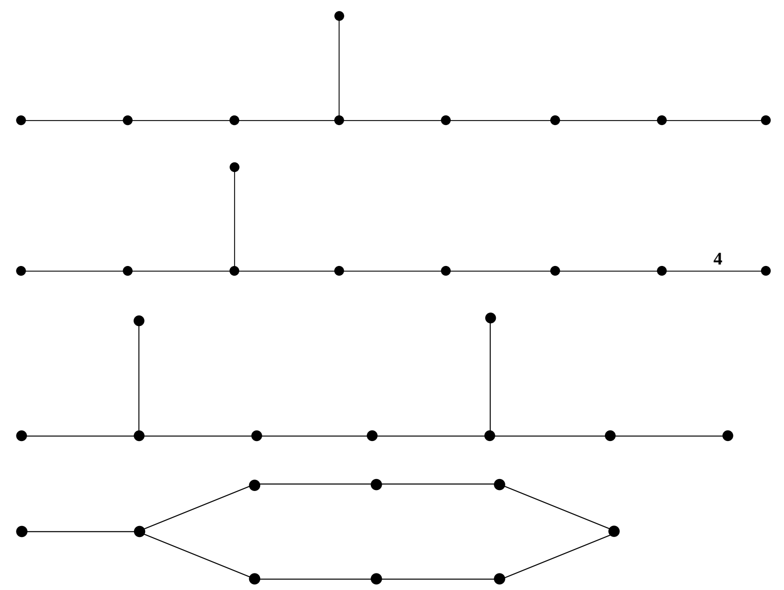
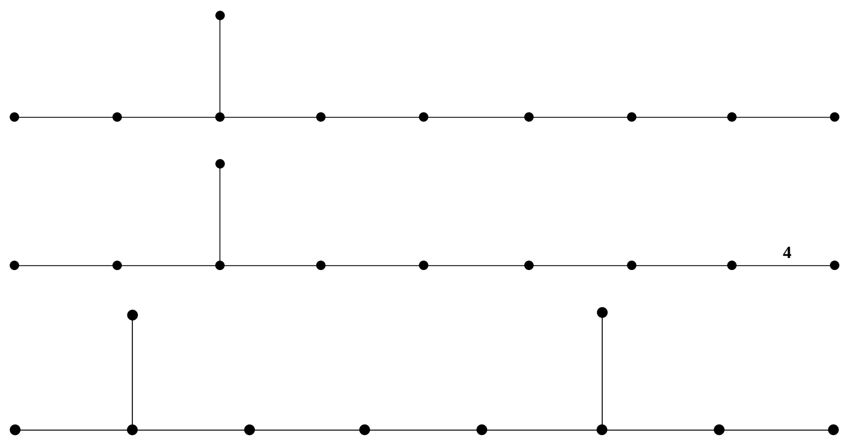


Table 9: Hyperbolic Coxeter groups of rank 10.



3.6 Crystallographic Coxeter groups

Among the Coxeter groups, only those that are crystallographic correspond to Weyl groups of Kac–Moody algebras. Therefore we now introduce this important concept. By definition, a Coxeter group is crystallographic if it stabilizes a lattice in V. This lattice need not be the lattice generated by the α_i's. As discussed in [107], a Coxeter group is crystallographic if and only if two conditions are satisfied: (i) The integers m_{ij} $(i \neq j)$ are restricted to be in the set $\{2, 3, 4, 6, \infty\}$, and (ii) for any closed circuit in the Coxeter graph of $\mathfrak{C}$, the number of edges labelled 4 or 6 is even.

Given a crystallographic Coxeter group, it is easy to exhibit a lattice L stabilized by it. We can construct a basis for that lattice as follows. The basis vectors μ_i of the lattice are multiples of the original simple roots, $\mu_i = c_i \alpha_i$ for some scalars c_i which we determine by applying the following rules:

- $m_{ij} = 3 \Rightarrow c_i = c_j$.
- $m_{ij} = 4 \Rightarrow c_i = \sqrt{2} c_j$ or $c_j = \sqrt{2} c_i$.
- $m_{ij} = 6 \Rightarrow c_i = \sqrt{3} c_j$ or $c_j = \sqrt{3} c_i$.
- $m_{ij} = \infty \Rightarrow c_i = c_j$.

One easily verifies that $\sigma_i(\mu_j) = \mu_j - d_{ij}\mu_i$ for some integers d_{ij}. Hence L is indeed stabilized. The integers d_{ij} are equal to $2\frac{B(\mu_i,\mu_j)}{B(\mu_i,\mu_i)}$.

The rules are consistent as can be seen by starting from an arbitrary node, say α_1, for which one takes $c_1 = 1$. One then proceeds to the next nodes in the (connected) Coxeter graph by applying the above rules. If there is no closed circuit, there is no consistency problem since there is only one way to proceed from α_1 to any given node. If there are closed circuits, one must make sure that one comes back to the same vector after one turn around any circuit. This can be arranged if the number of steps where one multiplies or divides by $\sqrt{2}$ (respectively, $\sqrt{3}$) is even.

Our construction shows that the lattice L is not unique. If there are only two different lengths for the lattice vectors μ_i, it is convenient to normalize the lengths so that the longest lattice vectors have length squared equal to two. This choice simplifies the factors $2\frac{B(\mu_i,\mu_j)}{B(\mu_i,\mu_i)}$.

The rank 10 hyperbolic Coxeter groups are all crystallographic. The lattices preserved by E_{10} and DE_{10} are unique up to an overall rescaling because the non-trivial m_{ij} $(i \neq j)$ are all equal to 3 and there is no choice in the ratios c_i/c_j, all equal to one (first rule above). The Coxeter group BE_{10} preserves two (dual) lattices.

On the normalization of roots and weights in the crystallographic case

Since the vectors μ_i and α_i are proportional, they generate identical reflections. Even though they do not necessarily have length squared equal to unity, the vectors μ_i are more convenient to work with because the lattice preserved by the Coxeter group is simply the lattice $\sum_i \mathbb{Z}\mu_i$ of points with integer coordinates in the basis $\{\mu_i\}$. For this reason, we shall call from now on "simple roots" the vectors μ_i and, to follow common practice, *will sometimes even rename them* α_i. Thus, in the crystallographic case, the (redefined) simple roots are appropriately normalized to the lattice structure. It turns out that it is with this normalization that simple roots of Coxeter groups correspond to simple roots of Kac–Moody algebras defined in the Section 4.6.3. A *root* is any point on the root lattice that is in the Coxeter orbit of some (redefined) simple root. It is these roots that coincide with the (real) roots of Kac–Moody algebras.

It is also useful to rescale the fundamental weights. The rescaled fundamental weights, of course proportional to ω_i, are denoted Λ_i. The convenient normalization is such that

$$(\Lambda_i|\mu_j) = \frac{(\mu_j|\mu_j)}{2}\delta_{ij}. \tag{3.42}$$

With this normalization, they coincide with the fundamental weights of Kac–Moody algebras, to be considered in Section 4.

4 Lorentzian Kac–Moody Algebras

The explicit appearance of infinite crystallographic Coxeter groups in the billiard limit suggests that gravitational theories might be invariant under a huge symmetry described by Lorentzian Kac–Moody algebras (defined in Section 4.1). Indeed, there is an intimate connection between crystallographic Coxeter groups and Kac–Moody algebras. This connection might be familiar in the finite case. For instance, it is well known that the finite symmetry group A_2 of the equilateral triangle (isomorphic to the group of permutations of 3 objects) and the corresponding hexagonal pattern of roots are related to the finite-dimensional Lie algebra $\mathfrak{sl}(3,\mathbb{R})$ (or $\mathfrak{su}(3)$). The group A_2 is in fact the Weyl group of $\mathfrak{sl}(3,\mathbb{R})$ (see Section 4.7).

This connection is not peculiar to the Coxeter group A_2 but is generally valid: Any crystallographic Coxeter group is the Weyl group of a Kac–Moody algebra traditionally denoted in the same way (see Section 4.7). This is the reason why it is expected that the Coxeter groups might signal a bigger symmetry structure. And indeed, there are indications that this is so since, as we shall discuss in Section 9, an attempt to reformulate the gravitational Lagrangians in a way that makes the conjectured symmetry manifest yields intriguing results.

The purpose of this section is to develop the mathematical concepts underlying Kac–Moody algebras and to explain the connection between Coxeter groups and Kac–Moody algebras. How this is relevant to gravitational theories will be discussed in Section 5.

4.1 Definitions

An $n \times n$ matrix A is called a "generalized Cartan matrix" (or just "Cartan matrix" for short) if it satisfies the following conditions[9]:

$$A_{ii} = 2 \qquad \forall i = 1, \cdots, n, \tag{4.1}$$
$$A_{ij} \in \mathbb{Z}_- \qquad (i \neq j), \tag{4.2}$$
$$A_{ij} = 0 \quad \Rightarrow \quad A_{ji} = 0, \tag{4.3}$$

where $\mathbb{Z}_-$ denotes the non-positive integers. One can encode the Cartan matrix in terms of a Dynkin diagram, which is obtained as follows:

1. For each $i = 1, \cdots, n$, one associates a node in the diagram.
2. One draws a line between the node i and the node j if $A_{ij} \neq 0$; if $A_{ij} = 0$ $(= A_{ji})$, one draws no line between i and j.
3. One writes the pair (A_{ij}, A_{ji}) over the line joining i to j. When the products $A_{ij} \cdot A_{ji}$ are all ≤ 4 (which is the only situation we shall meet in practice), this third rule can be replaced by the following rules:
   (a) one draws a number of lines between i and j equal to $\max(|A_{ij}|, |A_{ji}|)$;
   (b) one draws an arrow from j to i if $|A_{ij}| > |A_{ji}|$.

So, for instance, the Dynkin diagrams in Figure 16 correspond to the Cartan matrices

$$A[A_2] = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{4.4}$$

[9] We are employing the convention of Kac [116] for the Cartan matrix. There exists an alternative definition of Kac–Moody algebras in the literature, in which the transposed matrix A^T is used instead.

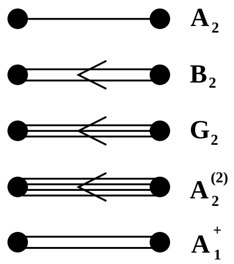


Figure 16: The Dynkin diagrams corresponding to the finite Lie algebras A_2, B_2 and G_2 and to the affine Kac–Moody algebras $A_2^{(2)}$ and A_1^+.

$$A[B_2] = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \tag{4.5}$$

$$A[G_2] = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \tag{4.6}$$

$$A[A_2^{(2)}] = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \tag{4.7}$$

$$A[A_1^+] = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \tag{4.8}$$

respectively. If the Dynkin diagram is connected, the matrix A is indecomposable. This is what shall be assumed in the following.

Although this is not necessary for developing the general theory, we shall impose two restrictions on the Cartan matrix. The first one is that $\det A \neq 0$; the second one is that A is symmetrizable. The restriction $\det A \neq 0$ excludes the important class of affine algebras and will be lifted below. We impose it at first because the technical definition of the Kac–Moody algebra when $\det A = 0$ is then slightly more involved.

The second restriction imposes that there exists an invertible diagonal matrix D with positive elements ϵ_i and a symmetric matrix S such that

$$A = DS. \tag{4.9}$$

The matrix S is called a symmetrization of A and is unique up to an overall positive factor because A is indecomposable. To prove this, choose the first (diagonal) element $\epsilon_1 > 0$ of D arbitrarily. Since A is indecomposable, there exists a nonempty set J_1 of indices j such that $A_{1j} \neq 0$. One has $A_{1j} = \epsilon_1 S_{1j}$ and $A_{j1} = \epsilon_j S_{j1}$. This fixes the ϵ_j's > 0 in terms of ϵ_1 since $S_{1j} = S_{j1}$. If not all the elements ϵ_j are determined at this first step, we pursue the same construction with the elements $A_{jk} = \epsilon_j S_{jk}$ and $A_{kj} = \epsilon_k S_{kj} = \epsilon_k S_{kj}$ with $j \in J_1$ and, more generally, at step p, with $j \in J_1 \cap J_2 \cdots \cap J_p$. As the matrix A is assumed to be indecomposable, all the elements ϵ_i of D and S_{ij} of S can be obtained, depending only on the choice of ϵ_1. One gets no contradicting values for the ϵ_j's because the matrix A is assumed to be symmetrizable.

In the symmetrizable case, one can characterize the Cartan matrix according to the signature of (any of) its symmetrization(s). One says that A is of finite type if S is of Euclidean signature, and that it is of Lorentzian type if S is of Lorentzian signature.

Given a Cartan matrix A (with $\det A \neq 0$), one defines the corresponding Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ as the algebra generated by $3n$ generators h_i, e_i, f_i subject to the following "Chevalley–Serre" relations (in addition to the Jacobi identity and anti-symmetry of the Lie bracket),

$$\begin{aligned} &[h_i, h_j] = 0, \\ &[h_i, e_j] = A_{ij} e_j \qquad \text{(no summation on } j\text{)}, \\ &[h_i, f_j] = -A_{ij} f_j \qquad \text{(no summation on } j\text{)}, \\ &[e_i, f_j] = \delta_{ij} h_j \qquad \text{(no summation on } j\text{)}, \end{aligned} \tag{4.10}$$

$$\mathrm{ad}_{e_i}^{1-A_{ij}}(e_j) = 0, \qquad \mathrm{ad}_{f_i}^{1-A_{ij}}(f_j) = 0, \qquad i \neq j. \tag{4.11}$$

The relations (4.11), called Serre relations, read explicitly

$$\underbrace{[e_i, [e_i, [e_i, \cdots, [e_i, e_j]] \cdots]}_{1-A_{ij} \text{ commutators}} = 0 \tag{4.12}$$

(and likewise for the f_k's).

Any multicommutator can be reduced, using the Jacobi identity and the above relations, to a multicommutator involving only the e_i's, or only the f_i's. Hence, the Kac–Moody algebra splits as a direct sum ("triangular decomposition")

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{4.13}$$

where $\mathfrak{n}_-$ is the subalgebra involving the multicommutators $[f_{i_1}, [f_{i_2}, \cdots, [f_{i_{k-1}}, f_{i_k}] \cdots]$, $\mathfrak{n}_+$ is the subalgebra involving the multicommutators $[e_{i_1}, [e_{i_2}, \cdots, [e_{i_{k-1}}, e_{i_k}] \cdots]$ and $\mathfrak{h}$ is the Abelian subalgebra containing the h_i's. This is called the *Cartan subalgebra* and its dimension n is the *rank* of the Kac–Moody algebra $\mathfrak{g}$. It should be stressed that the direct sum Equation (4.13) is a direct sum of $\mathfrak{n}_-$, $\mathfrak{h}$ and $\mathfrak{n}_+$ as vector spaces, not as subalgebras (since these subalgebras do not commute).

A priori, the numbers of the multicommutators

$$[f_{i_1}, [f_{i_2}, \cdots, [f_{i_{k-1}}, f_{i_k}] \cdots] \qquad \text{and} \qquad [e_{i_1}, [e_{i_2}, \cdots, [e_{i_{k-1}}, e_{i_k}] \cdots]$$

are infinite, even after one has taken into account the Jacobi identity. However, the Serre relations impose non-trivial relations among them, which, in some cases, make the Kac–Moody algebra finite-dimensional. Three worked examples are given in Section 4.4 to illustrate the use of the Serre relations. In fact, one can show [116] that the Kac–Moody algebra is finite-dimensional if and only if the symmetrization S of A is positive definite. In that case, the algebra is one of the finite-dimensional simple Lie algebras given by the Cartan classification. The list is given in Table 10.

When the Cartan matrix A is of Lorentzian signature the Kac–Moody algebra $\mathfrak{g}(A)$, constructed from A using the Chevalley–Serre relations, is called a *Lorentzian Kac–Moody algebra*. This is the case of main interest for the remainder of this paper.

4.2 Roots

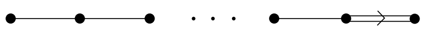
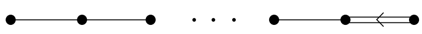
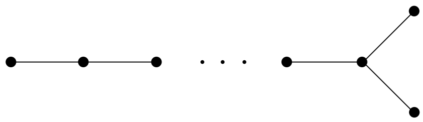
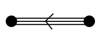
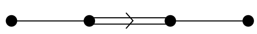
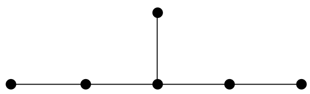
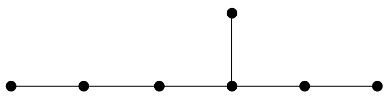
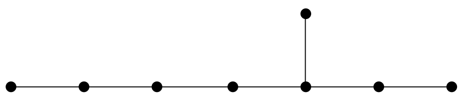
The adjoint action of the Cartan subalgebra on $\mathfrak{n}_+$ and $\mathfrak{n}_-$ is diagonal. Explicitly,

$$[h, e_i] = \alpha_i(h) e_i \qquad \text{(no summation on } i\text{)} \tag{4.14}$$

for any element $h \in \mathfrak{h}$, where α_i is the linear form on $\mathfrak{h}$ (i.e., the element of the dual $\mathfrak{h}^*$) defined by $\alpha_i(h_j) = A_{ji}$. The α_i's are called the simple roots. Similarly,

$$[h, [e_{i_1}, [e_{i_2}, \cdots, [e_{i_{k-1}}, e_{i_k}] \cdots]] = (\alpha_{i_1} + \alpha_{i_2} + \cdots \alpha_{i_k})(h)\, [e_{i_1}, [e_{i_2}, \cdots, [e_{i_{k-1}}, e_{i_k}] \cdots] \tag{4.15}$$

Table 10: Finite Lie algebras.

Name	Dynkin diagram
A_n	
B_n	
C_n	
D_n	
G_2	
F_4	
E_6	
E_7	
E_8	

and, if $[e_{i_1}, [e_{i_2}, \cdots, [e_{i_{k-1}}, e_{i_k}] \cdots]$ is non-zero, one says that $\alpha_{i_1} + \alpha_{i_2} + \cdots \alpha_{i_k}$ is a positive root. On the negative side, $\mathfrak{n}_-$, one has

$$[h, [f_{i_1}, [f_{i_2}, \cdots, [f_{i_{k-1}}, f_{i_k}] \cdots]] = -(\alpha_{i_1} + \alpha_{i_2} + \cdots \alpha_{i_k})(h) \, [f_{i_1}, [f_{i_2}, \cdots, [f_{i_{k-1}}, f_{i_k}] \cdots] \quad (4.16)$$

and $-(\alpha_{i_1} + \alpha_{i_2} + \cdots \alpha_{i_k})(h)$ is called a negative root when $[f_{i_1}, [f_{i_2}, \cdots, [f_{i_{k-1}}, f_{i_k}]$ is non-zero. This occurs if and only if $[e_{i_1}, [e_{i_2}, \cdots, [e_{i_{k-1}}, e_{i_k}] \cdots]$ is non-zero: $-\alpha$ is a negative root if and only if α is a positive root.

We see from the construction that the roots (linear forms α such that $[h, x] = \alpha(h)x$ has nonzero solutions x) are either positive (linear combinations of the simple roots α_i with integer non-negative coefficients) or negative (linear combinations of the simple roots with integer non-positive coefficients). The set of positive roots is denoted by Δ_+; that of negative roots by Δ_-. The set of all roots is Δ, so we have $\Delta = \Delta_+ \cup \Delta_-$. The simple roots are positive and form a basis of $\mathfrak{h}^*$. One sometimes denotes the h_i by $\alpha_i^\vee$ (and thus, $[\alpha_i^\vee, e_j] = A_{ij} e_j$ etc). Similarly, one also uses the notation $\langle \cdot, \cdot \rangle$ for the standard pairing between $\mathfrak{h}$ and its dual $\mathfrak{h}^*$, i.e., $\langle \alpha, h \rangle = \alpha(h)$. In this notation the entries of the Cartan matrix can be written as

$$A_{ij} = \alpha_j(\alpha_i^\vee) = \langle \alpha_j, \alpha_i^\vee \rangle \, . \quad (4.17)$$

Finally, the root lattice Q is the set of linear combinations with integer coefficients of the simple roots,

$$Q = \sum_i \mathbb{Z} \alpha_i. \quad (4.18)$$

All roots belong to the root lattice, of course, but the converse is not true: There are elements of Q that are not roots.

4.3 The Chevalley involution

The symmetry between the positive and negative subalgebras $\mathfrak{n}_+$ and $\mathfrak{n}_-$ of the Kac–Moody algebra can be rephrased formally as follows: The Kac–Moody algebra is invariant under the Chevalley involution τ, defined on the generators as

$$\tau(h_i) = -h_i, \qquad \tau(e_i) = -f_i, \qquad \tau(f_i) = -e_i. \quad (4.19)$$

The Chevalley involution is in fact an algebra automorphism that exchanges the positive and negative sides of the algebra.

Finally, we quote the following useful theorem.

Theorem: The Kac–Moody algebra $\mathfrak{g}$ defined by the relations (4.10, 4.11) is simple.
The proof may be found in Kac' book [116], page 12.

We note that invertibility and indecomposability of the Cartan matrix A are central ingredients in the proof. In particular, the theorem does not hold in the affine case, for which the Cartan matrix is degenerate and has non-trivial ideals[10] (see [116] and Section 4.5).

4.4 Three examples

To get a feeling for how the Serre relations work, we treat in detail three examples.

[10] We recall that an ideal $\mathfrak{i}$ is a subalgebra such that $[\mathfrak{i}, \mathfrak{g}] \subset \mathfrak{i}$. A simple algebra has no non-trivial ideals.

- A_2: We start with A_2, the Cartan matrix of which is Equation (4.4). The defining relations are then:

$$\begin{aligned} &[h_1, h_2] = 0, && [h_1, e_1] = 2e_1, && [h_1, e_2] = -e_2, \\ &[h_1, f_1] = -2f_1, && [h_1, f_2] = f_2, && [h_2, e_1] = -e_1, \\ &[h_2, e_2] = 2e_2, && [h_2, f_1] = f_1, && [h_2, f_2] = -2f_2, \\ &[e_1, [e_1, e_2]] = 0, && [e_2, [e_2, e_1]] = 0, && [f_1, [f_1, f_2]] = 0, \\ &[f_2, [f_2, f_1]] = 0 && [e_i, f_j] = \delta_{ij} h_j. && \end{aligned} \tag{4.20}$$

The commutator $[e_1, e_2]$ is not killed by the defining relations and hence is not equal to zero (the defining relations are *all* the relations). All the commutators with three (or more) e's are however zero. A similar phenomenon occurs on the negative side. Hence, the algebra A_2 is eight-dimensional and one may take as basis $\{h_1, h_2, e_1, e_2, [e_1, e_2], f_1, f_2, [f_1, f_2]\}$. The vector $[e_1, e_2]$ corresponds to the positive root $\alpha_1 + \alpha_2$.

- B_2: The algebra B_2, the Cartan matrix of which is Equation (4.5), is defined by the same set of generators, but the Serre relations are now $[e_1, [e_1, [e_1, e_2]]] = 0$ and $[e_2, [e_2, e_1]] = 0$ (and similar relations for the f's). The algebra is still finite-dimensional and contains, besides the generators, the commutators $[e_1, e_2]$, $[e_1, [e_1, e_2]]$, their negative counterparts $[f_1, f_2]$ and $[f_1, [f_1, f_2]]$, and nothing else. The triple commutator $[e_1, [e_1, [e_1, e_2]]]$ vanishes by the Serre relations. The other triple commutator $[e_2, [e_1, [e_1, e_2]]]$ vanishes also by the Jacobi identity and the Serre relations,

$$[e_2, [e_1, [e_1, e_2]]] = [[e_2, e_1], [e_1, e_2]] + [e_1, [e_2, [e_1, e_2]]] = 0.$$

(Each term on the right-hand side is zero: The first by antisymmetry of the bracket and the second because $[e_2, [e_1, e_2]] = -[e_2, [e_2, e_1]] = 0$.) The algebra is 10-dimensional and is isomorphic to $\mathfrak{so}(3, 2)$.

- A_1^+: We now turn to A_1^+, the Cartan matrix of which is Equation (4.8). This algebra is defined by the same set of generators as A_2, but with Serre relations given by

$$\begin{aligned} &[e_1, [e_1, [e_1, e_2]]] = 0, \\ &[e_2, [e_2, [e_2, e_1]]] = 0 \end{aligned} \tag{4.21}$$

(and similar relations for the f's). This innocent-looking change in the Serre relations has dramatic consequences because the corresponding algebra is infinite-dimensional. (We analyze here the algebra generated by the h's, e's and f's, which is in fact the derived Kac–Moody algebra – see Section 4.5 on affine Kac–Moody algebras. The derived algebra is already infinite-dimensional.) To see this, consider the $\mathfrak{sl}(2, \mathbb{R})$ current algebra, defined by

$$[J^a_m, J^b_n] = f^{ab}{}_c J^c_{m+n} + m k^{ab} c \delta_{m+n,0}, \tag{4.22}$$

where $a = 3, +, -$, $f^{ab}{}_c$ are the structure constants of $\mathfrak{sl}(2, \mathbb{R})$ and where k^{ab} is the invariant metric on $\mathfrak{sl}(2, \mathbb{R})$ which we normalize here so that $k^{-+} = 1$. The subalgebra with $n = 0$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$,

$$[J^3_0, J^+_0] = 2J^+_0, \qquad [J^3_0, J^-_0] = -2J^-_0, \qquad [J^+_0, J^-_0] = J^3_0.$$

The current algebra (4.22) is generated by J^a_0, c, J^-_1 and J^+_{-1} since any element can be written as a multi-commutator involving them. The map

$$\begin{aligned} &h_1 \to J^3_0, && h_2 \to -J^3_0 + c, \\ &e_1 \to J^+_0, && e_2 \to J^-_1, \\ &f_1 \to J^-_0, && f_2 \to J^+_{-1} \end{aligned} \tag{4.23}$$

preserves the defining relations of the Kac–Moody algebra and defines an isomorphism of the (derived) Kac–Moody algebra with the current algebra. The Kac–Moody algebra is therefore infinite-dimensional. One can construct non-vanishing infinite multi-commutators, in which e_1 and e_2 alternate:

$$\begin{aligned} &[e_1,[e_2,[e_1,\cdots,[e_1,e_2]\cdots]]] \sim J^3_n && (n\ e_1\text{'s and } n\ e_2\text{'s}),\\ &[e_1,[e_2,[e_1,\cdots,[e_2,e_1]\cdots]]] \sim J^+_n && (n+1\ e_1\text{'s and } n\ e_2\text{'s}),\\ &[e_2,[e_1,[e_2,\cdots,[e_1,e_2]\cdots]]] \sim J^-_{n+1} && (n\ e_1\text{'s and } n+1\ e_2\text{'s}). \end{aligned} \tag{4.24}$$

The Serre relations do not cut the chains of multi-commutators to a finite number.

We see from these examples that the exact consequences of the Serre relations might be intricate to derive explicitly. This is one of the difficulties of the theory.

4.5 The affine case

The affine case is characterized by the conditions that the Cartan matrix has vanishing determinant, is symmetrizable and is such that its symmetrization S is positive semi-definite (only one zero eigenvalue). As before, we also take the Cartan matrix to be indecomposable. By a reasoning analogous to what we did in Section 3.4 above, one can show that the radical of S is one-dimensional and that the ranks of S and A are equal to $n-1$.

One defines the corresponding Kac–Moody algebras in terms of $3n+1$ generators, which are the same generators h_i, e_i, f_i subject to the same conditions (4.10, 4.11) as above, plus one extra generator η which can be taken to fulfill

$$[\eta, h_i] = 0, \qquad [\eta, e_i] = \delta_{1i} e_1, \qquad [\eta, f_i] = -\delta_{1i} f_1. \tag{4.25}$$

The algebra admits the same triangular decomposition as above,

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{4.26}$$

but now the Cartan subalgebra $\mathfrak{h}$ has dimension $n+1$ (it contains the extra generator η).

Because the matrix A_{ij} has vanishing determinant, one can find a_i such that $\sum_i a_i A_{ij} = 0$. The element $c = \sum_i a_i h_i$ is in the center of the algebra. In fact, the center of the Kac–Moody algebra is one-dimensional and coincides with $\mathbb{C}c$ [116]. The derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the subalgebra generated by h_i, e_i, f_i and has codimension one (it does not contain η). One has

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathbb{C}\eta \tag{4.27}$$

(direct sum of vector spaces, not as algebras). The only proper ideals of the affine Kac–Moody algebra $\mathfrak{g}$ are $\mathfrak{g}'$ and $\mathbb{C}c$.

Affine Kac–Moody algebras appear in the BKL context as subalgebras of the relevant Lorentzian Kac–Moody algebras. Their complete list is known and is given in Tables 11 and 12.

4.6 The invariant bilinear form

4.6.1 Definition

To proceed, we assume, as announced above, that the Cartan matrix is invertible and symmetrizable since these are the only cases encountered in the billiards. Under these assumptions, an invertible, invariant bilinear form is easily defined on the algebra. We denote by ϵ_i the diagonal elements of D,

$$A = DS, \qquad D = \mathrm{diag}(\epsilon_1, \epsilon_2 \cdots, \epsilon_n). \tag{4.28}$$

Table 11: Untwisted affine Kac–Moody algebras.

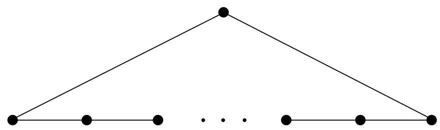
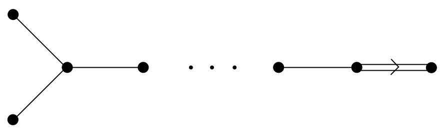
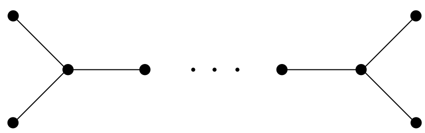
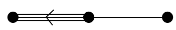
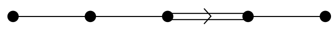
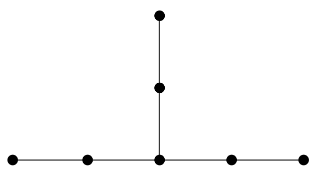
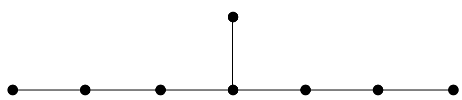
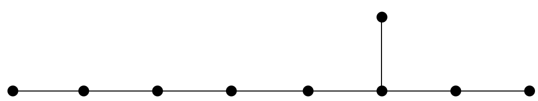
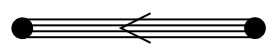
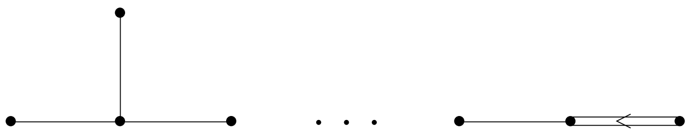
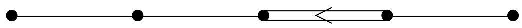
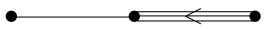
Name	Dynkin diagram
A_1^+	
A_n^+ $(n > 1)$	
B_n^+	
C_n^+	
D_n^+	
G_2^+	
F_4^+	
E_6^+	
E_7^+	
E_8^+	

Table 12: Twisted affine Kac–Moody algebras. We use the notation of Kac [116].

Name	Dynkin diagram
$A_2^{(2)}$	
$A_{2n}^{(2)}$ $(n \geq 2)$	
$A_{2n-1}^{(2)}$ $(n \geq 3)$	
$D_{n+1}^{(2)}$	
$E_6^{(2)}$	
$D_4^{(3)}$	

First, one defines an invertible bilinear form in the dual $\mathfrak{h}^*$ of the Cartan subalgebra. This is done by simply setting

$$(\alpha_i|\alpha_j) = S_{ij} \tag{4.29}$$

for the simple roots. It follows from $A_{ii} = 2$ that

$$\epsilon_i = \frac{2}{(\alpha_i|\alpha_i)} \tag{4.30}$$

and thus the Cartan matrix can be written as

$$A_{ij} = 2\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}. \tag{4.31}$$

It is customary to fix the normalization of S so that the longest roots have $(\alpha_i|\alpha_i) = 2$. As we shall now see, the definition (4.29) leads to an invariant bilinear form on the Kac–Moody algebra.

Since the bilinear form $(\cdot|\cdot)$ is nondegenerate on $\mathfrak{h}^*$, one has an isomorphism $\mu : \mathfrak{h}^* \to \mathfrak{h}$ defined by

$$\langle \alpha, \mu(\gamma) \rangle = (\alpha|\gamma). \tag{4.32}$$

This isomorphism induces a bilinear form on the Cartan subalgebra, also denoted by $(\cdot|\cdot)$. The inverse isomorphism is denoted by ν and is such that

$$\langle \nu(h), h' \rangle = (h|h'), \qquad h, h' \in \mathfrak{h}. \tag{4.33}$$

Since the Cartan elements $h_i \equiv \alpha_i^\vee$ obey

$$\langle \alpha_i, \alpha_j^\vee \rangle = A_{ji}, \tag{4.34}$$

it is clear from the definitions that

$$\nu(h_i) \equiv \nu(\alpha_i^\vee) = \epsilon_i \alpha_i \quad \Leftrightarrow \quad h_i \equiv \alpha_i^\vee = \frac{2\mu(\alpha_i)}{(\alpha_i|\alpha_i)}, \tag{4.35}$$

and thus also

$$(h_i|h_j) = \epsilon_i \epsilon_j S_{ij}. \tag{4.36}$$

The bilinear form $(\cdot|\cdot)$ can be uniquely extended from the Cartan subalgebra to the entire algebra by requiring that it is *invariant*, i.e., that it fulfills

$$([x,y]|z) = (x|[y,z]) \quad \forall\, x,y,z \in \mathfrak{g}. \tag{4.37}$$

For instance, for the e_i's and f_i's one finds

$$(h_i|e_j)A_{kj} = (h_i|[h_k,e_j]) = ([h_i,h_k]|e_j) = 0 \quad \Rightarrow \quad (h_i|e_j) = 0, \tag{4.38}$$

and similarly

$$(h_i|f_j) = 0. \tag{4.39}$$

In the same way we have

$$A_{ij}(e_j|f_k) = ([h_i,e_j]|f_k) = (h_i|[e_j,f_k]) = (h_i|h_j)\delta_{jk} = A_{ij}\epsilon_j\delta_{jk}, \tag{4.40}$$

and thus

$$(e_i|f_j) = \epsilon_i \delta_{ij}. \tag{4.41}$$

Quite generally, if e_α and e_γ are root vectors corresponding respectively to the roots α and γ,

$$[h,e_\alpha] = \alpha(h)e_\alpha, \qquad [h,e_\gamma] = \gamma(h)e_\gamma,$$

then $(e_\alpha|e_\gamma) = 0$ unless $\gamma = -\alpha$. Indeed, one has

$$\alpha(h)(e_\alpha|e_\gamma) = ([h,e_\alpha]|e_\gamma) = -(e_\alpha|[h,e_\gamma]) = -\gamma(h)(e_\alpha|e_\gamma),$$

and thus

$$(e_\alpha|e_\gamma) = 0 \qquad \text{if } \alpha + \gamma \neq 0. \tag{4.42}$$

It is proven in [116] that the invariance condition on the bilinear form defines it indeed consistently and that it is nondegenerate. Furthermore, one finds the relations

$$[h,x] = \alpha(h)x, \qquad [h,y] = -\alpha(h)y \quad \Rightarrow \quad [x,y] = (x|y)\mu(\alpha). \tag{4.43}$$

4.6.2 Real and imaginary roots

Consider the restriction $(\cdot|\cdot)_{\mathbb{R}}$ of the bilinear form to the real vector space $\mathfrak{h}^\star_{\mathbb{R}}$ obtained by taking the real span of the simple roots,

$$\mathfrak{h}^\star_{\mathbb{R}} = \sum_i \mathbb{R}\alpha_i. \tag{4.44}$$

This defines a scalar product with a definite signature. As we have mentioned, the signature is Euclidean if and only if the algebra is finite-dimensional [116]. In that case, all roots – and not just the simple ones – are spacelike, i.e., such that $(\alpha|\alpha) > 0$.

When the algebra is infinite-dimensional, the invariant scalar product does not have Euclidean signature. The spacelike roots are called "real roots", the non-spacelike ones are called "imaginary roots" [116]. While the real roots are nondegenerate (i.e., the corresponding eigenspaces, called "root spaces", are one-dimensional), this is not so for imaginary roots. In fact, it is a challenge to understand the degeneracy of imaginary roots for general indefinite Kac–Moody algebras, and, in particular, for Lorentzian Kac–Moody algebras.

Another characteristic feature of real roots, familiar from standard finite-dimensional Lie algebra theory, is that if α is a (real) root, no multiple of α is a root except $\pm\alpha$. This is not so for imaginary roots, where 2α (or other non-trivial multiples of α) can be a root even if α is. We shall provide explicit examples below.

Finally, while there are at most two different root lengths in the finite-dimensional case, this is no longer true even for real roots in the case of infinite-dimensional Kac–Moody algebras[11]. When all the real roots have the same length, one says that the algebra is "simply-laced". Note that the imaginary roots (if any) do not have the same length, except in the affine case where they all have length squared equal to zero.

4.6.3 Fundamental weights and the Weyl vector

The fundamental weights $\{\Lambda_i\}$ of the Kac–Moody algebra are vectors in the dual space $\mathfrak{h}^*$ of the Cartan subalgebra defined by

$$\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}. \tag{4.45}$$

This implies

$$(\Lambda_i|\alpha_j) = \frac{\delta_{ij}}{\epsilon_j}. \tag{4.46}$$

The Weyl vector $\rho \in \mathfrak{h}^*$ is defined by

$$(\rho|\alpha_j) = \frac{1}{\epsilon_j} \tag{4.47}$$

and is thus equal to

$$\rho = \sum_i \Lambda_i. \tag{4.48}$$

4.6.4 The generalized Casimir operator

From the invariant bilinear form, one can construct a generalized Casimir operator as follows.

We denote the eigenspace associated with α by $\mathfrak{g}_\alpha$. This is called the "root space" of α and is defined as

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,|\, [x,h] = \alpha(h)x, \quad \forall\, h \in \mathfrak{h}\}. \tag{4.49}$$

A representation of the Kac–Moody algebra is called restricted if for every vector v of the representation subspace V, one has $\mathfrak{g}_\alpha \cdot v = 0$ for all but a finite number of positive roots α.

Let $\{e_\alpha^K\}$ be a basis of $\mathfrak{g}_\alpha$ and let $\{e_{-\alpha}^K\}$ be the basis of $\mathfrak{g}_{-\alpha}$ dual to $\{e_\alpha^K\}$ in the B-metric,

$$(e_\alpha^K|e_{-\alpha}^L) = \delta^{KL}. \tag{4.50}$$

Similarly, let $\{u_i\}$ be a basis of $\mathfrak{h}$ and $\{u^i\}$ the dual basis of $\mathfrak{h}$ with respect to the bilinear form $(\cdot|\cdot)$,

$$(u_i|u^j) = \delta_i^j. \tag{4.51}$$

We set

$$\Omega = 2\mu(\rho) + \sum_i u^i u_i + 2\sum_{\alpha\in\Delta_+}\sum_K e_{-\alpha}^K e_\alpha^K, \tag{4.52}$$

where ρ is the Weyl vector. Recall from Section 4.6.1 that μ is an isomorphism from $\mathfrak{h}^\star$ to $\mathfrak{h}$, so, since $\rho \in \mathfrak{h}^\star$, the expression $\mu(\rho)$ belongs to $\mathfrak{h}$ as required. When acting on any vector of a restricted representation, Ω is well-defined since only a finite number of terms are different from zero.

It is proven in [116] that Ω commutes with all the operators of any restricted representation. For that reason, it is known as the (generalized) Casimir operator. It is quadratic in the generators[12].

[11] Imaginary roots may have arbitrarily negative length squared in general.

[12] The generalized Casimir operator Ω is the only known polynomial element of the center Z of the universal

Note

This definition – and, in particular, the presence of the linear term $\mu(\rho)$ – might seem a bit strange at first sight. To appreciate it, turn to a finite-dimensional simple Lie algebra. In the above notations, the usual expression for the quadratic Casimir operator reads

$$\Omega_{\text{finite}} = \sum_A \kappa^{AB} T_A T_B = \sum_i u^i u_i + \sum_{\alpha \in \Delta_+} (e_{-\alpha} e_\alpha + e_\alpha e_{-\alpha}) \tag{4.53}$$

(without degeneracy index K since the roots are nondegenerate in the finite-dimensional case). Here, κ^{AB} is the Killing metric and $\{T_A\}$ a basis of the Lie algebra. This expression is not "normal-ordered" because there are, in the last term, lowering operators standing on the right. We thus replace the last term by

$$\begin{aligned} \sum_{\alpha \in \Delta_+} e_\alpha e_{-\alpha} &= \sum_{\alpha \in \Delta_+} e_{-\alpha} e_\alpha + \sum_{\alpha \in \Delta_+} [e_\alpha, e_{-\alpha}] \\ &= \sum_{\alpha \in \Delta_+} e_{-\alpha} e_\alpha + \sum_{\alpha \in \Delta_+} \mu(\alpha). \end{aligned} \tag{4.54}$$

Using the fact that in a finite-dimensional Lie algebra, $\rho = (1/2) \sum_{\alpha \in \Delta_+} \alpha$, (see, e.g., [85]) one sees that the Casimir operator can be rewritten in "normal ordered" form as in Equation (4.52). The advantage of the normal-ordered form is that it makes sense also for infinite-dimensional Kac–Moody algebras in the case of restricted representations.

4.7 The Weyl group

The Weyl group $\mathfrak{W}[\mathfrak{g}]$ of a Kac–Moody algebra $\mathfrak{g}$ is a discrete group of transformations acting on $\mathfrak{h}^*$. It is defined as follows. One associates a "fundamental Weyl reflection" $r_i \in \mathfrak{W}[\mathfrak{g}]$ to each simple root through the formula

$$r_i(\lambda) = \lambda - 2 \frac{(\lambda|\alpha_i)}{(\alpha_i|\alpha_i)} \alpha_i. \tag{4.55}$$

The Weyl group is just the group generated by the fundamental Weyl reflections. In particular,

$$r_i(\alpha_j) = \alpha_j - A_{ij} \alpha_i \qquad \text{(no summation on } i\text{)}. \tag{4.56}$$

The Weyl group enjoys a number of interesting properties [116]:

- It preserves the scalar product on $\mathfrak{h}^*$.
- It preserves the root lattice and hence is crystallographic.
- Two roots that are in the same orbit have identical multiplicities.
- Any real root has in its orbit (at least) one simple root and hence, is nondegenerate.
- The Weyl group is a Coxeter group. The connection between the Coxeter exponents and the Cartan integers A_{ij} is given in Table 13 ($i \neq j$).

enveloping algebra $U(\mathfrak{g})$ of an indefinite Kac–Moody algebra $\mathfrak{g}$. However, Kac [115] has proven the existence of higher order non-polynomial Casimir operators which are elements of the center $Z_{\mathfrak{F}}$ of a suitable completion $U_{\mathfrak{F}}(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$. Recently, an explicit physics-inspired construction was made, following [115], for affine $\mathfrak{g}$ in terms of Wilson loops for WZW-models [1].

Table 13: Cartan integers and Coxeter exponents.

$A_{ij}A_{ji}$	m_{ij}
0	2
1	3
2	4
3	6
≥ 4	∞

This close relationship between Coxeter groups and Kac–Moody algebras is the reason for denoting both with the same notation (for instance, A_n denotes at the same time the Coxeter group with Coxeter graph of type A_n and the Kac–Moody algebra with Dynkin diagram A_n).

Note that different Kac–Moody algebras may have the same Weyl group. This is in fact already true for finite-dimensional Lie algebras, where dual algebras (obtained by reversing the arrows in the Dynkin diagram) have the same Weyl group. This property can be seen from the fact that the Coxeter exponents are related to the duality-invariant product $A_{ij}A_{ji}$. But, on top of this, one sees that whenever the product $A_{ij}A_{ji}$ exceeds four, which occurs only in the infinite-dimensional case, the Coxeter exponent m_{ij} is equal to infinity, independently of the exact value of $A_{ij}A_{ji}$. Information is thus clearly lost. For example, the Cartan matrices

$$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 2 & -9 & -8 \\ -4 & 2 & -5 \\ -3 & -7 & 2 \end{pmatrix} \tag{4.57}$$

lead to the same Weyl group, even though the corresponding Kac–Moody algebras are not isomorphic or even dual to each other.

Because the Weyl groups are (crystallographic) Coxeter groups, we can use the theory of Coxeter groups to analyze them. In the Kac–Moody context, the fundamental region is called "the fundamental Weyl chamber".

We also note that by (standard vector space) duality, one can define the action of the Weyl group in the Cartan subalgebra $\mathfrak{h}$, such that

$$\langle \gamma, r_i^\vee(h) \rangle = \langle r_i(\gamma), h \rangle \qquad \text{for } \gamma \in \mathfrak{h}^\star \text{ and } h \in \mathfrak{h}. \tag{4.58}$$

One has using Equations (4.30, 4.32, 4.33, 4.35),

$$r_i^\vee(h) = h - \langle \alpha_i, h \rangle h_i = h - 2\frac{(h|h_i)}{(h_i|h_i)} h_i. \tag{4.59}$$

Finally, we leave it to the reader to verify that when the products $A_{ij}A_{ji}$ are all ≤ 4, then the geometric action of the Coxeter group considered in Section 3.2.4 and the geometric action of the Weyl group considered here coincide. The (real) roots and the fundamental weights differ only in the normalization and, once this is taken into account, the metrics coincide. This is not the case when some products $A_{ij}A_{ji}$ exceed 4. It should be also pointed out that the imaginary roots of the Kac–Moody algebras do not have immediate analogs on the Coxeter side.

Examples

- Consider the Cartan matrices

$$A' = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad A'' = \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

As the first (respectively, second) Cartan matrix defines the Lie algebra A_1^{++} (respectively $A_2^{(2)+}$) introduced below in Section 4.9, we also write it as $A' \equiv A[A_1^{++}]$ (respectively, $A'' \equiv A[A_2^{(2)+}]$). We denote the associated sets of simple roots by $\{\alpha'_1, \alpha'_2, \alpha'_3\}$ and $\{\alpha''_1, \alpha''_2, \alpha''_3\}$, respectively. In both cases, the Coxeter exponents are $m_{12} = \infty$, $m_{13} = 2$, $m_{23} = 3$ and the metric B_{ij} of the geometric Coxeter construction is

$$A' = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}.$$

We associate the simple roots $\{\alpha_1, \alpha_2, \alpha_3\}$ with the geometric realisation of the Coxeter group $\mathfrak{B}$ defined by the matrix B. These roots may a priori differ by normalizations from the simple roots of the Kac–Moody algebras described by the Cartan matrices A' and A''.

Choosing the longest Kac–Moody roots to have squared length equal to two yields the scalar products

$$S' = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad S'' = \begin{pmatrix} \frac{1}{2} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Recall now from Section 3 that the fundamental reflections $\sigma_i \in \mathfrak{B}$ have the following geometric realisation

$$\sigma_i(\alpha_j) = \alpha_j - 2B_{ij}\alpha_i \qquad (i = 1, 2, 3), \tag{4.60}$$

which in this case becomes

$$\begin{array}{llll} \sigma_1: & \alpha_1 \to -\alpha_1, & \alpha_2 \to \alpha_2 + 2\alpha_1, & \alpha_3 \to \alpha_3, \\ \sigma_2: & \alpha_1 \to \alpha_1 + 2\alpha_2, & \alpha_2 \to -\alpha_2, & \alpha_3 \to \alpha_3 + \alpha_2, \\ \sigma_3: & \alpha_1 \to \alpha_1, & \alpha_2 \to \alpha_2 + \alpha_3, & \alpha_3 \to -\alpha_3. \end{array}$$

We now want to compare this geometric realisation of $\mathfrak{B}$ with the action of the Weyl groups of A' and A'' on the corresponding simple roots α'_i and α''_i. According to Equation (4.56), the Weyl group $\mathfrak{W}[A_1^{++}]$ acts as follows on the roots α'_i

$$\begin{array}{llll} r'_1: & \alpha'_1 \to -\alpha'_1, & \alpha'_2 \to \alpha'_2 + 2\alpha'_1, & \alpha'_3 \to \alpha'_3, \\ r'_2: & \alpha'_1 \to \alpha'_1 + 2\alpha'_2, & \alpha'_2 \to -\alpha'_2, & \alpha'_3 \to \alpha'_3 + \alpha'_2, \\ r'_3: & \alpha'_1 \to \alpha'_1, & \alpha'_2 \to \alpha'_2 + \alpha'_3, & \alpha'_3 \to -\alpha'_3, \end{array}$$

while the Weyl group $\mathfrak{W}[A_2^{(2)+}]$ acts as

$$\begin{array}{llll} r''_1: & \alpha''_1 \to -\alpha''_1, & \alpha''_2 \to \alpha''_2 + 4\alpha''_1, & \alpha''_3 \to \alpha''_3, \\ r''_2: & \alpha''_1 \to \alpha''_1 + \alpha''_2, & \alpha''_2 \to -\alpha''_2, & \alpha''_3 \to \alpha''_3 + \alpha''_2, \\ r''_3: & \alpha''_1 \to \alpha''_1, & \alpha''_2 \to \alpha''_2 + \alpha''_3, & \alpha''_3 \to -\alpha''_3. \end{array}$$

We see that the reflections coincide, $\sigma_1 = r'_1 = r''_1$, $\sigma_2 = r'_2 = r''_2$, $\sigma_3 = r'_3 = r''_3$, as well as the scalar products, provided that we set $2\alpha''_1 = \alpha'_1$, $\alpha''_2 = \alpha'_2$, $\alpha'_3 = \alpha_3$ and $\alpha'_i = \sqrt{2}\alpha_i$. The Coxeter group $\mathfrak{B}$ generated by the reflections thus preserves the lattices

$$Q' = \sum_i \mathbb{Z}\alpha'_i \qquad \text{and} \qquad Q'' = \sum_i \mathbb{Z}\alpha''_i, \tag{4.61}$$

showing explicitly that, in the present case, the lattices preserved by a Coxeter group are not unique – and might not even be dual to each other.

It follows, of course, that the Weyl groups of the Kac–Moody algebras A_1^{++} and $A_1^{(2)+}$ are the same,

$$\mathfrak{W}[A_1^{++}] = \mathfrak{W}[A_2^{(2)+}] = \mathfrak{B}. \tag{4.62}$$

- Consider now the Cartan matrix

$$A''' = \begin{pmatrix} 2 & -6 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

and its symmetrization

$$S''' = \begin{pmatrix} \frac{1}{3} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

The Weyl group $\mathfrak{W}[A''']$ of the corresponding Kac–Moody algebra is isomorphic to the Coxeter group $\mathfrak{B}$ above since, according to the rules, the Coxeter exponents are identical. But the action is now

$$\begin{array}{llll} r'''_1: & \alpha'''_1 \to -\alpha'''_1, & \alpha'''_2 \to \alpha'''_2 + 6\alpha'''_1, & \alpha'''_3 \to \alpha'''_3 \\ r'''_2: & \alpha'''_1 \to \alpha'''_1 + \alpha'''_2, & \alpha'''_2 \to -\alpha'''_2 & \alpha'''_3 \to \alpha'''_3 + \alpha'''_2 \\ r'''_3: & \alpha'''_1 \to \alpha'''_1, & \alpha'''_2 \to \alpha'''_2 + \alpha'''_3, & \alpha'''_3 \to -\alpha'''_3 \end{array}$$

and cannot be made to coincide with the previous action by rescalings of the α'''_i's. One can easily convince oneself of the inequivalence by computing the eigenvalues of the matrices S', S'' and S''' with respect to B.

4.8 Hyperbolic Kac–Moody algebras

Hyperbolic Kac–Moody algebras are by definition Lorentzian Kac–Moody algebras with the property that removing any node from their Dynkin diagram leaves one with a Dynkin diagram of affine or finite type. The Weyl group of hyperbolic Kac–Moody algebras is a crystallographic hyperbolic Coxeter group (as defined in Section 3.5). Conversely, any crystallographic hyperbolic Coxeter group is the Weyl group of at least one hyperbolic Kac–Moody algebra. Indeed, consider one of the lattices preserved by the Coxeter group as constructed in Section 3.6. The matrix with entries equal to the d_{ij} of that section is the Cartan matrix of a Kac–Moody algebra that has this given Coxeter group as Weyl group.

The hyperbolic Kac–Moody algebras have been classified in [154] and exist only up to rank 10 (see also [59]). In rank 10, there are four possibilities, known as $E_{10} \equiv E_8^{++}$, $BE_{10} \equiv B_8^{++}$, $DE_{10} \equiv D_8^{++}$ and $CE_{10} \equiv A_{15}^{(2)+}$, BE_{10} and CE_{10} being dual to each other and possessing the same Weyl group (the notation will be explained below).

4.8.1 The fundamental domain $\mathcal{F}$

For a hyperbolic Kac–Moody algebra, the fundamental weights Λ_i are timelike or null and lie within the (say) past lightcone. Similarly, the fundamental Weyl chamber $\mathcal{F}$ defined by $\{v \in \mathcal{F} \Leftrightarrow (v|\alpha_i) \geq 0\}$ also lies within the past lightcone and is a fundamental region for the action of the Weyl group on the Tits cone, which coincides in fact with the past light cone. All these properties carries over from our discussion of hyperbolic Coxeter groups in Section 3.

The positive imaginary roots α_K of the algebra fulfill $(\alpha_K|\Lambda_i) \geq 0$ (with, for any K, strict inequality for at least one i) and hence, since they are non-spacelike, must lie in the *future* light cone. Recall indeed that the scalar product of two non-spacelike vectors with the same time orientation is non-positive. For this reason, it is also of interest to consider the action of the Weyl group on the future lightcone, obtained from the action on the past lightcone by mere changes of signs. A fundamental region is clearly given by $-\mathcal{F}$. Any imaginary root is Weyl-conjugated to one that lies in $-\mathcal{F}$.

4.8.2 Roots and the root lattice

We have mentioned that not all points on the root lattice Q of a Kac–Moody algebras are actually roots. For hyperbolic algebras, there exists a simple criterion which enables one to determine whether a point on the root lattice is a root or not. We give it first in the case where all simple roots have equal length squared (assumed equal to two).

Theorem: Consider a hyperbolic Kac–Moody algebra such that $(\alpha_i|\alpha_i) = 2$ for all simple roots α_i. Then, any point α on the root lattice Q with $(\alpha|\alpha) \leq 2$ is a root (note that $(\alpha|\alpha)$ is even). In particular, the set of real roots is the set of points on the root lattice with $(\alpha|\alpha) = 2$, while the set of imaginary roots is the set of points on the root lattice (minus the origin) with $(\alpha|\alpha) \leq 0$.
For a proof, see [116], Chapter 5.

The version of this theorem applicable to Kac–Moody algebras with different simple root lengths is the following.

Theorem: Consider a hyperbolic algebra with root lattice Q. Let a be the smallest length squared of the simple roots, $a = \min_i(\alpha_i|\alpha_i)$. Then we have:

- The set of all short real roots is $\{\alpha \in Q \,|\, (\alpha|\alpha) = a\}$.
- The set of all real roots is
$$\left\{ \alpha = \sum_i k_i \alpha_i \in Q \,|\, (\alpha|\alpha) > 0 \text{ and } k_i \frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z}\ \forall i \right\}.$$
- The set of all imaginary roots is the set of points on the root lattice (minus the origin) with $(\alpha|\alpha) \leq 0$.

For a proof, we refer again to [116], Chapter 5.

We shall illustrate these theorems in the examples below. Note that it follows in particular from the theorems that if α is an imaginary root, all its integer multiples are also imaginary roots.

4.8.3 Examples

We discuss here briefly two examples, namely A_1^{++}, for which all simple roots have equal length, and $A_2^{(2)+}$, with respective Dynkin diagrams shown in Figures 17 and 18.

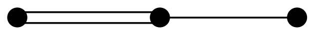


Figure 17: The Dynkin diagram of the hyperbolic Kac–Moody algebra A_1^{++}. This algebra is obtained through a standard overextension of the finite Lie algebra A_1.

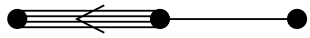


Figure 18: The Dynkin diagram of the hyperbolic Kac–Moody algebra $A_2^{(2)+}$. This algebra is obtained through a Lorentzian extension of the twisted affine Kac–Moody algebra $A_2^{(2)}$.

The Kac–Moody Algebra A_1^{++}

This is the algebra associated with vacuum four-dimensional Einstein gravity and the BKL billiard. We encountered its Weyl group $PGL(2,\mathbb{Z})$ already in Section 3.1.1. The algebra is also denoted AE_3, or H_3. The Cartan matrix is

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{4.63}$$

As it follows from our analysis in Section 3.1.1, the simple roots may be identified with the following linear forms $\alpha_i(\beta)$ in the three-dimensional space of the β^i's,

$$\alpha_1(\beta) = 2\beta^1, \qquad \alpha_2(\beta) = \beta^2 - \beta^1, \qquad \alpha_3(\beta) = \beta^3 - \beta^2 \tag{4.64}$$

with scalar product

$$(F|G) = \sum_i F_i G_i - \frac{1}{2}\left(\sum_i F_i\right)\left(\sum_i G_i\right) \tag{4.65}$$

for two linear forms $F = F_i\beta^i$ and $G = G_i\beta^i$. It is sometimes convenient to analyze the root system in terms of an "affine" level ℓ that counts the number of times the root α_3 occurs: The root $k\alpha_1 + m\alpha_2 + \ell\alpha_3$ has by definition level ℓ [13]. We shall consider here only positive roots for which $k, m, \ell \geq 0$.

Applying the first theorem, one easily verifies that the only positive roots at level zero are the roots $k\alpha_1 + m\alpha_2$, $|k-m| \leq 1$ $(k, m \geq 0)$ of the affine subalgebra A_1^+. When $k = m$, the root is imaginary and has length squared equal to zero. When $|k-m| = 1$, the root is real and has length squared equal to two.

Similarly, the only roots at level one are $(m+a)\alpha_1 + m\alpha_2 + \alpha_3$ with $a^2 \leq m$, i.e., $-[\sqrt{m}] \leq a \leq [\sqrt{m}]$. Whenever $\sqrt{m}$ is an integer, the roots $(m \pm \sqrt{m})\alpha_1 + m\alpha_2 + \alpha_3$ have squared length equal to two and are real. The roots $(m+a)\alpha_1 + m\alpha_2 + \alpha_3$ with $a^2 < m$ are imaginary and have squared length equal to $2(a^2 + 1 - m) \leq 0$. In particular, the root $m(\alpha_1 + \alpha_2) + \alpha_3$ has length squared equal to $2(1-m)$. Of all the roots at level one with $m > 1$, these are the only ones that are in the fundamental domain $-\mathfrak{F}$ (i.e., that fulfill $(\beta|\alpha_i) \leq 0$). When $m = 1$, none of the level-1 roots is in $-\mathfrak{F}$ and is either in the Weyl orbit of $\alpha_1 + \alpha_2$, or in the Weyl orbit of α_3.

We leave it to the reader to verify that the roots at level two that are in the fundamental domain $-\mathfrak{F}$ take the form $(m-1)\alpha_1 + m\alpha_2 + 2\alpha_3$ and $m(\alpha_1 + \alpha_2) + 2\alpha_3$ with $m \geq 4$. Further information on the roots of A_1^{++} may be found in [116], Chapter 11, page 215.

[13] We discuss in detail a different kind of level decomposition in Section 8.

The Kac–Moody Algebra $A_2^{(2)+}$

This is the algebra associated with the Einstein–Maxwell theory (see Section 7). The notation will be explained in Section 4.9. The Cartan matrix is

$$\begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \tag{4.66}$$

and there are now two lengths for the simple roots. The scalar products are

$$(\alpha_1|\alpha_1) = \frac{1}{2}, \qquad (\alpha_1|\alpha_2) = -1 = (\alpha_2|\alpha_1), \qquad (\alpha_2|\alpha_2) = 2. \tag{4.67}$$

One may realize the simple roots as the linear forms

$$\alpha_1(\beta) = \beta^1, \qquad \alpha_2(\beta) = \beta^2 - \beta^1, \qquad \alpha_3(\beta) = \beta^3 - \beta^2 \tag{4.68}$$

in the three-dimensional space of the β^i's with scalar product Equation (4.65).

The real roots, which are Weyl conjugate to one of the simple roots α_1 or α_2 (α_3 is in the same Weyl orbit as α_2), divide into long and short real roots. The long real roots are the vectors on the root lattice with squared length equal to two that fulfill the extra condition in the theorem. This condition expresses here that the coefficient of α_1 should be a multiple of 4. The short real roots are the vectors on the root lattice with length squared equal to one-half. The imaginary roots are all the vectors on the root lattice with length squared ≤ 0.

We define again the level ℓ as counting the number of times the root α_3 occurs. The positive roots at level zero are the positive roots of the twisted affine algebra $A_2^{(2)}$, namely, α_1 and $(2m + a)\alpha_1 + m\alpha_2$, $m = 1, 2, 3, \cdots$, with $a = -2, -1, 0, 1, 2$ for m odd and $a = -1, 0, 1$ for m odd. Although belonging to the root lattice and of length squared equal to two, the vectors $(2m \pm 2)\alpha_1 + m\alpha_2$ are not long real roots when m is even because they fail to satisfy the condition that the coefficient $(2m \pm 2)$ of α_1 is a multiple of 4. The roots at level zero are all real, except when $a = 0$, in which case the roots $m(2\alpha_1 + \alpha_2)$ have zero norm.

To get the long real roots at level one, we first determine the vectors $\alpha = \alpha_3 + k\alpha_1 + m\alpha_2$ of squared length equal to two. The condition $(\alpha|\alpha) = 2$ easily leads to $m = p^2$ for some integer $p \geq 0$ and $k = 2p^2 \pm 2p = 2p(p \pm 1)$. Since k is automatically a multiple of 4 for all $p = 0, 1, 2, 3, \cdots$, the corresponding vectors are all long real roots. Similarly, the short real roots at level one are found to be $(2p^2 + 1)\alpha_1 + (p^2 + p + 1)\alpha_2 + \alpha_3$ and $(2p^2 + 4p + 3)\alpha_1 + (p^2 + p + 1)\alpha_2 + \alpha_3$ for p a non-negative integer.

Finally, the imaginary roots at level one in the fundamental domain $-\mathfrak{F}$ read $(2m - 1)\alpha_1 + m\alpha_2 + \alpha_3$ and $2m\alpha_1 + m\alpha_2 + \alpha_3$ where m is an integer greater than or equal to 2. The first roots have length squared equal to $-2m + \frac{5}{2}$, the second have length squared equal to $-2m + 2$.

4.9 Overextensions of finite-dimensional Lie algebras

An interesting class of Lorentzian Kac–Moody algebras can be constructed by adding simple roots to finite-dimensional simple Lie algebras in a particular way which will be described below. These are called “overextensions”.

In this section, we let $\mathfrak{g}$ be a complex, finite-dimensional, simple Lie algebra of rank r, with simple roots $\alpha_1, \cdots, \alpha_r$. As stated above, normalize the roots so that the long roots have length squared equal to 2 (the short roots, if any, have then length squared equal to 1 (or 2/3 for G_2)). The roots of simply-laced algebras are regarded as long roots.

Let $\alpha = \sum_i n_i \alpha_i$, $n_i \geq 0$ be a positive root. One defines the *height* of α as

$$\mathrm{ht}(\alpha) = \sum_i n_i. \tag{4.69}$$

Among the roots of $\mathfrak{g}$, there is a unique one that has highest height, called the highest root. We denote it by θ. It is long and it fulfills the property that $(\theta|\alpha_i) \geq 0$ for all simple roots α_i, and

$$2\frac{(\alpha_i|\theta)}{(\theta|\theta)} \in \mathbb{Z}, \qquad 2\frac{(\theta|\alpha_i)}{(\alpha_i|\alpha_i)} \in \mathbb{Z} \tag{4.70}$$

(see, e.g., [85]). We denote by V the r-dimensional Euclidean vector space spanned by α_i ($i = 1, \cdots, r$). Let M_2 be the two-dimensional Minkowski space with basis vectors u and v so that $(u|u) = (v|v) = 0$ and $(u|v) = 1$. The metric in the space $V \oplus M_2$ has clearly Minkowskian signature $(-, +, +, \cdots, +)$ so that any Kac–Moody algebra whose simple roots span $V \oplus M_2$ is necessarily Lorentzian.

4.9.1 Untwisted overextensions

The standard overextensions $\mathfrak{g}^{++}$ are obtained by adding to the original roots of $\mathfrak{g}$ the roots

$$\alpha_0 = u - \theta, \qquad \alpha_{-1} = -u - v.$$

The matrix $A_{ij} = 2\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}$ where $i, j = -1, 0, 1, \cdots, r$ is a (generalized) Cartan matrix and defines indeed a Kac–Moody algebra.

The root α_0 is called the affine root and the algebra $\mathfrak{g}^+$ ($\mathfrak{g}^{(1)}$ in Kac's notations [116]) with roots $\alpha_0, \alpha_1, \cdots, \alpha_r$ is the untwisted affine extension of $\mathfrak{g}$. The root α_{-1} is known as the overextended root. One clearly has $\mathrm{rank}(\mathfrak{g}^{++}) = \mathrm{rank}(\mathfrak{g}) + 2$. The overextended root has vanishing scalar product with all other simple roots except α_0. One has explicitly $(\alpha_{-1}|\alpha_{-1}) = 2 = (\alpha_0|\alpha_0)$ and $(\alpha_{-1}|\alpha_0) = -1$, which shows that the overextended root is attached to the affine root (and only to the affine root) with a single link.

Of these Lorentzian algebras, the following ones are hyperbolic:

- A_k^{++} ($k \leq 7$),
- B_k^{++} ($k \leq 8$),
- C_k^{++} ($k \leq 4$),
- D_k^{++} ($k \leq 8$),
- G_2^{++},
- F_4^{++},
- E_k^{++} ($k = 6, 7, 8$).

The algebras B_8^{++}, D_8^{++} and E_8^{++} are also denoted BE_{10}, DE_{10} and E_{10}, respectively.

A special property of E_{10}

Of these maximal rank hyperbolic algebras, E_{10} plays a very special role. Indeed, one can verify that the determinant of its Cartan matrix is equal to -1. It follows that the lattice of E_{10} is self-dual, i.e., that the fundamental weights belong to the root lattice of E_{10}. In view of the above theorem on roots of hyperbolic algebras and of the hyperbolicity of E_{10}, the fundamental weights of E_{10} are actually (imaginary) roots since they are non-spacelike. The root lattice of E_{10} is the only Lorentzian, even, self-dual lattice in 10 dimensions (these lattices exist only in 2 mod 8 dimensions).

4.9.2 Root systems in Euclidean space

In order to describe the "twisted" overextensions, we need to introduce the concept of a "root system".

A *root system* in a real Euclidean space V is by definition a finite subset Δ of nonzero elements of V obeying the following two conditions:

$$\Delta \text{ spans } V, \tag{4.71}$$

$$\forall \alpha,\, \beta \in \Delta : \qquad \begin{cases} A_{\alpha,\beta} = 2\frac{(\alpha|\beta)}{(\beta|\beta)} \in \mathbb{Z}, \\ \beta - A_{\beta,\alpha}\, \alpha \in \Delta. \end{cases} \tag{4.72}$$

The elements of Δ are called the *roots*. From the definition one can prove the following properties [93]:

1. If $\alpha \in \Delta$, then $-\alpha \in \Delta$.

2. If $\alpha \in \Delta$, then the only elements of Δ proportional to α are $\pm\frac{1}{2}\alpha$, $\pm\alpha$, $\pm 2\alpha$. If only $\pm\alpha$ occurs (for all roots α), the root system is said to be *reduced* (*proper* in "Araki terminology" [5]).

3. If $\alpha,\, \beta \in \Delta$, then $0 \leq A_{\alpha,\beta}\, A_{\beta,\alpha} \leq 4$, i.e., $A_{\alpha,\beta} = 0, \pm 1, \pm 2, \pm 3, \pm 4$; the last occurrence appearing only for $\beta = \pm 2\alpha$, i.e., for nonreduced systems. (The proof of this point requires the use of the Schwarz inequality.)

4. If $\alpha,\, \beta \in \Delta$ are not proportional to each other and $(\alpha|\alpha) \leq (\beta|\beta)$ then $A_{\alpha,\beta} = 0, \pm 1$. Moreover if $(\alpha|\beta) \neq 0$, then $(\beta|\beta) = (\alpha|\alpha),\ 2\,(\alpha|\alpha),$ or $3\,(\alpha|\alpha)$.

5. If $\alpha,\, \beta \in \Delta$, but $\alpha - \beta \notin \Delta \cup 0$, then $(\alpha|\beta) \leq 0$ and, as a consequence, if $\alpha,\, \beta \in \Delta$ but $\alpha \pm \beta \notin \Delta \cup 0$ then $(\alpha|\beta) = 0$. That $(\alpha|\beta) \leq 0$ can be seen as follows. Clearly, α and β can be assumed to be linearly independent[14]. Now, assume $(\alpha|\beta) > 0$. By the previous point, $A_{\alpha,\beta} = 1$ or $A_{\beta,\alpha} = 1$. But then either $\alpha - A_{\alpha,\beta}\beta = \alpha - \beta \in \Delta$ or $-(\beta - A_{\beta,\alpha}\alpha) = \alpha - \beta \in \Delta$ by (4.72), contrary to the assumption. This proves that $(\alpha|\beta) \leq 0$.

Since Δ spans the vector space V, one can chose a basis $\{\alpha_i\}$ of elements of V within Δ. This can furthermore be achieved in such a way the α_i enjoy the standard properties of simple roots of Lie algebras so that in particular the concepts of positive, negative and highest roots can be introduced [93].

All the abstract root systems in Euclidean space have been classified (see, e.g., [93]) with the following results:

- The most general root system is obtained by taking a union of irreducible root systems. An irreducible root system is one that cannot be decomposed into two disjoint nonempty orthogonal subsets.

- The irreducible reduced root systems are simply the root systems of finite-dimensional simple Lie algebras (A_n with $n \geq 1$, B_n with $n \geq 3$, C_n with $n \geq 2$, D_n with $n \geq 4$, G_2, F_4, E_6, E_7 and E_8).

[14] If they were not, one would have by the second point above $\beta = \pm\frac{1}{2}\alpha$, $\beta = \pm\alpha$ or $\beta = \pm 2\alpha$. If the minus sign holds, then $(\alpha|\beta)$ is automatically < 0 and there is nothing to be proven. So we only need to consider the cases $\beta = +\frac{1}{2}\alpha$, $\beta = +\alpha$ or $\beta = +2\alpha$. In the first case, $\alpha - \beta = \beta \in \Delta$, in the second case $\alpha - \beta = 0$, and in the last case $\alpha - \beta = -\alpha \in \Delta$ so these three cases are in fact excluded by the assumption. We can therefore assume α and β to be linearly independent.

- Irreducible nonreduced root systems are all given by the so-called $(BC)_n$-systems. A $(BC)_n$-system is obtained by combining the root system of the algebra B_n with the root system of the algebra C_n in such a way that the long roots of B_n are the short roots of C_n. There are in that case three different root lengths. Explicitly Δ is given by the n unit vectors $\vec{e}_k$ and their opposite $-\vec{e}_k$ along the Cartesian axis of an n-dimensional Euclidean space, the $2n$ vectors $\pm 2\,\vec{e}_k$ obtained by multiplying the previous vectors by 2 and the $2n(n-1)$ diagonal vectors $\pm\vec{e}_k \pm \vec{e}_{k'}$, with $k \neq k'$ and $k, k' = 1, \dots, n$. The $n = 3$ case is pictured in Figure 19. The Dynkin diagram of $(BC)_r$ is the Dynkin diagram of B_r with a double circle ◎ over the simple short root, say α_1, to indicate that $2\alpha_1$ is also a root.

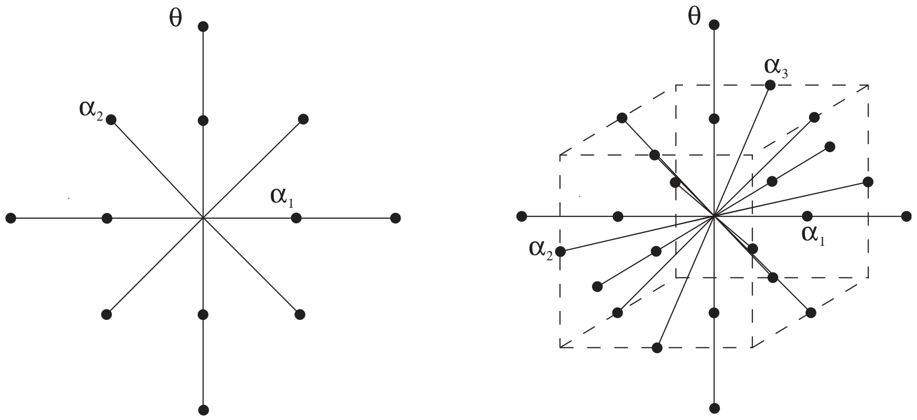


Figure 19: The nonreduced $(BC)_2$- and $(BC)_3$-root systems. In each case, the highest root θ is displayed.

It is sometimes convenient to rescale the roots by the factor $(1/\sqrt{2})$ so that the highest root $\theta = 2(\alpha_1 + \alpha_2 + \cdots + \alpha_r)$ [93] of the (BC)-system has length 2 instead of 4.

4.9.3 Twisted overextensions

We follow closely [95]. Twisted affine algebras are related to either the (BC)-root systems or to extensions by the highest short root (see [116], Proposition 6.4).

Twisted overextensions associated with the (BC)-root systems

These are the overextensions relevant for some of the gravitational billiards. The construction proceeds as for the untwisted overextensions, but the starting point is now the $(BC)_r$ root system with rescaled roots. The highest root has length squared equal to 2 and has non-vanishing scalar product only with α_r ($(\alpha_r|\theta) = 1$). The overextension procedure (defined by the same formulas as in the untwisted case) yields the algebra $(BC)_r^{++}$, also denoted $A_{2r}^{(2)+}$.

There is an alternative overextension $A_{2r}^{(2)'+}$ that can be defined by starting this time with the algebra C_r but taking *one-half* the highest root of C_r to make the extension (see [116], formula in Paragraph 6.4, bottom of page 84). The formulas for α_0 and α_{-1} are $2\alpha_0 = u - \theta$ and $2\alpha_{-1} = -u - v$ (where θ is now the highest root of C_r). The Dynkin diagram of $A_{2r}^{(2)'+}$ is dual to that of $A_{2r}^{(2)+}$. (Duality amounts to reversing the arrows in the Dynkin diagram, i.e., replacing the (generalized) Cartan matrix by its transpose.)

The algebras $A_{2r}^{(2)+}$ and $A_{2r}^{(2)'+}$ have rank $r + 2$ and are hyperbolic for $r \leq 4$. The intermediate affine algebras are in all cases the twisted affine algebras $A_{2r}^{(2)}$. We shall see in Section 7 that by coupling to three-dimensional gravity a coset model $\mathcal{G}/\mathcal{K}(\mathcal{G})$, where the so-called restricted root system (see Section 6) of the (real) Lie algebra $\mathfrak{g}$ of the Lie group $\mathcal{G}$ is of $(BC)_r$-type, one can realize all the $A_{2r}^{(2)+}$ algebras.

Twisted overextensions associated with the highest short root

We denote by θ_s the unique short root of heighest weight. It exists only for non-simply laced algebras and has length 1 (or 2/3 for G_2). The twisted overextensions are defined as the standard overextensions but one uses instead the highest short root θ_s. The formulas for the affine and overextended roots are

$$\alpha_0 = u - \theta_s, \qquad \alpha_{-1} = -u - \frac{1}{2}v, \qquad (\mathfrak{g} = B_r, C_r, F_4)$$

or

$$\alpha_0 = u - \theta_s, \qquad \alpha_{-1} = -u - \frac{1}{3}v, \qquad (\mathfrak{g} = G_2).$$

(We choose the overextended root to have the same length as the affine root and to be attached to it with a single link. This choice is motivated by considerations of simplicity and yields the fourth rank ten hyperbolic algebra when $\mathfrak{g} = C_8$.)

The affine extensions generated by $\alpha_0, \cdots, \alpha_r$ are respectively the twisted affine algebras $D_{r+1}^{(2)}$ ($\mathfrak{g} = B_r$), $A_{2r-1}^{(2)}$ ($\mathfrak{g} = C_r$), $E_6^{(2)}$ ($\mathfrak{g} = F_4$) and $D_4^{(3)}$ ($\mathfrak{g} = G_2$). These twisted affine algebras are related to external automorphisms of D_{r+1}, A_{2r-1}, E_6 and D_4, respectively (the same holds for $A_{2r}^{(2)}$ above) [116]. The corresponding twisted overextensions have the following features.

- The overextensions $D_{r+1}^{(2)+}$ have rank $r+2$ and are hyperbolic for $r \leq 4$.
- The overextensions $A_{2r-1}^{(2)+}$ have rank $r+2$ and are hyperbolic for $r \leq 8$. The last hyperbolic case, $r = 8$, yields the algebra $A_{15}^{(2)+}$, also denoted CE_{10}. It is the fourth rank-10 hyperbolic algebra, besides E_{10}, BE_{10} and DE_{10}.
- The overextensions $E_6^{(2)+}$ (rank 6) and $D_4^{(3)+}$ (rank 4) are hyperbolic.

We list in Table 14 the Dynkin diagrams of all twisted overextensions.

A satisfactory feature of the class of overextensions (standard *and* twisted) is that it is closed under duality. For instance, $A_{2r-1}^{(2)+}$ is dual to B_r^{++}. In fact, one could get the twisted overextensions associated with the highest short root from the standard overextensions precisely by requiring closure under duality. A similar feature already holds for the affine algebras.

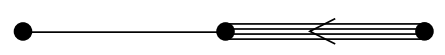
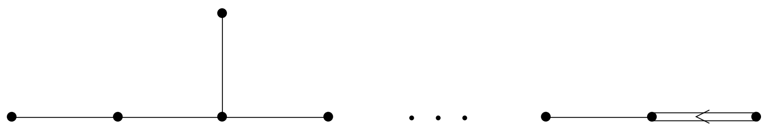
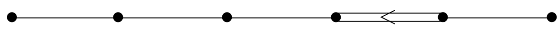
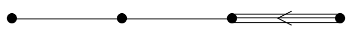
Note also that while not all hyperbolic Kac–Moody algebras are symmetrizable, the ones that are obtained through the process of overextension are.

4.9.4 Algebras of Gaberdiel–Olive–West type

One can further extend the overextended algebras to get "triple extensions" or "very extended algebras". This is done by adding a further simple root attached with a single link to the overextended root of Section 4.9. For instance, in the case of E_{10}, one gets E_{11} with the Dynkin diagram displayed in Figure 20. These algebras are Lorentzian, but not hyperbolic.

The very extended algebras belong to a more general class of algebras considered by Gaberdiel, Olive and West in [86]. These are defined to be algebras with a connected Dynkin diagram that possesses at least one node whose deletion yields a diagram with connected components that are of finite type except for at most one of affine type. For a hyperbolic algebra, the deletion of *any*

Table 14: Twisted overextended Kac–Moody algebras.

Name	Dynkin diagram
$A_2^{(2)+}$	
$A_2^{(2)'+}$	
$A_{2n}^{(2)+}$ $(n \geq 2)$	
$A_{2n}^{(2)'+}$ $(n \geq 2)$	
$A_{2n-1}^{(2)+}$ $(n \geq 3)$	
$D_{n+1}^{(2)+}$	
$E_6^{(2)+}$	
$D_4^{(3)+}$	

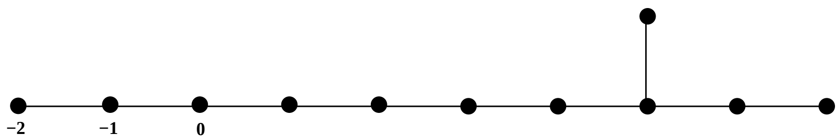


Figure 20: The Dynkin diagram of E_{11}. Labels $0, -1$ and -2 enumerate the nodes corresponding, respectively, to the affine root α_0, the overextended root α_{-1} and the "very extended" root α_{-2}.

node should fulfill this condition. The algebras of Gaberdiel, Olive and West are Lorentzian if not of finite or affine type [153, 86]. They include the overextensions of Section 4.9. The untwisted and twisted very extended algebras are clearly also of this type, since removing the affine root gives a diagram with the requested properties.

Higher order extensions with special additional properties have been investigated in [78].

4.10 Regular subalgebras of Kac–Moody algebras

This section is based on [96].

4.10.1 Definitions

Let $\mathfrak{g}$ be a Kac–Moody algebra, and let $\bar{\mathfrak{g}}$ be a subalgebra of $\mathfrak{g}$ with triangular decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$. We assume that $\bar{\mathfrak{g}}$ is canonically embedded in $\mathfrak{g}$, i.e., that the Cartan subalgebra $\bar{\mathfrak{h}}$ of $\bar{\mathfrak{g}}$ is a subalgebra of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $\bar{\mathfrak{h}} \subset \mathfrak{h}$, so that $\bar{\mathfrak{h}} = \bar{\mathfrak{g}} \cap \mathfrak{h}$. We shall say that $\bar{\mathfrak{g}}$ is regularly embedded in $\mathfrak{g}$ (and call it a "regular subalgebra") if and only if two conditions are fulfilled: (i) The root generators of $\bar{\mathfrak{g}}$ are root generators of $\mathfrak{g}$, and (ii) the simple roots of $\bar{\mathfrak{g}}$ are real roots of $\mathfrak{g}$. It follows that the Weyl group of $\bar{\mathfrak{g}}$ is a subgroup of the Weyl group of $\mathfrak{g}$ and that

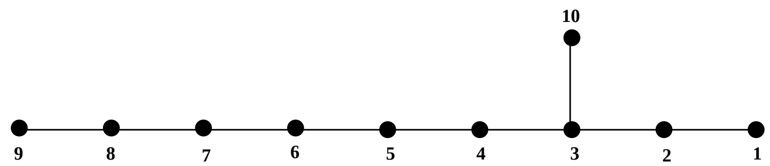


Figure 21: The Dynkin diagram of E_{10}. Labels $1, \cdots, 7$ and 10 enumerate the nodes corresponding the regular E_8 subalgebra discussed in the text.

the root lattice of $\bar{\mathfrak{g}}$ is a sublattice of the root lattice of $\mathfrak{g}$.

The second condition is automatic in the finite-dimensional case where there are only real roots. It must be separately imposed in the general case. Consider for instance the rank 2 Kac–Moody algebra $\mathfrak{g}$ with Cartan matrix

$$\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}.$$

Let

$$x = \frac{1}{\sqrt{3}}[e_1, e_2], \tag{4.73}$$

$$y = \frac{1}{\sqrt{3}}[f_1, f_2], \tag{4.74}$$

$$z = -(h_1 + h_2). \tag{4.75}$$

It is easy to verify that x, y, z define an A_1 subalgebra of $\mathfrak{g}$ since $[z, x] = 2x$, $[z, y] = -2y$ and $[x, y] = z$. Moreover, the Cartan subalgebra of A_1 is a subalgebra of the Cartan subalgebra of $\mathfrak{g}$, and the step operators of A_1 are step operators of $\mathfrak{g}$. However, the simple root $\alpha = \alpha_1 + \alpha_2$ of A_1 (which is an A_1-real root since A_1 is finite-dimensional), is an imaginary root of $\mathfrak{g}$: $\alpha_1 + \alpha_2$ has norm squared equal to -2. Even though the root lattice of A_1 (namely, $\{\pm\alpha\}$) is a sublattice of the root lattice of $\mathfrak{g}$, the reflection in α is not a Weyl reflection of $\mathfrak{g}$. According to our definition, this embedding of A_1 in $\mathfrak{g}$ is not a regular embedding.

4.10.2 Examples – Regular subalgebras of E_{10}

We shall describe some regular subalgebras of E_{10}. The Dynkin diagram of E_{10} is displayed in Figure 21.

$A_9 \subset \mathcal{B} \subset E_{10}$

A first, simple, example of a regular embedding is the embedding of A_9 in E_{10} which will be used to define the level when trying to reformulate eleven-dimensional supergravity as a nonlinear sigma model. This is not a maximal embedding since one can find a proper subalgebra $\mathcal{B}$ of E_{10} that contains A_9. One may take for $\mathcal{B}$ the Kac–Moody subalgebra of E_{10} generated by the operators at levels 0 and ± 2, which is a subalgebra of the algebra containing all operators of even level[15]. It is regularly embedded in E_{10}. Its Dynkin diagram is shown in Figure 22.

In terms of the simple roots of E_{10}, the simple roots of $\mathcal{B}$ are α_1 through α_9 and $\bar{\alpha}_{10} = 2\alpha_{10} + \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5$. The algebra $\mathcal{B}$ is Lorentzian but not hyperbolic. It can be identified with the "very extended" algebra E_7^{+++} [86].

[15]We thank Axel Kleinschmidt for an informative comment on this point.

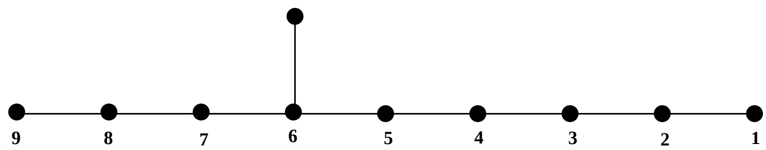


Figure 22: The Dynkin diagram of $\mathcal{B} \equiv E_7^{+++}$. The root without number is the root denoted $\bar{\alpha}_{10}$ in the text.

$DE_{10} \subset E_{10}$

In [67], Dynkin has given a method for finding all maximal regular subalgebras of finite-dimensional simple Lie algebras. The method is based on using the highest root and is not generalizable as such to general Kac–Moody algebras for which there is no highest root. Nevertheless, it is useful for constructing regular embeddings of overextensions of finite-dimensional simple Lie algebras. We illustrate this point in the case of E_8 and its overextension $E_{10} \equiv E_8^{++}$. In the notation of Figure 21, the simple roots of E_8 (which is regularly embedded in E_{10}) are $\alpha_1, \cdots, \alpha_7$ and α_{10}.

Applying Dynkin's procedure to E_8, one easily finds that D_8 can be regularly embedded in E_8. The simple roots of $D_8 \subset E_8$ are $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$, α_{10} and $\beta \equiv -\theta_{E_8}$, where

$$\theta_{E_8} = 3\alpha_{10} + 6\alpha_3 + 4\alpha_2 + 2\alpha_1 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 \tag{4.76}$$

is the highest root of E_8. One can replace this embedding, in which a simple root of D_8, namely β, is a negative root of E_8 (and the corresponding raising operator of D_8 is a lowering operator for E_8), by an equivalent one in which all simple roots of D_8 are positive roots of E_8.

This is done as follows. It is reasonable to guess that the searched-for Weyl element that maps the "old" D_8 on the "new" D_8 is some product of the Weyl reflections in the four E_8-roots orthogonal to the simple roots α_3, α_4, α_5, α_6 and α_7, expected to be shared (as simple roots) by E_8, the old D_8 and the new D_8 – and therefore to be invariant under the searched-for Weyl element. This guess turns out to be correct: Under the action of the product of the commuting E_8-Weyl reflections in the E_8-roots $\mu_1 = 2\alpha_1 + 3\alpha_2 + 5\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + 3\alpha_{10}$ and $\mu_2 = 2\alpha_1 + 4\alpha_2 + 5\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_{10}$, the set of D_8-roots $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_{10}, \beta\}$ is mapped on the equivalent set of positive roots $\{\alpha_{10}, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_2, \bar{\beta}\}$, where

$$\bar{\beta} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_{10}. \tag{4.77}$$

In this equivalent embedding, all raising operators of D_8 are also raising operators of E_8. What is more, the highest root of D_8,

$$\theta_{D_8} = \alpha_{10} + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_2 + \bar{\beta} \tag{4.78}$$

is equal to the highest root of E_8. Because of this, the affine root α_8 of the untwisted affine extension E_8^+ can be identified with the affine root of D_8^+, and the overextended root α_9 can also be taken to be the same. Hence, DE_{10} can be regularly embedded in E_{10} (see Figure 23).

The embedding just described is in fact relevant to string theory and has been discussed from various points of view in previous papers [125, 23]. By dimensional reduction of the bosonic sector of eleven-dimensional supergravity on a circle, one gets, after dropping the Kaluza–Klein vector and the 3-form, the bosonic sector of pure $\mathcal{N} = 1$ ten-dimensional supergravity. The simple roots of DE_{10} are the symmetry walls and the electric and magnetic walls of the 2-form and coincide with the positive roots given above [45]. A similar construction shows that A_8^{++} can be regularly embedded in E_{10}, and that DE_{10} can be regularly embedded in $BE_{10} \equiv B_8^{++}$. See [106] for a recent discussion of DE_{10} in the context of Type I supergravity.

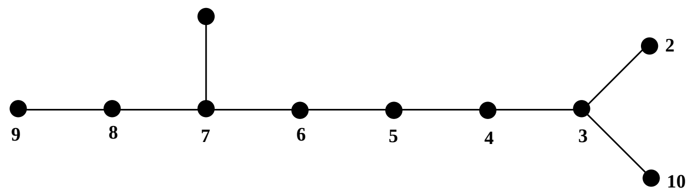


Figure 23: $DE_{10} \equiv D_8^{++}$ regularly embedded in E_{10}. Labels $2, \cdots, 10$ represent the simple roots $\alpha_2, \cdots, \alpha_{10}$ of E_{10} and the unlabeled node corresponds to the positive root $\bar{\beta} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_{10}$.

4.10.3 Further properties

As we have just seen, the raising operators of $\bar{\mathfrak{g}}$ might be raising or lowering operators of $\mathfrak{g}$. We shall consider here only the case when the positive (respectively, negative) root generators of $\bar{\mathfrak{g}}$ are also positive (respectively, negative) root generators of $\mathfrak{g}$, so that $\bar{\mathfrak{n}}_- = \mathfrak{n}_- \cap \bar{\mathfrak{g}}$ and $\bar{\mathfrak{n}}_+ = \mathfrak{n}_+ \cap \bar{\mathfrak{g}}$ ("positive regular embeddings"). This will always be assumed from now on.

In the finite-dimensional case, there is a useful criterion to determine regular algebras from subsets of roots. This criterion, which does not use the highest root, has been generalized to Kac–Moody algebras in [76]. It covers also non-maximal regular subalgebras and goes as follows:

Theorem: Let Φ^+_{real} be the set of positive real roots of a Kac–Moody algebra $\mathfrak{g}$. Let $\gamma_1, \cdots, \gamma_n \in \Phi^+_{\text{real}}$ be chosen such that none of the differences $\gamma_i - \gamma_j$ is a root of $\mathfrak{g}$. Assume furthermore that the γ_i's are such that the matrix $C = [C_{ij}] = [2\,(\gamma_i|\gamma_j)\,/\,(\gamma_i|\gamma_i)]$ has non-vanishing determinant. For each $1 \leq i \leq n$, choose non-zero root vectors E_i and F_i in the one-dimensional root spaces corresponding to the positive real roots γ_i and the negative real roots $-\gamma_i$, respectively, and let $H_i = [E_i, F_i]$ be the corresponding element in the Cartan subalgebra of $\mathfrak{g}$. Then, the (regular) subalgebra of $\mathfrak{g}$ generated by $\{E_i, F_i, H_i\}$, $i = 1, \cdots, n$, is a Kac–Moody algebra with Cartan matrix $[C_{ij}]$.

Proof: The proof of this theorem is given in [76]. Note that the Cartan integers $2\frac{(\gamma_i|\gamma_j)}{(\gamma_i|\gamma_i)}$ are indeed integers (because the γ_i's are positive real roots), which are non-positive (because $\gamma_i - \gamma_j$ is not a root), so that $[C_{ij}]$ is a Cartan matrix.

Comments

1. When the Cartan matrix is degenerate, the corresponding Kac–Moody algebra has nontrivial ideals [116]. Verifying that the Chevalley–Serre relations are fulfilled is not sufficient to guarantee that one gets the Kac–Moody algebra corresponding to the Cartan matrix $[C_{ij}]$ since there might be non-trivial quotients. Situations in which the algebra generated by the set $\{E_i, F_i, H_i\}$ is the quotient of the Kac–Moody algebra with Cartan matrix $[C_{ij}]$ by a non-trivial ideal were discussed in [96].

2. If the matrix $[C_{ij}]$ is decomposable, say $C = D \oplus E$ with D and E indecomposable, then the Kac–Moody algebra $\mathbb{KM}(C)$ generated by C is the direct sum of the Kac–Moody algebra $\mathbb{KM}(D)$ generated by D and the Kac–Moody algebra $\mathbb{KM}(E)$ generated by E. The subalgebras $\mathbb{KM}(D)$ and $\mathbb{KM}(E)$ are ideals. If C has non-vanishing determinant, then both D and E have non-vanishing determinant. Accordingly, $\mathbb{KM}(D)$ and $\mathbb{KM}(E)$ are simple [116] and hence, either occur faithfully or trivially. Because the generators E_i are linearly independent,

both $\mathbb{KM}(D)$ and $\mathbb{KM}(E)$ occur faithfully. Therefore, in the above theorem the only case that requires special treatment is when the Cartan matrix C has vanishing determinant.

As we have mentioned above, it is convenient to universally normalize the Killing form of Kac–Moody algebras in such a way that the long real roots have always the same squared length, conveniently taken equal to two. It is then easily seen that the Killing form of any regular Kac–Moody subalgebra of E_{10} coincides with the invariant form induced from the Killing form of E_{10} through the embedding since E_{10} is "simply laced". This property does not hold for non-regular embeddings as the example given in Section 4.1 shows: The subalgebra A_1 considered there has an induced form equal to minus the standard Killing form.

5 Kac–Moody Billiards I – The Case of Split Real Forms

In this section we will begin to explore in more detail the correspondence between Lorentzian Coxeter groups and the limiting behavior of the dynamics of gravitational theories close to a spacelike singularity.

We have seen in Section 2 that in the BKL-limit, the dynamics of gravitational theories is equivalent to a billiard dynamics in a region of hyperbolic space. In the generic case, the billiard region has no particular feature. However, we have seen in Section 3 that in the case of pure gravity in four spacetime dimensions, the billiard region has the remarkable property of being the fundamental domain of the Coxeter group $PGL(2, \mathbb{Z})$ acting on two-dimensional hyperbolic space.

This is not an accident. Indeed, this feature arises for all gravitational theories whose toroidal dimensional reduction to three dimensions exhibits hidden symmetries, in the sense that the reduced theory can be reformulated as three-dimensional gravity coupled to a nonlinear sigma-model based on $\mathcal{U}_3/\mathcal{K}(\mathcal{U}_3)$, where $\mathcal{K}(\mathcal{U}_3)$ is the maximal compact subgroup of $\mathcal{U}_3$. The "hidden" symmetry group $\mathcal{U}_3$ is also called, by a generalization of language, "the U-duality group" [142]. This situation covers the cases of pure gravity in any spacetime dimension, as well as all known supergravity models. In all these cases, the billiard region is the fundamental domain of a Lorentzian Coxeter group ("Coxeter billiard"). Furthermore, the Coxeter group in question is crystallographic and turns out to be the Weyl group of a Lorentzian Kac–Moody algebra. The billiard table is then the fundamental Weyl chamber of a Lorentzian Kac–Moody algebra [45, 46] and the billiard is also called a "Kac–Moody billiard". This enables one to reformulate the dynamics as a motion in the Cartan subalgebra of the Lorentzian Kac–Moody algebra, hinting at the potential – and still conjectural at this stage – existence of a deeper, infinite-dimensional symmetry of the theory.

The purpose of this section is threefold:

1. First, we exhibit other theories besides pure gravity in four dimensions which also lead to a Coxeter billiard. We stress further how exceptional these theories are in the space of all theories described by the action Equation (2.1).

2. Second, we show how to reformulate the dynamics as a motion in the Cartan subalgebra of a Lorentzian Kac–Moody algebra.

3. Finally, we connect the Lorentzian Kac–Moody algebra that appears in the BKL-limit to the "hidden" symmetry group $\mathcal{U}_3$ in the simplest case when the real Lie algebra $\mathfrak{u}_3$ of the group $\mathcal{U}_3$ is the split real form of the corresponding complexified Lie algebra $\mathfrak{u}_3^{\mathbb{C}}$. (These concepts will be defined below.) The general case will be dealt with in Section 7, after we have recalled the most salient features of the theory of real forms in Section 6.

5.1 More on Coxeter billiards

5.1.1 The Coxeter billiard of pure gravity in D spacetime dimensions

We start by providing other examples of theories leading to regular billiards, focusing first on pure gravity in any number of D (> 3) spacetime dimensions. In this case, there are $d = D - 1$ scale factors β^i and the relevant walls are the symmetry walls, Equation (2.48),

$$s_i(\beta) \equiv \beta^{i+1} - \beta^i = 0 \qquad (i = 1, 2, \cdots, d-1), \tag{5.1}$$

and the curvature wall, Equation (2.49),

$$r(\beta) \equiv 2\beta^1 + \beta^2 + \cdots + \beta^{d-2} = 0. \tag{5.2}$$

There are thus d relevant walls, which define a simplex in $(d-1)$-dimensional hyperbolic space $\mathcal{H}_{d-1}$. The scalar products of the linear forms defining these walls are easily computed. One finds as non-vanishing products

$$\begin{aligned} (s_i|s_i) &= 2 \qquad (i=1,\cdots,d-1), \\ (r|r) &= 2, \\ (s_{i+1}|s_i) &= -1 \qquad (i=2,\cdots,d-1) \\ (r|s_1) &= -1, \\ (r|s_{d-2}) &= -1. \end{aligned} \tag{5.3}$$

The matrix of the scalar products of the wall forms is thus the Cartan matrix of the (simply-laced) Lorentzian Kac–Moody algebra A^{++}_{d-2} with Dynkin diagram as in Figure 24. The roots of the underlying finite-dimensional algebra A_{d-2} are given by s_i $(i=1,\cdots,d-3)$ and r. The affine root is s_{d-2} and the overextended root is s_{d-1}.

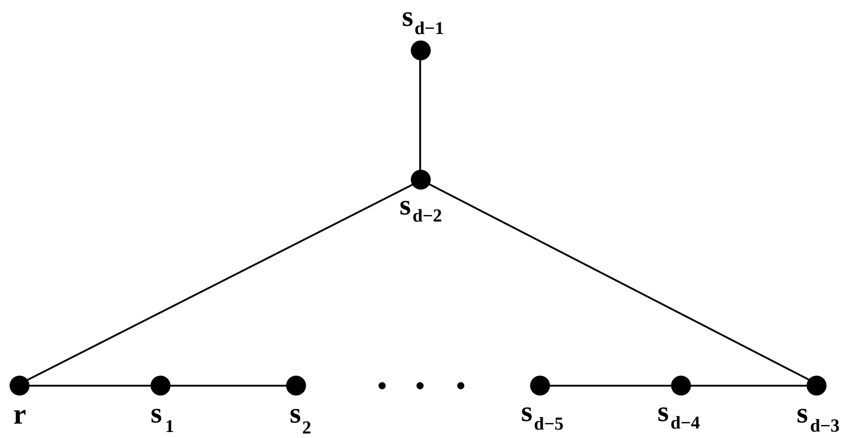


Figure 24: The Dynkin diagram of the hyperbolic Kac–Moody algebra A^{++}_{d-2} which controls the billiard dynamics of pure gravity in $D=d+1$ dimensions. The nodes $s_1,\cdots,s_{d-1}$ represent the "symmetry walls" arising from the off-diagonal components of the spatial metric, and the node r corresponds to a "curvature wall" coming from the spatial curvature. The horizontal line is the Dynkin diagram of the underlying A_{d-2}-subalgebra and the two topmost nodes, s_{d-2} and s_{d-1}, give the affine- and overextension, respectively.

Accordingly, in the case of pure gravity in any number of spacetime dimensions, one finds also that the billiard region is regular. This provides new examples of Coxeter billiards, with Coxeter groups A^{++}_{d-2}, which are also Kac–Moody billiards since the Coxeter groups are the Weyl groups of the Kac–Moody algebras A^{++}_{d-2}.

5.1.2 The Coxeter billiard for the coupled gravity-3-Form system

Coxeter polyhedra

Let us review the conditions that must be fulfilled in order to get a Kac–Moody billiard and let us emphasize how restrictive these conditions are. The billiard region computed from any theory coupled to gravity with n dilatons in $D=d+1$ dimensions always defines a convex polyhedron in a $(d+n-1)$-dimensional hyperbolic space $\mathcal{H}_{d+n-1}$. In the general case, the dihedral angles between adjacent faces of $\mathcal{H}_{d+n-1}$ can take arbitrary continuous values, which depend on the dilaton couplings, the spacetime dimensions and the ranks of the p-forms involved. However, only if the dihedral angles are integer submultiples of π (meaning of the form π/k for $k \in \mathbb{Z}_{\geq 2}$) do the reflections in the faces of $\mathcal{H}_{d+n-1}$ define a Coxeter group. In this special case the polyhedron is called a *Coxeter polyhedron*. This Coxeter group is then a (discrete) subgroup of the isometry group of $\mathcal{H}_{d+n-1}$.

In order for the billiard region to be identifiable with the fundamental Weyl chamber of a Kac–Moody algebra, the Coxeter polyhedron should be a *simplex*, i.e., bounded by $d+n$ walls in a $d+n-1$-dimensional space. In general, the Coxeter polyhedron need not be a simplex.

There is one additional condition. The angle ϑ between two adjacent faces i and j is given, in terms of the Coxeter exponents, by

$$\vartheta = \frac{\pi}{m_{ij}}. \tag{5.4}$$

Coxeter groups that correspond to Weyl groups of Kac–Moody algebras are the *crystallographic* Coxeter groups for which $m_{ij} \in \{2, 3, 4, 6, \infty\}$. So, the requirement for a gravitational theory to have a Kac–Moody algebraic description is not just that the billiard region is a Coxeter simplex but also that the angles between adjacent walls are such that the group of reflections in these walls is crystallographic.

These conditions are very restrictive and hence gravitational theories which can be mapped to a Kac–Moody algebra in the BKL-limit are rare.

The Coxeter billiard of eleven-dimensional supergravity

Consider for instance the action (2.1) for gravity coupled to a single three-form in $D = d+1$ spacetime dimensions. We assume $D \geq 6$ since in lower dimensions the 3-form is equivalent to a scalar ($D = 5$) or has no degree of freedom ($D < 5$).

Theorem: Whenever a p-form ($p \geq 1$) is present, the curvature wall is subdominant as it can be expressed as a linear combination with positive coefficients of the electric and magnetic walls of the p-forms. (These walls are all listed in Section 2.5.)

Proof: The dominant electric wall is (assuming the presence of a dilaton)

$$e_{1\cdots p}(\beta) \equiv \beta^1 + \beta^2 + \cdots + \beta^p - \frac{\lambda_p}{2}\phi = 0, \tag{5.5}$$

while one of the magnetic wall reads

$$m_{1,p+1,\cdots,d-2}(\beta) \equiv \beta^1 + \beta^{p+1} + \cdots + \beta^{d-2} + \frac{\lambda_p}{2}\phi = 0, \tag{5.6}$$

so that the dominant curvature wall is just the sum $e_{1\cdots p}(\beta) + m_{1,p+1,\cdots,d-2}(\beta)$.

It follows that in the case of gravity coupled to a single three-form in $D = d+1$ spacetime dimensions, the relevant walls are the symmetry walls, Equation (2.48),

$$s_i(\beta) \equiv \beta^{i+1} - \beta^i = 0, \qquad i = 1, 2, \cdots, d-1 \tag{5.7}$$

(as always) and the electric wall

$$e_{123}(\beta) \equiv \beta^1 + \beta^2 + \beta^3 = 0 \tag{5.8}$$

($D \geq 8$) or the magnetic wall

$$m_{1\cdots D-5}(\beta) \equiv \beta^1 + \beta^2 + \cdots \beta^{D-5} = 0 \tag{5.9}$$

($D \leq 8$). Indeed, one can express the magnetic walls as linear combinations with (in general non-integer) positive coefficients of the electric walls for $D \geq 8$ and vice versa for $D \leq 8$. Hence the billiard table is always a simplex (this would not be true had one a dilaton and various forms with different dilaton couplings).

However, it is only for $D = 11$ that the billiard is a Coxeter billiard. In all the other spacetime dimensions, the angle between the relevant p-form wall and the symmetry wall that does not intersect it orthogonally is not an integer submultiple of π. More precisely, the angle between

- the magnetic wall β^1 and the symmetry wall $\beta^2 - \beta^1$ ($D = 6$),
- the magnetic wall $\beta^1 + \beta^2$ and the symmetry wall $\beta^3 - \beta^2$ ($D = 7$), and
- the electric wall $\beta^1 + \beta^2 + \beta^3$ and the symmetry wall $\beta^4 - \beta^3$ ($D \geq 8$),

is easily verified to be an integer submultiple of π only for $D = 11$, for which it is equal to $\pi/3$.

From the point of view of the regularity of the billiard, the spacetime dimension $D = 11$ is thus privileged. This is of course also the dimension privileged by supersymmetry. It is quite intriguing that considerations *a priori* quite different (billiard regularity on the one hand, supersymmetry on the other hand) lead to the same conclusion that the gravity-3-form system is quite special in $D = 11$ spacetime dimensions.

For completeness, we here present the wall system relevant for the special case of $D = 11$. We obtain ten dominant wall forms, which we rename $\alpha_1, \cdots, \alpha_{10}$,

$$\begin{aligned} \alpha_m(\beta) &= \beta^{m+1} - \beta^m \qquad (m = 1, \cdots, 10), \\ \alpha_{10}(\beta) &= \beta^1 + \beta^2 + \beta^3. \end{aligned} \tag{5.10}$$

Then, defining a new collective index $i = (m, 10)$, we see that the scalar products between these wall forms can be organized into the matrix

$$A_{ij} = 2\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \tag{5.11}$$

which can be identified with the Cartan matrix of the hyperbolic Kac–Moody algebra E_{10} that we have encountered in Section 4.10.2. We again display the corresponding Dynkin diagram in Figure 25, where we point out the explicit relation between the simple roots and the walls of the Einstein–3-form theory. It is clear that the nine dominant symmetry wall forms correspond to the simple roots α_m of the subalgebra $\mathfrak{sl}(10, \mathbb{R})$. The enlargement to E_{10} is due to the tenth exceptional root realized here through the dominant electric wall form e_{123}.

5.2 Dynamics in the Cartan subalgebra

We have just learned that, in some cases, the group of reflections that describe the (possibly chaotic) dynamics in the BKL-limit is a Lorentzian Coxeter group $\mathfrak{C}$. In this section we fully exploit this algebraic fact and show that whenever $\mathfrak{C}$ is *crystallographic*, the dynamics takes place in the Cartan subalgebra $\mathfrak{h}$ of the Lorentzian Kac–Moody algebra $\mathfrak{g}$, for which $\mathfrak{C}$ is the Weyl group. Moreover, we show that the "billiard table" can be identified with the fundamental Weyl chamber in $\mathfrak{h}$.

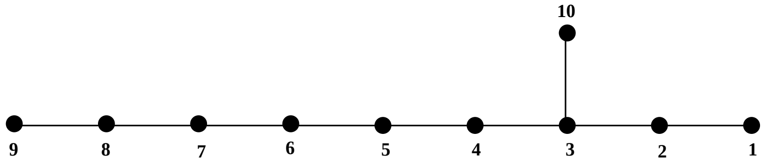


Figure 25: The Dynkin diagram of E_{10}. Labels $m = 1, \cdots, 9$ enumerate the nodes corresponding to simple roots, α_m, of the $\mathfrak{sl}(10, \mathbb{R})$ subalgebra and the exceptional node, labeled "10", is associated to the electric wall $\alpha_{10} = e_{123}$.

5.2.1 Billiard dynamics in the Cartan subalgebra

Scale factor space and the wall system

Let us first briefly review some of the salient features encountered so far in the analysis. In the following we denote by $\mathcal{M}_\beta$ the Lorentzian "scale factor"-space (or β-space) in which the billiard dynamics takes place. Recall that the metric in $\mathcal{M}_\beta$, induced by the Einstein–Hilbert action, is a flat Lorentzian metric, whose explicit form in terms of the (logarithmic) scale factors reads

$$G_{\mu\nu}\, d\beta^\mu\, d\beta^\nu = \sum_{i=1}^{d} d\beta^i\, d\beta^i - \left(\sum_{i=1}^{d} d\beta^i\right)\left(\sum_{j=1}^{d} d\beta^j\right) + d\phi\, d\phi, \tag{5.12}$$

where d counts the number of physical spatial dimensions (see Section 2.5). The role of all other "off-diagonal" variables in the theory is to interrupt the free-flight motion of the particle, by adding walls in $\mathcal{M}_\beta$ that confine the motion to a limited region of scale factor space, namely a convex cone bounded by timelike hyperplanes. When projected onto the unit hyperboloid, this region defines a simplex in hyperbolic space which we refer to as the "billiard table".

One has, in fact, more than just the walls. The theory provides these walls with a specific normalization through the Lagrangian, which is crucial for the connection to Kac–Moody algebras. Let us therefore discuss in somewhat more detail the geometric properties of the wall system. The metric, Equation (5.12), in scale factor space can be seen as an extension of a flat Euclidean metric in Cartesian coordinates, and reflects the Lorentzian nature of the vector space $\mathcal{M}_\beta$. In this space we may identify a pair of coordinates (β^i, ϕ) with the components of a vector $\beta \in \mathcal{M}_\beta$, with respect to a basis $\{\bar{u}_\mu\}$ of $\mathcal{M}_\beta$, such that

$$\bar{u}_\mu \cdot \bar{u}_\nu = G_{\mu\nu}. \tag{5.13}$$

The walls themselves are then defined by hyperplanes in this linear space, i.e., as linear forms $\omega = \omega_\mu \underline{\sigma}^\mu$, for which $\omega = 0$, where $\{\underline{\sigma}^\mu\}$ is the basis dual to $\{\bar{u}^\mu\}$. The pairing $\omega(\beta)$ between a vector $\beta \in \mathcal{M}_\beta$ and a form $\omega \in \mathcal{M}_\beta^\star$ is sometimes also denoted by $\langle \omega, \beta \rangle$, and for the two dual bases we have, of course,

$$\langle \underline{\sigma}^\mu, \bar{u}_\nu \rangle = \delta^\mu_\nu. \tag{5.14}$$

We therefore find that the walls can be written as linear forms in the scale factors:

$$\omega(\beta) = \sum_{\mu,\nu} \omega_\mu \beta^\nu \langle \underline{\sigma}^\mu, \bar{u}_\nu \rangle = \sum_\mu \omega_\mu \beta^\mu = \sum_{i=1}^{d} \omega_i \beta^i + \omega_\phi \phi. \tag{5.15}$$

We call $\omega(\beta)$ *wall forms*. With this interpretation they belong to the dual space $\mathcal{M}_\beta^\star$, i.e.,

$$\begin{aligned} \mathcal{M}_\beta^\star \ni \omega : \mathcal{M}_\beta &\longrightarrow \mathbb{R}, \\ \beta &\longmapsto \omega(\beta). \end{aligned} \tag{5.16}$$

From Equation (5.16) we may conclude that the walls bounding the billiard are the hyperplanes $\omega = 0$ through the origin in $\mathcal{M}_\beta$ which are orthogonal to the vector with components $\omega^\mu = G^{\mu\nu}\omega_\nu$.

It is important to note that it is the wall forms that the theory provides, as arguments of the exponentials in the potential, and not just the hyperplanes on which these forms ω vanish. The scalar products between the wall forms are computed using the metric in the dual space $\mathcal{M}_\beta^\star$, whose explicit form was given in Section 2.5,

$$(\omega|\omega') \equiv G^{\mu\nu}\omega_\mu\omega_\nu = \sum_{i=1}^{d}\omega_i\omega_i' - \frac{1}{d-1}\left(\sum_{i=1}^{d}\omega_i\right)\left(\sum_{j=1}^{d}\omega_j'\right) + \omega_\phi\omega_\phi', \qquad \omega,\omega' \in \mathcal{M}_\beta. \tag{5.17}$$

Scale factor space and the Cartan subalgebra

The crucial additional observation is that (for the "interesting" theories) the matrix A associated with the relevant walls ω_A,

$$A_{AB} = 2\frac{(\omega_A|\omega_B)}{(\omega_A|\omega_A)} \tag{5.18}$$

is a Cartan matrix, i.e., besides having 2's on its diagonal, which is rather obvious, it has as off-diagonal entries non-positive integers (with the property $A_{AB} \neq 0 \Rightarrow A_{BA} \neq 0$). This Cartan matrix is of course symmetrizable since it derives from a scalar product.

For this reason, one can usefully identify the space of the scale factors with the Cartan subalgebra $\mathfrak{h}$ of the Kac–Moody algebra $\mathfrak{g}(A)$ defined by A. In that identification, the wall forms become the simple roots, which span the vector space $\mathfrak{h}^\star = \text{span}\{\alpha_1, \cdots, \alpha_r\}$ dual to the Cartan subalgebra. The rank r of the algebra is equal to the number of scale factors β^μ, including the dilaton(s) if any $((\beta^\mu) \equiv (\beta^i, \phi))$. This number is also equal to the number of walls since we assume the billiard to be a simplex. So, both A and μ run from 1 to r. The metric in $\mathcal{M}_\beta$, Equation (5.12), can be identified with the invariant bilinear form of $\mathfrak{g}$, restricted to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The scale factors β^μ of $\mathcal{M}_\beta$ become then coordinates h^μ on the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}(A)$.

The Weyl group of a Kac–Moody algebra has been defined first in the space $\mathfrak{h}^\star$ as the group of reflections in the walls orthogonal to the simple roots. Since the metric is non degenerate, one can equivalently define by duality the Weyl group in the Cartan algebra $\mathfrak{h}$ itself (see Section 4.7). For each reflection r_i on $\mathfrak{h}^\star$ we associate a dual reflection $r_i^\vee$ as follows,

$$r_i^\vee(\beta) = \beta - \langle\alpha_i, \beta\rangle\, \alpha_i^\vee, \qquad \beta, \alpha_i^\vee \in \mathfrak{h}, \tag{5.19}$$

which is the reflection relative to the hyperplane $\alpha_i(\beta) = \langle\alpha_i, \beta\rangle = 0$. This expression can be rewritten (see Equation (4.59)),

$$r_i^\vee(\beta) = \beta - \frac{2(\beta|\alpha_i^\vee)}{(\alpha_i^\vee|\alpha_i^\vee)}\alpha_i^\vee, \tag{5.20}$$

or, in terms of the scale factor coordinates β^μ,

$$\beta^\mu \longrightarrow \beta^{\mu\prime} = \beta^\mu - \frac{2(\beta|\omega^\vee)}{(\omega^\vee|\omega^\vee)}\omega^{\vee\mu}. \tag{5.21}$$

This is precisely the billiard reflection Equation (2.45) found in Section 2.4.

Thus, we have the following correspondence:

$$\begin{aligned} \mathcal{M}_\beta &\equiv \mathfrak{h}, \\ \mathcal{M}_\beta^\star &\equiv \mathfrak{h}^\star, \\ \omega_A(\beta) &\equiv \alpha_A(h), \\ \text{billiard wall reflections} &\equiv \text{fundamental Weyl reflections.} \end{aligned} \tag{5.22}$$

As we have also seen, the Kac–Moody algebra $\mathfrak{g}(A)$ is Lorentzian since the signature of the metric Equation (5.12) is Lorentzian. This fact will be crucial in the analysis of subsequent sections and is due to the presence of gravity, where conformal rescalings of the metric define timelike directions in scale factor space.

We thereby arrive at the following important result [45, 46, 48]:

The dynamics of (a restricted set of) theories coupled to gravity can in the BKL-limit be mapped to a billiard motion in the Cartan subalgebra $\mathfrak{h}$ of a Lorentzian Kac–Moody algebra $\mathfrak{g}$.

5.2.2 The fundamental Weyl chamber and the billiard table

Let $\mathcal{B}_{\mathcal{M}_\beta}$ denote the region in scale factor space to which the billiard motion is confined,

$$\mathcal{B}_{\mathcal{M}_\beta} = \{\beta \in \mathcal{M}_\beta \,|\, \omega_A(\beta) \geq 0\}, \tag{5.23}$$

where the index A runs over all relevant walls. On the algebraic side, the fundamental Weyl chamber in $\mathfrak{h}$ is the closed convex (half) cone given by

$$\mathcal{W}_{\mathfrak{h}} = \{h \in \mathfrak{h} \,|\, \alpha_A(h) \geq 0;\ A = 1, \cdots, \mathrm{rank}\,\mathfrak{g}\}. \tag{5.24}$$

We see that the conditions $\alpha_A(h) \geq 0$ defining $\mathcal{W}_{\mathfrak{h}}$ are equivalent, upon examination of Equation (5.22), to the conditions $\omega_A(\beta) \geq 0$ defining the billiard table $\mathcal{B}_{\mathcal{M}_\beta}$.

We may therefore make the crucial identification

$$\mathcal{W}_{\mathfrak{h}} \equiv \mathcal{B}_{\mathcal{M}_\beta}, \tag{5.25}$$

which means that the particle geodesic is confined to move within the fundamental Weyl chamber of $\mathfrak{h}$. From the billiard analysis in Section 2 we know that the piecewise motion in scale-factor space is controlled by geometric reflections with respect to the walls $\omega_A(\beta) = 0$. By comparing with the dominant wall forms and using the correspondence in Equation (5.22) we may further conclude that the geometric reflections of the coordinates $\beta^\mu(\tau)$ are controlled by the Weyl group in the Cartan subalgebra of $\mathfrak{g}(A)$.

5.2.3 Hyperbolicity implies chaos

We have learned that the BKL dynamics is chaotic if and only if the billiard table is of finite volume when projected onto the unit hyperboloid. From our discussion of hyperbolic Coxeter groups in Section 3.5, we see that this feature is equivalent to hyperbolicity of the corresponding Kac–Moody algebra. This leads to the crucial statement [45, 46, 48]:

If the billiard region of a gravitational system in the BKL-limit can be identified with the fundamental Weyl chamber of a hyperbolic Kac–Moody algebra, then the dynamics is chaotic.

As we have also discussed above, hyperbolicity can be rephrased in terms of the fundamental weights Λ_i defined as

$$\langle \Lambda_j, \alpha_i^\vee \rangle = \frac{2(\Lambda_j|\alpha_i)}{(\alpha_i|\alpha_i)} \equiv \delta_{ij}, \qquad \alpha_i^\vee \in \mathfrak{h},\ \Lambda_i \in \mathfrak{h}^\star. \tag{5.26}$$

Just as the fundamental Weyl chamber in $\mathfrak{h}^\star$ can be expressed in terms of the fundamental weights (see Equation (3.40)), the fundamental Weyl chamber in $\mathfrak{h}$ can be expressed in a similar fashion in terms of the fundamental coweights:

$$\mathcal{W}_{\mathfrak{h}} = \{\beta \in \mathfrak{h} \,|\, \beta = \sum_i a_i \Lambda_i^\vee,\, a_i \in \mathbb{R}_{\geq 0}\}. \tag{5.27}$$

As we have seen (Sections 3.5 and 4.8), hyperbolicity holds if and only if none of the fundamental weights are spacelike,

$$(\Lambda_i|\Lambda_i) \leq 0, \tag{5.28}$$

for all $i \in \{1, \cdots, \mathrm{rank}\,\mathfrak{g}\}$.

Example: Pure gravity in $D = 3 + 1$ and A_1^{++}

Let us return once more to the example of pure four-dimensional gravity, i.e., the original "BKL billiard". We have already found in Section 3 that the three dominant wall forms give rise to the Cartan matrix of the hyperbolic Kac–Moody algebra A_1^{++} [46, 48]. Since the algebra is hyperbolic, this theory exhibits chaotic behavior. In this example, we verify that the Weyl chamber is indeed contained within the lightcone by computing explicitly the norms of the fundamental weights.

It is convenient to first write the simple roots in the β-basis as follows¿

$$\begin{aligned}\alpha_1 &= (2, 0, 0)\\ \alpha_2 &= (-1, 1, 0)\\ \alpha_3 &= (0, -1, 1).\end{aligned} \tag{5.29}$$

Since the Cartan matrix of A_1^{++} is symmetric, the relations defining the fundamental weights are

$$(\alpha_i|\Lambda_j) \equiv \delta_{ij}. \tag{5.30}$$

By solving these equations for Λ_i we deduce that the fundamental weights are

$$\begin{aligned}\Lambda_1 &= -\frac{3}{2}\alpha_1 - 2\alpha_2 - \alpha_3 = (-1, -1, -1),\\ \Lambda_2 &= -2\alpha_1 - 2\alpha_2 - 2\alpha_3 = (0, 1, -1),\\ \Lambda_3 &= -\alpha_1 - \alpha_2 = (-1, -1, 0),\end{aligned} \tag{5.31}$$

where in the last step we have written the fundamental weights in the β-basis. The norms may now be computed with the metric in root space and become

$$(\Lambda_1|\Lambda_1) = -\frac{3}{2}, \qquad (\Lambda_2|\Lambda_2) = -2, \qquad (\Lambda_3|\Lambda_3) = 0. \tag{5.32}$$

We see that Λ_1 and Λ_2 are timelike and that Λ_3 is lightlike. Thus, the Weyl chamber is indeed contained inside the lightcone, the algebra is hyperbolic and the billiard is of finite volume, in agreement with what we already found [46].

5.3 Understanding the emerging Kac–Moody algebra

We shall now relate the Kac–Moody algebra whose fundamental Weyl chamber emerges in the BKL-limit to the U-duality group that appears upon toroidal dimensional reduction to three spacetime dimensions. We shall do this first in the case when $\mathfrak{u}_3$ is a split real form. By this we mean that the real algebra $\mathfrak{u}_3$ possesses the same Chevalley–Serre presentation as $\mathfrak{u}_3^{\mathbb{C}}$, but with coefficients restricted to be real numbers. This restriction is mathematically consistent because the coefficients appearing in the Chevalley–Serre presentation are all reals (in fact, integers).

The fact that the billiard structure is preserved under reduction turns out to be very useful for understanding the emergence of "overextended" algebras in the BKL-limit. By computing the billiard in three spacetime dimensions instead of in maximal dimension, the relation to U-duality groups becomes particularly transparent and the computation of the billiard follows a similar pattern for all cases. We will see that if $\mathfrak{u}_3$ is the algebra representing the internal symmetry of the non-gravitational degrees of freedom in three dimensions, then the billiard is controlled by the Weyl group of the overextended algebra $\mathfrak{u}_3^{++}$. The analysis is somewhat more involved when $\mathfrak{u}_3$ is non-split, and we postpone a discussion of this until Section 7.

5.3.1 Invariance under toroidal dimensional reduction

It was shown in [41] that the structure of the billiard for any given theory is completely unaffected by dimensional reduction on a torus. In this section we illustrate this by an explicit example rather than in full generality. We consider the case of reduction of eleven-dimensional supergravity on a circle.

The compactification ansatz in the conventions of [35, 41] is

$$\mathrm{g}_{MN} = \begin{pmatrix} e^{-2(\frac{4}{3\sqrt{2}}\hat{\varphi})} & e^{-2(\frac{4}{3\sqrt{2}}\hat{\varphi})}\hat{\mathcal{A}}_\nu \\ e^{-2(\frac{4}{3\sqrt{2}}\hat{\varphi})}\hat{\mathcal{A}}_\mu & e^{-2(\frac{-1}{6\sqrt{2}}\hat{\varphi})}\hat{\mathrm{g}}_{\mu\nu} + e^{-2(\frac{4}{3\sqrt{2}}\hat{\varphi})}\hat{\mathcal{A}}_\mu\hat{\mathcal{A}}_\nu \end{pmatrix}, \tag{5.33}$$

where $\mu,\nu = 0,2,\cdots,10$, i.e., the compactification is performed along the first spatial direction[16]. We will refer to the new lower-dimensional fields $\hat{\varphi}$ and $\hat{\mathcal{A}}_\mu$ as the dilaton and the Kaluza–Klein (KK) vector, respectively. Quite generally, hatted fields are low-dimensional fields. The ten-dimensional Lagrangian becomes

$$\begin{aligned}\mathcal{L}^{\mathrm{SUGRA}_{11}}_{(10)} = R_{(10)} \star \mathbf{1} - \star d\hat{\varphi}\wedge d\hat{\varphi} - \frac{1}{2}e^{-2(\frac{3}{2\sqrt{2}}\hat{\varphi})} \star \hat{\mathcal{F}}^{(2)} \wedge \hat{\mathcal{F}}^{(2)} \\ -\frac{1}{2}e^{-2(\frac{1}{2\sqrt{2}}\hat{\varphi})} \star \hat{F}^{(4)} \wedge \hat{F}^{(4)} - \frac{1}{2}e^{-2(\frac{-1}{\sqrt{2}}\hat{\varphi})} \star \hat{F}^{(3)} \wedge \hat{F}^{(3)},\end{aligned} \tag{5.34}$$

where $\hat{\mathcal{F}}^{(2)} = d\hat{\mathcal{A}}^{(1)}$ and $\hat{F}^{(4)}, \hat{F}^{(3)}$ are the field strengths in ten dimensions originating from the eleven-dimensional 3-form field strength $F^{(4)} = dA^{(3)}$.

Examining the new form of the metric reveals that the role of the scale factor β^1, associated to the compactified dimension, is now instead played by the ten-dimensional dilaton, $\hat{\varphi}$. Explicitly we have

$$\beta^1 = \frac{4}{3\sqrt{2}}\hat{\varphi}. \tag{5.35}$$

The nine remaining eleven-dimensional scale factors, $\beta^2,\cdots,\beta^{10}$, may in turn be written in terms of the new scale factors, $\hat{\beta}^a$, associated to the ten-dimensional metric, $\hat{\mathrm{g}}_{\mu\nu}$, and the dilaton in the following way:

$$\beta^a = \hat{\beta}^a - \frac{1}{6\sqrt{2}}\hat{\varphi} \qquad (a = 2,\cdots,10). \tag{5.36}$$

We are interested in finding the dominant wall forms in terms of the new scale factors $\hat{\beta}_2,\cdots,\hat{\beta}_{10}$ and $\hat{\varphi}$. It is clear that we will have eight ten-dimensional symmetry walls,

$$\hat{s}_{\hat{m}}(\hat{\beta}) = \hat{\beta}^{\hat{m}+1} - \hat{\beta}^{\hat{m}} \qquad (\hat{m} = 2,\cdots,9), \tag{5.37}$$

[16] Taking the first spatial direction as compactification direction is convenient, for it does not change the conventions on the simple roots. More precisely, the Kaluza–Klein ansatz is compatible in that case with our Iwasawa decomposition (2.8) of the spatial metric with $\mathcal{N}$ an upper triangular matrix. The (equivalent) choice of the tenth direction as compactification direction would correspond to a different (equivalent) choice of $\mathcal{N}$.

which correspond to the eight simple roots of $\mathfrak{sl}(9,\mathbb{R})$. Using Equation (5.35) and Equation (5.36) one may also check that the symmetry wall $\beta^2-\beta^1$, that was associated with the compactified direction, gives rise to an electric wall of the Kaluza–Klein vector,

$$\hat{e}_2^{\hat{\mathcal{A}}}(\hat{\beta})=\hat{\beta}^2-\frac{3}{2\sqrt{2}}\hat{\varphi}. \tag{5.38}$$

The metric in the dual space gets modified in a natural way,

$$(\hat{\alpha}_k|\hat{\alpha}_l)=\sum_{i=2}^{10}\hat{\alpha}_{ki}\hat{\alpha}_{li}-\frac{1}{8}\left(\sum_{i=2}^{10}\hat{\alpha}_{ki}\right)\left(\sum_{j=2}^{10}\hat{\alpha}_{lj}\right)+\hat{\alpha}_{k\hat{\varphi}}\hat{\alpha}_{l\hat{\varphi}}, \tag{5.39}$$

i.e., the dilaton contributes with a flat spatial direction. Using this metric it is clear that $\hat{e}_2^{\hat{\mathcal{A}}}$ has non-vanishing scalar product only with the second symmetry wall $\hat{s}_2=\hat{\beta}^3-\hat{\beta}^2$, $(\hat{e}_2^{\hat{\mathcal{A}}}|\hat{s}_2)=-1$, and it follows that the electric wall of the Kaluza–Klein vector plays the role of the first simple root of $\mathfrak{sl}(10,\mathbb{R})$, $\hat{\alpha}_1\equiv\hat{e}_2^{\hat{\mathcal{A}}}$. The final wall form that completes the set will correspond to the exceptional node labeled "10" in Figure 25 and is now given by one of the electric walls of the NS-NS 2-form $\hat{A}^{(2)}$, namely

$$\hat{\alpha}_{10}\equiv\hat{e}_{23}^{\hat{A}^{(2)}}(\hat{\beta})=\hat{\beta}^2+\hat{\beta}^3+\frac{1}{\sqrt{2}}\hat{\varphi}. \tag{5.40}$$

It is then easy to verify that this wall form has non-vanishing scalar product only with the third simple root $\hat{\alpha}_3=\hat{s}_3$, $(\hat{e}_{23}^{\hat{A}^{(2)}}|\hat{s}_3)=-1$, as desired.

We have thus shown that the E_{10} structure is sufficiently rigid to withstand compactification on a circle with the new simple roots explicitly given by

$$\{\hat{\alpha}_1,\hat{\alpha}_2,\cdots,\hat{\alpha}_9,\hat{\alpha}_{10}\}=\{\hat{e}_2^{\hat{\mathcal{A}}},\hat{s}_2,\cdots,\hat{s}_9,\hat{e}_{23}^{\hat{A}^{(2)}}\}. \tag{5.41}$$

This result is in fact true also for the general case of compactification on tori, T^n. When reaching the limiting case of three dimensions, all the non-gravity wall forms correspond to the electric and magnetic walls of the axionic scalars. We will discuss this case in detail below.

For non-toroidal reductions the above analysis is drastically modified [166, 165]. The topology of the internal manifold affects the dominant wall system, and hence the algebraic structure in the lower-dimensional theory is modified. In many cases, the billiard of the effective compactified theory is described by a (non-hyperbolic) regular Lorentzian subalgebra of the original hyperbolic Kac–Moody algebra [98].

The walls are also invariant under dualization of a p-form into a $(D-p-2)$-form; this simply exchanges magnetic and electric walls.

5.3.2 Iwasawa decomposition for split real forms

We will now exploit the invariance of the billiard under dimensional reduction, by considering theories that – when compactified on a torus to three dimensions – exhibit "hidden" internal global symmetries $\mathcal{U}_3$. By this we mean that the three-dimensional reduced theory is described, after dualization of all vectors to scalars, by the sum of the Einstein–Hilbert Lagrangian coupled to the Lagrangian for the nonlinear sigma model $\mathcal{U}_3/\mathcal{K}(\mathcal{U}_3)$. Here, $\mathcal{K}(\mathcal{U}_3)$ is the maximal compact subgroup defining the "local symmetries". In order to understand the connection between the U-duality group $\mathcal{U}_3$ and the Kac–Moody algebras appearing in the BKL-limit, we must first discuss some important features of the Lie algebra $\mathfrak{u}_3$.

Let $\mathfrak{u}_3$ be a split real form, meaning that it can be defined in terms of the Chevalley–Serre presentation of the complexified Lie algebra $\mathfrak{u}_3^{\mathbb{C}}$ by simply restricting all linear combinations of generators to the real numbers $\mathbb{R}$. Let $\mathfrak{h}_3$ be the Cartan subalgebra of $\mathfrak{u}_3$ appearing in the Chevalley–Serre presentation, spanned by the generators $\alpha_1^\vee, \cdots, \alpha_n^\vee$. It is maximally noncompact (see Section 6). An *Iwasawa decomposition* of $\mathfrak{u}_3$ is a direct sum of vector spaces of the following form,

$$\mathfrak{u}_3 = \mathfrak{k}_3 \oplus \mathfrak{h}_3 \oplus \mathfrak{n}_3, \tag{5.42}$$

where $\mathfrak{k}_3$ is the "maximal compact subalgebra" of $\mathfrak{u}_3$, and $\mathfrak{n}_3$ is the nilpotent subalgebra spanned by the positive root generators E_α, $\forall\alpha \in \Delta_+$.

The corresponding Iwasawa decomposition at the group level enables one to write uniquely any group element as a product of an element of the maximally compact subgroup times an element in the subgroup whose Lie algebra is $\mathfrak{h}_3$ times an element in the subgroup whose Lie algebra is $\mathfrak{n}_3$. An arbitrary element $\mathcal{V}(x)$ of the coset $\mathcal{U}_3/\mathcal{K}(\mathcal{U}_3)$ is defined as the set of equivalence classes of elements of the group modulo elements in the maximally compact subgroup. Using the Iwasawa decomposition, one can go to the "Borel gauge", where the elements in the coset are obtained by exponentiating only generators belonging to the *Borel subalgebra*,

$$\mathfrak{b}_3 = \mathfrak{h}_3 \oplus \mathfrak{n}_3 \subset \mathfrak{u}_3. \tag{5.43}$$

In that gauge we have

$$\mathcal{V}(x) = \mathrm{Exp}\left[\phi(x)\cdot\mathfrak{h}_3\right] \mathrm{Exp}\left[\chi(x)\cdot\mathfrak{n}_3\right], \tag{5.44}$$

where ϕ and χ are (sets of) coordinates on the coset space $\mathcal{U}_3/\mathcal{K}(\mathcal{U}_3)$. A Lagrangian based on this coset will then take the generic form (see Section 9)

$$\mathcal{L}_{\mathcal{U}_3/\mathcal{K}(\mathcal{U}_3)} = \sum_{i=1}^{\dim\mathfrak{h}_3} \partial_x\phi^{(i)}(x)\partial_x\phi^{(i)}(x) + \sum_{\alpha\in\Delta_+} e^{2\alpha(\phi)}\Big[\partial_x\chi^{(\alpha)}(x)+\cdots\Big]\Big[\partial_x\chi^{(\alpha)}(x)+\cdots\Big], \tag{5.45}$$

where x denotes coordinates in spacetime and the "ellipses" denote correction terms that are of no relevance for our present purposes. We refer to the fields $\{\phi\}$ collectively as *dilatons* and the fields $\{\chi\}$ as *axions*. There is one axion field $\chi^{(\alpha)}$ for each positive root $\alpha \in \Delta_+$ and one dilaton field $\phi^{(i)}$ for each Cartan generator $\alpha_i^\vee \in \mathfrak{h}_3$.

The Lagrangian (5.45) coupled to the pure three-dimensional Einstein–Hilbert term is the key to understanding the appearance of the Lorentzian Coxeter group $\mathfrak{u}_3^{++}$ in the BKL-limit.

5.3.3 Starting at the bottom – Overextensions of finite-dimensional Lie algebras

To make the point explicit, we will again limit our analysis to the example of eleven-dimensional supergravity. Our starting point is then the Lagrangian for this theory compactified on an 8-torus, T^8, to $D = 2+1$ spacetime dimensions (after all form fields have been dualized into scalars),

$$\mathcal{L}_{(3)}^{\mathrm{SUGRA}_{11}} = R_{(3)} \star \mathbf{1} - \sum_{i=1}^{8} \star d\hat{\varphi}^{(i)} \wedge d\hat{\varphi}^{(i)} - \frac{1}{2}\sum_{q=1}^{120} e^{2\alpha_q(\hat{\varphi})} \star (d\hat{\chi}^{(q)}+\cdots) \wedge (d\hat{\chi}^{(q)}+\cdots). \tag{5.46}$$

The second two terms in this Lagrangian correspond to the coset model $\mathcal{E}_{8(8)}/(\mathrm{Spin}(16)/\mathbb{Z}_2)$, where $\mathcal{E}_{8(8)}$ denotes the group obtained by exponentiation of the split form $E_{8(8)}$ of the complex Lie algebra E_8 and $\mathrm{Spin}(16)/\mathbb{Z}_2$ is the maximal compact subgroup of $\mathcal{E}_{8(8)}$ [33, 134, 35]. The 8 dilatons $\hat{\varphi}$ and the 120 axions $\chi^{(q)}$ are coordinates on the coset space[17]. Furthermore, the $\alpha_q(\hat{\varphi})$

[17] This structure of $\mathcal{E}_{8(8)}$ can be understood as follows. The 248-dimensional Lie algebra $E_{8(8)}$ can be represented as $\mathfrak{so}(16) \oplus \mathcal{S}_{16}$ (direct sum of vector spaces), where $\mathcal{S}_{16}$ constitutes a 128-dimensional representation space of the group Spin(16), that transforms like Majorana–Weyl spinors. Using Dirac matrices $\Gamma^{\nu}_{\alpha\ \mu}$, the commutation relations read:

are linear forms on the elements of the Cartan subalgebra $h = \hat{\varphi}^i \alpha_i^\vee$ and they correspond to the positive roots of $E_{8(8)}$ [18]. As before, we do not write explicitly the corrections to the curvatures $d\hat{\chi}$ that appear in the compactification process. The entire set of positive roots can be obtained by taking linear combinations of the seven simple roots of $\mathfrak{sl}(8,\mathbb{R})$ (we omit the "hatted" notation on the roots since there is no longer any risk of confusion),

$$\begin{aligned} \alpha_1(\hat{\varphi}) &= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{7}}{2}\hat{\varphi}_2 - \frac{3}{2}\hat{\varphi}_1\right), & \alpha_2(\hat{\varphi}) &= \frac{1}{\sqrt{2}}\left(\frac{2\sqrt{3}}{\sqrt{7}}\hat{\varphi}_3 - \frac{4}{\sqrt{7}}\hat{\varphi}_2\right), \\ \alpha_3(\hat{\varphi}) &= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{5}}{\sqrt{3}}\hat{\varphi}_4 - \frac{\sqrt{7}}{\sqrt{3}}\hat{\varphi}_3\right), & \alpha_4(\hat{\varphi}) &= \frac{1}{\sqrt{2}}\left(\frac{2\sqrt{2}}{\sqrt{5}}\hat{\varphi}_5 - \frac{2\sqrt{3}}{\sqrt{5}}\hat{\varphi}_4\right), \\ \alpha_5(\hat{\varphi}) &= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{\sqrt{2}}\hat{\varphi}_6 - \frac{\sqrt{5}}{\sqrt{2}}\hat{\varphi}_5\right), & \alpha_6(\hat{\varphi}) &= \frac{1}{\sqrt{2}}\left(\frac{2}{\sqrt{3}}\hat{\varphi}_6 - \frac{2\sqrt{2}}{\sqrt{3}}\hat{\varphi}_5\right), \\ \alpha_7(\hat{\varphi}) &= \frac{1}{\sqrt{2}}\left(\hat{\varphi}_8 - \sqrt{3}\hat{\varphi}_7\right), \end{aligned} \tag{5.47}$$

and the exceptional root

$$\alpha_{10}(\hat{\varphi}) = \frac{1}{\sqrt{2}}\left(\hat{\varphi}_1 + \frac{3}{\sqrt{7}}\hat{\varphi}_2 + \frac{2\sqrt{3}}{\sqrt{7}}\hat{\varphi}_3\right). \tag{5.48}$$

These correspond exactly to the root vectors $\vec{b}_{i,i+1}$ and $\vec{a}_{123}$ as they appear in the analysis of [35], except for the additional factor of $\frac{1}{\sqrt{2}}$ needed to compensate for the fact that the aforementioned reference has an additional factor of 2 in the Killing form. Hence, using the Euclidean metric δ_{ij} $(i,j = 1,\cdots,8)$ one may check that the roots defined above indeed reproduce the Cartan matrix of E_8.

Next, we want to determine the billiard structure for this Lagrangian. As was briefly mentioned before, in the reduction from eleven to three dimensions all the non-gravity walls associated to the eleven-dimensional 3-form $A^{(3)}$ have been transformed, in the same spirit as for the example given above, into electric and magnetic walls of the axionic scalars $\hat{\chi}$. Since the terms involving the electric fields $\partial_t\hat{\chi}^{(i)}$ possess no spatial indices, the corresponding wall forms do not contain any of the remaining scale factors $\hat{\beta}^9, \hat{\beta}^{10}$, and are simply linear forms on the dilatons only. In fact the dominant electric wall forms are just the simple roots of E_8,

$$\begin{aligned} \hat{e}_a^{\hat{\chi}}(\hat{\varphi}) &= \alpha_a(\hat{\varphi}) \qquad (a = 1,\cdots,7), \\ \hat{e}_{10}^{\hat{\chi}}(\hat{\varphi}) &= \alpha_{10}(\hat{\varphi}). \end{aligned} \tag{5.49}$$

The magnetic wall forms naturally come with one factor of $\hat{\beta}$ since the magnetic field strength $\partial_i\hat{\chi}$ carries one spatial index. The dominant magnetic wall form is then given by

$$\hat{m}_9^{\hat{\chi}}(\hat{\beta},\hat{\varphi}) = \hat{\beta}^9 - \theta(\hat{\varphi}), \tag{5.50}$$

$$[M_{ab}, M_{cd}] = \delta_{ac}M_{bd} + \delta_{bd}M_{ac} - \text{ad}\, M_{bc} - \delta_{bc}M_{ad},$$

$$[M_{ab}, Q_\mu] = \frac{1}{2}\Gamma^{\ \ \nu}_{[ab]\mu}Q_\nu,$$

$$[Q_\mu, Q_\nu] = \Gamma^{[ab]}_{\mu\nu}M_{ab}.$$

For more information about $E_{8(8)}$ see [134], and for a general discussion of real forms of Lie algebras see Section 6.

[18] In the following we write simply E_8 and it is understood that we refer to the split real form $E_{8(8)}$.

where $\theta(\hat{\varphi})$ denotes the highest root of E_8 which takes the following form in terms of the simple roots,

$$\theta = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + 3\alpha_{10} = \sqrt{2}\,\hat{\varphi}_8. \tag{5.51}$$

Since we are in three dimensions there is no curvature wall and hence the only wall associated to the Einstein–Hilbert term is the symmetry wall

$$\hat{s}_9 = \hat{\beta}^{10} - \hat{\beta}^9, \tag{5.52}$$

coming from the three-dimensional metric $\hat{\mathrm{g}}_{\mu\nu}$ $(\mu, \nu = 0, 9, 10)$. We have thus found all the dominant wall forms in terms of the lower-dimensional variables.

The structure of the corresponding Lorentzian Kac–Moody algebra is now easy to establish in view of our discussion of overextensions in Section 4.9. The relevant walls listed above are the simple roots of the (untwisted) overextension E_8^{++}. Indeed, the relevant electric roots are the simple roots of E_8, the magnetic root of Equation (5.50) provides the affine extension, while the gravitational root of Equation (5.52) is the overextended root.

What we have found here in the case of eleven-dimensional supergravity also holds for the other theories with U-duality algebra $\mathfrak{u}_3$ in 3 dimensions when $\mathfrak{u}_3$ is a split real form. The Coxeter group and the corresponding Kac–Moody algebra are given by the untwisted overextension $\mathfrak{u}_3^{++}$. This overextension emerges as follows [41]:

- The dominant electric wall forms $\hat{e}^{\hat{\chi}}(\hat{\varphi})$ for the supergravity theory in question are in one-to-one correspondence with the simple roots of the associated U-duality algebra $\mathfrak{u}_3$.
- Adding the dominant magnetic wall form $\hat{m}^{\hat{\chi}}(\hat{\beta}, \hat{\varphi}) = \hat{\beta}^9 - \theta(\hat{\varphi})$ corresponds to an *affine extension* $\mathfrak{u}_3^{+}$ of $\mathfrak{u}_3$.
- Finally, completing the set of dominant wall forms with the symmetry wall $\hat{s}_9(\hat{\beta}) = \hat{\beta}^{10} - \hat{\beta}^9$, which is the only gravitational wall form existing in three dimensions, is equivalent to an *overextension* $\mathfrak{u}_3^{++}$ of $\mathfrak{u}_3$.

Thus we see that the appearance of overextended algebras in the BKL-analysis of supergravity theories is a generic phenomenon closely linked to hidden symmetries.

5.4 Models associated with split real forms

In this section we give a complete list of all theories whose billiard description can be given in terms of a Kac–Moody algebra that is the untwisted overextension of a split real form of the associated U-duality algebra (see Table 15). These are precisely the *maximally oxidized* theories introduced in [22] and further examined in [37]. These theories are completely classified by their global symmetry groups $\mathcal{U}_3$ arising in three dimensions. For the string-related theories the group $\mathcal{U}_3$ is the (classical version of) the U-duality symmetry obtained by combining the S- and T-dualities in three dimensions [142]. Thereof the notation $\mathcal{U}_3$ for the global symmetry group in three dimensions. We extend the classification to the non-split case in Section 7.

Let us also note here that, as shown in [55], the billiard analysis sheds light on the problem of oxidation, i.e., the problem of finding the maximum spacetime dimension in which a theory with a given duality group in three dimensions can be reformulated. More on this question can be found in [118, 119].

Table 15: We present here the complete list of theories that exhibit extended coset symmetries of split real Lie algebras upon compactification to three spacetime dimensions. In the leftmost column we give the coset space which is relevant in each case. We also list the Kac–Moody algebras that underlie the gravitational dynamics in the BKL-limit. These appear as overextensions of the finite Lie algebras found in three dimensions. Finally we indicate which of these theories are related to string/M-theory.

$\mathcal{U}_3/\mathcal{K}(\mathcal{U}_3)$	Lagrangian in maximal dimension	Kac–Moody algebra	String/M-theory
$\frac{SL(n+1,\mathbb{R})}{SO(n+1)}$	$\mathcal{L}_{n+3} = R \star \mathbf{1}$	$AE_{n+2} \equiv A_n^{++}$	No
$\frac{SO(n,n+1)}{SO(n)\times SO(n+1)}$	$\mathcal{L}_{n+2} = R\star\mathbf{1} - \star d\phi \wedge d\phi - \frac{1}{2}e^{2\frac{\sqrt{2}}{\sqrt{n}}\phi} \star G^{(3)} \wedge G^{(3)} - \frac{1}{2}e^{\frac{2}{\sqrt{n}}\phi} \star F^{(2)} \wedge F^{(2)}$, $G^{(3)} = dB^{(2)} + \frac{1}{2}A^{(1)} \wedge A^{(1)}$, $F^{(2)} = dA^{(1)}$	$BE_{n+2} \equiv B_n^{++}$	No
$\frac{Sp(n)}{U(n)}$	$\mathcal{L}_4 = R\star\mathbf{1} - \star d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2}\sum_\alpha e^{2\vec{\sigma}_\alpha\cdot\vec{\phi}} \star (d\chi^\alpha + \cdots) \wedge (d\chi^\alpha + \cdots) - \frac{1}{2}\sum_{a=1}^{n-1} e^{\vec{e}_a\cdot\vec{\phi}\sqrt{2}} \star dA^a_{(1)} \wedge dA^a_{(1)}$	$CE_{n+2} \equiv C_n^{++}$	No
$\frac{SO(n,n)}{SO(n)\times SO(n)}$	$\mathcal{L}_{n+2} = R\star\mathbf{1} - \star d\phi \wedge d\phi - \frac{1}{2}e^{\frac{4}{\sqrt{n}}\phi} \star dB^{(2)} \wedge dB^{(2)}$	$DE_{n+2} \equiv D_n^{++}$	type I ($n = 8$) / bosonic string ($n = 24$)
$\frac{G_{2(2)}}{SU(8)}$	$\mathcal{L}_5 = R\star\mathbf{1} - \frac{1}{2}\star F^{(2)} \wedge F^{(2)} + \frac{1}{3\sqrt{3}}F^{(2)} \wedge F^{(2)} \wedge A^{(1)}$, $F^{(2)} = dA^{(1)}$	G_2^{++}	No
$\frac{F_{4(4)}}{Sp(3)\times SU(3)}$	$\mathcal{L}_6 = R\star\mathbf{1} - \star d\phi \wedge d\phi - \frac{1}{2}e^{2\phi} \star d\chi \wedge d\chi - \frac{1}{2}e^{-2\phi} \star H^{(3)} \wedge H^{(3)} - \frac{1}{2}\star G^{(3)} \wedge G^{(3)} - \frac{1}{2}e^{\phi} \star F^+_{(2)} \wedge F^+_{(2)} - \frac{1}{2}e^{-\phi} \star F^-_{(2)} \wedge F^-_{(2)} - \frac{1}{\sqrt{2}}\chi H^{(3)} \wedge G^{(3)} - \frac{1}{2}A^+_{(1)} \wedge F^+_{(2)} \wedge H^{(3)} - \frac{1}{2}A^+_{(1)} \wedge F^-_{(2)} \wedge G^{(3)}$, $F^+_{(2)} = dA^+_{(1)} + \frac{1}{\sqrt{2}}\chi dA^-_{(1)}$, $F^-_{(2)} = dA^-_{(1)}$, $H^{(3)} = dB^{(2)} + \frac{1}{2}A^-_{(1)} \wedge dA^-_{(1)}$, $G^{(3)} = dC^{(2)} - \frac{1}{\sqrt{2}}\chi H^{(3)} - \frac{1}{2}A^+_{(1)} \wedge dA^-_{(1)}$	F_4^{++}	No
$\frac{E_{6(6)}}{Sp(4)/\mathbb{Z}_2}$	$\mathcal{L}_8 = R\star\mathbf{1} - \star d\phi \wedge d\phi - \frac{1}{2}e^{2\sqrt{2}\phi} \star d\chi \wedge d\chi - \frac{1}{2}e^{-\sqrt{2}\phi} \star G^{(4)} \wedge G^{(4)} + \chi G^{(4)} \wedge G^{(4)}$, $G^{(4)} = dC^{(3)}$	E_6^{++}	No
$\frac{E_{7(7)}}{SU(8)/\mathbb{Z}_2}$	$\mathcal{L}_9 = R\star\mathbf{1} - \star d\phi \wedge d\phi - \frac{1}{2}e^{\frac{2\sqrt{2}}{\sqrt{7}}\phi} \star dC^{(3)} \wedge dC^{(3)} - \frac{1}{2}e^{-\frac{4\sqrt{2}}{\sqrt{7}}\phi} \star dA^{(1)} \wedge dA^{(1)} - \frac{1}{2}dC^{(3)} \wedge dC^{(3)} \wedge A^{(1)}$	E_7^{++}	No
$\frac{E_{8(8)}}{\mathrm{Spin}(16)/\mathbb{Z}_2}$	$\mathcal{L}_{11} = R\star\mathbf{1} - \frac{1}{2}\star dC^{(3)} \wedge dC^{(3)} - \frac{1}{6}dC^{(3)} \wedge dC^{(3)} \wedge C^{(3)}$	$E_{10} \equiv E_8^{++}$	M-theory, type IIA and type IIB string theory

6 Finite-Dimensional Real Lie Algebras

In this section we explain the basic theory of real forms of finite-dimensional Lie algebras. This material is somewhat technical and may therefore be skipped at a first reading. The theory of real forms of Lie algebras is required for a complete understanding of Section 7, which deals with the general case of Kac–Moody billiards for non-split real forms. However, for the benefit of the reader who wishes to proceed directly to the physical applications, we present a brief summary of the main points in the beginning of Section 7.

Our intention with the following presentation is to provide an accessible reference on the subject, directed towards physicists. We therefore consider this section to be somewhat of an entity of its own, which can be read independently of the rest of the paper. Consequently, we introduce Lie algebras in a rather different manner compared to the presentation of Kac–Moody algebras in Section 4, emphasizing here more involved features of the general structure theory of real Lie algebras rather than relying entirely on the Chevalley–Serre basis and its properties. In the subsequent section, the reader will then see these two approaches merged, and used simultaneously to describe the billiard structure of theories whose U-duality algebras in three dimensions are given by arbitrary real forms.

We have adopted a rather detailed and explicit presentation. We do not provide all proofs, however, referring the reader to [93, 129, 133, 94] for more information (including definitions of basic Lie algebra theory concepts).

There are two main approaches to the classification of real forms of finite-dimensional Lie algebras. One focuses on the maximal compact Cartan subalgebra and leads to Vogan diagrams. The other focuses on the maximal noncompact Cartan subalgebra and leads to Tits–Satake diagrams. It is this second approach that is of direct use in the billiard analysis. However, we have chosen to present here both approaches as they mutually enlighten each other.

6.1 Definitions

Lie algebras are usually, in a first step at least, considered as complex, i.e., as complex vector spaces, structured by an antisymmetric internal bilinear product, the Lie bracket, obeying the Jacobi identity. If $\{T_\alpha\}$ denotes a basis of such a complex Lie algebra $\mathfrak{g}$ of dimension n (over $\mathbb{C}$), we may also consider $\mathfrak{g}$ as a real vector space of double dimension $2\,n$ (over $\mathbb{R}$), a basis being given by $\{T_\alpha,\, iT_\alpha\}$. Conversely, if $\mathfrak{g}_0$ is a real Lie algebra, by extending the field of scalars from $\mathbb{R}$ to $\mathbb{C}$, we obtain the complexification of $\mathfrak{g}_0$, denoted by $\mathfrak{g}^{\mathbb{C}}$, defined as:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}. \tag{6.1}$$

Note that $(\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ and $\dim_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}} = 2\,\dim_{\mathbb{R}}(\mathfrak{g}_0)$. When a complex Lie algebra $\mathfrak{g}$, considered as a real algebra, has a decomposition

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus i\,\mathfrak{g}_0, \tag{6.2}$$

with $\mathfrak{g}_0$ being a real Lie algebra, we say that $\mathfrak{g}_0$ is a real form of the complex Lie algebra $\mathfrak{g}$. In other words, a real form of a complex algebra exists if and only if we may choose a basis of the complex algebra such that all the structure constants become real. Note that while $\mathfrak{g}^{\mathbb{R}}$ is a real space, multiplication by a complex number is well defined on it since $\mathfrak{g}_0 \oplus i\mathfrak{g}_0 = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. As we easily see from Equation (6.2),

$$\mathbb{C} \times \mathfrak{g}^{\mathbb{R}} \quad \rightarrow \quad \mathfrak{g}^{\mathbb{R}} : (a + i\,b, X_0 + i\,Y_0) \mapsto (aX_0 - bY_0) + i\,(aY_0 + bX_0),, \tag{6.3}$$

where $a, b \in \mathbb{R}$ and $X_0, Y_0 \in \mathfrak{g}_0$.

The Killing form is defined by

$$B(X,Y) = \mathrm{Tr}(\mathrm{ad}\,X\,\mathrm{ad}\,Y). \tag{6.4}$$

The Killing forms on $\mathfrak{g}^{\mathbb{R}}$ and $\mathfrak{g}^{\mathbb{C}}$ or $\mathfrak{g}_0$ are related as follows. If we split an arbitrary generator Z of $\mathfrak{g}$ according to Equation (6.2) as $Z = X_0 + i\,Y_0$, we may write:

$$B_{\mathfrak{g}^{\mathbb{R}}}(Z,\, Z') = 2\,\mathrm{Re}\,B_{\mathfrak{g}^{\mathbb{C}}}(Z,\, Z') = 2\left(B_{\mathfrak{g}_0}(X_0,\, X_0') - B_{\mathfrak{g}_0}(Y_0,\, Y_0')\right). \tag{6.5}$$

Indeed, if $\mathrm{ad}_{\mathfrak{g}}\,Z$ is a complex $n \times n$ matrix, $\mathrm{ad}_{\mathfrak{g}^{\mathbb{R}}}(X_0 + i\,Y_0)$ is a real $2n \times 2n$ matrix:

$$\mathrm{ad}_{\mathfrak{g}^{\mathbb{R}}}(X_0 + i\,Y_0) = \begin{pmatrix} \mathrm{ad}_{\mathfrak{g}_0}\,X_0 & -\,\mathrm{ad}_{\mathfrak{g}_0}\,Y_0 \\ \mathrm{ad}_{\mathfrak{g}_0}\,Y_0 & \mathrm{ad}_{\mathfrak{g}_0}\,X_0 \end{pmatrix}. \tag{6.6}$$

6.2 A preliminary example: $\mathfrak{sl}(2,\mathbb{C})$

Before we proceed to develop the general theory of real forms, we shall in this section discuss in detail some properties of the real forms of $A_1 = \mathfrak{sl}(2,\mathbb{C})$. This is a nice example, which exhibits many properties that turn out not to be specific just to the case at hand, but are, in fact, valid also in the general framework of semi-simple Lie algebras. The main purpose of subsequent sections will then be to show how to extend properties that are immediate in the case of $\mathfrak{sl}(2,\mathbb{C})$, to general semi-simple Lie algebras.

6.2.1 Real forms of $\mathfrak{sl}(2,\mathbb{C})$

The complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ can be represented as the space of complex linear combinations of the three matrices

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{6.7}$$

which satisfy the well known commutation relations

$$[h,\, e] = 2\,e, \qquad [h,\, f] = -2\,f, \qquad [e,\, f] = h. \tag{6.8}$$

A crucial property of these commutation relations is that the structure constants defined by the brackets are all real. Thus by restricting the scalars in the linear combinations from the complex to the real numbers, we still obtain closure for the Lie bracket on real combinations of h, e and f, defining thereby a real form of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$: the real Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ [19]. As we have indicated above, this real form of $\mathfrak{sl}(2,\mathbb{C})$ is called the "split real form".

Another choice of $\mathfrak{sl}(2,\mathbb{C})$ generators that, similarly, leads to a real Lie algebra consists in taking i times the Pauli matrices σ^x, σ^y, σ^z, i.e.,

$$\tau^x = i(e+f) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \tau^y = (e-f) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tau^z = ih = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{6.9}$$

The real linear combinations of these matrices form the familiar $\mathfrak{su}(2)$ Lie algebra (a real Lie algebra, even if some of the matrices using to represent it are complex). This real Lie algebra is non-isomorphic (as a real algebra) to $\mathfrak{sl}(2,\mathbb{R})$ as there is no real change of basis that maps $\{h, e, f\}$ on a basis with the $\mathfrak{su}(2)$ commutation relations. Of course, the two algebras are isomorphic over the complex numbers.

[19] Actually, the structure constants are integers and thus allows for defining the arithmetic subgroup $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$.

6.2.2 Cartan subalgebras

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{sl}(2,\mathbb{R})$. We say that $\mathfrak{h}$ is a *Cartan subalgebra* of $\mathfrak{sl}(2,\mathbb{R})$ if it is a Cartan subalgebra of $\mathfrak{sl}(2,\mathbb{C})$ when the real numbers are replaced by the complex numbers. Two Cartan subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ of $\mathfrak{sl}(2,\mathbb{R})$ are said to be equivalent (as Cartan subalgebras of $\mathfrak{sl}(2,\mathbb{R})$) if there is an automorphism a of $\mathfrak{sl}(2,\mathbb{R})$ such that $a(\mathfrak{h}_1) = \mathfrak{h}_2$.

The subspace $\mathbb{R}h$ constitutes clearly a Cartan subalgebra of $\mathfrak{sl}(2,\mathbb{R})$. The adjoint action of h is diagonal in the basis $\{e,\, f,\, h\}$ and can be represented by the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.10}$$

Another Cartan subalgebra of $\mathfrak{sl}(2,\mathbb{R})$ is given by $\mathbb{R}(e-f) \equiv \mathbb{R}\tau^y$, whose adjoint action with respect to the same basis is represented by the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}. \tag{6.11}$$

Contrary to the matrix representing ad_h, in addition to 0 this matrix has two imaginary eigenvalues: $\pm 2\,i$. Thus, there can be no automorphism a of $\mathfrak{sl}(2,\mathbb{R})$ such that $\tau^y = \lambda a(h)$, $\lambda \in \mathbb{R}$ since $\mathrm{ad}_{a(h)}$ has the same eigenvalues as ad_h, implying that the eigenvalues of $\lambda\ \mathrm{ad}_{a(h)}$ are necessarily real ($\lambda \in \mathbb{R}$).

Consequently, even though they are equivalent over the complex numbers since there is an automorphism in $SL(2,\mathbb{C})$ that connects the complex Cartan subalgebras $\mathbb{C}\,h$ and $\mathbb{C}\,\tau^y$, we obtain

$$\tau^y = i\ \mathrm{Ad}\left(\mathrm{Exp}\left[i\frac{\pi}{4}(e+f)\right]\right) h, \qquad h = i\ \mathrm{Ad}\left(\mathrm{Exp}\left[\frac{\pi}{4}\tau^x\right]\right)\tau^y. \tag{6.12}$$

The real Cartan subalgebras generated by h and τ^y are non-isomorphic over the real numbers.

6.2.3 The Killing form

The Killing form of $SL(2,\mathbb{R})$ reads explicitly

$$B = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix} \tag{6.13}$$

in the basis $\{e,\, f,\, h\}$. The Cartan subalgebra $\mathbb{R}h$ is spacelike while the Cartan subalgebra $\mathbb{R}\tau^y$ is timelike. This is another way to see that these are inequivalent since the automorphisms of $\mathfrak{sl}(2,\mathbb{R})$ preserve the Killing form. The group $\mathrm{Aut}[\mathfrak{sl}(2,\mathbb{R})]$ of automorphisms of $\mathfrak{sl}(2,\mathbb{R})$ is $SO(2,1)$, while the subgroup $\mathrm{Int}[\mathfrak{sl}(2,\mathbb{R})] \subset \mathrm{Aut}[\mathfrak{sl}(2,\mathbb{R})]$ of inner automorphisms is the connected component $SO(2,1)^+$ of $SO(2,1)$. All spacelike directions are equivalent, as are all timelike directions, which shows that all the Cartan subalgebras of $\mathfrak{sl}(2,\mathbb{R})$ can be obtained by acting on these two inequivalent particular ones by $\mathrm{Int}[\mathfrak{sl}(2,\mathbb{R})]$, i.e., the adjoint action of the group $SL(2,\mathbb{R})$. The lightlike directions do not define Cartan subalgebras because the adjoint action of a lighlike vector is non-diagonalizable. In particular $\mathbb{R}e$ and $\mathbb{R}f$ are not Cartan subalgebras even though they are Abelian.

By exponentiation of the generators h and τ^y, we obtain two subgroups, denoted $\mathcal{A}$ and $\mathcal{K}$:

$$\mathcal{A} = \left\{\mathrm{Exp}[t\,h] = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} |t \in \mathbb{R}\right\} \simeq \mathbb{R}, \tag{6.14}$$

$$\mathcal{K} = \left\{\mathrm{Exp}[t\,\tau^y] = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} |t \in [0, 2\pi[\ \right\} \simeq \mathbb{R}/\mathbb{Z}. \tag{6.15}$$

The subgroup defined by Equation (6.14) is noncompact, the one defined by Equation (6.15) is compact; consequently the generator h is also said to be noncompact while τ^y is called compact.

6.2.4 The compact real form $\mathfrak{su}(2)$

The Killing metric on the group $\mathfrak{su}(2)$ is negative definite. In the basis $\{\tau^x, \tau^y, \tau^z\}$, it reads

$$B = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix}. \tag{6.16}$$

The corresponding group obtained by exponentiation is $SU(2)$, which is isomorphic to the 3-sphere and which is accordingly compact. All directions in $\mathfrak{su}(2)$ are equivalent and hence, all Cartan subalgebras are $SU(2)$ conjugate to $\mathbb{R}\tau^y$. Any generator provides by exponentiation a group isomorphic to $\mathbb{R}/\mathbb{Z}$ and is thus compact.

Accordingly, while $\mathfrak{sl}(2,\mathbb{R})$ admits both compact and noncompact Cartan subalgebras, the Cartan subalgebras of $\mathfrak{su}(2)$ are all compact. The real algebra $\mathfrak{su}(2)$ is called the compact real form of $\mathfrak{sl}(2,\mathbb{C})$. One often denotes the real forms by their signature. Adopting Cartan's notation A_1 for $\mathfrak{sl}(2,\mathbb{C})$, one has $\mathfrak{sl}(2,\mathbb{R}) \equiv A_{1\,(1)}$ and $\mathfrak{su}(2) \equiv A_{1\,(-3)}$. We shall verify before that there are no other real forms of $\mathfrak{sl}(2,\mathbb{C})$.

6.2.5 $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$ compared and contrasted – The Cartan involution

Within $\mathfrak{sl}(2,\mathbb{C})$, one may express the basis vectors of one of the real subalgebras $\mathfrak{su}(2)$ or $\mathfrak{sl}(2,\mathbb{R})$ in terms of those of the other. We obtain, using the notations $t = (e-f)$ and $x = (e+f)$:

$$\begin{aligned} x &= -i\,\tau^x, & \tau^x &= i\,x, \\ h &= -i\,\tau^z, & \tau^z &= i\,h, \\ t &= \tau^y, & \tau^y &= t. \end{aligned} \tag{6.17}$$

Let us remark that, in terms of the generators of $\mathfrak{su}(2)$, the noncompact generators x and h of $\mathfrak{sl}(2,\mathbb{R})$ are purely imaginary but the compact one t is real.

More precisely, if τ denotes the conjugation[20] of $\mathfrak{sl}(2,\mathbb{C})$ that fixes $\{\tau^x,\ \tau^y,\ \tau^z\}$, we obtain:

$$\tau(x) = -x, \qquad \tau(t) = +t, \qquad \tau(h) = -h, \tag{6.18}$$

or, more generally,

$$\forall X \in \mathfrak{sl}(2,\mathbb{C}) : \tau(X) = -X^\dagger. \tag{6.19}$$

Conversely, if we denote by σ the conjugation of $\mathfrak{sl}(2,\mathbb{C})$ that fixes the previous $\mathfrak{sl}(2,\mathbb{R})$ Cartan subalgebra in $\mathfrak{sl}(2,\mathbb{C})$, we obtain the usual complex conjugation of the matrices:

$$\sigma(X) = \overline{X}. \tag{6.20}$$

The two conjugations τ and σ of $\mathfrak{sl}(2,\mathbb{C})$ associated with the real subalgebras $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$ of $\mathfrak{sl}(2,\mathbb{C})$ commute with each other. Each of them, trivially, fixes pointwise the algebra defining it and globally the other algebra, where it constitutes an involutive automorphism ("involution").

The Killing form is neither positive definite nor negative definite on $\mathfrak{sl}(2,\mathbb{R})$: The symmetric matrices have positive norm squared, while the antisymmetric ones have negative norm squared. Thus, by changing the relative sign of the contributions associated with symmetric and antisymmetric matrices, one can obtain a bilinear form which is definite. Explicitly, the involution θ of $\mathfrak{sl}(2,\mathbb{R})$ defined by $\theta(X) = -X^t$ has the feature that

$$B^\theta(X,Y) = -B(X,\theta Y) \tag{6.21}$$

[20] A conjugation on a complex Lie algebra is an antilinear involution, preserving the Lie algebra structure.

is positive definite. An involution of a real Lie algebra with that property is called a "Cartan involution" (see Section 6.4.3 for the general definition).

The Cartan involution θ is just the restriction to $\mathfrak{sl}(2,\mathbb{R})$ of the conjugation τ associated with the compact real form $\mathfrak{su}(2)$ since for real matrices $X^\dagger = X^t$. One says for that reason that the compact real form $\mathfrak{su}(2)$ and the noncompact real form $\mathfrak{sl}(2,\mathbb{R})$ are "aligned".

Using the Cartan involution θ, one can split $\mathfrak{sl}(2,\mathbb{R})$ as the direct sum

$$\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}, \tag{6.22}$$

where $\mathfrak{k}$ is the subspace of antisymmetric matrices corresponding to the eigenvalue $+1$ of the Cartan involution while $\mathfrak{p}$ is the subspace of symmetric matrices corresponding to the eigenvalue -1. These are also eigenspaces of τ and given explicitly by $\mathfrak{k} = \mathbb{R}t$ and $\mathfrak{p} = \mathbb{R}x \oplus \mathbb{R}h$. One has

$$\mathfrak{su}(2) = \mathfrak{k} \oplus i\mathfrak{p}, \tag{6.23}$$

i.e., the real form $\mathfrak{sl}(2,\mathbb{R})$ is obtained from the compact form $\mathfrak{su}(2)$ by inserting an "i" in front of the generators in $\mathfrak{p}$.

6.2.6 Concluding remarks

Let us close these preliminaries with some remarks.

1. The conjugation τ allows to define a Hermitian form on $\mathfrak{sl}(2,\mathbb{C})$:

$$X \bullet Y = -\operatorname{Tr}(Y\tau(X)). \tag{6.24}$$

2. Any element of the group $SL(2,\mathbb{R})$ can be written as a product of elements belonging to the subgroups $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{N} = \operatorname{Exp}[\mathbb{R}e]$ (Iwasawa decomposition),

$$\operatorname{Exp}[\theta\, t]\ \operatorname{Exp}[a\, h]\ \operatorname{Exp}[n\, e] = \begin{pmatrix} e^a\cos\theta & n\, e^a\cos\theta + e^{-a}\sin\theta \\ -e^a\sin\theta & e^{-a}\cos\theta - n\, e^a\sin\theta \end{pmatrix}. \tag{6.25}$$

3. Any element of $\mathfrak{p}$ is conjugated via $\mathcal{K}$ to a multiple of h ,

$$\rho(\cos\alpha\, h + \sin\alpha\, x) = \begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \rho\, h \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}, \tag{6.26}$$

so, denoting by $\mathfrak{a} = \mathbb{R}\, h$ the (maximal) noncompact Cartan subalgebra of $\mathfrak{sl}(2,\mathbb{R})$, we obtain

$$\mathfrak{p} = \operatorname{Ad}(\mathcal{K})\mathfrak{a}. \tag{6.27}$$

4. Any element of $SL(2,\mathbb{R})$ can be written as the product of an element of $\mathcal{K}$ and an element of $\operatorname{Exp}[\mathfrak{p}]$. Thus, as a consequence of the previous remark, we have $SL(2,\mathbb{R}) = \mathcal{K}\mathcal{A}\mathcal{K}$ (Cartan)[21].

5. When the Cartan subalgebra of $\mathfrak{sl}(2,\mathbb{R})$ is chosen to be $\mathbb{R}\, h$, the root vectors are e and f. We obtain the compact element t, generating a non-equivalent Cartan subalgebra by taking the combination

$$t = e + \theta(e). \tag{6.28}$$

[21] This decomposition is just the "standard" decomposition of any $2+1$ Lorentz transformation, into the product of a rotation followed by a boost in a fixed direction and finally followed by yet another rotation.

Similarly, the normalized root vectors associated with t are (up to a complex phase) $E_{\pm 2i} = \frac{1}{2}(h \mp ix)$:

$$[t,\, E_{2i}] = 2i\, E_{2i}, \qquad [t,\, E_{-2i}] = -2i\, E_{-2i}, \qquad [E_{2i},\, E_{-2i}] = i\, t. \tag{6.29}$$

Note that both the real and imaginary components of $E_{\pm 2i}$ are noncompact. They allow to obtain the noncompact Cartan generators h, x by taking the combinations

$$\cos\alpha\, h + \sin\alpha\, x = e^{i\alpha} E_{2i} + e^{-i\alpha} E_{-2i}. \tag{6.30}$$

6.3 The compact and split real forms of a semi-simple Lie algebra

We shall consider here only semi-simple Lie algebras. Over the complex numbers, Cartan subalgebras are "unique"[22]. These subalgebras may be defined as maximal Abelian subalgebras $\mathfrak{h}$ such that the transformations in $\mathrm{ad}[\mathfrak{h}]$ are simultaneously diagonalizable (over $\mathbb{C}$). Diagonalizability is an essential ingredient in the definition. There might indeed exist Abelian subalgebras of dimension higher than the rank (= dimension of Cartan subalgebras), but these would involve non-diagonalizable elements and would not be Cartan subalgebras[23].

We denote the set of nonzero roots as Δ. One may complete the Chevalley generators into a full basis, the so-called *Cartan–Weyl basis*, such that the following commutation relations hold:

$$[H, E_\alpha] = \alpha(H)\, E_\alpha, \tag{6.31}$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\,\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \in \Delta, \\ H_\alpha & \text{if } \alpha+\beta = 0, \\ 0 & \text{if } \alpha+\beta \notin \Delta, \end{cases} \tag{6.32}$$

where H_α is defined by duality thanks to the Killing form $B(X, Y) = \mathrm{Tr}(\mathrm{ad}\, X\ \mathrm{ad}\, Y)$, which is non-singular on semi-simple Lie algebras:

$$\forall H \in \mathfrak{h} : \alpha(H) = B(H_\alpha, H), \tag{6.33}$$

and the generators are normalized according to (see Equation (6.43))

$$B(E_\alpha,\, E_\beta) = \delta_{\alpha+\beta,0}. \tag{6.34}$$

The generators E_α associated with the roots α (where α need not be a simple root) may be chosen such that the structure constants $N_{\alpha,\,\beta}$ satisfy the relations

$$N_{\alpha,\beta} = -N_{\beta,\alpha} = -N_{-\alpha,-\beta} = N_{\beta,-\alpha-\beta}, \tag{6.35}$$

$$N^2_{\alpha,\,\beta} = \frac{1}{2} q(p+1)(\alpha|\alpha), \qquad p,\, q \in \mathbb{N}_0, \tag{6.36}$$

where the scalar product between roots is defined as

$$(\alpha|\beta) = B(H_\alpha, H_\beta). \tag{6.37}$$

The non-negative integers p and q are such that the string of all vectors $\beta + n\,\alpha$ belongs to Δ for $-p \le n \le q$; they also satisfy the equation $p - q = 2(\beta|\alpha)/(\alpha|\alpha)$. A standard result states that for semi-simple Lie algebras

$$(\alpha|\beta) = \sum_{\gamma\in\Delta} (\alpha|\gamma)(\gamma|\beta) \in \mathbb{Q}, \tag{6.38}$$

[22] We say that an object is "unique" when it is unique up to an internal automorphism.

[23] For example, for the split form $E_{8(8)}$ of E_8, the 8 level 3-elements and the 28 level 2-elements form an Abelian subalgebra since there are no elements at levels > 3 (the level is defined in Section 8). This Abelian subalgebra has dimension 36, which is clearly much greater than the rank (8). We thank Bernard Julia for a discussion on this example. Note that for subgroups of the unitary group, diagonalizability is automatic.

from which we notice that the roots are real when evaluated on an H_β-generator,

$$\alpha(H_\beta) = (\alpha|\beta). \tag{6.39}$$

An important consequence of this discussion is that in Equation (6.32), the structure constants of the commutations relations may all be chosen real. Thus, if we restrict ourselves to real scalars we obtain a real Lie algebra $\mathfrak{s}_0$, which is called the *split real form* because it contains the maximal number of noncompact generators. This real form of $\mathfrak{g}$ reads explicitly

$$\mathfrak{s}_0 = \bigoplus_{\alpha\in\Delta} \mathbb{R}H_\alpha \oplus \bigoplus_{\alpha\in\Delta} \mathbb{R}E_\alpha. \tag{6.40}$$

The signature of the Killing form on $\mathfrak{s}_0$ (which is real) is easily computed. First, it is positive definite on the real linear span $\mathfrak{h}_0$ of the H_α generators. Indeed,

$$B(H_\alpha,\, H_\alpha) = (\alpha|\alpha) = \sum_{\gamma\in\Delta} (\alpha|\gamma)^2 > 0. \tag{6.41}$$

Second, the invariance of the Killing form fixes the normalization of the E_α generators to one,

$$B(E_\alpha, E_{-\alpha}) = 1, \tag{6.42}$$

since[24]

$$B([E_\alpha,\, E_{-\alpha}]\,, H_\alpha) = (\alpha|\alpha) = -B(E_{-\alpha}, [E_\alpha,\, H_\alpha]) = (\alpha|\alpha) B(E_\alpha, E_{-\alpha}). \tag{6.43}$$

Moreover, one has $B(\mathbf{g}_\alpha,\, \mathbf{g}_\beta) = 0$ if $\alpha + \beta \neq 0$. Indeed $\mathrm{ad}[\mathbf{g}_\alpha]\ \mathrm{ad}[\mathbf{g}_\beta]$ maps $\mathbf{g}_\mu$ into $\mathbf{g}_{\mu+\alpha+\beta}$, i.e., in matrix terms $\mathrm{ad}[\mathbf{g}_\alpha]\ \mathrm{ad}[\mathbf{g}_\beta]$ has zero elements on the diagonal when $\alpha + \beta \neq 0$. Hence, the vectors $E_\alpha + E_{-\alpha}$ are spacelike and orthogonal to the vectors $E_\alpha - E_{-\alpha}$, which are timelike. This implies that the signature of the Killing form is

$$\left(\frac{1}{2}(\dim \mathfrak{s}_0 + \mathrm{rank}\, \mathfrak{s}_0)\Big|_+ ,\ \frac{1}{2}(\dim \mathfrak{s}_0 - \mathrm{rank}\, \mathfrak{s}_0)\Big|_- \right). \tag{6.44}$$

The split real form $\mathfrak{s}_0$ of $\mathfrak{g}$ is "unique".

On the other hand, it is not difficult to check that the linear span

$$\mathfrak{c}_0 = \bigoplus_{\alpha\in\Delta} \mathbb{R}(i\, H_\alpha) \oplus \bigoplus_{\alpha\in\Delta} \mathbb{R}(E_\alpha - E_{-\alpha}) \oplus \bigoplus_{\alpha\in\Delta} \mathbb{R}\, i\, (E_\alpha + E_{-\alpha}) \tag{6.45}$$

also defines a real Lie algebra. An important property of this real form is that the Killing form is negative definite on it. Its signature is

$$(0|_+, \dim \mathfrak{c}_0|_-). \tag{6.46}$$

This is an immediate consequence of the previous discussion and of the way $\mathfrak{c}_0$ is constructed. Hence, this real Lie algebra is compact[25]. For this reason, $\mathfrak{c}_0$ is called the "compact real form" of $\mathfrak{g}$. It is also "unique".

[24] Quite generally, if X_α is a vector in $\mathbf{g}_\alpha$ and $Y_{-\alpha}$ is a vector in $\mathbf{g}_{-\alpha}$, then one has $[X_\alpha, Y_{-\alpha}] = B(X_\alpha, Y_{-\alpha}) H_\alpha$.

[25] An algebra is said to be compact if its group of internal automorphisms is compact in the topological sense. A classic theorem states that a semi-simple algebra is compact if and only if its Killing form is negative definite.

6.4 Classical decompositions

6.4.1 Real forms and conjugations

The compact and split real Lie algebras constitute the two ends of a string of real forms that can be inferred from a given complex Lie algebra. As announced, this section is devoted to the systematic classification of these various real forms.

If $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$, it defines a conjugation on $\mathfrak{g}$. Indeed we may express any $Z \in \mathfrak{g}$ as $Z = X_0 + i\,Y_0$ with $X_0 \in \mathfrak{g}_0$ and $i\,Y_0 \in i\,\mathfrak{g}_0$, and the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ is given by

$$Z \mapsto \overline{Z} = X_0 - i\,Y_0. \tag{6.47}$$

Using Equation (6.3), it is immediate to verify that this involutive map is antilinear: $\overline{\lambda Z} = \overline{\lambda}\,\overline{Z}$, where $\overline{\lambda}$ is the complex conjugate of the complex number λ.

Conversely, if σ is a conjugation on $\mathfrak{g}$, the set $\mathfrak{g}_\sigma$ of elements of $\mathfrak{g}$ fixed by σ provides a real form of $\mathfrak{g}$. Then σ constitutes the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_\sigma$. Thus, on $\mathfrak{g}$, real forms and conjugations are in one-to-one correspondence. The strategy used to classify and describe the real forms of a given complex simple algebra consists of obtaining all the nonequivalent possible conjugations it admits.

6.4.2 The compact real form aligned with a given real form

Let $\mathfrak{g}_0$ be a real form of the complex semi-simple Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. Consider a compact real form $\mathfrak{c}_0$ of $\mathfrak{g}^{\mathbb{C}}$ and the respective conjugations τ and σ associated with $\mathfrak{c}_0$ and $\mathfrak{g}_0$. It may or it may not be that τ and σ commute. When they do, τ leaves $\mathfrak{g}_0$ invariant,

$$\tau(\mathfrak{g}_0) \subset \mathfrak{g}_0$$

and, similarly, σ leaves $\mathfrak{c}_0$ invariant,

$$\sigma(\mathfrak{c}_0) \subset \mathfrak{c}_0.$$

In that case, one says that the real form $\mathfrak{g}_0$ and the compact real form $\mathfrak{c}_0$ are "aligned".

Alignment is not automatic. For instance, one can always de-align a compact real form by applying an automorphism to it while keeping $\mathfrak{g}_0$ unchanged. However, there is a theorem that states that given a real form $\mathfrak{g}_0$ of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, there is always a compact real form $\mathfrak{c}_0$ associated with it [93, 129]. As this result is central to the classification of real forms, we provide a proof in Appendix B, where we also prove the uniqueness of the Cartan involution.

We shall from now on always consider the compact real form aligned with the real form under study.

6.4.3 Cartan involution and Cartan decomposition

A Cartan involution θ of a real Lie algebra $\mathfrak{g}_0$ is an involutive automorphism such that the symmetric, bilinear form B^θ defined by

$$B^\theta(X,Y) = -B(X,\theta Y) \tag{6.48}$$

is positive definite. If the algebra $\mathfrak{g}_0$ is compact, a Cartan involution is trivially given by the identity.

A Cartan involution θ of the real semi-simple Lie algebra $\mathfrak{g}_0$ yields the direct sum decomposition (called Cartan decomposition)

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \tag{6.49}$$

where $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are the θ-eigenspaces of eigenvalues $+1$ and -1. Explicitly, the decomposition of a Lie algebra element is given by

$$X = \frac{1}{2}(X + \theta[X]) + \frac{1}{2}(X - \theta[X]). \tag{6.50}$$

The eigenspaces obey the commutation relations

$$[\mathfrak{k}_0,\, \mathfrak{k}_0] \subset \mathfrak{k}_0, \qquad [\mathfrak{k}_0,\, \mathfrak{p}_0] \subset \mathfrak{p}_0, \qquad [\mathfrak{p}_0,\, \mathfrak{p}_0] \subset \mathfrak{k}_0, \tag{6.51}$$

from which we deduce that $B(\mathfrak{k}_0, \mathfrak{p}_0) = 0$ because the mappings $\mathrm{ad}[\mathfrak{k}_0]\ \mathrm{ad}[\mathfrak{p}_0]$ map $\mathfrak{p}_0$ on $\mathfrak{k}_0$ and $\mathfrak{k}_0$ on $\mathfrak{p}_0$. Moreover $\theta[\mathfrak{k}_0] = +\mathfrak{k}_0$ and $\theta[\mathfrak{p}_0] = -\mathfrak{p}_0$, and hence $B^\theta(\mathfrak{k}_0,\, \mathfrak{p}_0) = 0$. In addition, since B^θ is positive definite, the Killing form B is negative definite on $\mathfrak{k}_0$ (which is thus a compact subalgebra) but is positive definite on $\mathfrak{p}_0$ (which is not a subalgebra).

Define in $\mathfrak{g}^{\mathbb{C}}$ the algebra $\mathfrak{c}_0$ by

$$\mathfrak{c}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0. \tag{6.52}$$

It is clear that $\mathfrak{c}_0$ is also a real form of $\mathfrak{g}^{\mathbb{C}}$ and is furthermore compact since the Killing form restricted to it is negative definite. The conjugation τ that fixes $\mathfrak{c}_0$ is such that $\tau(X) = X$ $(X \in \mathfrak{k}_0)$, $\tau(iY) = iY$ $(Y \in \mathfrak{p}_0)$ and hence $\tau(Y) = -Y$ $(Y \in \mathfrak{p}_0)$. It leaves $\mathfrak{g}_0$ invariant, which shows that $\mathfrak{c}_0$ is aligned with $\mathfrak{g}_0$. One has

$$\mathfrak{c}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0, \qquad \mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{c}_0, \qquad \mathfrak{p}_0 = \mathfrak{g}_0 \cap i\,\mathfrak{c}_0. \tag{6.53}$$

Conversely, let $\mathfrak{c}_0$ be a compact real form aligned with $\mathfrak{g}_0$ and τ the corresponding conjugation. The restriction θ of τ to $\mathfrak{g}_0$ is a Cartan involution. Indeed, one can decompose $\mathfrak{g}_0$ as in Equation (6.49), with Equation (6.51) holding since θ is an involution of $\mathfrak{g}_0$. Furthermore, one has also Equation (6.53), which shows that $\mathfrak{k}_0$ is compact and that B_θ is positive definite.

This shows, in view of the result invoked above that an aligned compact real form always exists, that any real form possesses a Cartan involution and a Cartan decomposition. If there are two Cartan involutions, θ and θ', defined on a real semi-simple Lie algebra, one can show that they are conjugated by an internal automorphism [93, 129]. It follows that any real semi-simple Lie algebra possesses a "unique" Cartan involution.

On the matrix algebra $\mathrm{ad}[\mathfrak{g}_0]$, the Cartan involution is nothing else than minus the transposition with respect to the scalar product B^θ,

$$\mathrm{ad}\,\theta X = -(\mathrm{ad}\,X)^T. \tag{6.54}$$

Indeed, remembering that the transpose of a linear operator with respect to B^θ is defined by $B^\theta(X, AY) = B^\theta(A^T X, Y)$, one gets

$$\begin{aligned} B^\theta(\mathrm{ad}\,\theta X(Y),\, Z) &= -B([\theta X,\, Y]\,,\, \theta Z) = B(Y,\, [\theta X,\, \theta Z]) \\ &= B(Y,\, \theta\,[X,\, Z]) = -B^\theta(Y,\, \mathrm{ad}\,X(Z)) = -B^\theta((\mathrm{ad}\,X)^T(Y),\, Z). \end{aligned} \tag{6.55}$$

Since B_θ is positive definite, this implies, in particular, that the operator $\mathrm{ad}\,Y$, with $Y \in \mathfrak{p}_0$, is diagonalizable over the real numbers since it is symmetric, $\mathrm{ad}\,Y = (\mathrm{ad}\,Y)^T$.

An important consequence of this [93, 129] is that any real semi-simple Lie algebra can be realized as a real matrix Lie algebra, closed under transposition. One can also show [93, 129] that the Cartan decomposition of the Lie algebra of a semi-simple group can be lifted to the group via a diffeomorphism between $\mathfrak{k}_0 \times \mathfrak{p}_0 \mapsto \mathcal{G} = \mathcal{K} \exp[\mathfrak{p}_0]$, where $\mathcal{K}$ is a closed subgroup with $\mathfrak{k}_0$ as Lie algebra. It is this subgroup that contains all the topology of $\mathcal{G}$.

6.4.4 Restricted roots

Let $\mathfrak{g}_0$ be a real semi-simple Lie algebra. It admits a Cartan involution θ that allows to split it into eigenspaces $\mathfrak{k}_0$ of eigenvalue $+1$ and $\mathfrak{p}_0$ of eigenvalue -1. We may choose in $\mathfrak{p}_0$ a maximal Abelian subalgebra $\mathfrak{a}_0$ (because the dimension of $\mathfrak{p}_0$ is finite). The set $\{\mathrm{ad}\, H | H \in \mathfrak{a}_0\}$ is a set of symmetric transformations that can be simultaneously diagonalized on $\mathbb{R}$. Accordingly we may decompose $\mathfrak{g}_0$ into a direct sum of eigenspaces labelled by elements of the dual space $\mathfrak{a}_0^*$:

$$\mathfrak{g}_0 = \bigoplus_\lambda \mathbf{g}_\lambda, \qquad \mathbf{g}_\lambda = \{X \in \mathfrak{g}_0 | \forall H \in \mathfrak{a}_0 : \mathrm{ad}\, H(X) = \lambda(H)\, X\}. \tag{6.56}$$

One, obviously non-vanishing, subspace is $\mathbf{g}_0$, which contains $\mathfrak{a}_0$. The other nontrivial subspaces define the *restricted root spaces* of $\mathfrak{g}_0$ with respect to $\mathfrak{a}_0$, of the pair $(\mathfrak{g}_0, \mathfrak{a}_0)$. The λ that label these subspaces $\mathbf{g}_\lambda$ are the *restricted roots* and their elements are called *restricted root vectors*. The set of all λ is called the *restricted root system*. By construction the different $\mathbf{g}_\lambda$ are mutually B^θ-orthogonal. The Jacobi identity implies that $[\mathbf{g}_\lambda, \mathbf{g}_\mu] \subset \mathbf{g}_{\lambda+\mu}$, while $\mathfrak{a}_0 \subset \mathfrak{p}_0$ implies that $\theta \mathbf{g}_\lambda = \mathbf{g}_{-\lambda}$. Thus if λ is a restricted root, so is $-\lambda$.

Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}_0$. The space $\mathbf{g}_0$ is given by

$$\mathbf{g}_0 = \mathfrak{a}_0 \oplus \mathfrak{m}. \tag{6.57}$$

If $\mathfrak{t}_0$ is a maximal Abelian subalgebra of $\mathfrak{m}$, the subalgebra $\mathfrak{h}_0 = \mathfrak{a}_0 \oplus \mathfrak{t}_0$ is a Cartan subalgebra of the real algebra $\mathfrak{g}_0$ in the sense that its complexification $\mathfrak{h}^\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}^\mathbb{C}$. Accordingly we may consider the set of nonzero roots Δ of $\mathfrak{g}^\mathbb{C}$ with respect to $\mathfrak{h}^\mathbb{C}$ and write

$$\mathfrak{g}^\mathbb{C} = \mathfrak{h}^\mathbb{C} \bigoplus_{\alpha \in \Delta} (\mathbf{g}_\alpha)^\mathbb{C}. \tag{6.58}$$

The restricted root space $\mathbf{g}_\lambda$ is given by

$$\mathbf{g}_\lambda = \mathfrak{g}_0 \cap \bigoplus_{\substack{\alpha \in \Delta \\ \alpha|_{\mathfrak{a}_0} = \lambda}} (\mathbf{g}_\alpha)^\mathbb{C} \tag{6.59}$$

and similarly

$$\mathfrak{m}^\mathbb{C} = \mathfrak{t}^\mathbb{C} \bigoplus_{\substack{\alpha \in \Delta \\ \alpha|_{\mathfrak{a}_0} = 0}} (\mathbf{g}_\alpha)^\mathbb{C}. \tag{6.60}$$

Note that the multiplicities of the restricted roots λ might be nontrivial even though the roots α are nondegenerate, because distinct roots α might yield the same restricted root when restricted to $\mathfrak{a}_0$.

Let us denote by Σ the subset of nonzero restricted roots and by V_Σ the subspace of $\mathfrak{a}_0^*$ that they span. One can show [93, 129] that Σ is a root system as defined in Section 4. This root system need not be reduced. As for all root systems, one can choose a way to split the roots into positive and negative ones. Let Σ^+ be the set of positive roots and

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathbf{g}_\lambda. \tag{6.61}$$

As Σ^+ is finite, $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}_0$ and $\mathfrak{a}_0 \oplus \mathfrak{n}$ is a solvable subalgebra.

6.4.5 Iwasawa and $\mathcal{KAK}$ decompositions

The Iwasawa decomposition provides a global factorization of any semi-simple Lie group in terms of closed subgroups. It can be viewed as the generalization of the Gram–Schmidt orthogonalization process.

At the level of the Lie algebra, the Iwasawa decomposition theorem states that

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}. \tag{6.62}$$

Indeed any element X of $\mathfrak{g}_0$ can be decomposed as

$$X = X_0 + \sum_{\lambda} X_\lambda = X_0 + \sum_{\lambda\in\Sigma^+} (X_{-\lambda} + \theta X_{-\lambda}) + \sum_{\lambda\in\Sigma^+} (X_\lambda - \theta X_{-\lambda}). \tag{6.63}$$

The first term X_0 belongs to $\mathbf{g}_0 = \mathfrak{a}_0 \oplus \mathfrak{m} \subset \mathfrak{a}_0 \oplus \mathfrak{k}_0$, while the second term belongs to $\mathfrak{k}_0$, the eigenspace subspace of θ-eigenvalue $+1$. The third term belongs to $\mathfrak{n}$ since $\theta X_{-\lambda} \in \mathbf{g}_\lambda$. The sum is furthermore direct. This is because one has obviously $\mathfrak{k}_0 \cap \mathfrak{a}_0 = 0$ as well as $\mathfrak{a}_0 \cap \mathfrak{n} = 0$. Moreover, $\mathfrak{k}_0 \cap \mathfrak{n}$ also vanishes because $\theta\mathfrak{n} \cap \mathfrak{n} = 0$ as a consequence of $\theta\mathfrak{n} = \bigoplus_{\lambda\in\Sigma^+} \mathbf{g}_{-\lambda}$.

The Iwasawa decomposition of the Lie algebra differs from the Cartan decomposition and is tilted with respect to it, in the sense that $\mathfrak{n}$ is neither in $\mathfrak{k}_0$ nor in $\mathfrak{p}_0$. One of its virtues is that it can be elevated from the Lie algebra $\mathfrak{g}_0$ to the semi-simple Lie group $\mathcal{G}$. Indeed, it can be shown [93, 129] that the map

$$(k, a, n) \in \mathcal{K} \times \mathcal{A} \times \mathcal{N} \mapsto k\,a\,n \in \mathcal{G} \tag{6.64}$$

is a global diffeomorphism. Here, the subgroups $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{N}$ have respective Lie algebras $\mathfrak{k}_0$, $\mathfrak{a}_0$, $\mathfrak{n}$. This decomposition is "unique".

There is another useful decomposition of $\mathcal{G}$ in terms of a product of subgroups. Any two generators of $\mathfrak{p}_0$ are conjugate via internal automorphisms of the compact subgroup $\mathcal{K}$. As a consequence writing an element $g \in \mathcal{G}$ as a product $g = k\ \mathrm{Exp}[\mathfrak{p}_0]$, we may write $\mathcal{G} = \mathcal{K}\,\mathcal{A}\,\mathcal{K}$, which constitutes the so-called $\mathcal{KAK}$ decomposition of the group (also sometimes called the Cartan decomposition of the group although it is not the exponention of the Cartan decomposition of the algebra). Here, however, the writing of an element of $\mathcal{G}$ as product of elements of $\mathcal{K}$ and $\mathcal{A}$ is, in general, not unique.

6.4.6 θ-stable Cartan subalgebras

As in the previous sections, $\mathfrak{g}_0$ is a real form of the complex semi-simple algebra $\mathfrak{g}$, σ denotes the conjugation it defines, τ the conjugation that commutes with σ, $\mathfrak{c}_0$ the associated compact aligned real form of $\mathfrak{g}$ and θ the Cartan involution. It is also useful to introduce the involution of $\mathfrak{g}$ given by the product $\sigma\tau$ of the commuting conjugations. We denote it also by θ since it reduces to the Cartan involution when restricted to $\mathfrak{g}_0$. Contrary to the conjugations σ and τ, θ is linear over the complex numbers. Accordingly, if we complexify the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, to

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{6.65}$$

with $\mathfrak{k} = \mathfrak{k}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{k}_0 \oplus i\mathfrak{k}_0$ and $\mathfrak{p} = \mathfrak{p}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{p}_0 \oplus i\mathfrak{p}_0$, the involution θ fixes $\mathfrak{k}$ pointwise while $\theta(X) = -X$ for $X \in \mathfrak{p}$.

Let $\mathfrak{h}_0$ be a θ-stable Cartan subalgebra of $\mathfrak{g}_0$, i.e., a subalgebra such that (i) $\theta(\mathfrak{h}_0) \subset \mathfrak{h}_0$, and (ii) $\mathfrak{h} \equiv \mathfrak{h}_0^{\mathbb{C}}$ is a Cartan subalgebra of the complex algebra $\mathfrak{g}$. One can decompose $\mathfrak{h}_0$ into compact and noncompact parts,

$$\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0, \qquad \mathfrak{t}_0 = \mathfrak{h}_0 \cap \mathfrak{k}_0, \qquad \mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p}_0. \tag{6.66}$$

We have seen that for real Lie algebras, the Cartan subalgebras are not all conjugate to each other; in particular, even though the dimensions of the Cartan subalgebras are all equal to the rank of $\mathfrak{g}$, the dimensions of the compact and noncompact subalgebras depend on the choice of $\mathfrak{h}_0$. For example, for $\mathfrak{sl}(2,\mathbb{R})$, one may take $\mathfrak{h}_0 = \mathbb{R}t$, in which case $\mathfrak{t}_0 = 0$, $\mathfrak{a}_0 = \mathfrak{h}_0$. Or one may take $\mathfrak{h}_0 = \mathbb{R}\tau^y$, in which case $\mathfrak{t}_0 = \mathfrak{h}_0$, $\mathfrak{a}_0 = 0$.

One says that the θ-stable Cartan subalgebra $\mathfrak{h}_0$ is *maximally compact* if the dimension of its compact part $\mathfrak{t}_0$ is as large as possible; and that it is *maximally noncompact* if the dimension of its noncompact part $\mathfrak{a}_0$ is as large as possible. The θ-stable Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ used above to introduce restricted roots, where $\mathfrak{a}_0$ is a maximal Abelian subspace of $\mathfrak{p}_0$ and $\mathfrak{t}_0$ a maximal Abelian subspace of its centralizer $\mathfrak{m}$, is maximally noncompact. If $\mathfrak{m} = 0$, the Lie algebra $\mathfrak{g}_0$ constitutes a split real form of $\mathfrak{g}^{\mathbb{C}}$. The *real rank* of $\mathfrak{g}_0$ is the dimension of its maximally noncompact Cartan subalgebras (which can be shown to be conjugate, as are the maximally compact ones [129]).

6.4.7 Real roots – Compact and non-compact imaginary roots

Consider a general θ-stable Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, which need not be maximally compact or maximally non compact. A consequence of Equation (6.54) is that the matrices of the real linear transformations $\operatorname{ad} H$ are real symmetric for $H \in \mathfrak{a}_0$ and real antisymmetric for $H \in \mathfrak{t}_0$. Accordingly, the eigenvalues of $\operatorname{ad} H$ are real (and $\operatorname{ad} H$ can be diagonalized over the real numbers) when $H \in \mathfrak{a}_0$, while the eigenvalues of $\operatorname{ad} H$ are imaginary (and $\operatorname{ad} H$ cannot be diagonalized over the real numbers although it can be diagonalized over the complex numbers) when $H \in \mathfrak{t}_0$.

Let α be a root of $\mathfrak{g}$, i.e., a non-zero eigenvalue of $\operatorname{ad} \mathfrak{h}$ where $\mathfrak{h}$ is the complexification of the θ-stable Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{h}_0 \oplus i\mathfrak{h}_0$. As the values of the roots acting on a given H are the eigenvalues of $\operatorname{ad} H$, we find that the roots are real on $\mathfrak{a}_0$ and imaginary on $\mathfrak{t}_0$. One says that a root is *real* if it takes real values on $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, i.e., if it vanishes on $\mathfrak{t}_0$. It is *imaginary* if it takes imaginary values on $\mathfrak{h}_0$, i.e., if it vanishes on $\mathfrak{a}_0$, and *complex* otherwise. These notions of "real" and "imaginary" roots should not be confused with the concepts with similar terminology introduced in Section 4 in the context of non-finite-dimensional Kac–Moody algebras.

If $\mathfrak{h}_0$ is a θ-stable Cartan subalgebra, its complexification $\mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{h}_0 \oplus i\mathfrak{h}_0$ is stable under the involutive authormorphism $\theta = \tau\,\sigma$. One can extend the action of θ from $\mathfrak{h}$ to $\mathfrak{h}^*$ by duality. Denoting this transformation by the same symbol θ, one has

$$\forall H \in \mathfrak{h} \text{ and } \forall \alpha \in \mathfrak{h}^*, \qquad \theta(\alpha)(H) = \alpha(\theta^{-1}(H)), \tag{6.67}$$

or, since $\theta^2 = 1$,

$$\theta(\alpha)(H) = \alpha(\theta H). \tag{6.68}$$

Let E_α be a nonzero root vector associated with the root α and consider the vector θE_α. One has

$$[H,\, \theta E_\alpha] = \theta\,[\theta H,\, E_\alpha] = \alpha(\theta H)\,\theta E_\alpha = \theta(\alpha)(H)\,\theta E_\alpha, \tag{6.69}$$

i.e., $\theta(\mathbf{g}_\alpha) = \mathbf{g}_{\theta(\alpha)}$ because the roots are nondegenerate, i.e., all root spaces are one-dimensional.

Consider now an imaginary root α. Then for all $h \in \mathfrak{h}_0$ and $a \in \mathfrak{a}_0$ we have $\alpha(h+a) = \alpha(h)$, while $\theta(\alpha)\,(h+a) = \alpha(\theta(h+a)) = \alpha(h-a) = \alpha(h)$; accordingly $\alpha = \theta(\alpha)$. Moreover, as the roots are nondegenerate, one has $\theta E_\alpha = \pm E_\alpha$. Writing E_α as

$$E_\alpha = X_\alpha + i\,Y_\alpha \qquad \text{with } X_\alpha,\, Y_\alpha \in \mathfrak{g}_0, \tag{6.70}$$

it is easy to check that $\theta E_\alpha = +E_\alpha$ implies that X_α and Y_α belong to $\mathfrak{k}_0$, while both are in $\mathfrak{p}_0$ if $\theta E_\alpha = -E_\alpha$. Accordingly, $\mathbf{g}_\alpha$ is completely contained either in $\mathfrak{k} = \mathfrak{k}_0 \oplus i\,\mathfrak{k}_0$ or in $\mathfrak{p} = \mathfrak{p}_0 \oplus i\,\mathfrak{p}_0$. If $\mathbf{g}_\alpha \subset \mathfrak{k}$, the imaginary root is said to be *compact*, and if $\mathbf{g}_\alpha \subset \mathfrak{p}$ it is said to be *noncompact*.

6.4.8 Jumps between Cartan subalgebras – Cayley transformations

Suppose that β is an imaginary noncompact root. Consider a β-root vector $E_\beta \in \mathbf{g}_\beta \subset \mathfrak{p}$. If this root is expressed according to Equation (6.70), then its conjugate, with respect to (the conjugation σ defined by) $\mathfrak{g}_0$, is

$$\sigma E_\beta = X_\beta - i\,Y_\beta \qquad \text{with } X_\beta,\, Y_\beta \in \mathfrak{p}_0. \tag{6.71}$$

It belongs to $\mathbf{g}_{-\beta}$ because (using $\forall H \in \mathfrak{h}_0 : \sigma H = H$)

$$[H,\, \sigma E_\beta] = \sigma\,[\sigma H,\, E_\beta] = \overline{\beta(\sigma H)}\,\sigma E_\beta = -\beta(H)\,\sigma E_\beta. \tag{6.72}$$

Hereafter, we shall denote σE_β by $\overline{E}_\beta$. The commutator

$$\left[E_\beta,\, \overline{E}_\beta\right] = B(E_\beta,\, \overline{E}_\beta)\, H_\beta \tag{6.73}$$

belongs to $i\,\mathfrak{k}_0$ since $\sigma([E_\beta, \overline{E}_\beta]) = [\overline{E}_\beta, E_\beta] = -[E_\beta, \overline{E}_\beta]$ and can be written, after a renormalization of the generators E_β, as

$$\left[E_\beta,\, \overline{E}_\beta\right] = \frac{2}{(\beta|\beta)}\, H_\beta = H'_\beta \qquad \in i\,\mathfrak{k}_0. \tag{6.74}$$

Indeed as $E_\beta \in \mathfrak{p}$, we have $\overline{E}_\beta \in \mathfrak{p}$ and thus $\theta \overline{E}_\beta = -\overline{E}_\beta$. This implies

$$B(E_\beta\,, \overline{E}_\beta) = -B(E_\beta,\, \theta \overline{E}_\beta) = B^\theta(E_\beta, \overline{E}_\beta) > 0.$$

The three generators $\{H_{\beta'},\, E_\beta,\, \overline{E}_\beta\}$ therefore define an $\mathfrak{sl}(2,\mathbb{C})$ subalgebra:

$$\left[E_\beta,\, \overline{E}_\beta\right] = H_{\beta'}, \qquad [H_{\beta'},\, E_\beta] = 2\, E_\beta, \qquad \left[H_{\beta'},\, \overline{E}_\beta\right] = -2\, \overline{E}_\beta. \tag{6.75}$$

We may change the basis and take

$$h = E_\beta + \overline{E}_\beta, \qquad e = \frac{i}{2}(E_\beta - \overline{E}_\beta - H_{\beta'}), \qquad f = \frac{i}{2}(E_\beta - \overline{E}_\beta + H_{\beta'}), \tag{6.76}$$

whose elements belong to $\mathfrak{g}_0$ (since they are fixed by σ) and satisfy the commutation relations (6.8)

$$[e,\, f] = h, \qquad [h,\, e] = 2\, e, \qquad [h,\, f] = -2\, f. \tag{6.77}$$

The subspace

$$\mathfrak{h}'_0 = \ker(\beta|\mathfrak{h}_0) \oplus \mathbb{R} h \tag{6.78}$$

constitutes a new real Cartan subalgebra whose intersection with $\mathfrak{p}_0$ has one more dimension.

Conversely, if β is a real root then $\theta(\beta) = -\beta$. Let E_β be a root vector. Then $\overline{E}_\beta$ is also in $\mathbf{g}_\beta$ and hence proportional to E_β. By adjusting the phase of E_β, we may assume that E_β belongs to $\mathfrak{g}_0$. At the same time, θE_β, also in $\mathfrak{g}_0$, is an element of $\mathbf{g}_{-\beta}$. Evidently, $B(E_\beta,\, \theta E_\beta) = -B^\theta(E_\beta,\, E_\beta)$ is negative. Introducing $H_{\beta'} = 2/(\beta|\beta) H_\beta$ (which is in $\mathfrak{p}_0$), we obtain the $\mathfrak{sl}(2,\mathbb{R})$ commutation relations

$$[E_\beta,\, \theta E_\beta] = -H_{\beta'} \in \mathfrak{p}_0, \qquad [H_{\beta'},\, E_\beta] = 2\, E_\beta, \qquad [H_{\beta'},\, \theta E_\beta] = -2\, \theta E_\beta. \tag{6.79}$$

Defining the compact generator $E_\beta + \theta E_\beta$, which obviously belongs to $\mathfrak{g}_0$, we may build a new Cartan subalgebra of $\mathfrak{g}_0$:

$$\mathfrak{h}'_0 = \ker(\beta|\mathfrak{h}_0) \oplus \mathbb{R}\,(E_\beta + \theta E_\beta), \tag{6.80}$$

whose noncompact subspace is now one dimension less than previously.

These two kinds of transformations – called *Cayley transformations* – allow, starting from a θ-stable Cartan subalgebra, to transform it into new ones with an increasing number of noncompact dimensions, as long as noncompact imaginary roots remain; or with an increasing number of compact dimensions, as long as real roots remain. Exploring the algebra in this way, we obtain all the Cartan subalgebras up to conjugacy. One can prove that the endpoints are maximally noncompact and maximally compact, respectively.

Theorem: Let $\mathfrak{h}_0$ be a θ stable Cartan subalgebra of $\mathfrak{g}_0$. Then there are no noncompact imaginary roots if and only if $\mathfrak{h}_0$ is maximally noncompact, and there are no real roots if and only if $\mathfrak{h}_0$ is maximally compact [129].

For a proof of this, note that we have already proven that if there are imaginary noncompact (respectively, real) roots, then $\mathfrak{h}_0$ is not maximally noncompact (respectively, compact). The converse is demonstrated in [129].

6.5 Vogan diagrams

Let $\mathfrak{g}_0$ be a real semi-simple Lie algebra, $\mathfrak{g}$ its complexification, θ a Cartan involution leading to the Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \tag{6.81}$$

and $\mathfrak{h}_0$ a Cartan θ-stable subalgebra of $\mathfrak{g}_0$. Using, if necessary, successive Cayley transformations, we may build a maximally compact θ-stable Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, with complexification $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. As usual we denote by Δ the set of (nonzero) roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. This set does not contain any real root, the compact dimension being assumed to be maximal.

From Δ we may define a positive subset Δ^+ by choosing the first set of indices from a basis of $i\,\mathfrak{t}_0$, and then the next set from a basis of $\mathfrak{a}_0$. Since there are no real roots, the roots in Δ^+ have at least one non-vanishing component along $i\,\mathfrak{t}_0$, and the first non-zero one of these components is strictly positive. Since $\theta = +1$ on $\mathfrak{t}_0$, and since there are no real roots: $\theta\Delta^+ = \Delta^+$. Thus θ permutes the simple roots, fixes the imaginary roots and exchanges in 2-tuples the complex roots: it constitutes an involutive automorphism of the Dynkin diagram of $\mathfrak{g}$.

A Vogan diagram is associated to the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ as follows. It corresponds to the standard Dynkin diagram of Δ^+, with additional information: the 2-element orbits under θ are exhibited by joining the correponding simple roots by a double arrow and the 1-element orbit is painted in black (respectively, not painted), if the corresponding imaginary simple root is noncompact (respectively, compact).

6.5.1 Illustration – The $\mathfrak{sl}(5,\mathbb{C})$ case

The complex Lie algebra $\mathfrak{sl}(5,\mathbb{C})$ can be represented as the algebra of traceless 5×5 complex matrices, the Lie bracket being the usual commutator. It has dimension 24. In principle, in order to compute the Killing form, one needs to handle the 24×24 matrices of the adjoint representation. Fortunately, the uniqueness (up to an overall factor) of the bi-invariant quadratic form on a simple Lie algebra leads to the useful relation

$$B(X,Y) = \mathrm{Tr}(\mathrm{ad}\,X\ \mathrm{ad}\,Y) = 10\ \mathrm{Tr}(XY). \tag{6.82}$$

The coefficient 10 appearing in this relation is known as the Coxeter index of $\mathfrak{sl}(5,\mathbb{C})$.

A Cartan–Weyl basis is obtained by taking the 20 nilpotent generators $K^p{}_q$ (with $p \neq q$) corresponding to matrices, all elements of which are zero except the one located at the intersection of row p and column q, which is equal to 1,

$$(K^p{}_q)^\alpha_\beta = \delta^{\alpha\, p}\delta_{\beta\, q} \tag{6.83}$$

and the four diagonal ones,

$$
\begin{aligned}
H_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad
H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
H_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad
H_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},
\end{aligned}
\tag{6.84}
$$

which constitute a Cartan subalgebra $\mathfrak{h}$.

The root space is easily described by introducing the five linear forms ϵ_p, acting on diagonal matrices $d = \mathrm{diag}(d_1, \dots, d_5)$ as follows:

$$\epsilon_p(d) = d_p. \tag{6.85}$$

In terms of these, the dual space $\mathfrak{h}^\star$ of the Cartan subalgebra may be identified with the subspace

$$\left\{ \epsilon = \sum_p A^p \, \epsilon_p \,|\, \sum_p A^p = 0 \right\}. \tag{6.86}$$

The 20 matrices ${K^p}_q$ are root vectors,

$$[H_k, {K^p}_q] = (\epsilon_p[H_k] - \epsilon_q[H_k]){K^p}_q, \tag{6.87}$$

i.e., ${K^p}_q$ is a root vector associated to the root $\epsilon_p - \epsilon_q$.

$\mathfrak{sl}(5,\mathbb{R})$ and $\mathfrak{su}(5)$

By restricting ourselves to real combinations of these generators we obtain the real Lie algebra $\mathfrak{sl}(5,\mathbb{R})$. The conjugation η that it defines on $\mathfrak{sl}(5,\mathbb{C})$ is just the usual complex conjugation. This $\mathfrak{sl}(5,\mathbb{R})$ constitutes the split real form $\mathfrak{s}_0$ of $\mathfrak{sl}(5,\mathbb{C})$. Applying the construction given in Equation (6.45) to the generators of $\mathfrak{sl}(5,\mathbb{R})$, we obtain the set of antihermitian matrices

$$i\,H_k, \qquad {K^p}_q - {K^q}_p, \qquad i({K^p}_q + {K^q}_p) \qquad (p > q), \tag{6.88}$$

defining a basis of the real subalgebra $\mathfrak{su}(5)$. This is the compact real form $\mathfrak{c}_0$ of $\mathfrak{sl}(5,\mathbb{C})$. The conjugation associated to this algebra (denoted by τ) is minus the Hermitian conjugation,

$$\tau(X) = -X^\dagger. \tag{6.89}$$

Since $[\eta, \tau] = 0$, τ induces a Cartan involution θ on $\mathfrak{sl}(5,\mathbb{R})$, providing a Euclidean form on the previous $\mathfrak{sl}(5,\mathbb{R})$ subalgebra

$$B^\theta(X, Y) = 10\ \mathrm{Tr}(XY^t), \tag{6.90}$$

which can be extended to a Hermitian form on $\mathfrak{sl}(5,\mathbb{C})$,

$$B^\theta(X, Y) = 10\ \mathrm{Tr}(XY^\dagger). \tag{6.91}$$

Note that the generators $i\,H_k$ and $i({K^p}_q + {K^q}_p)$ are real generators (although described by complex matrices) since, e.g., $(i\,H_k)^\dagger = -i\,H_k^\dagger$, i.e., $\tau(i\,H_k) = i\,H_k$.

The other real forms

The real forms of $\mathfrak{sl}(5,\mathbb{C})$ that are not isomorphic to $\mathfrak{sl}(5,\mathbb{R})$ or $\mathfrak{su}(5)$ are isomorphic either to $\mathfrak{su}(3,2)$ or $\mathfrak{su}(4,1)$. In terms of matrices these algebras can be represented as

$$\begin{aligned} X &= \begin{pmatrix} A & \Gamma \\ \Gamma^\dagger & B \end{pmatrix} && \text{where } A = -A^\dagger \in \mathbb{C}^{p\times p}, \quad B = -B^\dagger \in \mathbb{C}^{q\times q}, \\ \operatorname{Tr} A + \operatorname{Tr} B &= 0, \quad \Gamma \in \mathbb{C}^{p\times q} && \text{with } p = 3 \text{ (respectively 4) and } q = 2 \text{ (respectively 1).} \end{aligned} \tag{6.92}$$

We shall call these ways of describing $\mathfrak{su}(p,q)$ the "natural" descriptions of $\mathfrak{su}(p,q)$. Introducing the diagonal matrix

$$I_{p,\,q} = \begin{pmatrix} Id^{p\times p} & \\ & -Id^{q\times q} \end{pmatrix}, \tag{6.93}$$

the conjugations defined by these subalgebras are given by:

$$\sigma_{p,\,q}(X) = -I_{p,\,q} X^\dagger I_{p,\,q}. \tag{6.94}$$

Vogan diagrams

The Dynkin diagram of $\mathfrak{sl}(5,\mathbb{C})$ is of A_4 type (see Figure 26).

Figure 26: The A_4 Dynkin diagram.

Let us first consider an $\mathfrak{su}(3,2)$ subalgebra. Diagonal matrices define a Cartan subalgebra whose all elements are compact. Accordingly all associated roots are imaginary. If we define the positive roots using the natural ordering $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4 > \epsilon_5$, the simple roots $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$, $\alpha_4 = \epsilon_4 - \epsilon_5$ are compact but $\alpha_3 = \epsilon_3 - \epsilon_4$ is noncompact. The corresponding Vogan diagram is displayed in Figure 27.

Figure 27: A Vogan diagram associated to $\mathfrak{su}(3,2)$.

However, if instead of the natural order we define positive roots by the rule $\epsilon_1 > \epsilon_2 > \epsilon_4 > \epsilon_5 > \epsilon_3$, the simple positive roots are $\tilde\alpha_1 = \epsilon_1 - \epsilon_2$ and $\tilde\alpha_3 = \epsilon_4 - \epsilon_5$ which are compact, and $\tilde\alpha_2 = \epsilon_2 - \epsilon_4$ and $\tilde\alpha_4 = \epsilon_5 - \epsilon_3$ which are noncompact. The associated Vogan diagram is shown in Figure 28.

Figure 28: Another Vogan diagram associated to $\mathfrak{su}(3,2)$.

Alternatively, the choice of order $\epsilon_1 > \epsilon_5 > \epsilon_3 > \epsilon_4 > \epsilon_2$ leads to the diagram in Figure 29.

There remain seven other possibilities, all describing the same subalgebra $\mathfrak{su}(3,2)$. These are displayed in Figure 30.

In a similar way, we obtain four different Vogan diagrams for $\mathfrak{su}(4,1)$, displayed in Figure 31.

Finally we have two non-isomorphic Vogan diagrams associated with $\mathfrak{su}(5)$ and $\mathfrak{sl}(5,\mathbb{R})$. These are shown in Figure 32.

Figure 29: Yet another Vogan diagram associated to $\mathfrak{su}(3,2)$.

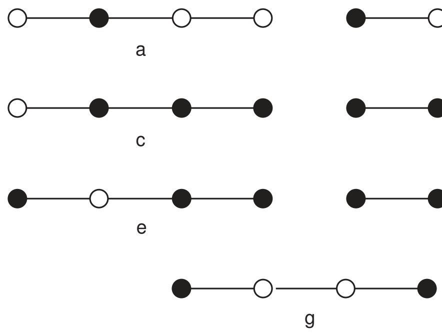


Figure 30: The remaining Vogan diagrams associated to $\mathfrak{su}(3,2)$.

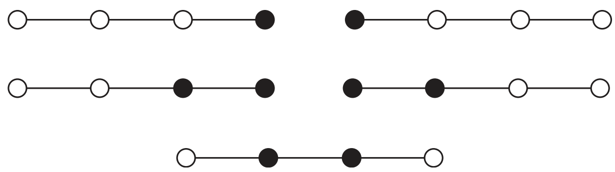


Figure 31: The four Vogan diagrams associated to $\mathfrak{su}(4,1)$.

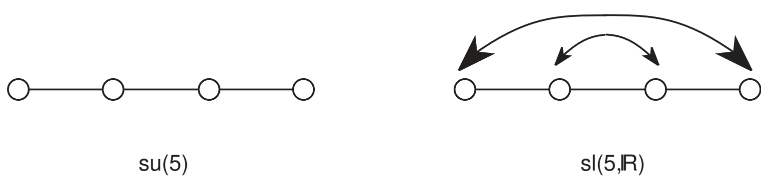


Figure 32: The Vogan diagrams for $\mathfrak{su}(5)$ and $\mathfrak{sl}(5,\mathbb{R})$.

6.5.2 The Borel and de Siebenthal theorem

As we just saw, the same real Lie algebra may yield different Vogan diagrams only by changing the definition of positive roots. But fortunately, a theorem of Borel and de Siebenthal tells us that we may always choose the simple roots such that at most one of them is noncompact [129]. In other words, we may always assume that a Vogan diagram possesses at most one black dot.

Furthermore, assume that the automorphism associated with the Vogan diagram is the identity (no complex roots). Let $\{\alpha_p\}$ be the basis of simple roots and $\{\Lambda_q\}$ its dual basis, i.e., $(\Lambda_q|\alpha_p) = \delta_{p\,q}$. Then the single painted simple root α_p may be chosen so that there is no q with $(\Lambda_p - \Lambda_q|\Lambda_q) > 0$. This remark, which limits the possible simple root that can be painted, is particularly helpful when analyzing the real forms of the exceptional groups. For instance, from the Dynkin diagram of E_8 (see Figure 33), it is easy to compute the dual basis and the matrix of scalar products $B_{p\,q} = (\Lambda_p - \Lambda_q|\Lambda_q)$.

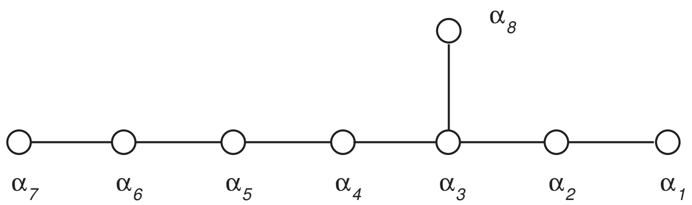


Figure 33: The Dynkin diagram of E_8. Seen as a Vogan diagram, it corresponds to the maximally compact form of E_8.

We obtain

$$(B_{p\,q}) = \begin{pmatrix} -0 & -7 & -20 & -12 & -6 & -2 & -0 & -3 \\ -3 & -0 & -10 & -4 & -0 & -2 & -2 & -2 \\ -6 & -6 & -0 & -4 & -6 & -6 & -4 & -7 \\ -4 & -2 & -6 & -0 & -3 & -4 & -3 & -4 \\ -2 & -2 & -12 & -5 & -0 & -2 & -2 & -1 \\ -0 & -6 & -18 & -10 & -4 & -0 & -1 & -2 \\ -2 & -10 & -24 & -15 & -8 & -3 & -0 & -5 \\ -1 & -4 & -15 & -8 & -3 & -0 & -1 & -0 \end{pmatrix}, \tag{6.95}$$

from which we see that there exist, besides the compact real form, only two other non-isomorphic real forms of E_8, described by the Vogan diagrams in Figure 34[26].

6.5.3 Cayley transformations in $\mathfrak{su}(3,2)$

Let us now illustrate the Cayley transformations. For this purpose, consider again $\mathfrak{su}(3,2)$ with the imaginary diagonal matrices as Cartan subalgebra and the natural ordering of the ϵ_k defining the positive roots. As we have seen, $\alpha_3 = \epsilon_3 - \epsilon_4$ is an imaginary noncompact root. The associated $\mathfrak{sl}(2,\mathbb{C})$ generators are

$$E_{\alpha_3} = K^3_4, \qquad \overline{E_{\alpha_3}} = \sigma K^3_4 = K^4_3, \qquad i\,H_3. \tag{6.96}$$

From the action of α_3 on the Cartan subalgebra $D = \mathrm{span}\{i\,H_k,\ k = 1,\dots,4\}$, we may check that

$$\ker(\alpha_3|D) = \mathrm{span}\{i\,H_1,\ i(2\,H_2 + H_3),\ i(2\,H_4 + H_3)\}, \tag{6.97}$$

[26] In the notation $H_{r(\sigma)}$ for a real form of the simple complex Lie algebra H_r, with the integer σ referring to the signature.

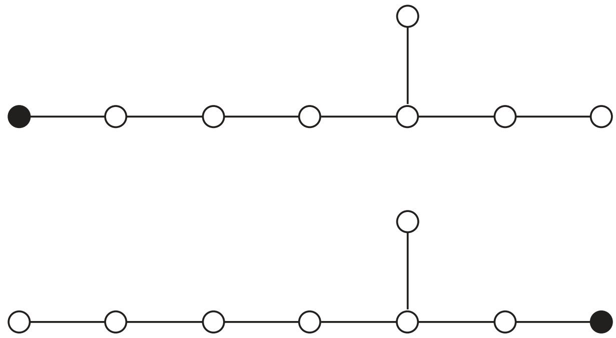


Figure 34: Vogan diagrams of the two different noncompact real forms of E_8: $E_{8(-24)}$ and $E_{8(8)}$. The lower one corresponds to the split real form.

and that $H' = (E_{\alpha_3} + \overline{E_{\alpha_3}}) = (K^3_{\ 4} + K^4_{\ 3})$ is such that $\theta H' = -H'$ and $\sigma H' = H'$. Moreover H' commutes with $\ker(\alpha_3|D)$. Thus

$$C = \ker(\alpha_3|D) \oplus \mathbb{R}\, H' \tag{6.98}$$

constitutes a θ-stable Cartan subalgebra with one noncompact dimension H'. Indeed, we have $B(H', H') = 20$. If we compute the roots with respect to this new Cartan subalgebra, we obtain twelve complex roots (expressed in terms of their components in the basis dual to the one implicitly defined by Equations (6.97) and (6.98),

$$\pm(i, -3i, i, \pm 1), \qquad \pm(0, i, -3i, \pm 1), \qquad \pm(i, i, -i, \pm 1), \tag{6.99}$$

six imaginary roots

$$\pm i(2, -2, 0, 0), \qquad \pm i(1, -2, -2, 0), \qquad \pm i(1, 0, 2, 0), \tag{6.100}$$

and a pair of real roots $\pm(0, 0, 0, 2)$.

Let us first consider the Cayley transformation obtained using, for instance, the real root $(0, 0, 0, 2)$. An associated root vector, belonging to $\mathfrak{g}_0$, reads

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & -\frac{i}{2} & 0 \\ 0 & 0 & \frac{i}{2} & -\frac{i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.101}$$

The corresponding compact Cartan generator is

$$h = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.102}$$

which, together with the three generators in Equation (6.97), provide a compact Cartan subalgebra of $\mathfrak{su}(3, 2)$.

If we consider instead the imaginary roots, we find for instance that $K^5_{\ 2} = -\tilde{\theta} K^5_{\ 2}$ is a noncompact complex root vector corresponding to the root $\beta = i(1, -2, -2, 0)$. It provides the noncompact generator $K^2_{\ 5} + K^5_{\ 2}$ which, together with

$$\ker(\beta|C) = \mathrm{span}\{i(2\, H_1 + 2\, H_2 + H_3),\ i(2\, H_1 + H_3 + 2\, H_4),\ K^3_{\ 4} + K^4_{\ 3}\}, \tag{6.103}$$

generates a maximally noncompact Cartan subalgebra of $\mathfrak{su}(3,2)$. A similar construction can be done using, for instance, the roots $\pm i(1,0,2,0)$, but not with the roots $\pm i(2,-2,0,0)$ as their corresponding root vectors K^1_2 and K^2_1 are fixed by $\tilde{\theta}$ and thus are compact.

6.5.4 Reconstruction

We have seen that every real Lie algebra leads to a Vogan diagram. Conversely, every Vogan diagram defines a real Lie algebra. We shall sketch the reconstruction of the real Lie algebras from the Vogan diagrams here, referring the reader to [129] for more detailed information.

Given a Vogan diagram, the reconstruction of the associated real Lie algebra proceeds as follows. From the diagram, which is a Dynkin diagram with extra information, we may first construct the associated complex Lie algebra, select one of its Cartan subalgebras and build the corresponding root system. Then we may define a compact real subalgebra according to Equation (6.45).

We know the action of θ on the simple roots. This implies that the set Δ of all roots is invariant under θ. This is proven inductively on the level of the roots, starting from the simple roots (level 1). Suppose we have proven that the image under θ of all the positive roots, up to level n are in Δ. If γ is a root of level $n+1$, choose a simple root α such that $(\gamma|\alpha) > 0$. Then the Weyl transformed root $s_\alpha \gamma = \beta$ is also a positive root, but of smaller level. Since $\theta(\alpha)$ and $\theta(\beta)$ are then known to be in Δ, and since the involution acts as an isometry, $\theta(\gamma) = s_{\theta(\alpha)}(\theta(\beta))$ is also in Δ.

One can transfer by duality the action of θ on $\mathfrak{h}^*$ to the Cartan subalgebra $\mathfrak{h}$, and then define its action on the root vectors associated to the simple roots according to the rules

$$\theta E_\alpha = \begin{cases} E_\alpha & \text{if } \alpha \text{ is unpainted and invariant,} \\ -E_\alpha & \text{if } \alpha \text{ is painted and invariant,} \\ -E_{\theta[\alpha]} & \text{if } \alpha \text{ belongs to a 2-cycle.} \end{cases} \tag{6.104}$$

These rules extend θ to an involution of $\mathfrak{g}$.

This involution is such that $\theta E_\alpha = a_\alpha E_{\theta[\alpha]}$, with $a_\alpha = \pm 1$ [27]. Furthermore it globally fixes $\mathfrak{c}_0$, $\theta\mathfrak{c}_0 = \mathfrak{c}_0$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ or -1 eigenspaces of θ in $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Define $\mathfrak{k}_0 = \mathfrak{c}_0 \cap \mathfrak{k}$ and $\mathfrak{p}_0 = i(\mathfrak{c}_0 \cap \mathfrak{p})$ so that $\mathfrak{c}_0 = \mathfrak{k}_0 \oplus i\,\mathfrak{p}_0$. Set

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0. \tag{6.105}$$

Using $\theta\mathfrak{c}_0 = \mathfrak{c}_0$, one then verifies that $\mathfrak{g}_0$ constitutes the desired real form of $\mathfrak{g}$ [129].

6.5.5 Illustrations: $\mathfrak{sl}(4,\mathbb{R})$ versus $\mathfrak{sl}(2,\mathbb{H})$

We shall exemplify the reconstruction of real algebras from Vogan diagrams by considering two examples of real forms of $\mathfrak{sl}(4,\mathbb{C})$. The diagrams are shown in Figure 35.

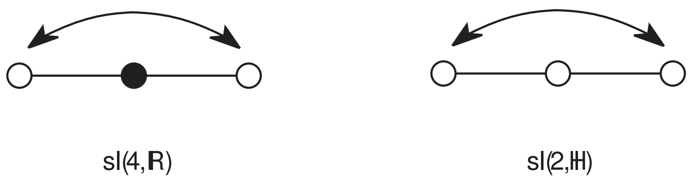


Figure 35: The Vogan diagrams associated to a $\mathfrak{sl}(4\mathbb{R})$ and $\mathfrak{sl}(2\mathbb{H})$ subalgebra.

[27] The coefficients a_α are determined from the commutation relations as follows: $N_{\alpha,\beta} a_{\alpha+\beta} = N_{\theta[\alpha],\theta[\beta]} a_\alpha a_\beta$. Moreover, because θ is an automorphism of the root lattice we have $N^2_{\alpha,\beta} = N^2_{\theta[\alpha],\theta[\beta]}$, and so if a_α and a_β are equal to ± 1, then so is $a_{\alpha+\beta}$. But since this is true for the simple roots it remains true for all roots.

The θ involutions they describe are (the upper signs correspond to the left-hand side diagram, the lower signs to the right-hand side diagram):

$$\begin{aligned} \theta H_{\alpha_1} &= H_{\alpha_3}, & H_{\alpha_2} &= H_{\alpha_2}, & \theta H_{\alpha_3} &= H_{\alpha_1} \\ \theta E_{\alpha_1} &= E_{\alpha_3}, & \theta E_{\alpha_2} &= \mp E_{\alpha_2}, & \theta E_{\alpha_3} &= E_{\alpha_1}. \end{aligned} \tag{6.106}$$

Using the commutations relations

$$\begin{aligned} [E_{\alpha_1}, E_{\alpha_2}] &= E_{\alpha_1+\alpha_2}, \\ [E_{\alpha_2}, E_{\alpha_3}] &= E_{\alpha_2+\alpha_3}, \\ [E_{\alpha_1+\alpha_2}, E_{\alpha_3}] &= E_{\alpha_1+\alpha_2+\alpha_3} = [E_{\alpha_1}, E_{\alpha_2+\alpha_3}] \end{aligned} \tag{6.107}$$

we obtain

$$\theta E_{\alpha_1+\alpha_2} = \pm E_{\alpha_2+\alpha_3}, \qquad \theta E_{\alpha_2+\alpha_3} = \pm E_{\alpha_1+\alpha_2}, \qquad \theta E_{\alpha_1+\alpha_2+\alpha_3} = \mp E_{\alpha_1+\alpha_2+\alpha_3}. \tag{6.108}$$

Let us consider the left-hand side diagram. The corresponding $+1$ θ-eigenspace $\mathfrak{k}$ has the following realisation,

$$\mathfrak{k} = \mathrm{span}\left\{H_{\alpha_1} + H_{\alpha_3}, H_{\alpha_2}, E_{\alpha_1} + E_{\alpha_3}, E_{-\alpha_1} + E_{-\alpha_3}, E_{\alpha_1+\alpha_2} + E_{\alpha_2+\alpha 3}, E_{-\alpha_1-\alpha_2} + E_{-\alpha_2-\alpha 3}\right\}, \tag{6.109}$$

and the -1 θ-eigenspace $\mathfrak{p}$ is given by

$$\begin{aligned} \mathfrak{p} = \mathrm{span}\{&H_{\alpha_1} - H_{\alpha_3}, E_{\pm\alpha_2}, E_{\alpha_1} - E_{\alpha_3}, E_{-\alpha_1} - E_{-\alpha_3}, \\ &E_{\alpha_1+\alpha_2} - E_{\alpha_2+\alpha_3}, E_{-\alpha_1-\alpha_2} - E_{-\alpha_2-\alpha_3}, E_{\pm(\alpha_1+\alpha_2+\alpha_3)}\}. \end{aligned} \tag{6.110}$$

The intersection $\mathfrak{c}_0 \cap \mathfrak{k}$ then leads to the $\mathfrak{so}(4,\mathbb{R}) = \mathfrak{so}(3,\mathbb{R}) \oplus \mathfrak{so}(3,\mathbb{R})$ algebra

$$\begin{aligned} \mathfrak{k}_0 = &\mathrm{span}\{i(H_{\alpha_1} + H_{\alpha_3}), (E_{\alpha_1} + E_{\alpha_3} - E_{-\alpha_1} - E_{-\alpha_3}), i(E_{\alpha_1} + E_{\alpha_3} + E_{-\alpha_1} + E_{-\alpha_3})\} \\ &\oplus \mathrm{span}\left\{i(H_{\alpha_1} + 2\,H_{\alpha_2} + H_{\alpha_3}), (E_{\alpha_1+\alpha_2} + E_{\alpha_2+\alpha_3} - E_{-(\alpha_1+\alpha_2)} - E_{-(\alpha_2+\alpha_3)}),\right. \\ &\qquad \left. i(E_{\alpha_1+\alpha_2} + E_{\alpha_2+\alpha_3} + E_{-(\alpha_1+\alpha_2)} + E_{-(\alpha_2+\alpha_3)})\right\}, \end{aligned} \tag{6.111}$$

and the remaining noncompact generator subspace $\mathfrak{p}_0 = i(\mathfrak{c}_0 \cap \mathfrak{p})$ becomes

$$\begin{aligned} \mathfrak{p}_0 = \mathrm{span}\{&H_{\alpha_1} - H_{\alpha_3}, (E_{\alpha_1} - E_{\alpha_3} + E_{-\alpha_1} - E_{-\alpha_3}), i(E_{\alpha_1} - E_{\alpha_3} - E_{-\alpha_1} + E_{-\alpha_3}), \\ &(E_{\alpha_1+\alpha_2} - E_{\alpha_2+\alpha_3} + E_{-(\alpha_1+\alpha_2)} - E_{-(\alpha_2+\alpha_3)}), \\ &i(E_{\alpha_1+\alpha_2} - E_{\alpha_2+\alpha_3} - E_{-(\alpha_1+\alpha_2)} + E_{-(\alpha_2+\alpha_3)}), E_{\alpha_2} + E_{-\alpha_2}, i(E_{\alpha_2} - E_{-\alpha_2}), \\ &E_{\alpha_1+\alpha_2+\alpha_3} + E_{-(\alpha_1+\alpha_2+\alpha_3)}, i(E_{\alpha_1+\alpha_2+\alpha_3} - E_{-(\alpha_1+\alpha_2+\alpha_3)})\}. \end{aligned} \tag{6.112}$$

Doing the same exercise for the second diagram, we obtain the real algebra $\mathfrak{sl}(2,\mathbb{H})$ with $\mathfrak{k}_0 = \mathfrak{so}(5,\mathbb{R}) = \mathfrak{sp}(4,\mathbb{R})$, which is a 10-parameter compact subalgebra, and $\mathfrak{p}_0$ given by

$$\begin{aligned} \mathfrak{p}_0 = \mathrm{span}\{&H_{\alpha_1} - H_{\alpha_3}, (E_{\alpha_1} - E_{\alpha_3} + E_{-\alpha_1} - E_{-\alpha_3}), i(E_{\alpha_1} - E_{\alpha_3} - E_{-\alpha_1} + E_{-\alpha_3}), \\ &(E_{\alpha_1+\alpha_2} + E_{\alpha_2+\alpha_3} + E_{-(\alpha_1+\alpha_2)} + E_{-(\alpha_2+\alpha_3)}), \\ &i(E_{\alpha_1+\alpha_2} + E_{\alpha_2+\alpha_3} - E_{-(\alpha_1+\alpha_2)} - E_{-(\alpha_2+\alpha_3)})\}. \end{aligned} \tag{6.113}$$

6.5.6 A pictorial summary – All real simple Lie algebras (Vogan diagrams)

The following tables provide all real simple Lie algebras and the corresponding Vogan diagrams. The restrictions imposed on some of the Lie algebra parameters eliminate the consideration of isomorphic algebras. See [129] for the derivation.

Table 16: Vogan diagrams (A_n series)

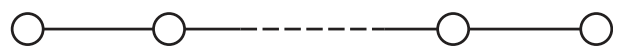
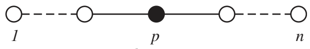
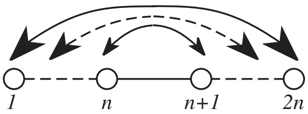
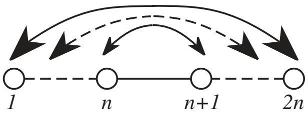
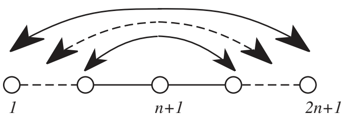
A_n series, $n \geq 1$	Vogan diagram	Maximal compact subalgebra
$\mathfrak{su}(n+1)$	 No painted root	$\mathfrak{su}(n+1)$
$\mathfrak{su}(p,q)$	 Only the p^{th} root is painted	$\mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus u(1)$
$\mathfrak{sl}(2n,\mathbb{R})$	 Odd number of roots	$\mathfrak{so}(2n)$
$\mathfrak{sl}(2n+1,\mathbb{R})$	 Even number of (unpainted) roots	$\mathfrak{so}(2n+1)$
$\mathfrak{sl}(n+1,\mathbb{H})$	 Odd number of (unpainted) roots	$\mathfrak{sp}(n+1)$

Table 17: Vogan diagrams (B_n series)

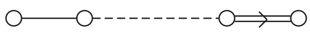
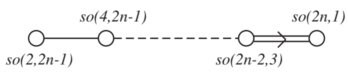
B_n series, $n \geq 2$	Vogan diagram	Maximal compact subalgebra
$\mathfrak{so}(2n+1)$	 No painted root	$\mathfrak{so}(2n+1)$
$\mathfrak{so}(p,q)$ $p \leq n - \frac{1}{2}$, $q = 2n+1-p$	 Each of the roots, once painted, leads to the algebra mentioned under it.	$\mathfrak{so}(p) \oplus \mathfrak{so}(q)$

Table 18: Vogan diagrams (C_n series)

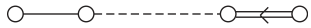
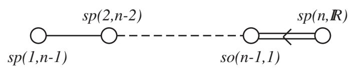
C_n series, $n \geq 3$	Vogan diagram	Maximal compact subalgebra
$\mathfrak{sp}(n)$	 No painted root,	$\mathfrak{sp}(n)$
$\mathfrak{sp}(p, q)$ $0 < p \leq \frac{n}{2},\ q = n - p$ $\mathfrak{sp}(n, \mathbb{R})$	 Each of the roots, once painted, corresponds to the algebra mentioned near it.	$\mathfrak{sp}(p) \oplus \mathfrak{sp}(q)$ $\mathfrak{u}(n)$

Table 19: Vogan diagrams (D_n series)

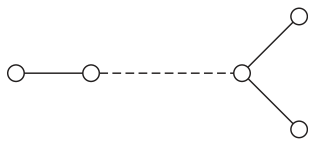
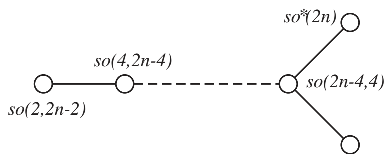
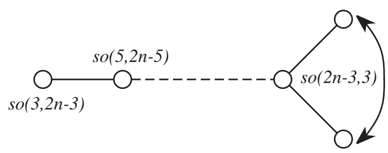
D_n series, $n \geq 4$	Vogan diagram	Maximal compact subalgebra
$\mathfrak{so}(2n)$	 No painted root	$\mathfrak{so}(2n)$
$\mathfrak{so}(2p, 2q)$ $0 < p \leq \frac{n}{2}, q = n - p$ $\mathfrak{so}^*(2n)$	 Each of the roots, once painted, corresponds to the algebra mentioned near it.	$\mathfrak{so}(2p) \oplus \mathfrak{so}(2q)$ $\mathfrak{u}(n)$
$\mathfrak{so}(2p+1, 2q+1)$ $0 < p \leq \frac{n-1}{2}$, $q = n - p - 1$	 Each of the roots, once painted, corresponds to the algebra mentioned near it. No root painted corresponds to $\mathfrak{so}(1, 2n-1)$.	$\mathfrak{so}(2p+1) \oplus \mathfrak{so}(2q+1)$

Table 20: Vogan diagrams (G_2 series)

G_2	Vogan diagram	Maximal compact subalgebra
G_2	 No painted root, providing the real compact form	G_2
$G_{2(2)}$		$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

Table 21: Vogan diagrams (F_4 series)

F_4 series	Vogan diagram	Maximal compact subalgebra
F_4	 No painted root, providing the real compact form	F_4
$F_{4(4)}$		$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$
$F_{4(-20)}$		$\mathfrak{so}(9)$

Table 22: Vogan diagrams (E_6 series)

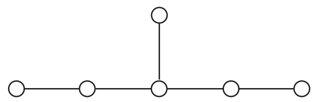
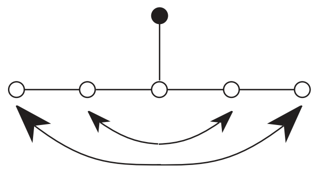
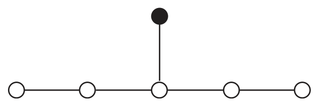
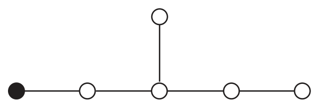
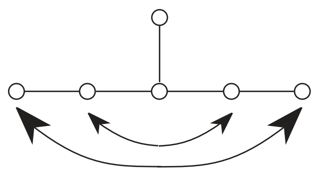
E_6	Vogan diagram	Maximal compact subalgebra
E_6	 No painted root, providing the real compact form	E_6
$E_{6(6)}$		$\mathfrak{sp}(4)$
$E_{6(2)}$		$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$
$E_{6(-14)}$		$\mathfrak{su}(10) \oplus \mathfrak{u}(1)$
$E_{6(-26)}$		F_4

Table 23: Vogan diagrams (E_7 series)

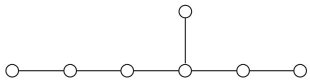
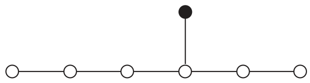
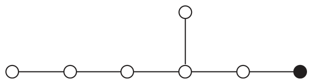
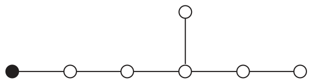
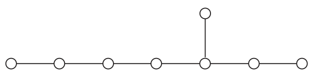
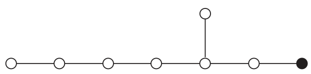
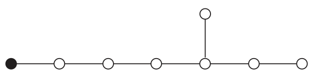
E_7	Vogan diagram	Maximal compact subalgebra
E_7	 No painted root, providing the real compact form	E_7
$E_{7(7)}$		$\mathfrak{su}(8)$
$E_{7(43)}$		$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$
$E_{7(-25)}$		$E_6 \oplus \mathfrak{u}(1)$

Table 24: Vogan diagrams (E_8 series)

E_8	Vogan diagram	Maximal compact subalgebra
E_8	 No painted root, providing the real compact form	E_8
$E_{8(8)}$		$\mathfrak{so}(16)$
$E_{8(-24)}$		$E_7 \oplus \mathfrak{su}(2)$

Using these diagrams, the matrix $I_{p,q}$ defined by Equation (6.93), and the three matrices

$$J_n = \begin{pmatrix} 0 & Id^{n\times n} \\ -Id^{n\times n} & 0 \end{pmatrix}, \tag{6.114}$$

$$K_{p,q} = \begin{pmatrix} Id^{p\times p} & 0 & 0 & 0 \\ 0 & -Id^{q\times q} & 0 & 0 \\ 0 & 0 & Id^{p\times p} & 0 \\ 0 & 0 & 0 & -Id^{q\times q} \end{pmatrix}, \tag{6.115}$$

$$L_{p,q} = K_{p,q}\, J_{p+q} = \begin{pmatrix} 0 & 0 & Id^{p\times p} & 0 \\ 0 & 0 & 0 & -Id^{q\times q} \\ -Id^{p\times p} & 0 & 0 & 0 \\ 0 & Id^{q\times q} & 0 & 0 \end{pmatrix}, \tag{6.116}$$

we may check that the involutive automorphisms of the classical Lie algebras are all conjugate to one of the types listed in Table 25.

For completeness we remind the reader of the definitions of matrix algebras ($\mathfrak{su}(p,\,q)$ has been defined in Equation (6.93)):

$$\begin{aligned} \mathfrak{su}^*(2n) &= \left\{X \,\middle|\, XJ_n - J_n\overline{X} = 0,\ \mathrm{Tr}\,X = 0,\ X \in \mathbb{C}^{2n\times 2n}\right\} \\ &= \left\{\begin{pmatrix} A & C \\ -\overline{C} & \overline{A} \end{pmatrix} \,\middle|\, \begin{matrix} A,\, C \in \mathbb{C}^{n\times n} \\ \mathrm{Re}[\mathrm{Tr}\,A] = 0 \end{matrix}\right\}, \end{aligned} \tag{6.117}$$

$$\begin{aligned} \mathfrak{so}(p,\,q) &= \left\{X \,\middle|\, XI_{p,q} + I_{p,q}X^t = 0,\ X = -X^t,\ X \in \mathbb{R}^{(p+q)\times(p+q)}\right\} \\ &= \left\{\begin{pmatrix} A & C \\ C^t & B \end{pmatrix} \,\middle|\, \begin{matrix} A = -A^t \in \mathbb{R}^{p\times p},\ B = -B^t \in \mathbb{R}^{q\times q}, \\ C \in \mathbb{R}^{p\times q} \end{matrix}\right\}, \end{aligned} \tag{6.118}$$

$$\begin{aligned} \mathfrak{so}^*(2n) &= \left\{X \,\middle|\, X^tJ_n + J_n\overline{X} = 0,\ X = -X^t,\ X \in \mathbb{C}^{2n\times 2n}\right\} \\ &= \left\{\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \,\middle|\, A = -A^t,\ B = B^\dagger \in \mathbb{C}^{n\times n}\right\}, \end{aligned} \tag{6.119}$$

$$\begin{aligned} \mathfrak{sp}(n,\,\mathbb{R}) &= \left\{X \,\middle|\, X^tJ_n + J_nX = 0,\ \mathrm{Tr}\,X = 0,\ X \in \mathbb{R}^{2n\times 2n}\right\} \\ &= \left\{\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \,\middle|\, A,\ B = B^t,\ C = C^t \in \mathbb{R}^{n\times n}\right\}, \end{aligned} \tag{6.120}$$

$$\begin{aligned} \mathfrak{sp}(n,\,\mathbb{C})) &= \left\{X \,\middle|\, X^tJ_n + J_nX = 0,\ \mathrm{Tr}\,X = 0,\ X \in \mathbb{C}^{2n\times 2n}\right\} \\ &= \left\{\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \,\middle|\, A,\ B = B^t,\ C = C^t \in \mathbb{C}^{n\times n}\right\}, \end{aligned} \tag{6.121}$$

$$\begin{aligned} \mathfrak{sp}(p,\,q) &= \left\{X \,\middle|\, X^tK_{p,q} + K_{p,q}\overline{X} = 0,\ \mathrm{Tr}\,X = 0,\ X \in \mathbb{C}^{(p+q)\times(p+q)}\right\} \\ &= \left\{\begin{pmatrix} A & P & Q & R \\ P^\dagger & B & R^t & S \\ -\overline{Q} & \overline{R} & \overline{A} & -\overline{P} \\ R^\dagger & -\overline{S} & -P^t & \overline{B} \end{pmatrix} \,\middle|\, \begin{matrix} A,\, Q \in \mathbb{C}^{p\times p} \\ P,\, R \in \mathbb{C}^{p\times q},\ S \in \mathbb{C}^{q\times p} \\ A = -A^\dagger,\ B = -B^\dagger \\ Q = Q^t,\ S = S^t \end{matrix}\right\}, \end{aligned} \tag{6.122}$$

$$\mathfrak{usp}(2p,\,2q) = \mathfrak{su}(2p,\,2q) \cap \mathfrak{sp}(2p + 2q). \tag{6.123}$$

Table 25: List of all involutive automorphisms (up to conjugation) of the classical compact real Lie algebras [93]. The first column gives the complexification $\mathfrak{u}_0^{\mathbb{C}}$ of the compact real algebra $\mathfrak{u}_0$, the second $\mathfrak{u}_0$, the third the involution τ that $\mathfrak{u}_0$ defines in $\mathfrak{u}^{\mathbb{C}}$, and the fourth a non-compact real subalgebra $\mathfrak{g}_0$ of $\mathfrak{u}^{\mathbb{C}}$ aligned with the compact one. In the second table, the second column displays the involution that $\mathfrak{g}_0$ defines on $\mathfrak{u}^{\mathbb{C}}$, the third the involutive automorphism of $\mathfrak{u}_0$, i.e, the Cartan conjugation $\theta = \sigma\tau$, and the last column indicates the common compact subalgebra $\mathfrak{k}_0$ of $\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\,\mathfrak{p}_0$ and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$.

$\mathfrak{u}^{\mathbb{C}}$	$\mathfrak{u}_0$	τ	$\mathfrak{g}_0$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{su}(n)$	$-X^\dagger$	$A\,I\ \mathfrak{sl}(n,\mathbb{R})$
$\mathfrak{sl}(2n,\mathbb{C})$	$\mathfrak{su}(2n)$	$-X^\dagger$	$A\,II\ \mathfrak{su}^*(2n)$
$\mathfrak{sl}(p+q,\mathbb{C})$	$\mathfrak{su}(p+q)$	$-X^\dagger$	$A\,III\ \mathfrak{su}(p,q)$
$\mathfrak{so}(p+q,\mathbb{C})$	$\mathfrak{so}(p+q,\mathbb{R})$	$\overline{X}$	$B\,I,\ D\,I\ \mathfrak{so}(p,q)$
$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}(2n,\mathbb{R})$	$\overline{X}$	$D\,III\ \mathfrak{so}^*(2n)$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{usp}(n)$	$-J_n\overline{X}J_n$	$C\,I\ \mathfrak{sp}(n,\mathbb{R})$
$\mathfrak{sp}(p+q,\mathbb{C})$	$\mathfrak{usp}(p+q)$	$-J_{p+q}\overline{X}J_{p+q}$	$C\,III\ \mathfrak{sp}(p,q)$

$\mathfrak{u}^{\mathbb{C}}$	σ	θ	$\mathfrak{k}_0$
$\mathfrak{sl}(n,\mathbb{C})$	$\overline{X}$	$-X^t$	$\mathfrak{so}(n,\mathbb{R})$
$\mathfrak{sl}(2n,\mathbb{C})$	$-J_n\overline{X}J_n$	$J_nX^tJ_n$	$\mathfrak{usp}(2n)$
$\mathfrak{sl}(p+q,\mathbb{C})$	$-I_{p,q}X^\dagger I_{p,q}$	$I_{p,q}XI_{p,q}$	$\mathfrak{so}(n,\mathbb{R})$
$\mathfrak{so}(p+q,\mathbb{C})$	$I_{p,q}\overline{X}I_{p,q}$	$I_{p,q}XI_{p,q}$	$\mathfrak{so}(p,\mathbb{R})\oplus\mathfrak{so}(q,\mathbb{R})$
$\mathfrak{so}(2n,\mathbb{C})$	$-J_n\overline{X}J_n$	$-J_nXJ_n$	$\mathfrak{su}(n)\oplus\mathfrak{u}(1)$
$\mathfrak{sp}(n,\mathbb{C})$	$\overline{X}$	$-J_nXJ_n$	$\mathfrak{su}(n)\oplus\mathfrak{u}(1)$
$\mathfrak{sp}(p+q,\mathbb{C})$	$-K_{p,q}X^\dagger K_{p,q}$	$L_{p,q}X^tL_{p,q}$	$\mathfrak{sp}(p)\oplus\mathfrak{sp}(q)$

Alternative definitions are:

$$\begin{aligned}
\mathfrak{sp}(p,q) &= \{X \in \mathfrak{gl}(p+q,\mathbb{H}) | \overline{X}\, I_{p,q} + I_{p,q}\, X = 0\},\\
\mathfrak{sp}(n,\mathbb{R}) &= \{X \in \mathfrak{gl}(2\,n,\mathbb{R}) | X^t\, J_n + J_n\, X = 0\},\\
\mathfrak{sl}(n,\mathbb{H}) &= \{X \in \mathfrak{gl}(n,\mathbb{H}) | \overline{X} + X = 0\},\\
\mathfrak{so}^*(2\,n) &= \{X \in \mathfrak{su}(n,n) | X^t\, K_n + K_n\, X = 0\}.
\end{aligned} \tag{6.124}$$

For small dimensions we have the following isomorphisms:

$$\begin{aligned}
&\mathfrak{so}(1,2) \simeq \mathfrak{su}(1,1) \simeq \mathfrak{sp}(1,\mathbb{R}) \simeq \mathfrak{sl}(2,\mathbb{R}),\\
&\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{so}(1,3),\\
&\mathfrak{so}^*(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(1,1),\\
&\mathfrak{so}^*(6) \simeq \mathfrak{su}(3,1),\\
&\mathfrak{sp}(1,1) \simeq \mathfrak{so}(1,4),\\
&\mathfrak{sl}(2,\mathbb{H}) \simeq \mathfrak{so}(1,5),\\
&\mathfrak{su}(2,2) \simeq \mathfrak{so}(2,4),\\
&\mathfrak{sl}(4,\mathbb{R}) \simeq \mathfrak{so}(3,3),\\
&\mathfrak{so}^*(8) \simeq \mathfrak{so}(2,6).
\end{aligned} \tag{6.125}$$

6.6 Tits–Satake diagrams

The classification of real forms of a semi-simple Lie algebra, using Vogan diagrams, rests on the construction of a maximally compact Cartan subalgebra. On the other hand, the Iwasawa decomposition emphasizes the role of a maximally noncompact Cartan subalgebra. The consideration of these Cartan subalgebras leads to another way to classify real forms of semi-simple Lie algebras, developed mainly by Araki [5], and based on so-called Tits–Satake diagrams [161, 155].

6.6.1 Example 1: $\mathfrak{su}(3,2)$

Diagonal description

At the end of Section 6.5.3, we obtained a matrix representation of a maximally noncompact Cartan subalgebra of $\mathfrak{su}(3,2)$ in terms of the natural description of this algebra. To facilitate the forthcoming discussion, we find it useful to use an equivalent description, in which the matrices representing this Cartan subalgebra are diagonal, as this subalgebra will now play a central role. It is obtained by performing a similarity transformation $X \mapsto S^T X\, S$, where

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}}\\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{6.126}$$

In this new "diagonal" description, the conjugation σ (see Equation (6.94)) becomes

$$\sigma(X) = -\tilde{I}_{3,2} X^\dagger \tilde{I}_{3,2}, \tag{6.127}$$

where

$$\tilde{I}_{3,2} = S^T I_{3,2} S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{6.128}$$

The Cartan involution has the following realisation:

$$\theta(X) = \tilde{I}_{3,2} X \, \tilde{I}_{3,2}. \tag{6.129}$$

In terms of the four matrices introduced in Equation (6.84), the generators defining this Cartan subalgebra $\mathfrak{h}$ reads

$$\begin{aligned} h_1 &= H_3, & h_2 &= H_2 + H_3 + H_4, \\ h_3 &= i(2\,H_1 + 2\,H_2 + H_3), & h_4 &= i(2\,H_1 + H_2 + H_3 + H_4). \end{aligned} \tag{6.130}$$

Let us emphasize that we have numbered the basis generators of $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ by first choosing those in $\mathfrak{a}$, then those in $\mathfrak{t}$.

Cartan involution and roots

The standard matrix representation of $\mathfrak{su}(5)$ constitutes a compact real Lie subalgebra of $\mathfrak{sl}(5, \mathbb{C})$ aligned with the diagonal description of the real form $\mathfrak{su}(3,2)$. Moreover, its Cartan subalgebra $\mathfrak{h}_0$ generated by purely imaginary combinations of the four diagonal matrices H_k is such that its complexification $\mathfrak{h}^{\mathbb{C}}$ contains $\mathfrak{h}$. Accordingly, the roots it defines act both on $\mathfrak{h}_0$ and $\mathfrak{h}$. Note that on $\mathfrak{h}_{\mathbb{R}} = i\,\mathfrak{h}_0$, the roots take only real values.

Our first task is to compute the action of the Cartan involution θ on the root lattice. With this aim in view, we introduce two distinct bases on $\mathfrak{h}^*_{\mathbb{R}}$. The first one is $\{F^1, F^2, F^3, F^4\}$, which is dual to the basis $\{H_1, H_2, H_3, H_4\}$ and is adapted to the relation $\mathfrak{h}_{\mathbb{R}} = i\,\mathfrak{h}_0$. The second one is $\{f^1, f^2, f^3, f^4\}$, dual to the basis $\{h_1,\, h_2,\, -ih_3,\, -ih_4\}$, which is adapted to the decomposition $\mathfrak{h}_{\mathbb{R}} = \mathfrak{a} \oplus i\,\mathfrak{t}$. The Cartan involution acts on these root space bases as

$$\theta\{f^1, f^2, f^3, f^4\} = \{-f^1, -f^2, f^3, f^4\}. \tag{6.131}$$

From the relations (6.130) it is easy to obtain the expression of the $\{F^k\}$ $(k = 1, \cdots, 4)$ in terms of the $\{f^k\}$ and thus also the expressions for the simple roots $\alpha_1 = 2F^1 - F^2$, $\alpha_2 = -F^1 + 2F^2 - F^3$, $\alpha_3 = -F^2 + 2F^3 - F^4$ and $\alpha_4 = -F^3 + 2F^4$, defined by $\mathfrak{h}_0$,

$$\begin{aligned} \alpha_1 &= -f^2 + 2\,f^3 + 3\,f^4, \\ \alpha_2 &= -f^1 + f^2 + f^3 - f^4, \\ \alpha_3 &= 2\,f^1, \\ \alpha_4 &= -f^1 + f^2 - f^3 + f^4. \end{aligned} \tag{6.132}$$

It is then straightforward to obtain the action of θ on the roots, which, when expressed in terms of the $\mathfrak{h}_0$ simple roots themselves, is given by

$$\begin{aligned} \theta[\alpha_1] &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \\ \theta[\alpha_2] &= -\alpha_4, \\ \theta[\alpha_3] &= -\alpha_3, \\ \theta[\alpha_4] &= -\alpha_2. \end{aligned} \tag{6.133}$$

We see that the root α_3 is real while α_1, α_2 and α_4 are complex. As a check of these results, we may, for instance, verify that

$$\theta E_{\alpha_1} = \tilde{I}_{3,2}\, K^1_2\, \tilde{I}_{3,2} = K^1_5 = E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}. \tag{6.134}$$

In fact, this kind of computation provides a simpler way to obtain Equation (6.133).

The basis $\{f^1, f^2, f^3, f^4\}$ allows to define a different ordering on the root lattice, merely by considering the corresponding lexicographic order. In terms of this new ordering we obtain for instance $\alpha_1 < 0$ since the first nonzero component of α_1 (in this case -1 along f^2) is strictly negative. Similarly, one finds $\alpha_2 < 0$, $\alpha_3 > 0$, $\alpha_4 < 0$, $\alpha_1 + \alpha_2 < 0$, $\alpha_2 + \alpha_3 > 0$, $\alpha_3 + \alpha_4 > 0$, $\alpha_1 + \alpha_2 + \alpha_3 > 0$, $\alpha_2 + \alpha_3 + \alpha_4 > 0$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > 0$. A basis of simple roots, according to this ordering, is given by

$$\begin{aligned} \tilde{\alpha}_1 &= -\alpha_4 = f^1 - f^2 + f^3 - f^4, \\ \tilde{\alpha}_2 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = f^2 + 2\,f^3 + 3\,f^4, \\ \tilde{\alpha}_3 &= -\alpha_1 = f^2 - 2\,f^3 - 3\,f^4, \\ \tilde{\alpha}_4 &= -\alpha_2 = f^1 - f^2 - f^3 + f^4. \end{aligned} \tag{6.135}$$

(We have put $\tilde{\alpha}_4$ in fourth position, rather than in second, to follow usual conventions.) The action of θ on this basis reads

$$\theta[\tilde{\alpha}_1] = -\tilde{\alpha}_4, \qquad \theta[\tilde{\alpha}_2] = -\tilde{\alpha}_3, \qquad \theta[\tilde{\alpha}_3] = -\tilde{\alpha}_2, \qquad \theta[\tilde{\alpha}_4] = -\tilde{\alpha}_1. \tag{6.136}$$

These new simple roots are now all complex.

Restricted roots

The restricted roots are obtained by considering only the action of the roots on the noncompact Cartan generators h_1 and h_2. The two-dimensional vector space spanned by the restricted roots can be identified with the subspace spanned by f_1 and f_2; one simply projects out f_3 and f_4. In the notations $\beta_1 = f_1 - f_2$ and $\beta_2 = f_2$, one gets as positive restricted roots:

$$\beta_1, \qquad \beta_2, \qquad \beta_1 + \beta_2, \qquad \beta_1 + 2\beta_2, \qquad 2\beta_2, \qquad 2(\beta_1 + \beta_2), \tag{6.137}$$

which are the positive roots of the $(BC)_2$ (non-reduced) root system. The first four roots are degenerate twice, while the last two roots are nondegenerate. For instance, the two simple roots $\tilde{\alpha}_1$ and $\tilde{\alpha}_4$ project on the same restricted root β_1, while the two simple roots $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ project on the same restricted root β_2.

Counting multiplicities, there are ten restricted roots – the same number as the number of positive roots of $\mathfrak{sl}(5,\mathbb{C})$. No root of $\mathfrak{sl}(5,\mathbb{C})$ projects onto zero. The centralizer of $\mathfrak{a}$ consists only of $\mathfrak{a} \oplus \mathfrak{t}$.

6.6.2 Example 2: $\mathfrak{su}(4,1)$

Diagonal description

Let us now perform the same analysis within the framework of $\mathfrak{su}(4,1)$. Starting from the natural description (6.92) of $\mathfrak{su}(4,1)$, we first make a similarity transformation using the matrix

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{6.138}$$

so that a maximally noncompact Cartan subalgebra can be taken to be diagonal and is explicitly given by

$$h_1 = H_4, \qquad h_2 = i\,H_1, \qquad h_3 = i\,H_2, \qquad h_4 = i(2\,H_3 + H_4). \tag{6.139}$$

The corresponding $\mathfrak{su}(4,1)$ in the $\mathfrak{sl}(5,\mathbb{C})$ algebra is still aligned with the natural matrix representation of $\mathfrak{su}(5)$. The Cartan involution is given by $X \mapsto \tilde{I}_{4,1} X\, \tilde{I}_{4,1}$ where $\tilde{I}_{4,1} = S^T\, I_{4,1}\, S$. One has $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ where the noncompact part $\mathfrak{a}$ is one-dimensional and spanned by h_1, while the compact part $\mathfrak{t}$ is three-dimensional and spanned by h_2, h_3 and h_4.

Cartan involution and roots

In terms of the f^i's, the standard simple roots now read

$$\begin{aligned}
\alpha_1 &= 2\, f^2 - f^3, \\
\alpha_2 &= -f^2 + 2\, f^3 - 2\, f^4, \\
\alpha_3 &= -f^1 - f^2 + 3\, f^4, \\
\alpha_4 &= 2\, f^1.
\end{aligned} \tag{6.140}$$

The Cartan involution acts as

$$\begin{aligned}
\theta[\alpha_1] &= \alpha_1, \\
\theta[\alpha_2] &= \alpha_2, \\
\theta[\alpha_3] &= \alpha_3 + \alpha_4, \\
\theta[\alpha_4] &= -\alpha_4,
\end{aligned} \tag{6.141}$$

showing that α_1 and α_2 are imaginary, α_4 is real, while α_3 is complex.

A calculation similar to the one just described above, using as ordering rules the lexicographic ordering defined by the dual of the basis in Equation (6.139), leads to the new system of simple roots,

$$\begin{aligned}
\tilde{\alpha}_1 &= -\alpha_1 - \alpha_2 - \alpha_3, \\
\tilde{\alpha}_2 &= \alpha_1 + \alpha_2, \\
\tilde{\alpha}_3 &= -\alpha_2, \\
\tilde{\alpha}_4 &= \alpha_2 + \alpha_3 + \alpha_4,
\end{aligned} \tag{6.142}$$

which transform as

$$\begin{aligned}
\theta[\tilde{\alpha}_1] &= -\tilde{\alpha}_4 - \tilde{\alpha}_2 - \tilde{\alpha}_3, \\
\theta[\tilde{\alpha}_2] &= \tilde{\alpha}_2, \\
\theta[\tilde{\alpha}_3] &= \tilde{\alpha}_3, \\
\theta[\tilde{\alpha}_4] &= -\tilde{\alpha}_1 - \tilde{\alpha}_2 - \tilde{\alpha}_3
\end{aligned} \tag{6.143}$$

under the Cartan involution. Note that in this system, the simple roots $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ are imaginary and hence fixed by the Cartan involution, while the other two simple roots are complex.

Restricted roots

The restricted roots are obtained by considering the action of the roots on the single noncompact Cartan generator h_1. The one-dimensional vector space spanned by the restricted roots can be identified with the subspace spanned by f_1; one now simply projects out f_2, f_3 and f_4. With the notation $\beta_1 = f_1$, we get as positive restricted roots

$$\beta_1, \qquad 2\beta_1, \tag{6.144}$$

which are the positive roots of the $(BC)_1$ (non-reduced) root system. The first root is six times degenerate, while the second one is nondegenerate. The simple roots $\tilde{\alpha}_1$ and $\tilde{\alpha}_4$ project on the same restricted root β_1, while the imaginary root $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ project on zero (as does also the non-simple, positive, imaginary root $\tilde{\alpha}_2 + \tilde{\alpha}_3$).

Let us finally emphasize that the centralizer of $\mathfrak{a}$ in $\mathfrak{su}(4,1)$ is now given by $\mathfrak{a} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the center of $\mathfrak{a}$ in $\mathfrak{k}$ (i.e., the subspace generated by the compact generators that commute with H_4) and contains more than just the three compact Cartan generators h_2, h_3 and h_4. In fact, $\mathfrak{m}$ involves also the root vectors E_β whose roots restrict to zero. Explicitly, expressed in the basis of Equation (6.85), these roots read $\beta = \epsilon_p - \epsilon_q$ with $p, q = 1, 2,$ or 3 and are orthogonal to α_4. The algebra $\mathfrak{m}$ constitutes a rank 3, 9-dimensional Lie algebra, which can be identified with $\mathfrak{su}(3) \oplus \mathfrak{u}(1)$.

6.6.3 Tits–Satake diagrams: Definition

We may associate with each of the constructions of these simple root bases a Tits–Satake diagram as follows. We start with a Dynkin diagram of the complex Lie algebra and paint in black (•) the imaginary simple roots, i.e., the ones fixed by the Cartan involution. The others are represented by a white vertex (◦). Moreover, some double arrows are introduced in the following way. It can be easily proven (see Section 6.6.4) that for real semi-simple Lie algebras, there always exists a basis of simple roots B that can be split into two subsets: $B_0 = \{\alpha_{r+1}, \dots, \alpha_l\}$ whose elements are fixed by θ (they correspond to the black vertices) and $B \setminus B_0 = \{\alpha_1, \dots, \alpha_r\}$ (corresponding to white vertices) such that

$$\forall \alpha_k \in B \setminus B_0 : \theta[\alpha_k] = -\alpha_{\boldsymbol{\pi}(k)} + \sum_{j=r+1}^{l} a_k^j \, \alpha_j, \tag{6.145}$$

where $\boldsymbol{\pi}$ is an involutive permutation of the r indices of the elements of $B \setminus B_0$. Accordingly, $\boldsymbol{\pi}$ contains cycles of one or two elements. In the Tits–Satake diagram, we connect by a double arrow all pairs of distinct simple roots α_k and $\alpha_{\boldsymbol{\pi}(k)}$ in the same two-cycle orbit. For instance, for $\mathfrak{su}(3,2)$ and $\mathfrak{su}(4,1)$, we obtain the diagrams in Figure 36.

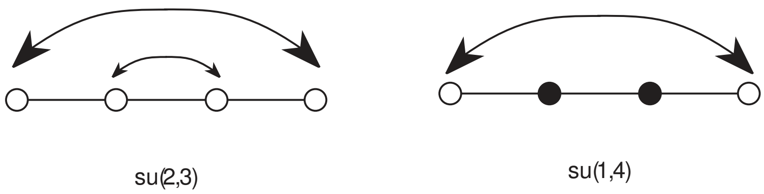


Figure 36: Tits–Satake diagrams for $\mathfrak{su}(3,2)$ and $\mathfrak{su}(4,1)$.

6.6.4 Formal considerations

Tits–Satake diagrams provide a lot of information about real semi-simple Lie algebras. For instance, we can read from them the full action of the Cartan involution as we now briefly pass to show. More information may be found in [5, 93].

The Cartan involution allows one to define a closed subsystem[28] Δ_0 of Δ:

$$\Delta_0 = \{\alpha \in \Delta | \theta[\alpha] = \alpha\}, \tag{6.146}$$

which is the system of imaginary roots. These project to zero when restricted to the maximally noncompact Cartan subalgebra. As we have seen in the examples, it is useful to use an ordering adapted to the Cartan involution. This can be obtained by considering a basis of $\mathfrak{h}$ constituted firstly by elements of $\mathfrak{a}$ followed by elements of $\mathfrak{t}$. If we use the lexicographic order defined by the dual of this basis, we obtain a root ordering such that if $\alpha \notin \Delta_0$ is positive, $\theta[\alpha]$ is negative since the real part comes first and changes sign. Let B be a simple root basis built with respect to this ordering and let $B_0 = B \cap \Delta_0$. Then we have

$$B = \{\alpha_1, \dots, \alpha_l\} \qquad \text{and} \qquad B_0 = \{\alpha_{r+1}, \dots, \alpha_l\}. \tag{6.147}$$

The subset B_0 is a basis for Δ_0. To see this, let $B \setminus B_0 = \{\alpha_1, \dots, \alpha_r\}$. If $\beta = \sum_{k=1}^{l} b^k \, \alpha_k$ is, say, a positive root (i.e., with coefficients $b^k \geq 0$) belonging to Δ_0, then $\beta - \theta[\beta] = 0$ is given by a sum

[28] A system Δ is closed if α, $\beta \in \Delta$ implies that $-\alpha \in \Delta$ and $\alpha + \beta \in \Delta$.

of positive roots, weighted by non-negative coefficients, $\sum_{k=1}^{r} b^k\,(\alpha_k - \theta[\alpha_k])$. As a consequence, the coefficients b^k are all zero for $k = 1, \cdots, r$ and B_0 constitutes a basis of Δ_0, as claimed.

To determine completely θ we just need to know its action on a basis of simple roots. For those belonging to B_0 it is the identity, while for the other ones we have to compute the coefficients a_k^j in Equation (6.145). These are obtained by solving the linear system given by the scalar products of these equations with the elements of B_0,

$$(\theta[\alpha_k] + \alpha_{\pi(k)}|\alpha_q) = \sum_{j=r+1}^{l} a_k^j\,(\alpha_j|\alpha_q). \tag{6.148}$$

Solving these equations for the unknown coefficients a_k^j is always possible because the Killing metric is nondegenerate on B_0.

The black roots of a Tits–Satake diagram represent B_0 and constitute the Dynkin diagram of the compact part $\mathfrak{m}$ of the centralizer of $\mathfrak{a}$. Because $\mathfrak{m}$ is compact, it is the direct sum of a semi-simple compact Lie algebra and one-dimensional, Abelian $\mathfrak{u}(1)$ summands. The rank of $\mathfrak{m}$ (defined as the dimension of its maximal Abelian subalgebra; diagonalizability is automatic here because one is in the compact case) is equal to the sum of the rank of its semi-simple part and of the number of $\mathfrak{u}(1)$ terms, while the dimension of $\mathfrak{m}$ is equal to the dimension of its semi-simple part and of the number of $\mathfrak{u}(1)$ terms. The Dynkin diagram of $\mathfrak{m}$ reduces to the Dynkin diagram of its semi-simple part.

The rank of the compact subalgebra $\mathfrak{m}$ is given by

$$\operatorname{rank}\mathfrak{m} = \operatorname{rank}\mathfrak{g} - \operatorname{rank}\mathfrak{p}, \tag{6.149}$$

where $\operatorname{rank}\mathfrak{p}$, called as we have indicated above the *real rank* of $\mathfrak{g}$, is given by the number of cycles of the permutation $\boldsymbol{\pi}$ (since two simple white roots joined by a double-arrow project on the same simple restricted root [5, 93]). These two sets of data allow one to determine the dimension of $\mathfrak{m}$ (without missing $\mathfrak{u}(1)$ generators) [5, 93]. Another useful information, which can be directly read off from the Tits–Satake diagrams is the dimension of the noncompact subspace $\mathfrak{p}$ appearing in the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. It is given (see Section 6.6.6) by

$$\dim\mathfrak{p} = \frac{1}{2}(\dim\mathfrak{g} - \dim\mathfrak{m} + \operatorname{rank}\mathfrak{p}). \tag{6.150}$$

This can be illustrated in the two previous examples. For $\mathfrak{su}(3,2)$, one gets $\dim\mathfrak{g} = 24$, $\operatorname{rank}\mathfrak{g} = 4$ and $\operatorname{rank}\mathfrak{p} = 2$. It follows that $\operatorname{rank}\mathfrak{m} = 2$ and since $\mathfrak{m}$ has no semi-simple part (no black root), it reduces to $\mathfrak{m} = \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ and has dimension 2. This yields $\dim\mathfrak{p} = 12$, and, by substraction, $\dim\mathfrak{k} = 12$ ($\mathfrak{k}$ is easily verified to be equal to $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$). Similarly, for $\mathfrak{su}(4,1)$, one gets $\dim\mathfrak{g} = 24$, $\operatorname{rank}\mathfrak{g} = 4$ and $\operatorname{rank}\mathfrak{p} = 1$. It follows that $\operatorname{rank}\mathfrak{m} = 3$ and since the semi-simple part of $\mathfrak{m}$ is read from the black roots to be $\mathfrak{su}(3)$, which has rank two, one deduces $\mathfrak{m} = \mathfrak{su}(3) \oplus \mathfrak{u}(1)$ and $\dim\mathfrak{m} = 9$. This yields $\dim\mathfrak{p} = 8$, and, by substraction, $\dim\mathfrak{k} = 16$ ($\mathfrak{k}$ is easily verified to be equal to $\mathfrak{su}(4) \oplus \mathfrak{u}(1)$ in this case).

Finally, from the knowledge of θ, we may obtain the restricted root space by projecting the root space according to

$$\Delta \to \overline{\Delta} : \alpha \mapsto \bar{\alpha} = \frac{1}{2}(\alpha - \theta[\alpha]) \tag{6.151}$$

and restricting their action on $\mathfrak{a}$ since α and $-\theta(\alpha)$ project on the same restricted root [5, 93].

6.6.5 Illustration: F_4

The Lie algebra F_4 is a 52-dimensional simple Lie algebra of rank 4. Its root vectors can be expressed in terms of the elements of an orthonormal basis $\{e_k | k = 1, \ldots, 4\}$ of a four-dimensional

Euclidean space:

$$\Delta_{F_4} = \left\{ \pm e_i \pm e_j | i < j \} \cup \{\pm e_i\} \cup \{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \right\}. \tag{6.152}$$

A basis of simple roots is

$$\alpha_1 = e_2 - e_3, \qquad \alpha_2 = e_3 - e_4, \qquad \alpha_3 = e_4, \qquad \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4). \tag{6.153}$$

The corresponding Dynkin diagram can be obtained from Figure 37 by ignoring the painting of the vertices. To the real Lie algebra, denoted $F\,II$ in [28], is associated the Tits–Satake diagram of the left hand side of Figure 37. We immediately obtain from this diagram the following information:

$$\operatorname{rank}\mathfrak{p} = 1, \qquad \operatorname{rank}\mathfrak{m} = 3, \qquad \mathfrak{m} = \mathfrak{so}(7), \qquad \dim\mathfrak{p} = \frac{1}{2}(52 - 21 + 1) = 16. \tag{6.154}$$

Accordingly, $F\,II$ has signature -21 (compact part) $+\,1$ (rank of $\mathfrak{p}$) $= -20$ and is denoted $F_{4(-20)}$. Moreover, solving a system of three equations, we obtain: $\theta[\alpha_4] = -\alpha_4 - \alpha_1 - 2\,\alpha_2 - 3\,\alpha_3$, i.e.,

$$\theta[e_1] = -e_1 \qquad \text{and} \qquad \theta[e_k] = e_k \quad \text{if } k = 2,3,4. \tag{6.155}$$

This shows that the projection defining the reduced root system Σ consists of projecting any given root orthogonally onto its e_1 component. Thus we obtain $\Sigma = \{\pm\frac{1}{2}e_1, \pm e_1\}$, with multiplicity 8 for $\frac{1}{2}e_1$ (resulting from the projection of the eight roots $\{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$) and 7 for e_1 (resulting from the projection of the seven roots $\{e_1 \pm e_k | k = 2,3,4\} \cup \{e_1\}$).

Figure 37: On the left, the Tits–Satake diagram of the real form $F_{4(-20)}$. On the right, a non-admissible Tits–Satake diagram.

Let us mention that, contrary to the Vogan diagrams, any "formal Tits–Satake diagram" is not admissible. For instance if we consider the right hand side diagram of Figure 37 we get

$$\theta[e_1] = -e_2,\ \theta[e_2] = -e_1, \qquad \text{and} \qquad \theta[e_k] = e_k \qquad \text{if } k = 3 \text{ or } 4. \tag{6.156}$$

But this means that for the root $\alpha = e_1$, $\alpha + \theta^*[\alpha] = e_1 - e_2$ is again a root, which is impossible as we shall see below.

6.6.6 Some more formal considerations

Let us recall some crucial aspects of the discussion so far. Let $\mathfrak{g}_\sigma$ be a real form of the complex semi-simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and σ be the conjugation it defines. We have seen that there always exists a compact real Lie algebra $\mathfrak{u}_\tau$ such that the corresponding conjugation τ commutes with σ. Moreover, we may choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{u}_\tau$ such that its complexification $\mathfrak{h}^{\mathbb{C}}$ is invariant under σ, i.e., $\sigma(\mathfrak{h}^{\mathbb{C}}) = \mathfrak{h}^{\mathbb{C}}$. Then the real form $\mathfrak{g}_\sigma$ is said to be *normally related* to $(\mathfrak{u}_\theta, \mathfrak{h})$. As previously, we denote by the same letter θ the involution defined by duality on $(\mathfrak{h}^{\mathbb{C}})^*$ (and also on the root lattice with respect to $\mathfrak{h}^{\mathbb{C}}$: Δ) by $\theta = \tau\sigma$.

When $\mathfrak{g}_\sigma$ and $\mathfrak{u}_\tau$ are normally related, we may decompose the former into compact and non-compact components $\mathfrak{g}_\sigma = \mathfrak{k} \oplus \mathfrak{p}$ such that $\mathfrak{u}_\tau = \mathfrak{k} \oplus i\,\mathfrak{p}$. As mentioned, the starting point consists of choosing a maximally Abelian noncompact subalgebra $\mathfrak{a} \subset \mathfrak{p}$ and extending it to a Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$, where $\mathfrak{t} \subset \mathfrak{k}$. This Cartan subalgebra allows one to consider the real Cartan subalgebra

$$\mathfrak{h}^{\mathbb{R}} = i\mathfrak{t} \oplus \mathfrak{p} = \sum_{\alpha\in\Delta} \mathbb{R}\, H_\alpha. \tag{6.157}$$

Let us remind the reader that, in this case, the Cartan involution $\theta = \sigma\,\tau = \tau\,\sigma$ is such that $\theta|_{\mathfrak{k}} = +1$ and $\theta|_{\mathfrak{p}} = -1$. From Equation (6.69) we obtain

$$\theta(E_\alpha) = \rho_\alpha\, E_{\theta[\alpha]}, \tag{6.158}$$

and using $\theta^2 = 1$ we deduce that

$$\rho_\alpha\, \rho_{\theta[\alpha]} = 1. \tag{6.159}$$

Furthermore, Equation (6.32) and the fact that the structure constants are rational yield the following relations:

$$\begin{aligned} \rho_\alpha\, \rho_\beta N_{\theta[\alpha],\,\theta[\beta]} &= \rho_{\alpha+\beta} N_{\alpha,\,\beta}, \\ \theta(H_\alpha) &= H_{\theta[\alpha]}, \\ \rho_\alpha\, \rho_{-\alpha} &= 1. \end{aligned} \tag{6.160}$$

On the other hand, the commutativity of τ and σ implies

$$\sigma(H_\alpha) = -H_{\sigma[\alpha]}, \qquad \sigma(E_\alpha) = \kappa_\alpha\, E_{\sigma[\alpha]}, \tag{6.161}$$

with

$$\kappa_\alpha = -\bar{\rho}_\alpha, \qquad \sigma[\alpha] = -\theta[\alpha]. \tag{6.162}$$

In particular, if the root α belongs to Δ_0, defined in Equation (6.146), then $\theta[\alpha] = \alpha$ and thus $\rho_\alpha^2 = 1$, i.e.,

$$\rho_\alpha = -\kappa_\alpha = \pm 1. \tag{6.163}$$

Let us denote by $\Delta_{0,\,-}$ and $\Delta_{0,\,+}$ the subsets of Δ_0 corresponding to the imaginary noncompact and imaginary compact roots, respectively. We have

$$\Delta_{0,\,-} = \{\alpha \in \Delta_0 | \rho_\alpha = -1\} \qquad \text{and} \qquad \Delta_{0,\,+} = \{\alpha \in \Delta_0 | \rho_\alpha = +1\}. \tag{6.164}$$

Obviously, for $\alpha \in \Delta_{0,\,-}$, E_α belongs to $\mathfrak{p}^{\mathbb{C}}$, while for $\alpha \in \Delta_{0,\,+}$, E_α belongs to $\mathfrak{k}^{\mathbb{C}}$. Moreover, if $\alpha \in \Delta \setminus \Delta_0$ we find

$$E_\alpha + \theta(E_\alpha) \in \mathfrak{k}^{\mathbb{C}} \qquad \text{and} \qquad E_\alpha - \theta(E_\alpha) \in \mathfrak{p}^{\mathbb{C}}. \tag{6.165}$$

These remarks lead to the following explicit constructions of the complexifications of $\mathfrak{k}$ and $\mathfrak{p}$:

$$\begin{aligned} \mathfrak{k}^{\mathbb{C}} &= \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta_{0,\,+}} \mathbb{C}\, E_\alpha \oplus \bigoplus_{\alpha\in\Delta\setminus\Delta_0} \mathbb{C}\,(E_\alpha + \theta(E_\alpha)), \\ \mathfrak{p}^{\mathbb{C}} &= \mathfrak{a}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta_{0,\,-}} \mathbb{C}\, E_\alpha \oplus \bigoplus_{\alpha\in\Delta\setminus\Delta_0} \mathbb{C}\,(E_\alpha - \theta(E_\alpha)). \end{aligned} \tag{6.166}$$

Furthermore, since θ fixes all the elements of Δ_0, the subspace $\bigoplus_{\alpha\in\Delta_{0,\,-}} \mathbb{C}\, E_\alpha$ belongs to the centralizer[29] of $\mathfrak{a}$ and thus is empty if $\mathfrak{a}$ is maximally Abelian in $\mathfrak{p}$. Taking this remark into account, we immediately obtain the dimension formulas (6.149, 6.150).

[29] Geometrically, this results from the orthogonality of roots α and β such that $H_\alpha \in \mathfrak{k}$ and $H_\beta \in \mathfrak{p}$, or, equivalently, because $\alpha(H_\beta) = \theta[\alpha](H_\beta) = \alpha(\theta(H_\beta)) = -\alpha(H_\beta)$.

Using, as before, the basis in Equation (6.147) we obtain for the roots belonging to $B \setminus B_0$, i.e., for an index $i \leq r$:

$$-\theta[\alpha_i] = \sum_{j=1,\dots,r} p_i^j \alpha_j + \sum_{j=r+1,\dots,l} q_i^j \alpha_j \qquad \text{with} \qquad p_i^j,\, q_i^j \in \mathbb{N}. \tag{6.167}$$

Thus

$$\alpha_i = (-\theta)^2[\alpha_i] = \sum_{\substack{j=1,\dots,r \\ k=1,\dots,r}} p_i^j p_j^k \alpha_k + \sum_{\substack{j=1,\dots,r \\ k=r+1,\dots,l}} p_i^j q_j^k \alpha_k - \sum_{j=r+1,\dots,l} q_i^j \alpha_j. \tag{6.168}$$

As $\sum_{j=1,\dots,r} p_i^j p_j^k = \delta_i^j$, where the coefficients p_i^j are non-negative integers, the matrix (p_i^j) must be a permutation matrix and it follows that

$$\theta[\alpha_i] = -\alpha_{\boldsymbol{\pi}(i)} \qquad (\text{mod } \Delta_0), \tag{6.169}$$

where $\boldsymbol{\pi}$ is an involutive permutation of $\{1,\dots,r\}$.

A fundamental property of Δ is

$$\forall \alpha \in \Delta : \theta[\alpha] + \alpha \notin \Delta. \tag{6.170}$$

To show this, note that if $\alpha \in \Delta_0$, it would imply that $2\,\alpha$ belongs to Δ, which is impossible for the root lattice of a semi-simple Lie algebra. If $\alpha \in \Delta \setminus \Delta_0$ and $\theta[\alpha] + \alpha \in \Delta$, then $\theta[\alpha] + \alpha \in \Delta_0$. Thus we obtain using Equation (6.35) and taking into account that $\mathfrak{a}$ is maximal Abelian in $\mathfrak{p}$, that $\rho_\alpha = +1$, i.e.,

$$\sigma(E_{\sigma[\alpha]-\alpha}) = +E_{\theta[\alpha]-\alpha} \tag{6.171}$$

and

$$\begin{aligned} [E_\alpha,\, \sigma(E_{-\alpha})] &= \rho_{-\alpha}\, N_{\alpha,-\sigma[\alpha]}\, E_{\alpha-\sigma[\alpha]}, \\ [\sigma(E_\alpha),\, E_{-\alpha}] &= \overline{\rho_{-\alpha}}\, N_{\alpha,-\sigma[\alpha]}\, E_{\sigma[\alpha]-\alpha} \\ &= \rho_{-\alpha}\, N_{\sigma[\alpha],-\alpha}\, E_{\sigma[\alpha]-\alpha}. \end{aligned} \tag{6.172}$$

From this result we deduce

$$\rho_\alpha\, N_{\sigma[\alpha],-\alpha} = \overline{\rho_{-\alpha}}\, N_{\alpha,-\sigma[\alpha]} = -\rho_\alpha\, N_{\alpha,-\sigma[\alpha]}, \tag{6.173}$$

i.e., $\rho_\alpha = -\overline{\rho_{-\alpha}}$ which is incompatible with equation (6.160). Thus, the statement (6.170) follows.

6.7 The real semi-simple algebras $\mathfrak{so}(k,l)$

The dimensional reduction from 10 to 3 dimensions of $\mathcal{N} = 1$ supergravity coupled to m Maxwell multiplets leads to a nonlinear sigma model $\mathcal{G}/\mathcal{K}(\mathcal{G})$ with Lie($\mathcal{G}$)=$\mathfrak{so}(8, 8 + m)$ (see Section 7). To investigate the geometry of these cosets, we shall construct their Tits–Satake diagrams.

The $\mathfrak{so}(n,\,\mathbb{C})$ Lie algebra can be represented by $n \times n$ antisymmetric complex matrices. The compact real form is $\mathfrak{so}(k + l,\, \mathbb{R})$, naturally represented as the set of $n \times n$ antisymmetric real matrices. One way to describe the real subalgebras $\mathfrak{so}(k,l)$, aligned with the compact form $\mathfrak{so}(k + l,\, \mathbb{R})$, is to consider $\mathfrak{so}(k,l)$ as the set of infinitesimal rotations expressed in Pauli coordinates, i.e., to represent the hyperbolic space on which they act as a Euclidean space whose first k coordinates, x^a, are real while the last l coordinates y^b are purely imaginary. Writing the matrices of $\mathfrak{so}(k,l)$ in block form as

$$X = \begin{pmatrix} A & i\,C \\ -i\,C^t & B \end{pmatrix}, \tag{6.174}$$

where

$$A = -A^t \in \mathbb{R}^{k\times k}, \qquad B = -B^t \in \mathbb{R}^{l\times l}, \qquad C \in \mathbb{R}^{k\times l}, \tag{6.175}$$

we may obtain a maximal Abelian subspace $\mathfrak{a}$ by allowing C to have nonzero elements only on its diagonal, i.e., to be of the form:

$$C = \begin{pmatrix} a_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & a_l \\ \vdots & \cdots & \vdots \end{pmatrix} \qquad \text{or} \qquad C = \begin{pmatrix} a_1 & \cdots & & \cdots & 0 \\ & \ddots & & & \\ 0 & \cdots & a_k & \cdots & 0 \end{pmatrix}, \tag{6.176}$$

with $k > l$ or $l < k$, respectively.

To proceed, let us denote by H_j the matrices whose entries are everywhere vanishing except for a 2×2 block,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

on the diagonal. These matrices have the following realisation in terms of the $K^i{}_j$ (defined in Equation (6.83)):

$$H_j = K^{2j-1}{}_{2j} - K^{2j}{}_{2j-1}. \tag{6.177}$$

They constitute a set of $\mathfrak{so}(k+l)$ commuting generators that provide a Cartan subalgebra; it will be the Cartan subalgebra fixed by the Cartan involution defined by the real forms that we shall now discuss.

6.7.1 Dimensions $l = 2\,q + 1 < k = 2\,p$

Motivated by the dimensional reduction of supergravity, we shall assume $k = 2p$, even. We first consider $l = 2\,q + 1 < k$. Then by reordering the coordinates as follows,

$$\{x_1, y_1; \cdots; x_l, y_l; x_{l+1}, x_{l+2}; \cdots; x_{2p-2}, x_{2p-1}; x_{2p}\}, \tag{6.178}$$

we obtain a Cartan subalgebra of $\mathfrak{so}(2q+1, 2p)$, with noncompact generators first, and aligned with the one introduced in Equation (6.177) by considering the basis $\{i\,H_1, \cdots, i\,H_l, H_{l+1}, \cdots, H_{q+p}\}$ [30]. These generators are all orthogonal to each other. Let us denote the elements of the dual basis by $\{f_A | A = 1, \cdots, p+q\}$, and split them into two subsets: $\{f_a | a = 1, \cdots, 2q+1\}$ and $\{f_\alpha | \alpha = 2q+2, \cdots, p+q\}$. The action of the Cartan involution on these generators is very simple,

$$\theta[f_a] = -f_a, \qquad \text{and} \qquad \theta[f_\alpha] = +f_\alpha. \tag{6.179}$$

The root system of $\mathfrak{so}(2q+1, 2p)$ is $B_{(p+q)}$, represented by $\Delta = \{\pm f_A \pm f_B | A < B = 1, \cdots, p+q\} \cup \{\pm f_A | A1, \cdots, p+q\}$. A simple root basis can be taken as:

$$\{\alpha_1 = f_1 - f_2, \cdots, \alpha_{p+q-1} = f_{p+q-1} - f_{p+q}, \alpha_{p+q} = f_{p+q}\}.$$

It is then straigthforward to obtain the action of the Cartan involution on the simple roots:

$$\begin{aligned} \theta[\alpha_A] &= -\alpha_A && \text{for } A = 1, \cdots, 2q, \\ \theta[\alpha_{2q+1}] &= -\alpha_{2q+1} - 2(\alpha_{2q+2} + \cdots + \alpha_{q+p}), && \\ \theta[\alpha_A] &= +\alpha_A && \text{for } A = 2q+2, \cdots, q+p. \end{aligned}$$

The corresponding Tits–Satake diagrams are displayed in Figure 38.

From Equation (6.179) we also obtain without effort that the set of restricted roots consists of the $4q(2q+1)$ roots $\{\pm f_a \pm f_b\}$, each of multiplicity one, and the $4q+2$ roots $\{\pm f_a\}$, each of multiplicity $2(p-q)-1$. These constitute a B_{2q+1} root system.

[30] If $p = q + 1$, this basis consists only of noncompact generators.

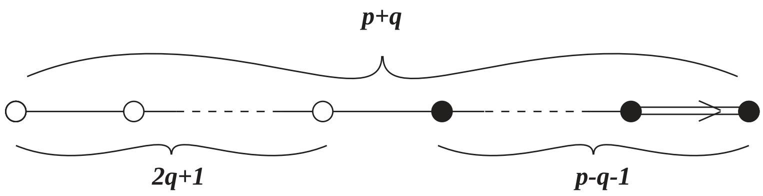


Figure 38: Tits–Satake diagrams for the $\mathfrak{so}(2p, 2q+1)$ Lie algebra with $q < p$. If $p = q+1$, all nodes are white.

6.7.2 Dimensions $l = 2q + 1 > k = 2p$

Following the same procedure as for the previous case, we obtain a Cartan subalgebra consisting of $2p$ noncompact generators and $q-p$ compact generators. The corresponding Tits–Satake diagrams are displayed in Figure 39.

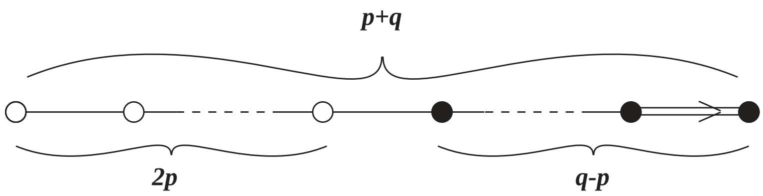


Figure 39: Tits–Satake diagrams for the $\mathfrak{so}(2p, 2q+1)$ Lie algebra with $q \geq p$. If $q = p$, all nodes are white.

The restricted root system is now of type B_{2p}, with $4p(2p-1)$ long roots of multiplicity one and $4p$ short roots of multiplicity $2(q-p)+1$.

6.7.3 Dimensions $l = 2q$, $k = 2p$

Here the root system is of type D_{p+q}, represented by $\Delta = \{\pm f_A \pm f_B | \, A < B = 1, \cdots, p+q\}$, where the orthonormal vectors f_A again constitute a basis dual to the natural Cartan subalgebra of $\mathfrak{so}(k+l)$. Now, $k = 2p$ and $l = 2q$ are both assumed even, and we may always suppose $k \geq l$. The Cartan involution to be considered acts as previously on the f_A:

$$\theta[f_a] = -f_a, \qquad a = 1, \cdots, 2q \tag{6.180}$$

and

$$\theta[f_\alpha] = +f_\alpha, \qquad \alpha = 2q+1, \cdots, p+q \qquad \text{for } q < p. \tag{6.181}$$

The simple roots can be chosen as

$$\{\alpha_1 = f_1 - f_2, \cdots, \alpha_{p+q-1} = f_{p+q-1} - f_{p+q}, \, \alpha_{p+q} = f_{p+q-1} + f_{p+q}\},$$

on which the Cartan involution has the following action:

- For $q = p$

$$\theta[\alpha_A] = -\alpha_A \qquad \text{for } A = 1, \cdots, q+p. \tag{6.182}$$

- For $q = p-1$

$$\begin{aligned} \theta[\alpha_A] &= -\alpha_A && \text{for } A = 1, \cdots, 2q = q+p-1, \\ \theta[\alpha_{q+p-1}] &= -\alpha_{q+p}, \\ \theta[\alpha_{q+p}] &= -\alpha_{q+p-1}. \end{aligned} \tag{6.183}$$

- For $q < p-1$

$$\begin{aligned} \theta[\alpha_A] &= -\alpha_A & A &= 1, \cdots, 2q-1, \\ \theta[\alpha_{2q}] &= -\alpha_{2q} - 2(\alpha_{2q+1} + \cdots, \alpha_{q+p-2}) \\ &\quad -\alpha_{q+p-1} - \alpha_{q+p}, \\ \theta[\alpha_A] &= +\alpha_A & A &= 2q+1, \cdots, q+p, \end{aligned} \tag{6.184}$$

The corresponding Tits–Satake diagrams are obtained in the same way as before and are displayed in Figure 40.

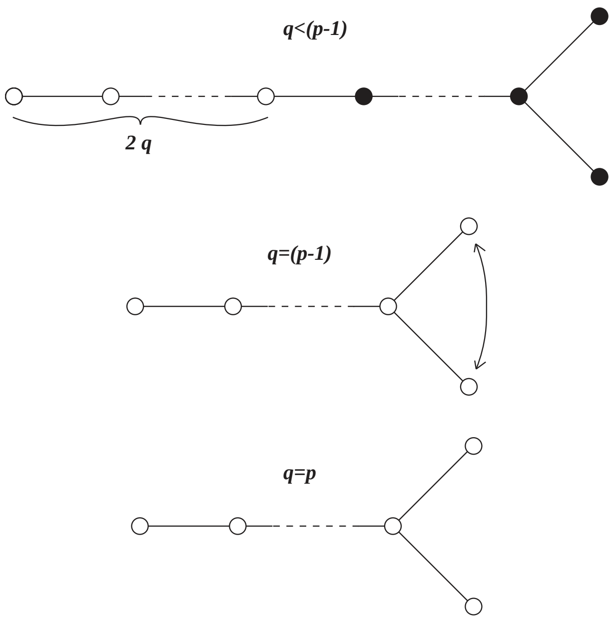


Figure 40: Tits–Satake diagrams for the $\mathfrak{so}(2p, 2q)$ Lie algebra with $q < p-1$, $q = p-1$ and $q = p$.

When $q < p$, the restricted root system is again of type B_{2q}, with $4q(2q-1)$ long roots of multiplicity one and $4q$ short roots of multiplicity $2(p-q)$. For $p = q$, the short roots disappear and the restricted root system is of D_{2p} type, with all roots having multiplicity one.

6.8 Summary – Tits–Satake diagrams for non-compact real forms

To summarize the analysis, we provide the Tits–Satake diagrams for all noncompact real forms of all simple Lie algebras [5, 93]. We do not give explicitly the Tits–Satake diagrams of the compact real forms as these are simply obtained by painting in black all the roots of the standard Dynkin diagrams.

Theorem: The simple real Lie algebras are:

- The Lie algebras $\mathfrak{g}^{\mathbb{R}}$ where $\mathfrak{g}$ is one of the complex simple Lie algebras A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, G_2, F_4, E_6, E_7, or E_8, and the compact real forms of these.
- The classical real Lie algebras of types $\mathfrak{su}$, $\mathfrak{so}$, $\mathfrak{sp}$ and $\mathfrak{sl}$. These are listed in Table 26.
- The twelve exceptional real Lie algebras, listed in Table 27 (our conventions are due to Cartan).

Table 26: All classical real Lie algebras of $\mathfrak{su}$, $\mathfrak{so}$, $\mathfrak{sp}$ and $\mathfrak{sl}$ type.

Algebra		Real rank	Restricted root lattice
$\mathfrak{su}(p,q)$	$p \geq q > 0\, p+q \geq 2$	q	$(BC)_q$ if $p > q$, C_q if $p = q$
$\mathfrak{so}(p,q)$	$p > q > 0\, p+q = 2n+1 \geq 5$	q	B_q
	$p \geq q > 0\, p+q = 2n \geq 8$	q	B_q if $p > q$, D_q if $p = q$
$\mathfrak{sp}(p,q)$	$p \geq q > 0\, p+q \geq 3$	q	$(BC)_q$ if $p > q$, C_q if $p = q$
$\mathfrak{sp}(n,\mathbb{R})$	$n \geq 3$	n	C_n
$\mathfrak{so}^*(2n)$	$n \geq 5$	$[n/2]$	$C_{\frac{n}{2}}$ if n even, $(BC)_{\frac{n-1}{2}}$ if n odd
$\mathfrak{sl}(n,\mathbb{R})$	$n \geq 3$	$n-1$	A_{n-1}
$\mathfrak{sl}(n,\mathbb{H})$	$n \geq 2$	$n-1$	A_{n-1}

Table 27: All exceptional real Lie algebras.

Algebra	Real rank	Restricted root lattice
G	2	G_2
$F\ I$	4	F_4
$F\ II$	1	$(BC)_1$
$E\ I$	6	E_6
$E\ II$	4	F_4
$E\ III$	2	$(BC)_2$
$E\ IV$	2	A_2
$E\ V$	7	E_7
$E\ VI$	4	F_4
$E\ VII$	3	C_3
$E\ VIII$	8	E_8
$E\ IX$	4	F_4

Table 28: Tits–Satake diagrams (A_n series)

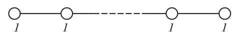
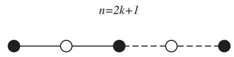
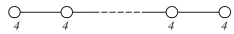
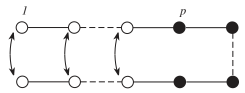
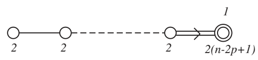
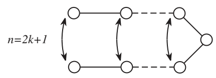
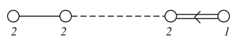
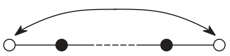
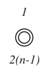
A_n series $n \geq 1$	Tits–Satake diagram	Restricted root system
$\mathfrak{sl}(n, \mathbb{R})$, $n \geq 3$ *A I*		 A_n
$\mathfrak{su}^*(n+1)$, $n = 2k+1$ *A II*	 (k+1) black and k white roots alternate.	 A_{2k}
$\mathfrak{su}(p, n+1-p)$ *A III*	 The $p(>0)$ first and p last roots are white and connected.	 BC_p
$\mathfrak{su}(\frac{n+1}{2}, \frac{n+1}{2})$, $n = 2k+1$ *A III*		 $C_{(k+1)}$
$\mathfrak{su}(1, n-1)$, $n \geq 3$ *A IV*	 Only the first and last roots are white and connected.	 A_1

Table 29: Tits–Satake diagrams (B_n series)

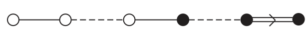
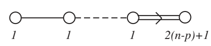
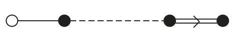
B_n series $n \geq 4$	Tits–Satake diagram	Restricted root system
$\mathfrak{so}(p, 2n-p+1)$, $p \geq 1$ *B I*	 The $p(\geq 2)$ first roots are white.	 B_p
$\mathfrak{so}(1, 2n)$ *B II*	 Only the first root is white.	 A_1

Table 30: Tits–Satake diagrams (C_n series)

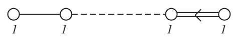
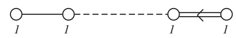
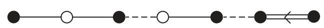
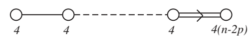
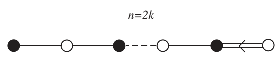
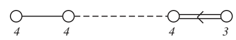
C_n series $n \geq 3$	Tits–Satake diagram	Restricted root system
$\mathfrak{sp}(n, \mathbb{R})$ $C\ I$		 C_n
$\mathfrak{sp}(p, n-p)$ $C\ II$	 The $2p$ first roots are alternatively white and black, the $n-2p$ remaining are black	 B_p
$\mathfrak{sp}(\frac{n}{2}, \frac{n}{2})$, $n = 2k$ $C\ II$		 $C_{\frac{n}{2}}$

Table 31: Tits–Satake diagrams (D_n series)

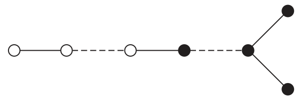
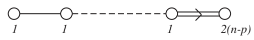
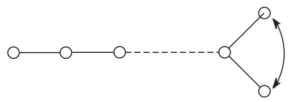
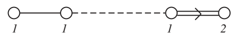
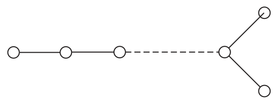
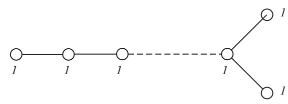
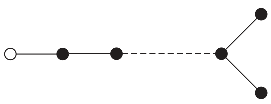
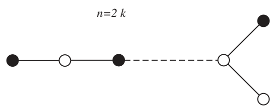
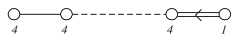
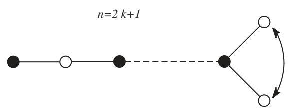
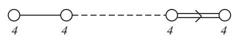
D_n series $n \geq 4$	Tits–Satake diagram	Restricted root system
$\mathfrak{so}(p, 2n-p),\ p \leq n-2$ $D\ I$	 The $p \leq n-2$ first roots are white.	 B_p
$\mathfrak{so}(n-1, n+1)$ $D\ I$		 $B_{(n-1)}$
$\mathfrak{so}(n, n)$ $D\ I$		 D_n
$\mathfrak{so}(1, 2n-1)$ $D\ II$		 A_1
$\mathfrak{so}(^*(2\,n)),\ n = 2k$ $D\ III$		 C_{2k-1}
$\mathfrak{so}(^*(2n)),\ n = 2k+1$ $D\ III$		 BC_{2k}

Table 32: Tits–Satake diagrams (G_2 series)

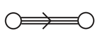
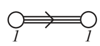
G_2 series	Tits–Satake diagram	Restricted root system
$G_{2(2)}$ G		

Table 33: Tits–Satake diagrams (F_4 series)

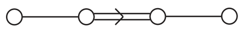
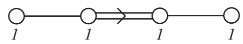
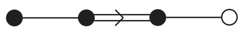
F_4 series	Tits–Satake diagram	Restricted root system
$F_{4(4)}$ $F\ I$		
$F_{4(-20)}$ $F\ II$		

Table 34: Tits–Satake diagrams (E_6 series)

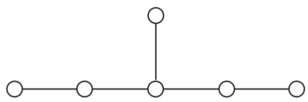
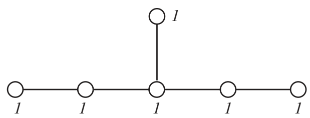
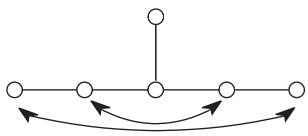
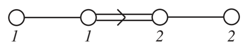
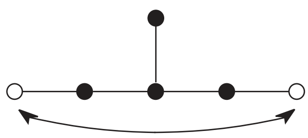
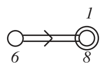
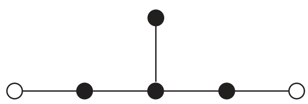
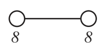
E_6 series	Tits–Satake diagram	Restricted root system
$E_{6(6)}$ $E\ I$		
$E_{6(2)}$ $E\ II$		
$E_{6(-14)}$ $E\ III$		
$E_{6(-26)}$ $E\ IV$		

Table 35: Tits–Satake diagrams (E_7 series)

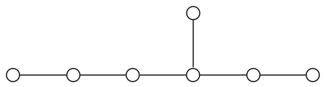
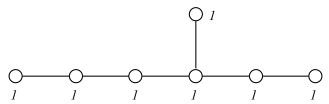
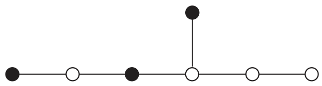
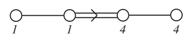
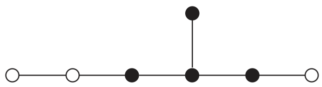
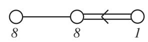
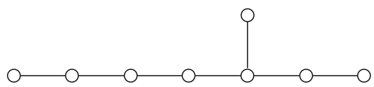
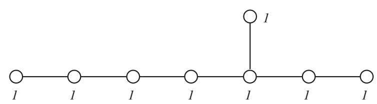
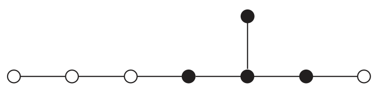
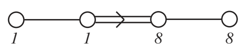
E_7 series	Tits–Satake diagram	Restricted root system
$E_{7(7)}$ $E\ V$		
$E_{7(-5)}$ $E\ VI$		
$E_{7(-25)}$ $E\ VII$		

Table 36: Tits–Satake diagrams (E_8 series)

E_8 series	Tits–Satake diagram	Restricted root system
$E_{8(8)}$ $E\ VIII$		
$E_{8(-24)}$ $E\ IX$		

7 Kac–Moody Billiards II – The Case of Non-Split Real Forms

We will now make use of the results from the previous section to extend the analysis of Kac–Moody billiards to include also theories whose U-duality symmetries are described by algebras $\mathfrak{u}_3$ that are non-split. The key concepts are that of *restricted root systems*, *restricted Weyl group* – and the associated concept of *maximal split subalgebra* – as well as the Iwasawa decomposition already encountered above. These play a prominent role in our discussion as they determine the billiard structure. We mainly follow [95].

7.1 The restricted Weyl group and the maximal split "subalgebra"

Let $\mathfrak{u}_3$ be any real form of the complex Lie algebra $\mathfrak{u}_3^{\mathbb{C}}$, θ its Cartan involution, and let

$$\mathfrak{u}_3 = \mathfrak{k}_3 \oplus \mathfrak{p}_3 \tag{7.1}$$

be the corresponding Cartan decomposition. Furthermore, let

$$\mathfrak{h}_3 = \mathfrak{t}_3 \oplus \mathfrak{a}_3 \tag{7.2}$$

be a maximal noncompact Cartan subalgebra, with $\mathfrak{t}_3$ (respectively, $\mathfrak{a}_3$) its compact (respectively, noncompact) part. The real rank of $\mathfrak{u}_3$ is, as we have seen, the dimension of $\mathfrak{a}_3$. Let now Δ denote the root system of $\mathfrak{u}_3^{\mathbb{C}}$, Σ the restricted root system and m_λ the multiplicity of the restricted root λ.

As explained in Section 4.9.2, the restricted root system of the real form $\mathfrak{u}_3$ can be either reduced or non-reduced. If it is reduced, it corresponds to one of the root systems of the finite-dimensional simple Lie algebras. On the other hand, if the restricted root system is non-reduced, it is necessarily of $(BC)_n$-type [93] (see Figure 19 for a graphical presentation of the BC_3 root system).

The restricted Weyl group

By definition, the restricted Weyl group is the Coxeter group generated by the fundamental reflections, Equation (4.55), with respect to the simple roots of the restricted root system. The restricted Weyl group preserves multiplicities [93].

The maximal split "subalgebra" $\mathfrak{f}$

Although multiplicities are an essential ingredient for describing the full symmetry $\mathfrak{u}_3$, they turn out to be irrelevant for the construction of the gravitational billiard. For this reason, it is useful to consider the *maximal split "subalgebra"* $\mathfrak{f}$, which is defined as the real, semi-simple, split Lie algebra with the same root system as the restricted root system as $\mathfrak{u}_3$ (in the $(BC)_n$-case, we choose for definiteness the root system of $\mathfrak{f}$ to be of B_n-type). The real rank of $\mathfrak{f}$ coincides with the rank of its complexification $\mathfrak{f}^{\mathbb{C}}$, and one can find a Cartan subalgebra $\mathfrak{h}_\mathfrak{f}$ of $\mathfrak{f}$, consisting of all generators of $\mathfrak{h}_3$ which are diagonalizable over the reals. This subalgebra $\mathfrak{h}_\mathfrak{f}$ has the same dimension as the maximal noncompact subalgebra $\mathfrak{a}_3$ of the Cartan subalgebra $\mathfrak{h}_3$ of $\mathfrak{u}_3$.

By construction, the root space decomposition of $\mathfrak{f}$ with respect to $\mathfrak{h}_\mathfrak{f}$ provides the same root system as the restricted root space decomposition of $\mathfrak{u}_3$ with respect to $\mathfrak{a}_3$, except for multiplicities, which are all trivial (i.e., equal to one) for $\mathfrak{f}$. In the $(BC)_n$-case, there is also the possibility that twice a root of $\mathfrak{f}$ might be a root of $\mathfrak{u}_3$. It is only when $\mathfrak{u}_3$ is itself split that $\mathfrak{f}$ and $\mathfrak{u}_3$ coincide.

One calls $\mathfrak{f}$ the "split symmetry algebra". It contains as we shall see all the information about the billiard region [95]. How $\mathfrak{f}$ can be embedded as a subalgebra of $\mathfrak{u}_3$ is not a question that shall be of our concern here.

The Iwasawa decomposition and scalar coset Lagrangians

The purpose of this section is to use the Iwasawa decomposition for $\mathfrak{u}_3$ described in Section 6.4.5 to derive the scalar Lagrangian based on the coset space $\mathcal{U}_3/\mathcal{K}(\mathcal{U}_3)$. The important point is to understand the origin of the similarities between the two Lagrangians in Equation (5.45) and Equation (7.8) below.

The full algebra $\mathfrak{u}_3$ is subject to the root space decomposition

$$\mathfrak{u}_3 = \mathbf{g}_0 \oplus \bigoplus_{\lambda\in\Sigma} \mathbf{g}_\lambda \tag{7.3}$$

with respect to the restricted root system. For each restricted root λ, the space $\mathbf{g}_\lambda$ has dimension m_λ. The nilpotent algebra $\mathfrak{n}_3 \subset \mathfrak{u}_3$, consisting of positive root generators only, is the direct sum

$$\mathfrak{n}_3 = \bigoplus_{\lambda\in\Sigma^+} \mathbf{g}_\lambda \tag{7.4}$$

over positive roots. The Iwasawa decomposition of the U-duality algebra $\mathfrak{u}_3$ reads

$$\mathfrak{u}_3 = \mathfrak{k}_3 \oplus \mathfrak{a}_3 \oplus \mathfrak{n}_3 \tag{7.5}$$

(see Section 6.4.5). It is $\mathfrak{a}_3$ that appears in Equation (7.5) and *not* the full Cartan subalgebra $\mathfrak{h}_3$ since the compact part of $\mathfrak{h}_3$ belongs to $\mathfrak{k}_3$.

This implies that when constructing a Lagrangian based on the coset space $\mathcal{U}_3/\mathcal{K}(\mathcal{U}_3)$, the only part of $\mathfrak{u}_3$ that will show up in the Borel gauge is the Borel subalgebra

$$\mathfrak{b}_3 = \mathfrak{a}_3 \oplus \mathfrak{n}_3. \tag{7.6}$$

Thus, there will be a number of dilatons equal to the dimension of $\mathfrak{a}_3$, i.e., equal to the real rank of $\mathfrak{u}_3$, and axion fields for the restricted roots (with multiplicities).

More specifically, an (x-dependent) element of the coset space $\mathcal{U}_3/\mathcal{K}(\mathcal{U}_3)$ takes the form

$$\mathcal{V}(x) = \mathrm{Exp}\left[\phi(x)\cdot\mathfrak{a}_3\right]\,\mathrm{Exp}\left[\chi(x)\cdot\mathfrak{n}_3\right], \tag{7.7}$$

where the dilatons ϕ and the axions χ are coordinates on the coset space, and where x denotes an arbitrary set of parameters on which the coset element might depend. The corresponding Lagrangian becomes

$$\mathcal{L}_{\mathcal{U}_3/\mathcal{K}(\mathcal{U}_3)} = \sum_{i=1}^{\dim\mathfrak{a}_3} \partial_x\phi^{(i)}(x)\partial_x\phi^{(i)}(x) + \sum_{\alpha\in\Sigma^+}\sum_{s_\alpha=1}^{\mathrm{mult}\,\alpha} e^{2\alpha(\phi)}\left[\partial_x\chi^{(\alpha)}_{[s_\alpha]}(x)+\cdots\right]\left[\partial_x\chi^{(\alpha)}_{[s_\alpha]}(x)+\cdots\right], \tag{7.8}$$

where the sums over $s_\alpha = 1,\cdots,\mathrm{mult}\,\alpha$ are sums over the multiplicities of the positive restricted roots α.

By comparing Equation (7.8) with the corresponding expression (5.45) for the split case, it is clear why it is the maximal split subalgebra of the U-duality algebra that is relevant for the gravitational billiard. Were it not for the additional sum over multiplicities, Equation (7.8) would exactly be the Lagrangian for the coset space $\mathcal{F}/\mathcal{K}(\mathcal{F})$, where $\mathfrak{k}_\mathfrak{f} = \mathrm{Lie}\,\mathcal{K}(\mathcal{F})$ is the maximal compact subalgebra of $\mathfrak{f}$ (note that $\mathfrak{k}_\mathfrak{f} \neq \mathfrak{k}_3$). Recall now that from the point of view of the billiard, the positive roots correspond to walls that deflect the particle motion in the Cartan subalgebra. Therefore, multiplicities of roots are irrelevant since these will only result in several walls stacked on top of each other without affecting the dynamics. (In the $(BC)_n$-case, the wall associated with 2λ is furthermore subdominant with respect to the wall associated with λ when both λ and 2λ are restricted roots, so one can keep only the wall associated with λ. This follows from the fact that in the $(BC)_n$-case the root system of $\mathfrak{f}$ is taken to be of B_n-type.)

7.2 "Split symmetry controls chaos"

The main point of this section is to illustrate and explain the statement "split symmetry controls chaos" [95]. To this end, we will now extend the analysis of Section 5 to non-split real forms, using the Iwasawa decomposition. As we have seen, there are two main cases to be considered:

- The restricted root system Σ of $\mathfrak{u}_3$ is of *reduced* type, in which case it is one of the standard root systems for the Lie algebras $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7$ or E_8.
- The restricted root system, Σ, of $\mathfrak{u}_3$ is of *non-reduced* type, in which case it is of $(BC)_n$-type.

In the first case, the billiard is governed by the overextended algebra $\mathfrak{f}^{++}$, where $\mathfrak{f}$ is the "maximal split subalgebra" of $\mathfrak{u}_3$. Indeed, the coupling to gravity of the coset Lagrangian of Equation (7.8) will introduce, besides the simple roots of $\mathfrak{f}$ (electric walls) the affine root of $\mathfrak{f}$ (dominant magnetic wall) and the overextended root (symmetry wall), just as in the split case (but for $\mathfrak{f}$ instead of $\mathfrak{u}_3$). This is therefore a straightforward generalization of the analysis in Section 5.

The second case, however, introduces a new phenomenon, the *twisted overextensions* of Section 4. This is because the highest root of the $(BC)_n$ system differs from the highest root of the B_n system. Hence, the dominant magnetic wall will provide a twisted affine root, to which the symmetry wall will attach itself as usual [95].

We illustrate the two possible cases in terms of explicit examples. The first one is the simplest case for which a twisted overextension appears, namely the case of pure four-dimensional gravity coupled to a Maxwell field. This is the bosonic sector of $\mathcal{N} = 2$ supergravity in four dimensions, which has the non-split real form $\mathfrak{su}(2,1)$ as its U-duality symmetry. The restricted root system of $\mathfrak{su}(2,1)$ is the non-reduced $(BC)_1$-system, and, consequently, as we shall see explicitly, the billiard is governed by the twisted overextension $A_2^{(2)+}$.

The second example is that of heterotic supergravity, which exhibits an $SO(8,24)/(SO(8) \times SO(24))$ coset symmetry in three dimensions. The U-duality algebra is thus $\mathfrak{so}(8,24)$, which is non-split. In this example, however, the restricted root system is B_8, which is reduced, and so the billiard is governed by a standard overextension of the maximal split subalgebra $\mathfrak{so}(8,9) \subset \mathfrak{so}(8,24)$.

7.2.1 $(BC)_1$ and $\mathcal{N} = 2$, $D = 4$ pure supergravity

We consider $\mathcal{N} = 2$ supergravity in four dimensions where the bosonic sector consists of gravity coupled to a Maxwell field. It is illuminating to compare the construction of the billiard in the two limiting dimensions, $D = 4$ and $D = 3$.

In maximal dimension the metric contains three scale factors, β^1, β^2 and β^3, which give rise to three symmetry wall forms,

$$s_{21}(\beta) = \beta^2 - \beta^1, \qquad s_{32}(\beta) = \beta^3 - \beta^2, \qquad s_{31}(\beta) = \beta^3 - \beta^1, \tag{7.9}$$

where only s_{21} and s_{32} are dominant. In four dimensions the curvature walls read

$$c_{123}(\beta) \equiv c_1(\beta) = 2\beta^1, \qquad c_{231}(\beta) \equiv c_2(\beta) = 2\beta^2, \qquad c_{312}(\beta) \equiv c_3(\beta) = 2\beta^3. \tag{7.10}$$

Finally we have the electric and magnetic wall forms of the Maxwell field. These are equal because there is no dilaton. Hence, the wall forms are

$$e_1(\beta) = m_1(\beta) = \beta^1, \qquad e_2(\beta) = m_2(\beta) = \beta^2, \qquad e_3(\beta) = m_3(\beta) = \beta^3. \tag{7.11}$$

The billiard region $\mathcal{B}_{\mathcal{M}_\beta}$ is defined by the set of dominant wall forms,

$$\mathcal{B}_{\mathcal{M}_\beta} = \{\beta \in \mathcal{M}_\beta \,|\, e_1(\beta), s_{21}(\beta), s_{32}(\beta) \ > \ 0\}. \tag{7.12}$$

The first dominant wall form, $e_1(\beta)$, is twice degenerate because it occurs once as an electric wall form and once as a magnetic wall form. Because of the existence of the curvature wall, $c_1(\beta) = 2\beta^1$, we see that $2\alpha_1$ is also a root.

The same billiard emerges after reduction to three spacetime dimensions, where the algebraic structure is easier to exhibit. As before, we perform the reduction along the first spatial direction. The associated scale factor is then replaced by the Kaluza–Klein dilaton $\hat{\varphi}$ such that

$$\beta^1 = \frac{1}{\sqrt{2}}\hat{\varphi}. \tag{7.13}$$

The remaining scale factors change accordingly,

$$\beta^2 = \hat{\beta}^2 - \frac{1}{\sqrt{2}}\hat{\varphi}, \qquad \beta^3 = \hat{\beta}^3 - \frac{1}{\sqrt{2}}\hat{\varphi}, \tag{7.14}$$

and the two symmetry walls become

$$s_{21}(\hat{\beta}, \hat{\varphi}) = \hat{\beta}^2 - \sqrt{2}\hat{\varphi}, \qquad \hat{s}_{32}(\hat{\beta}) = \hat{\beta}^3 - \hat{\beta}^2. \tag{7.15}$$

In addition to the dilaton $\hat{\varphi}$, there are three axions: one $(\hat{\chi})$ arising from the dualization of the Kaluza–Klein vector, one $(\hat{\chi}^E)$ coming from the component A_1 of the Maxwell vector potential and one $(\hat{\chi}^C)$ coming from dualization of the Maxwell vector potential in 3 dimensions (see, e.g., [35] for a review). There are then a total of four scalars. These parametrize the coset space $SU(2,1)/S(U(2) \times U(1))$ [113].

The Einstein–Maxwell Lagrangian in four dimensions yields indeed in three dimensions the Einstein–scalar Lagrangian, where the Lagrangian for the scalar fields is given by

$$\mathcal{L}_{SU(2,1)/S(U(2)\times U(1))} = \partial_\mu\hat{\varphi}\partial^\mu\hat{\varphi} + e^{2e_1(\hat{\varphi})}\left(\partial_\mu\hat{\chi}^E\partial^\mu\hat{\chi}^E + \partial_\mu\hat{\chi}^C\partial^\mu\hat{\chi}^C\right) + e^{4e_1(\hat{\varphi})}\left(\partial_\mu\hat{\chi}\partial^\mu\hat{\chi}\right) + \cdots \tag{7.16}$$

with

$$e_1(\hat{\varphi}) = \frac{1}{\sqrt{2}}\hat{\varphi}.$$

Here, the ellipses denotes terms that are not relevant for understanding the billiard structure. The U-duality algebra of $\mathcal{N} = 2$ supergravity compactified to three dimensions is therefore

$$\mathfrak{u}_3 = \mathfrak{su}(2,1), \tag{7.17}$$

which is a non-split real form of the complex Lie algebra $\mathfrak{sl}(3,\mathbb{C})$. This is in agreement with Table 1 of [113]. The restricted root system of $\mathfrak{su}(2,1)$ is of $(BC)_1$-type (see Table 28 in Section 6.8) and has four roots: α_1, $2\alpha_1$, $-\alpha_1$ and $-2\alpha_1$. One may take α_1 to be the simple root, in which case $\Sigma_+ = \{\alpha_1, 2\alpha_1\}$ and $2\alpha_1$ is the highest root. The short root α_1 is degenerate twice while the long root $2\alpha_1$ is nondegenerate. The Lagrangian (7.16) coincides with the Lagrangian (7.8) for $\mathfrak{su}(2,1)$ with the identification

$$\hat{\alpha}_1 \equiv e_1. \tag{7.18}$$

We clearly see from the Lagrangian that the simple root $\hat{\alpha}_1$ has multiplicity 2 in the restricted root system, since the corresponding wall appears twice. The maximal split subalgebra may be taken to be $A_1 \equiv \mathfrak{su}(1,1)$ with root system $\{\hat{\alpha}_1, -\hat{\alpha}_1\}$.

Let us now see how one goes from $\mathfrak{su}(2,1)$ described by the scalar Lagrangian to the full algebra, by including the gravitational scale factors. Let us examine in particular how the twist arises. For the standard root system of A_1 the highest root is just $\hat{\alpha}_1$. However, as we have seen, for the $(BC)_1$ root system the highest root is $\theta_{(BC)_1} = 2\hat{\alpha}_1$, with

$$(\theta_{(BC)_1}|\theta_{(BC)_1}) = 4(\hat{\alpha}_1|\hat{\alpha}_1) = 2. \tag{7.19}$$

So we see that because of $(\hat{\alpha}_1|\hat{\alpha}_1) = \frac{1}{2}$, the highest root $\theta_{(BC)_1}$ already comes with the desired normalization. The affine root is therefore

$$\hat{\alpha}_2(\hat{\varphi}, \hat{\beta}) = \hat{m}_2^{\hat{\chi}}(\hat{\beta}, \hat{\varphi}) = \hat{\beta}^2 - \theta_{(BC)_1} = \hat{\beta}^2 - \sqrt{2}\hat{\varphi}, \tag{7.20}$$

whose norm is

$$(\hat{\alpha}_2|\hat{\alpha}_2) = 2. \tag{7.21}$$

The scalar product between $\hat{\alpha}_1$ and $\hat{\alpha}_2$ is $(\hat{\alpha}_1|\hat{\alpha}_2) = -1$ and the Cartan matrix at this stage becomes $(i, j = 1, 2)$

$$A_{ij}[A_2^{(2)}] = 2\frac{(\hat{\alpha}_i|\hat{\alpha}_j)}{(\hat{\alpha}_i|\hat{\alpha}_i)} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \tag{7.22}$$

which may be identified not with the affine extension of A_1 but with the Cartan matrix of the *twisted* affine Kac–Moody algebra $A_2^{(2)}$. It is the underlying $(BC)_1$ root system that is solely responsible for the appearance of the twist. Because of the fact that $\theta_{(BC)_1} = 2\hat{\alpha}_1$ the two simple roots of the affine extension come with different length and hence the asymmetric Cartan matrix in Equation (7.22). It remains to include the overextended root

$$\hat{\alpha}_3(\hat{\beta}) = \hat{s}_{32}(\hat{\beta}) = \hat{\beta}^3 - \hat{\beta}^2, \tag{7.23}$$

which has non-vanishing scalar product only with $\hat{\alpha}_2$, $(\hat{\alpha}_2|\hat{\alpha}_3) = -1$, and so its node in the Dynkin diagram is attached to the second node by a single link. The complete Cartan matrix is

$$A[A_2^{(2)+}] = \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \tag{7.24}$$

which is the Cartan matrix of the Lorentzian extension $A_2^{(2)+}$ of $A_2^{(2)}$ henceforth referred to as the *twisted overextension* of A_1. Its Dynkin diagram is displayed in Figure 41.

The algebra $A_2^{(2)+}$ was already analyzed in Section 4, where it was shown that its Weyl group coincides with the Weyl group of the algebra A_1^{++}. Thus, in the BKL-limit the dynamics of the coupled Einstein–Maxwell system in four-dimensions is equivalent to that of pure four-dimensional gravity, although the set of dominant walls are different. Both theories are chaotic.

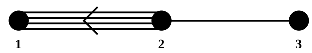


Figure 41: The Dynkin diagram of $A_2^{(2)+}$. Label 1 denotes the simple root $\hat{\alpha}_{(1)}$ of the restricted root system of $\mathfrak{u}_3 = \mathfrak{su}(2,1)$. Labels 2 and 3 correspond to the affine and overextended roots, respectively. The arrow points towards the short root which is normalized such that $(\hat{\alpha}_1|\hat{\alpha}_1) = \frac{1}{2}$.

7.2.2 Heterotic supergravity and $\mathfrak{so}(8, 24)$

Pure $\mathcal{N} = 1$ supergravity in $D = 10$ dimensions has a billiard description in terms of the hyperbolic Kac–Moody algebra $DE_{10} = D_8^{++}$ [45]. This algebra is the overextension of the U-duality algebra, $\mathfrak{u}_3 = D_8 \equiv \mathfrak{so}(8,8)$, appearing upon compactification to three dimensions. In this case, $\mathfrak{so}(8,8)$ is the split form of the complex Lie algebra D_8, so we have $\mathfrak{f} = \mathfrak{u}_3$.

By adding one Maxwell field to the theory we modify the billiard to the hyperbolic Kac–Moody algebra $BE_{10} = B_8^{++}$, which is the overextension of the split form $\mathfrak{so}(8,9)$ of B_8 [45]. This is the case relevant for (the bosonic sector of) Type I supergravity in ten dimensions. In both these cases the relevant Kac–Moody algebra is the overextension of a split real form and so falls under the classification given in Section 5.

Let us now consider an interesting example for which the relevant U-duality algebra is non-split. For the heterotic string, the bosonic field content of the corresponding supergravity is given by pure gravity coupled to a dilaton, a 2-form and an $E_8 \times E_8$ Yang–Mills gauge field. Assuming the gauge field to be in the Cartan subalgebra, this amounts to adding 16 $\mathcal{N} = 1$ vector multiplets in the bosonic sector, i.e, to adding 16 Maxwell fields to the ten-dimensional theory discussed above. Geometrically, these 16 Maxwell fields correspond to the Kaluza–Klein vectors arising from the compactification on T^{16} of the 26-dimensional bosonic left-moving sector of the heterotic string [89].

Consequently, the relevant U-duality algebra is $\mathfrak{so}(8, 8+16) = \mathfrak{so}(8, 24)$ which is a non-split real form. But we know that the billiard for the heterotic string is governed by the same Kac–Moody algebra as for the Type I case mentioned above, namely $BE_{10} \equiv \mathfrak{so}(8, 9)^{++}$, and not $\mathfrak{so}(8, 24)^{++}$ as one might have expected [45]. The only difference is that the walls associated with the one-forms are degenerate 16 times. We now want to understand this apparent discrepancy using the machinery of non-split real forms exhibited in previous sections. The same discussion applies to the $SO(32)$-superstring.

In three dimensions the heterotic supergravity Lagrangian is given by a pure three-dimensional Einstein–Hilbert term coupled to a nonlinear sigma model for the coset $SO(8, 24)/(SO(8) \times SO(24))$. This Lagrangian can be understood by analyzing the Iwasawa decomposition of $\mathfrak{so}(8, 24) = \text{Lie}\,[SO(8, 24)]$. The maximal compact subalgebra is

$$\mathfrak{k}_3 = \mathfrak{so}(8) \oplus \mathfrak{so}(24). \tag{7.25}$$

This subalgebra does not appear in the sigma model since it is divided out in the coset construction (see Equation (7.7)) and hence we only need to investigate the Borel subalgebra $\mathfrak{a}_3 \oplus \mathfrak{n}_3$ of $\mathfrak{so}(8, 24)$ in more detail.

As was emphasized in Section 7.1, an important feature of the Iwasawa decomposition is that the full Cartan subalgebra $\mathfrak{h}_3$ does not appear explicitly but only the maximal noncompact Cartan subalgebra $\mathfrak{a}_3$, associated with the restricted root system. This is the maximal Abelian subalgebra of $\mathfrak{u}_3 = \mathfrak{so}(8, 24)$, whose adjoint action can be diagonalized over the reals. The remaining Cartan generators of $\mathfrak{h}_3$ are compact and so their adjoint actions have imaginary eigenvalues. The general case of $\mathfrak{so}(2q, 2p)$ was analyzed in detail in Section 6.7 where it was found that if $q < p$, the restricted root system is of type B_{2q}. For the case at hand we have $q = 4$ and $p = 12$ which implies that the restricted root system of $\mathfrak{so}(8, 24)$ is (modulo multiplicities) $\Sigma_{\mathfrak{so}(8,24)} = B_8$.

The root system of B_8 is eight-dimensional and hence there are eight Cartan generators that may be simultaneously diagonalized over the real numbers. Therefore the real rank of $\mathfrak{so}(8, 24)$ is eight, i.e.,

$$\text{rank}_{\mathbb{R}}\, \mathfrak{u}_3 = \dim \mathfrak{a}_3 = 8. \tag{7.26}$$

Moreover, it was shown in Section 6.7 that the restricted root system of $\mathfrak{so}(2q, 2p)$ has $4q(2q - 1)$ long roots which are nondegenerate, i.e., with multiplicity one, and $4q$ long roots with multiplicities $2(p-q)$. In the example under consideration this corresponds to seven nondegenerate simple roots $\alpha_1, \cdots, \alpha_7$ and one short simple root α_8 with multiplicity 16. The Dynkin diagram for the restricted root system $\Sigma_{\mathfrak{so}(8,24)}$ is displayed in Figure 42 with the multiplicity indicated in brackets over the short root. It is important to note that the restricted root system $\Sigma_{\mathfrak{so}(8,24)}$ differs from the standard root system of $\mathfrak{so}(8, 9)$ precisely because of the multiplicity 16 of the simple root α_8.

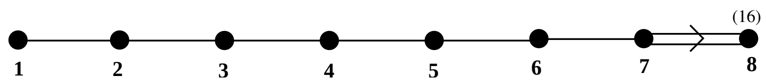


Figure 42: The Dynkin diagram representing the restricted root system $\Sigma_{\mathfrak{so}(8,24)}$ of $\mathfrak{so}(8, 24)$. Labels $1, \cdots, 7$ denote the long simple roots that are nondegenerate while the eighth simple root is short and has multiplicity 16.

Because of these properties of $\mathfrak{so}(8,24)$ the Lagrangian for the coset

$$\frac{SO(8,24)}{SO(8)\times SO(24)} \tag{7.27}$$

takes a form very similar to the Lagrangian for the coset

$$\frac{SO(8,9)}{SO(8)\times SO(9)}. \tag{7.28}$$

The algebra constructed from the restricted root system B_8 is the maximal split subalgebra

$$\mathfrak{f} = \mathfrak{so}(8,9). \tag{7.29}$$

Let us now take a closer look at the Lagrangian in three spacetime dimensions. We parametrize an element of the coset by

$$\mathcal{V}(x^\mu) = \mathrm{Exp}\left[\sum_{i=1}^{8}\phi^{(i)}(x^\mu)\alpha_i^\vee\right]\mathrm{Exp}\left[\sum_{\gamma\in\Delta_+}\chi^{(\gamma)}(x^\mu)E_\gamma\right] \in \frac{SO(8,24)}{SO(8)\times SO(24)}, \tag{7.30}$$

where x^μ ($\mu=0,1,2$) are the coordinates of the external three-dimensional spacetime, $\alpha_i^\vee$ are the noncompact Cartan generators and Δ_+ denotes the full set of positive roots of $\mathfrak{so}(8,24)$.

The Lagrangian constructed from the coset representative in Equation (7.30) becomes (again, neglecting corrections to the single derivative terms of the form "$\partial_x\chi$")

$$\begin{aligned}\mathcal{L}_{\mathcal{U}_3/\mathcal{K}(\mathcal{U}_3)} = &\sum_{i=1}^{8}\partial_\mu\phi^{(i)}(x)\,\partial^\mu\phi^{(i)}(x) + \sum_{j=1}^{7}e^{\alpha_j(\phi)}\,\partial_\mu\chi^{(j)}(x)\,\partial^\mu\chi^{(j)}(x)\\ &+e^{\alpha_8(\phi)}\left(\sum_{k=1}^{16}\partial_\mu\chi^{(8)}_{[k]}(x)\,\partial^\mu\chi^{(8)}_{[k]}(x)\right)+\sum_{\alpha\in\tilde{\Sigma}^+}\sum_{s_\alpha=1}^{\mathrm{mult}(\alpha)}e^{\alpha(\phi)}\,\partial_\mu\chi^{(\alpha)}_{[s_\alpha]}(x)\,\partial^\mu\chi^{(\alpha)}_{[s_\alpha]}(x),\end{aligned} \tag{7.31}$$

where $\tilde{\Sigma}^+$ denotes all non-simple positive roots of Σ, i.e.,

$$\tilde{\Sigma}^+ = \Sigma^+/\bar{B} \tag{7.32}$$

with

$$\bar{B} = \{\alpha_1,\cdots,\alpha_8\}. \tag{7.33}$$

This Lagrangian is equivalent to the Lagrangian for $SO(8,9)/(SO(8)\times SO(9))$ except for the existence of the non-trivial root multiplicities.

The billiard for this theory can now be computed with the same methods that were treated in detail in Section 5.3.3. In the BKL-limit, the simple roots $\alpha_1,\cdots,\alpha_8$ become the non-gravitational dominant wall forms. In addition to this we get one magnetic and one gravitational dominant wall form:

$$\begin{aligned}\alpha_0 &= \beta^1-\theta(\phi),\\ \alpha_{-1} &= \beta^2-\beta^1,\end{aligned} \tag{7.34}$$

where $\theta(\phi)$ is the highest root of $\mathfrak{so}(8,9)$:

$$\theta = \alpha_1+2\alpha_2+\cdots+2\alpha_7+\alpha_8. \tag{7.35}$$

The affine root α_0 attaches with a single link to the second simple root α_2 in the Dynkin diagram of B_8. Similarly the overextended root α_{-1} attaches to α_0 with a single link so that the resulting

Dynkin diagram corresponds to BE_{10} (see Figure 43). It is important to note that the underlying root system is still an overextension of the restricted root system and hence the multiplicity of the simple short root α_8 remains equal to 16. Of course, this does not affect the dynamics in the BKL-limit because the multiplicity of α_8 simply translates to having multiple electric walls on top of each other and this does not alter the billiard motion.

This analysis again showed explicitly how it is always the split symmetry that controls the chaotic behavior in the BKL-limit. It is important to point out that when going beyond the strict BKL-limit, one expects more and more roots of the algebra to play a role. Then it is no longer sufficient to study only the maximal split subalgebra $\mathfrak{so}(8,9)^{++}$ but instead the symmetry of the theory is believed to contain the full algebra $\mathfrak{so}(8,24)^{++}$. In the spirit of [47] one may then conjecture that the dynamics of the heterotic supergravity should be equivalent to a null geodesic on the coset space $SO(8,24)^{++}/\mathcal{K}(SO(8,24)^{++})$ [42].

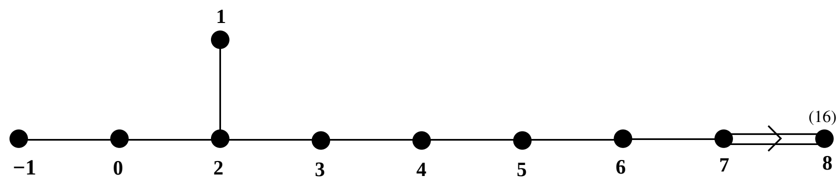


Figure 43: The Dynkin diagram representing the overextension B_8^{++} of the restricted root system $\Sigma = B_8$ of $\mathfrak{so}(8,24)$. Labels $-1, 0, 1, \cdots, 7$ denote the long simple roots that are nondegenerate while the eighth simple root is short and has multiplicity 16.

7.3 Models associated with non-split real forms

In this section we provide a list of all theories coupled to gravity which, upon compactification to three dimensions, display U-duality algebras that are *not* maximal split [95]. This therefore completes the classification of Section 5.

One can classify the various theories through the number $\mathcal{N}$ of supersymmetries that they possess in $D = 4$ spacetime dimensions. All p-forms can be dualized to scalars or to 1-forms in four dimensions so the theories all take the form of pure supergravities coupled to collections of Maxwell multiplets. The analysis performed for the split forms in Section 5.3 were all concerned with the cases of $\mathcal{N} = 0$ or $\mathcal{N} = 8$ supergravity in $D = 4$. We consider all pure four-dimensional supergravities ($\mathcal{N} = 1, \cdots, 8$) as well as pure $\mathcal{N} = 4$ supergravity coupled to k Maxwell multiplets.

As we have pointed out, the main new feature in the non-split cases is the possible appearance of so-called *twisted overextensions*. These arise when the restricted root system of $\mathcal{U}_3$ is of non-reduced type hence yielding a twisted affine Kac–Moody algebra in the affine extension of $\mathfrak{f} \subset \mathfrak{u}_3$. It turns out that the only cases for which the restricted root system is of non-reduced ((BC)-type) is for the pure $\mathcal{N} = 2, 3$ and $\mathcal{N} = 5$ supergravities. The example of $\mathcal{N} = 2$ was already discussed in detail before, where it was found that the U-duality algebra is $\mathfrak{u}_3 = \mathfrak{su}(2,1)$ whose restricted root system is $(BC)_1$, thus giving rise to the twisted overextension $A_2^{(2)+}$. It turns out that for the $\mathcal{N} = 3$ case the same twisted overextension appears. This is due to the fact that the U-duality algebra is $\mathfrak{u}_3 = \mathfrak{su}(4,1)$ which has the same restricted root system as $\mathfrak{su}(2,1)$, namely $(BC)_1$. Hence, $A_1^{(2)+}$ controls the BKL-limit also for this theory.

The case of $\mathcal{N} = 5$ follows along similar lines. In $D = 3$ the non-split form $E_{6(-14)}$ of E_6 appears, whose maximal split subalgebra is $\mathfrak{f} = C_2$. However, the relevant Kac–Moody algebra is not C_2^{++} but rather $A_4^{(2)+}$ because the restricted root system of $E_{6(-14)}$ is $(BC)_2$.

In Table 37 we display the algebraic structure for all pure supergravities in four dimensions as well as for $\mathcal{N} = 4$ supergravity with k Maxwell multiplets. We give the relevant U-duality

Table 37: Classification of theories whose U-duality symmetry in three dimensions is described by a non-split real form $\mathfrak{u}_3$. The leftmost column indicates the number $\mathcal{N}$ of supersymmetries that the theories possess when compactified to four dimensions, and the associated number k of Maxwell multiplets. The middle column gives the restricted root system Σ of $\mathfrak{u}_3$ and to the right of this we give the maximal split subalgebras $\mathfrak{f} \subset \mathfrak{u}_3$, constructed from a basis of Σ. Finally, the rightmost column displays the overextended Kac–Moody algebras that governs the billiard dynamics.

$(\mathcal{N}, k)$	$\mathfrak{u}_3$	Σ	$\mathfrak{f}$	$\mathfrak{g}$
(1,0)	$\mathfrak{sl}(2,\mathbb{R})$	A_1	A_1	A_1^{++}
(2,0)	$\mathfrak{su}(2,1)$	$(BC)_1$	A_1	$A_2^{(2)+}$
(3,0)	$\mathfrak{su}(4,1)$	$(BC)_1$	A_1	$A_1^{(2)+}$
(4,0)	$\mathfrak{so}(8,2)$	C_2	C_2	C_2^{++}
$(4, k<6)$	$\mathfrak{so}(8,k+2)$	B_{k+2}	B_{k+2}	B_{k+2}^{++}
$(4,6)$	$\mathfrak{so}(8,8)$	D_8	D_8	$DE_{10} = D_8^{++}$
$(4, k>6)$	$\mathfrak{so}(8,k+2)$	B_8	B_8	$BE_{10} = B_8^{++}$
(5,0)	$E_{6(-14)}$	$(BC)_2$	C_2	$A_4^{(2)+}$
(6,0)	$E_{7(-5)}$	F_4	F_4	F_4^{++}
(8,0)	$E_{8(+8)}$	E_8	E_8	$E_{10} = E_8^{++}$

algebras $\mathfrak{u}_3$, the restricted root systems Σ, the maximal split subalgebras $\mathfrak{f}$ and, finally, the resulting overextended Kac–Moody algebras $\mathfrak{g}$.

Let us end this section by noting that the study of real forms of hyperbolic Kac–Moody algebras has been pursued in [17].

8 Level Decomposition in Terms of Finite Regular Subalgebras

We have shown in the previous sections that Weyl groups of Lorentzian Kac–Moody algebras naturally emerge when analysing gravity in the extreme BKL regime. This has led to the conjecture that the corresponding Kac–Moody algebra is in fact a symmetry of the theory (most probably enlarged with new fields) [46]. The idea is that the BKL analysis is only the "revelator" of that huge symmetry, which would exist independently of that limit, without making the BKL truncations. Thus, if this conjecture is true, there should be a way to rewrite the gravity Lagrangians in such a way that the Kac–Moody symmetry is manifest. This conjecture itself was made previously (in this form or in similar ones) by other authors on the basis of different considerations [113, 139, 167]. To explore this conjecture, it is desirable to have a concrete method of dealing with the infinite-dimensional structure of a Lorentzian Kac–Moody algebra $\mathfrak{g}$. In this section we present such a method.

The method by which we shall deal with the infinite-dimensional structure of a Lorentzian Kac–Moody algebra $\mathfrak{g}$ is based on a certain gradation of $\mathfrak{g}$ into finite-dimensional subspaces $\mathfrak{g}_\ell$. More precisely, we will define a so-called *level decomposition* of the adjoint representation of $\mathfrak{g}$ such that each level ℓ corresponds to a finite number of representations of a finite regular subalgebra $\mathfrak{r}$ of $\mathfrak{g}$. Generically the decomposition will take the form of the adjoint representation of $\mathfrak{r}$ plus a (possibly infinite) number of additional representations of $\mathfrak{r}$. This type of expansion of $\mathfrak{g}$ will prove to be very useful when considering sigma models invariant under $\mathfrak{g}$ for which we may use the level expansion to consistently truncate the theory to any finite level ℓ (see Section 9).

We begin by illustrating these ideas for the finite-dimensional Lie algebra $\mathfrak{sl}(3,\mathbb{R})$ after which we generalize the procedure to the indefinite case in Sections 8.2, 8.3 and 8.4.

8.1 A finite-dimensional example: $\mathfrak{sl}(3,\mathbb{R})$

The rank 2 Lie algebra $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ is characterized by the Cartan matrix

$$A[\mathfrak{sl}(3,\mathbb{R})] = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{8.1}$$

whose Dynkin diagram is displayed in Figure 44.

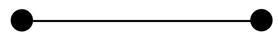


Figure 44: The Dynkin diagram of $\mathfrak{sl}(3,\mathbb{R})$.

Recall from Section 6 that $\mathfrak{sl}(3,\mathbb{R})$ is the split real form of $\mathfrak{sl}(3,\mathbb{C}) \equiv A_2$, and is thus defined through the same Chevalley–Serre presentation as for $\mathfrak{sl}(3,\mathbb{C})$, but with all coefficients restricted to the real numbers.

The Cartan generators $\{h_1, h_2\}$ will indifferently be denoted by $\{\alpha_1^\vee, \alpha_2^\vee\}$. As we have seen, they form a basis of the Cartan subalgebra $\mathfrak{h}$, while the simple roots $\{\alpha_1, \alpha_2\}$, associated with the raising operators e_1 and e_2, form a basis of the dual root space $\mathfrak{h}^\star$. Any root $\gamma \in \mathfrak{h}^\star$ can thus be decomposed in terms of the simple roots as follows,

$$\gamma = m\alpha_1 + \ell\alpha_2, \tag{8.2}$$

and the only values of (m, n) are $(1,0)$, $(0,1)$, $(1,1)$ for the positive roots and minus these values for the negative ones.

The algebra $\mathfrak{sl}(3,\mathbb{R})$ defines through the adjoint action a representation of $\mathfrak{sl}(3,\mathbb{R})$ itself, called the adjoint representation, which is eight-dimensional and denoted **8**. The weights of the adjoint representation are the roots, plus the weight $(0,0)$ which is doubly degenerate. The lowest weight of the adjoint representation is

$$\Lambda_{\mathfrak{g}} = -\alpha_1 - \alpha_2, \tag{8.3}$$

corresponding to the generator $[f_1, f_2]$. We display the weights of the adjoint representation in Figure 45.

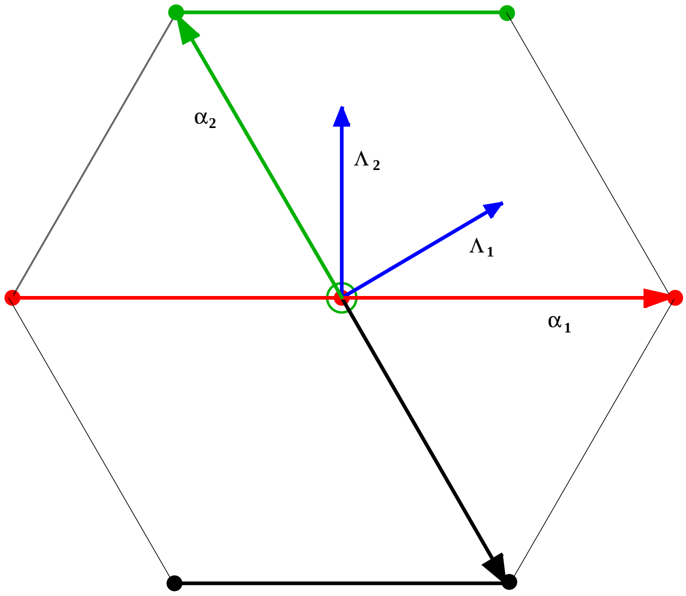


Figure 45: Level decomposition of the adjoint representation $\mathcal{R}_{\mathrm{ad}} = \mathbf{8}$ of $\mathfrak{sl}(3,\mathbb{R})$ into representations of the subalgebra $\mathfrak{sl}(2,\mathbb{R})$. The labels 1 and 2 indicate the simple roots α_1 and α_2. Level zero corresponds to the horizontal axis where we find the adjoint representation $\mathcal{R}^{(0)}_{\mathrm{ad}} = \mathbf{3}_0$ of $\mathfrak{sl}(2,\mathbb{R})$ (red nodes) and the singlet representation $\mathcal{R}^{(0)}_s = \mathbf{1}_0$ (green circle about the origin). At level one we find the two-dimensional representation $\mathcal{R}^{(1)} = \mathbf{2}_1$ (green nodes). The black arrow denotes the negative level root $-\alpha_2$ and so gives rise to the level $\ell = -1$ representation $\mathcal{R}^{(-1)} = \mathbf{2}_{(-1)}$. The blue arrows represent the fundamental weights Λ_1 and Λ_2.

The idea of the level decomposition is to decompose the adjoint representation into representations of one of the regular $\mathfrak{sl}(2,\mathbb{R})$-subalgebras associated with one of the two simple roots α_1 or α_2, i.e., either $\{e_1, \alpha_1^\vee, f_1\}$ or $\{e_2, \alpha_2^\vee, f_2\}$. For definiteness we choose the level to count the number ℓ of times the root α_2 occurs, as was anticipated by the notation in Equation (8.2). Consider the subspace of the adjoint representation spanned by the vectors with a fixed value of ℓ. This subspace is invariant under the action of the subalgebra $\{e_1, \alpha_1^\vee, f_1\}$, which only changes the value of m. Vectors at a definite level transform accordingly in a representation of the regular $\mathfrak{sl}(2,\mathbb{R})$-subalgebra

$$\mathfrak{r} \equiv \mathbb{R}e_1 \oplus \mathbb{R}\alpha_1^\vee \oplus \mathbb{R}f_1. \tag{8.4}$$

Let us begin by analyzing states at level $\ell = 0$, i.e., with weights of the form $\gamma = m\alpha_1$. We see from Figure 45 that we are restricted to move along the horizontal axis in the root diagram. By the defining Lie algebra relations we know that $\mathrm{ad}_{f_1}(f_1) = 0$, implying that $\Lambda^{(0)}_{\mathrm{ad}} = -\alpha_1$ is a lowest weight of the $\mathfrak{sl}(2,\mathbb{R})$-representation. Here, the superscript 0 indicates that this is a level $\ell = 0$ representation. The corresponding complete irreducible module is found by acting on f_1 with e_1, yielding

$$[e_1, f_1] = \alpha_1^\vee, \qquad [e_1, \alpha_1^\vee] = -2e_1, \qquad [e_1, e_1] = 0. \tag{8.5}$$

We can then conclude that $\Lambda^{(0)}_{\text{ad}} = -\alpha_1$ is the lowest weight of the three-dimensional adjoint representation $\mathbf{3}_0$ of $\mathfrak{sl}(2,\mathbb{R})$ with weights $\{\Lambda^{(0)}_{\text{ad}}, 0, -\Lambda^{(0)}_{\text{ad}}\}$, where the subscript on $\mathbf{3}_0$ again indicates that this representation is located at level $\ell = 0$ in the decomposition. The module for this representation is $\mathcal{L}(\Lambda^{(0)}_{\text{ad}}) = \text{span}\{f_1, \alpha_1^\vee, e_1\}$.

This is, however, not the complete content at level zero since we must also take into account the Cartan generator $\alpha_2^\vee$ which remains at the origin of the root diagram. We can combine $\alpha_2^\vee$ with $\alpha_1^\vee$ into the vector

$$h = \alpha_1^\vee + 2\alpha_2^\vee, \tag{8.6}$$

which constitutes the one-dimensional singlet representation $\mathbf{1}_0$ of $\mathfrak{r}$ since it is left invariant under all generators of $\mathfrak{r}$,

$$[e_1, h] = [f_1, h] = [\alpha_1^\vee, h] = 0, \tag{8.7}$$

as follows trivially from the Chevalley relations. Thus level zero contains the representations $\mathbf{3}_0$ and $\mathbf{1}_0$.

Note that the vectors at level 0 not only transform in a (reducible) representation of $\mathfrak{sl}(2,\mathbb{R})$, but also form a subalgebra since the level is additive under taking commutators. The algebra in question is $\mathfrak{gl}(2,\mathbb{R}) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$. Accordingly, if the generator $\alpha_2^\vee$ is added to the subalgebra $\mathfrak{r}$, through the combination in Equation (8.6), so as to take the entire $\ell = 0$ subspace, $\mathfrak{r}$ is enlarged from $\mathfrak{sl}(2,\mathbb{R})$ to $\mathfrak{gl}(2,\mathbb{R})$, the generator h being somehow the "trace" part of $\mathfrak{gl}(2,\mathbb{R})$. This fact will prove to be important in subsequent sections.

Let us now ascend to the next level, $\ell = 1$. The weights of $\mathfrak{r}$ at level 1 take the general form $\gamma = m\alpha_1 + \alpha_2$ and the lowest weight is $\Lambda^{(1)} = \alpha_2$, which follows from the vanishing of the commutator

$$[f_1, e_2] = 0. \tag{8.8}$$

Note that $m \geq 0$ whenever $\ell > 0$ since $m\alpha_1 + \ell\alpha_2$ is then a positive root. The complete representation is found by acting on the lowest weight $\Lambda^{(1)}$ with e_1 and we get that the commutator $[e_1, e_2]$ is allowed by the Serre relations, while $[e_1, [e_1, e_2]]$ is killed, i.e.,

$$\begin{aligned} [e_1, e_2] &\neq 0, \\ [e_1, [e_1, e_2]] &= 0. \end{aligned} \tag{8.9}$$

The non-vanishing commutator $e_\theta \equiv [e_1, e_2]$ is the vector associated with the highest root θ of $\mathfrak{sl}(3,\mathbb{R})$ given by

$$\theta = \alpha_1 + \alpha_2. \tag{8.10}$$

This is just the negative of the lowest weight $\Lambda_{\mathfrak{g}}$. The only representation at level one is thus the two-dimensional representation $\mathbf{2}_1$ of $\mathfrak{r}$ with weights $\{\Lambda^{(1)}, \theta\}$. The decomposition stops at level one for $\mathfrak{sl}(3,\mathbb{R})$ because any commutator with two e_2's vanishes by the Serre relations. The negative level representations may be found simply by applying the Chevalley involution and the result is the same as for level one.

Hence, the total level decomposition of $\mathfrak{sl}(3,\mathbb{R})$ in terms of the subalgebra $\mathfrak{sl}(2,\mathbb{R})$ is given by

$$\mathbf{8} = \mathbf{3}_0 \oplus \mathbf{1}_0 \oplus \mathbf{2}_1 \oplus \mathbf{2}_{(-1)}. \tag{8.11}$$

Although extremely simple (and familiar), this example illustrates well the situation encountered with more involved cases below. In the following analysis we will not mention the negative levels any longer because these can always be obtained simply through a reflection with respect to the $\ell = 0$ "hyperplane", using the Chevalley involution.

8.2 Some formal considerations

Before we proceed with a more involved example, let us formalize the procedure outlined above. We mainly follow the excellent treatment given in [124], although we restrict ourselves to the cases where $\mathfrak{r}$ is a *finite* regular subalgebra of $\mathfrak{g}$.

In the previous example, we performed the decomposition of the roots (and the ensuing decomposition of the algebra) with respect to one of the simple roots which then defined the level. In general, one may consider a similar decomposition of the roots of a rank r Kac–Moody algebra with respect to an arbitrary number $s < r$ of the simple roots and then the level ℓ is generalized to the "multilevel" $\ell = (\ell_1, \cdots, \ell_s)$.

8.2.1 Gradation

We consider a Kac–Moody algebra $\mathfrak{g}$ of rank r and we let $\mathfrak{r} \subset \mathfrak{g}$ be a finite regular rank $m < r$ subalgebra of $\mathfrak{g}$ whose Dynkin diagram is obtained by deleting a set of nodes $\mathcal{N} = \{n_1, \cdots, n_s\}$ $(s = r - m)$ from the Dynkin diagram of $\mathfrak{g}$.

Let γ be a root of $\mathfrak{g}$,

$$\gamma = \sum_{i \notin \mathcal{N}} m_i \alpha_i + \sum_{a \in \mathcal{N}} \ell_a \alpha_a. \tag{8.12}$$

To this decomposition of the roots corresponds a decomposition of the algebra, which is called a *gradation* of $\mathfrak{g}$ and which can be written formally as

$$\mathfrak{g} = \bigoplus_{\ell \in \mathbb{Z}^s} \mathfrak{g}_\ell, \tag{8.13}$$

where for a given ℓ, $\mathfrak{g}_\ell$ is the subspace spanned by all the vectors e_γ with that definite value ℓ of the multilevel,

$$[h, e_\gamma] = \gamma(h) e_\gamma, \qquad l_a(\gamma) = \ell_a. \tag{8.14}$$

Of course, if $\mathfrak{g}$ is finite-dimensional this sum terminates for some finite level, as in Equation (8.11) for $\mathfrak{sl}(3, \mathbb{R})$. However, in the following we shall mainly be interested in cases where Equation (8.13) is an infinite sum.

We note for further reference that the following structure is inherited from the gradation:

$$[\mathfrak{g}_\ell, \mathfrak{g}_{\ell'}] \subseteq \mathfrak{g}_{\ell + \ell'}. \tag{8.15}$$

This implies that for $\ell = 0$ we have

$$[\mathfrak{g}_0, \mathfrak{g}_{\ell'}] \subseteq \mathfrak{g}_{\ell'}, \tag{8.16}$$

which means that $\mathfrak{g}_{\ell'}$ is a representation of $\mathfrak{g}_0$ under the adjoint action. Furthermore, $\mathfrak{g}_0$ is a subalgebra. Now, the algebra $\mathfrak{r}$ is a subalgebra of $\mathfrak{g}_0$ and hence we also have

$$[\mathfrak{r}, \mathfrak{g}_{\ell'}] \subseteq \mathfrak{g}_{\ell'}, \tag{8.17}$$

so that *the subspaces* $\mathfrak{g}_\ell$ *at definite values of the multilevel are invariant subspaces under the adjoint action of* $\mathfrak{r}$. In other words, the action of $\mathfrak{r}$ on $\mathfrak{g}_\ell$ does not change the coefficients ℓ_a.

At level zero, $\ell = (0, \cdots, 0)$, the representation of the subalgebra $\mathfrak{r}$ in the subspace $\mathfrak{g}_0$ contains the adjoint representation of $\mathfrak{r}$, just as in the case of $\mathfrak{sl}(3, \mathbb{R})$ discussed in Section 8.1. All positive and negative roots of $\mathfrak{r}$ are relevant. Level zero contains in addition s singlets for each of the Cartan generator associated to the set $\mathcal{N}$.

Whenever one of the ℓ_a's is positive, all the other ones must be non-negative for the subspace $\mathfrak{g}_\ell$ to be nontrivial and only positive roots appear at that value of the multilevel.

8.2.2 Weights of $\mathfrak{g}$ and weights of $\mathfrak{r}$

Let V be the module of a representation $\mathcal{R}(\mathfrak{g})$ of $\mathfrak{g}$ and $\Lambda \in \mathfrak{h}_{\mathfrak{g}}^\star$ be one of the weights occurring in the representation. We define the action of $h \in \mathfrak{h}_{\mathfrak{g}}$ in the representation $\mathcal{R}(\mathfrak{g})$ on $x \in V$ as

$$h \cdot x = \Lambda(h)x \tag{8.18}$$

(we consider representations of $\mathfrak{g}$ for which one can speak of "weights" [116]). Any representation of $\mathfrak{g}$ is also a representation of $\mathfrak{r}$. When restricted to the Cartan subalgebra $\mathfrak{h}_{\mathfrak{r}}$ of $\mathfrak{r}$, Λ defines a weight $\bar{\Lambda} \in \mathfrak{h}_{\mathfrak{r}}^\star$, which one can realize geometrically as follows.

The dual space $\mathfrak{h}_{\mathfrak{r}}^\star$ may be viewed as the m-dimensional subspace Π of $\mathfrak{h}_{\mathfrak{g}}^\star$ spanned by the simple roots α_i, $i \notin \mathcal{N}$. The metric induced on that subspace is positive definite since $\mathfrak{r}$ is finite-dimensional. This implies, since we assume that the metric on $\mathfrak{h}_{\mathfrak{g}}^\star$ is nondegenerate, that $\mathfrak{h}_{\mathfrak{g}}^\star$ can be decomposed as the direct sum

$$\mathfrak{h}_{\mathfrak{g}}^\star = \mathfrak{h}_{\mathfrak{r}}^\star \oplus \Pi^\perp. \tag{8.19}$$

To that decomposition corresponds the decomposition

$$\Lambda = \Lambda^\parallel + \Lambda^\perp \tag{8.20}$$

of any weight, where $\Lambda^\parallel \in \mathfrak{h}_{\mathfrak{r}}^\star \equiv \Pi$ and $\Lambda^\perp \in \Pi^\perp$. Now, let $h = \sum k_i \alpha_i^\vee \in \mathfrak{h}_{\mathfrak{r}}$ $(i \notin \mathcal{N})$. One has $\Lambda(h) = \Lambda^\parallel(h) + \Lambda^\perp(h) = \Lambda^\parallel(h)$ because $\Lambda^\perp(h) = 0$: The component perpendicular to $\mathfrak{h}_{\mathfrak{r}}^\star$ drops out. Indeed, $\Lambda^\perp(\alpha_i^\vee) = \frac{2(\Lambda^\perp|\alpha_i)}{(\alpha_i|\alpha_i)} = 0$ for $i \notin \mathcal{N}$.

It follows that one can identify the weight $\bar{\Lambda} \in \mathfrak{h}_{\mathfrak{r}}^\star$ with the orthogonal projection $\Lambda^\parallel \in \mathfrak{h}_{\mathfrak{r}}^\star$ of $\Lambda \in \mathfrak{h}_{\mathfrak{g}}^\star$ on $\mathfrak{h}_{\mathfrak{r}}^\star$. This is true, in particular, for the fundamental weights Λ_i. The fundamental weights Λ_i project on 0 for $i \in \mathcal{N}$ and project on the fundamental weights $\bar{\Lambda}_i$ of the subalgebra $\mathfrak{r}$ for $i \notin \mathcal{N}$. These are also denoted λ_i. For a general weight, one has

$$\Lambda = \sum_{i \notin \mathcal{N}} p_i \Lambda_i + \sum_{a \in \mathcal{N}} k_a \Lambda_a \tag{8.21}$$

and

$$\bar{\Lambda} = \Lambda^\parallel = \sum_{i \notin \mathcal{N}} p_i \lambda_i. \tag{8.22}$$

The coefficients p_i can easily be extracted by taking the scalar product with the simple roots,

$$p_i = \frac{2}{(\alpha_i|\alpha_i)}(\alpha_i|\Lambda), \tag{8.23}$$

a formula that reduces to

$$p_i = (\alpha_i|\Lambda) \tag{8.24}$$

in the simply-laced case. Note that $(\Lambda^\parallel|\Lambda^\parallel) > 0$ even when Λ is non-spacelike.

8.2.3 Outer multiplicity

There is an interesting relationship between root multiplicities in the Kac–Moody algebra $\mathfrak{g}$ and weight multiplicites of the corresponding $\mathfrak{r}$-weights, which we will explore here.

For finite Lie algebras, the roots always come with multiplicity one. This is in fact true also for the real roots of indefinite Kac–Moody algebras. However, as pointed out in Section 4, the imaginary roots can have arbitrarily large multiplicity. This must therefore be taken into account in the sum (8.13).

Let $\gamma \in \mathfrak{h}_{\mathfrak{g}}^\star$ be a root of $\mathfrak{g}$. There are two important ingredients:

- The multiplicity $\mathrm{mult}(\gamma)$ of each $\gamma \in \mathfrak{h}_{\mathfrak{g}}^\star$ at level ℓ as a *root* of $\mathfrak{g}$.
- The multiplicity $\mathrm{mult}_{\mathcal{R}_\gamma^{(\ell)}}(\gamma)$ of the corresponding weight $\bar{\gamma} \in \mathfrak{h}_{\mathfrak{r}}^\star$ at level ℓ as a *weight* in the representation $\mathcal{R}_\gamma^{(\ell)}$ of $\mathfrak{r}$. (Note that two distinct roots at the same level project on two distinct $\mathfrak{r}$-weights, so that given the $\mathfrak{r}$-weight and the level, one can reconstruct the root.)

It follows that the root multiplicity of γ is given as a sum over its multiplicities as a weight in the various representations $\{\mathcal{R}_q^{(\ell)} \,|\, q = 1, \cdots, N_\ell\}$ at level ℓ. Some representations can appear more than once at each level, and it is therefore convenient to introduce a new measure of multiplicity, called the *outer multiplicity* $\mu(\mathcal{R}_q^{(\ell)})$, which counts the number of times each representation $\mathcal{R}_q^{(\ell)}$ appears at level ℓ. So, for each representation at level ℓ we must count the individual weight multiplicities in that representation and also the number of times this representation occurs. The total multiplicity of γ can then be written as

$$\mathrm{mult}(\gamma) = \sum_{q=1}^{N_\ell} \mu(\mathcal{R}_q^{(\ell)})\, \mathrm{mult}_{\mathcal{R}_q^{(\ell)}}(\gamma). \tag{8.25}$$

This simple formula might provide useful information on which representations of $\mathfrak{r}$ are allowed within $\mathfrak{g}$ at a given level. For example, if γ is a real root of $\mathfrak{g}$, then it has multiplicity one. This means that in the formula (8.25), only the representations of $\mathfrak{r}$ for which γ has weight multiplicity equal to one are permitted. The others have $\mu(\mathcal{R}_q^{(\ell)}) = 0$. Furthermore, only one of the permitted representations does actually occur and it has necessarily outer multiplicity equal to one, $\mu(\mathcal{R}_q^{(\ell)}) = 1$.

The subspaces $\mathfrak{g}_\ell$ can now be written explicitly as

$$\mathfrak{g}_\ell = \bigoplus_{q=1}^{N_\ell} \left[\bigoplus_{k=1}^{\mu(\mathcal{R}_q^{(\ell)})} \mathcal{L}^{[k]}(\Lambda_q^{(\ell)}) \right], \tag{8.26}$$

where $\mathcal{L}(\Lambda_q^{(\ell)})$ denotes the module of the representation $\mathcal{R}_q^{(\ell)}$ and N_ℓ is the number of inequivalent representations at level ℓ. It is understood that if $\mu(\mathcal{R}_q^{(\ell)}) = 0$ for some ℓ and q, then $\mathcal{L}(\Lambda_q^{(\ell)})$ is absent from the sum. Note that the superscript $[k]$ labels multiple modules associated to the same representation, e.g., if $\mu(\mathcal{R}_q^{(\ell)}) = 3$ this contributes to the sum with a term

$$\mathcal{L}^{[1]}(\Lambda_q^{(\ell)}) \oplus \mathcal{L}^{[2]}(\Lambda_q^{(\ell)}) \oplus \mathcal{L}^{[3]}(\Lambda_q^{(\ell)}). \tag{8.27}$$

Finally, we mention that the multiplicity $\mathrm{mult}(\alpha)$ of a root $\alpha \in \mathfrak{h}^\star$ can be computed recursively using the *Peterson recursion relation*, defined as [116]

$$(\alpha|\alpha - 2\,\rho)c_\alpha = \sum_{\substack{\gamma + \gamma' = \alpha \\ \gamma, \gamma' \in Q_+}} (\gamma|\gamma')c_\gamma c_{\gamma'}, \tag{8.28}$$

where Q_+ denotes the set of all positive integer linear combinations of the simple roots, i.e., the positive part of the root lattice, and ρ is the Weyl vector (defined in Section 4). The coefficients c_γ are defined as

$$c_\gamma = \sum_{k \geq 1} \frac{1}{k} \,\mathrm{mult}\left(\frac{\gamma}{k}\right), \tag{8.29}$$

and, following [19], we call this the *co-multiplicity*. Note that if γ/k is not a root, this gives no contribution to the co-multiplicity. Another feature of the co-multiplicity is that even if the multiplicity of some root γ is zero, the associated co-multiplicity c_γ does not necessarily vanish. Taking advantage of the fact that all real roots have multiplicity one it is possible, in principle, to compute recursively the multiplicity of any imaginary root. Since no closed formula exists for the outer multiplicity μ, one must take a detour via the Peterson relation and Equation (8.25) in order to find the outer multiplicity of each representation at a given level. We give in Table 38 a list of root multiplicities and co-multiplicities of roots of AE_3 up to height 10.

8.3 Level decomposition of AE_3

The Kac–Moody algebra $AE_3 = A_1^{++}$ is one of the simplest hyperbolic algebras and so provides a nice testing ground for investigating general properties of hyperbolic Kac–Moody algebras. From a physical point of view, it is the Weyl group of AE_3 which governs the chaotic behavior of pure four-dimensional gravity close to a spacelike singularity [46], as we have explained. Moreover, as we saw in Section 3, the Weyl group of AE_3 is isomorphic with the well-known arithmetic group $PGL(2,\mathbb{Z})$ which has interesting properties [75].

The level decomposition of $\mathfrak{g} = AE_3$ follows a similar route as for $\mathfrak{sl}(3,\mathbb{R})$ above, but the result is much more complicated due to the fact that AE_3 is infinite-dimensional. This decomposition has been treated before in [48]. Recall that the Cartan matrix for AE_3 is given by

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \tag{8.30}$$

and the associated Dynkin diagram is given in Figure 46.

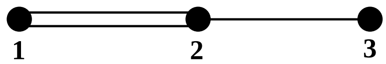


Figure 46: The Dynkin diagram of the hyperbolic Kac–Moody algebra $AE_3 \equiv A_1^{++}$. The labels indicate the simple roots α_1, α_2 and α_3. The nodes "2" and "3" correspond to the subalgebra $\mathfrak{r} = \mathfrak{sl}(3,\mathbb{R})$ with respect to which we perform the level decomposition.

We see that there exist three rank 2 regular subalgebras that we can use for the decomposition: $A_2, A_1 \oplus A_1$ or A_1^+. We will here focus on the decomposition into representations of $\mathfrak{r} = A_2 = \mathfrak{sl}(3,\mathbb{R})$ because this is the one relevant for pure gravity in four dimensions [46][31]. The level ℓ is then the coefficient in front of the simple root α_1 in an expansion of an arbitrary root $\gamma \in \mathfrak{h}_\mathfrak{g}^\star$, i.e.,

$$\gamma = \ell\alpha_1 + m_2\alpha_2 + m_3\alpha_3. \tag{8.31}$$

We restrict henceforth our analysis to positive levels only, $\ell \geq 0$. Before we begin, let us develop an intuitive idea of what to expect. We know that at each level we will have a set of finite-dimensional representations of the subalgebra $\mathfrak{r}$. The corresponding weight diagrams will then be represented in a Euclidean two-dimensional lattice in exactly the same way as in Figure 45 above. The level ℓ can be understood as parametrizing a third direction that takes us into the full three-dimensional root space of AE_3. We display the level decomposition up to positive level two in Figure 47[32].

From previous sections we recall that AE_3 is hyperbolic so its root space is of Lorentzian signature. This implies that there is a lightcone in $\mathfrak{h}_\mathfrak{g}^\star$ whose origin lies at the origin of the root diagram for the adjoint representation of $\mathfrak{r}$ at level $\ell = 0$. The lightcone separates real roots from

[31] The decomposition of AE_3 into representations of A_1^+ was done in [75].

[32] D.P. would like to thank Bengt E.W. Nilsson and Jakob Palmkvist for helpful discussions during the creation of Figure 47.

Table 38: Multiplicities $m_\alpha = \mathrm{mult}(\alpha)$ and co-multiplicities c_α of all roots α of AE_3 up to height 10.

ℓ	m_1	m_2	c_α	m_α	α^2
0	0	1	1	1	2
0	0	$k>1$	$1/k$	0	$2\,k^2$
0	1	0	1	1	2
0	$k>1$	1	$1/k$	0	$2\,k^2$
1	0	0	1	1	2
$k>0$	0	0	$1/k$	0	$2\,k^2$
0	1	1	1	1	2
0	$k>1$	$k>1$	$1/k$	0	$2\,k^2$
1	1	0	1	1	0
2	2	0	3/2	1	0
3	3	0	4/3	1	0
4	4	0	7/4	1	0
5	5	0	6/5	1	0
1	1	1	1	1	0
2	2	2	3/2	1	0
3	3	3	4/3	1	0
1	2	0	1	1	2
2	4	0	1/2	0	8
3	6	0	1/3	0	2
2	1	0	1	1	2
4	2	0	1/2	0	8
6	3	0	1/3	0	18
1	2	1	1	1	0
2	4	2	3/2	1	0
2	1	1	1	1	2
4	2	2	1/2	0	8
1	2	2	1	1	2
2	4	4	1/2	0	8
2	2	1	2	2	-2
4	4	2	8	7	-8
2	3	0	1	1	2
4	6	0	1/2	0	8
3	2	0	1	1	2
6	4	0	1/2	0	8
2	3	1	2	2	-2
3	2	1	1	1	0
2	4	1	1	1	2
2	3	2	2	2	-2
3	2	2	1	1	2
3	3	1	3	3	-4
3	4	0	1	1	2
4	3	0	1	1	2
2	3	3	1	1	2
3	4	1	3	3	-4
2	4	3	1	1	2
3	3	2	3	3	-4
4	3	1	2	2	-2
3	4	2	5	5	-6
3	5	1	1	1	0
4	3	2	2	2	-2
4	4	1	5	5	-6
4	5	0	1	1	2
5	4	0	1	1	2
3	4	3	3	3	-4
3	5	2	3	3	-4
4	3	3	1	1	2
4	5	1	5	5	-6
5	4	1	3	3	-4

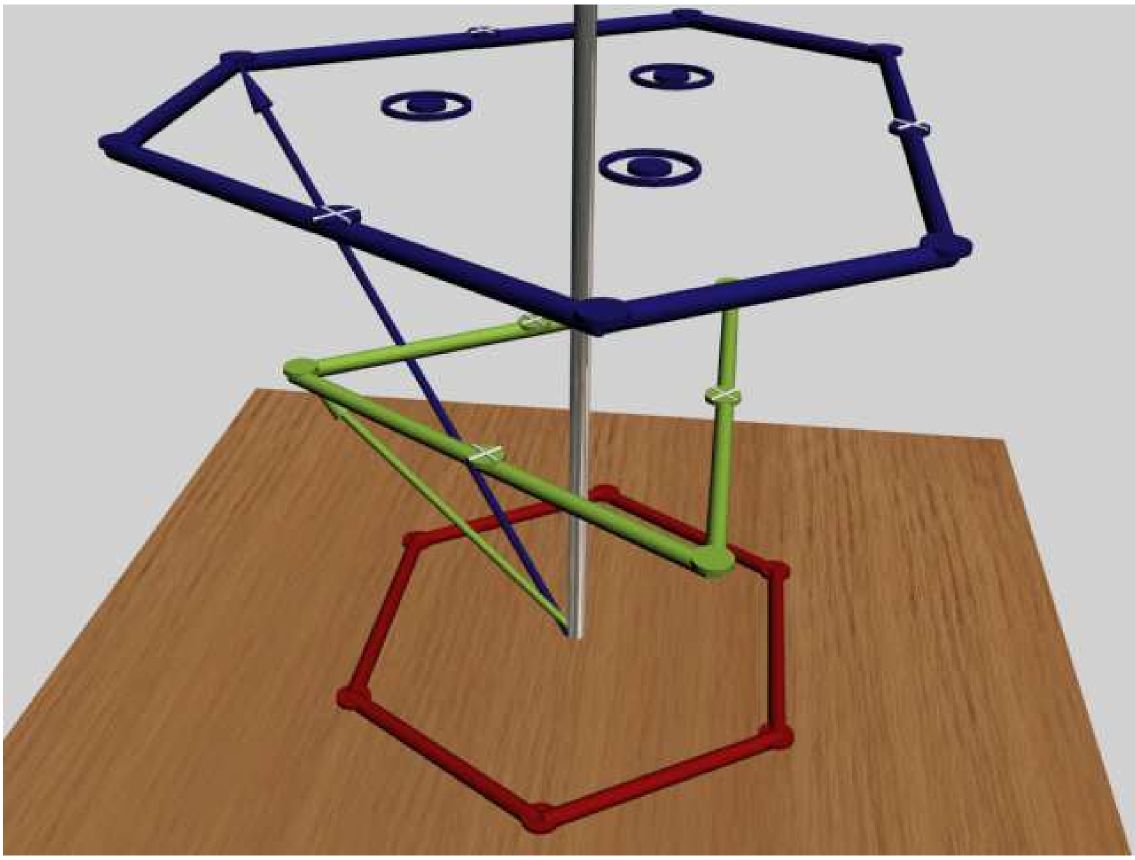


Figure 47: Level decomposition of the adjoint representation of AE_3. We have displayed the decomposition up to positive level $\ell = 2$. At level zero we have the adjoint representation $\mathcal{R}_1^{(0)} = \mathbf{8}_0$ of $\mathfrak{sl}(3, \mathbb{R})$ and the singlet representation $\mathcal{R}_2^{(0)} = \mathbf{1}_0$ defined by the simple Cartan generator $\alpha_1^\vee$. Ascending to level one with the root α_1 (green vector) gives the lowest weight $\Lambda^{(1)}$ of the representation $\mathcal{R}^{(1)} = \mathbf{6}_1$. The weights of $\mathcal{R}^{(1)}$ labelled by white crosses are on the lightcone and so their norm squared is zero. At level two we find the lowest weight $\Lambda^{(2)}$ (blue vector) of the 15-dimensional representation $\mathcal{R}^{(2)} = \mathbf{15}_2$. Again, the white crosses label weights that are on the lightcone. The three innermost weights are inside of the lightcone and the rings indicate that these all have multiplicity 2 as weights of $\mathcal{R}^{(2)}$. Since these also have multiplicity 2 as *roots* of $\mathfrak{h}_{\mathfrak{g}}^\star$ we find that the outer multiplicity of this representation is one, $\mu(\mathcal{R}^{(2)}) = 1$.

imaginary roots and so it is clear that if a representation at some level ℓ intersects the walls of the lightcone, this means that some weights in the representation will correspond to imaginary roots of $\mathfrak{h}^\star_{\mathfrak{g}}$ but will be real as weights of $\mathfrak{h}^\star_{\mathfrak{r}}$. On the other hand if a weight lies outside of the lightcone it will be real both as a root of $\mathfrak{h}^\star_{\mathfrak{g}}$ and as a weight of $\mathfrak{h}^\star_{\mathfrak{r}}$.

8.3.1 Level $\ell = 0$

Consider first the representation content at level zero. Given our previous analysis we expect to find the adjoint representation of $\mathfrak{r}$ with the additional singlet representation from the Cartan generator $\alpha_1^\vee$. The Chevalley generators of $\mathfrak{r}$ are $\{e_2, f_2, e_3, f_3, \alpha_2^\vee, \alpha_3^\vee\}$ and the generators associated to the root defining the level are $\{e_1, f_1, \alpha_1^\vee\}$. As discussed previously, the additional Cartan generator $\alpha_1^\vee$ that sits at the origin of the root space enlarges the subalgebra from $\mathfrak{sl}(3,\mathbb{R})$ to $\mathfrak{gl}(3,\mathbb{R})$. A canonical realisation of $\mathfrak{gl}(3,\mathbb{R})$ is obtained by defining the Chevalley generators in terms of the matrices $K^i{}_j$ $(i,j = 1,2,3)$ whose commutation relations are

$$[K^i{}_j, K^k{}_l] = \delta^k_j K^i{}_l - \delta^i_l K^k{}_j. \tag{8.32}$$

All the defining Lie algebra relations of $\mathfrak{gl}(3,\mathbb{R})$ are then satisfied if we make the identifications

$$\begin{aligned} & & & & \alpha_1^\vee &= K^1{}_1 - K^2{}_2 - K^3{}_3, \\ e_2 &= K^2{}_1, & f_2 &= K^1{}_2, & \alpha_2^\vee &= K^2{}_2 - K^1{}_1, \\ e_3 &= K^3{}_2, & f_3 &= K^2{}_3, & \alpha_3^\vee &= K^3{}_3 - K^2{}_2. \end{aligned} \tag{8.33}$$

Note that the trace $K^1{}_1 + K^2{}_2 + K^3{}_3$ is equal to $-4\alpha_2^\vee - 2\alpha_3^\vee - 3\alpha_1^\vee$. The generators e_1 and f_1 can of course not be realized in terms of the matrices $K^i{}_j$ since they do not belong to level zero. The invariant bilinear form $(\ |\ )$ at level zero reads

$$(K^i{}_j | K^k{}_l) = \delta^i_l \delta^k_j - \delta^i_j \delta^k_l, \tag{8.34}$$

where the coefficient in front of the second term on the right hand side is fixed to -1 through the embedding of $\mathfrak{gl}(3,\mathbb{R})$ in AE_3.

The commutation relations in Equation (8.32) characterize the adjoint representation of $\mathfrak{gl}(3,\mathbb{R})$ as was expected at level zero, which decomposes as the representation $\mathcal{R}^{(0)}_{\text{ad}} \oplus \mathcal{R}^{(0)}_s$ of $\mathfrak{sl}(3,\mathbb{R})$ with $\mathcal{R}^{(0)}_{\text{ad}} = \mathbf{8}_0$ and $\mathcal{R}^{(0)}_s = \mathbf{1}_0$.

8.3.2 Dynkin labels

It turns out that at each positive level ℓ, the weight that is easiest to identify is the lowest weight. For example, at level one, the lowest weight is simply α_1 from which one builds all the other weights by adding appropriate positive combinations of the roots α_2 and α_3. It will therefore turn out to be convenient to characterize the representations at each level by their (conjugate) Dynkin labels p_2 and p_3 defined as the coefficients of minus the (projected) lowest weight $-\bar{\Lambda}^{(\ell)}_{\text{lw}}$ expanded in terms of the fundamental weights λ_2 and λ_3 of $\mathfrak{sl}(3,\mathbb{R})$ (blue arrows in Figure 48),

$$-\bar{\Lambda}^{(\ell)}_{\text{lw}} = p_2\lambda_2 + p_3\lambda_3. \tag{8.35}$$

Note that for any weight Λ we have the inequality

$$(\Lambda|\Lambda) \leq (\bar{\Lambda}|\bar{\Lambda}) \tag{8.36}$$

since $(\Lambda|\Lambda) = (\bar{\Lambda}|\bar{\Lambda}) - |(\Lambda^\perp|\Lambda^\perp)|$.

The Dynkin labels can be computed using the scalar product $(\ |\ )$ in $\mathfrak{h}^\star_{\mathfrak{g}}$ in the following way:

$$p_2 = -(\alpha_2|\Lambda^{(\ell)}_{\text{lw}}), \qquad p_3 = -(\alpha_3|\Lambda^{(\ell)}_{\text{lw}}). \tag{8.37}$$

For the level zero sector we therefore have

$$\begin{aligned} \mathbf{8}_0 &: [p_2, p_3] = [1, 1], \\ \mathbf{1}_0 &: [p_2, p_3] = [0, 0]. \end{aligned} \tag{8.38}$$

The module for the representation $\mathbf{8}_0$ is realized by the eight traceless generators $K^i{}_j$ of $\mathfrak{sl}(3, \mathbb{R})$ and the module for the representation $\mathbf{1}_0$ corresponds to the "trace" $\alpha_1^\vee$.

Note that the highest weight Λ_{hw} of a given representation of $\mathfrak{r}$ is not in general equal to minus the lowest weight Λ of the same representation. In fact, $-\Lambda_{\text{hw}}$ is equal to the lowest weight of the *conjugate* representation. This is the reason our Dynkin labels are really the conjugate Dynkin labels in standard conventions. It is only if the representation is self-conjugate that we have $\Lambda_{\text{hw}} = -\Lambda$. This is the case for example in the adjoint representation $\mathbf{8}_0$.

It is interesting to note that since the weights of a representation at level ℓ are related by Weyl reflections to weights of a representation at level $-\ell$, it follows that the negative of a lowest weight $\Lambda^{(\ell)}$ at level ℓ is actually equal to the *highest* weight $\Lambda^{(-\ell)}_{\text{hw}}$ of the conjugate representation at level $-\ell$. Therefore, the Dynkin labels at level ℓ as defined here are the standard Dynkin labels of the representations at level $-\ell$.

8.3.3 Level $\ell = 1$

We now want to exhibit the representation content at the next level $\ell = 1$. A generic level one commutator is of the form $[e_1, [\cdots[\cdots]]]$, where the ellipses denote (positive) level zero generators. Hence, including the generator e_1 implies that we step upwards in root space, i.e., in the direction of the forward lightcone. The root vector e_1 corresponds to a lowest weight of $\mathfrak{r}$ since it is annihilated by f_2 and f_3,

$$\begin{aligned} \text{ad}_{f_2}(e_1) &= [f_2, e_1] = 0, \\ \text{ad}_{f_3}(e_1) &= [f_3, e_1] = 0, \end{aligned} \tag{8.39}$$

which follows from the defining relations of AE_3.

Explicitly, the root associated to e_1 is simply the root α_1 that defines the level expansion. Therefore the lowest weight of this level one representation is

$$\Lambda^{(1)}_{\text{lw}} = \bar{\alpha}_1, \tag{8.40}$$

Although α_1 is a real *positive* root of $\mathfrak{h}^\star_{\mathfrak{g}}$, its projection $\bar{\alpha}_{(1)}$ is a *negative* weight of $\mathfrak{h}^\star_{\mathfrak{r}}$. Note that since the lowest weight $\Lambda^{(1)}_1$ is real, the representation $\mathcal{R}^{(1)}$ has outer multiplicity one, $\mu(\mathcal{R}^{(1)}) = 1$.

Acting on the lowest weight state with the raising operators of $\mathfrak{r}$ yields the six-dimensional representation $\mathcal{R}^{(1)} = \mathbf{6}_1$ of $\mathfrak{sl}(3, \mathbb{R})$. The root α_1 is displayed as the green vector in Figure 47, taking us from the origin at level zero to the lowest weight of $\mathcal{R}^{(1)}$. The Dynkin labels of this representation are

$$\begin{aligned} p_2(\mathcal{R}^{(1)}) &= -(\alpha_2|\alpha_1) = 2, \\ p_3(\mathcal{R}^{(1)}) &= -(\alpha_3|\alpha_1) = 0, \end{aligned} \tag{8.41}$$

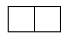
which follows directly from the Cartan matrix of AE_3. Three of the weights in $\mathcal{R}^{(1)}$ correspond to roots that are located on the lightcone in root space and so are null roots of $\mathfrak{h}^\star_{\mathfrak{g}}$. These are $\alpha_1 + \alpha_2$, $\alpha_1 + \alpha_2 + \alpha_3$ and $\alpha_1 + 2\alpha_2 + \alpha_3$ and are labelled with white crosses in Figure 47. The other roots present in the representation, in addition to α_1, are $\alpha_1 + 2\alpha_2$ and $\alpha_1 + 2\alpha_2 + 2\alpha_3$, which are real. This representation therefore contains no weights inside the lightcone.

The $\mathfrak{gl}(3,\mathbb{R})$-generator encoding this representation is realized as a symmetric 2-index tensor E^{ij} which indeed carries six independent components. In general we can easily compute the dimensionality of a representation given its Dynkin labels using the *Weyl dimension formula* which for $\mathfrak{sl}(3,\mathbb{R})$ takes the form [84]

$$d_{\Lambda_{\text{hw}}}\left(\mathfrak{sl}(3,\mathbb{R})\right) = (p_2+1)(p_3+1)\left(\frac{1}{2}(p_2+p_3)+1\right). \tag{8.42}$$

In particuar, for $(p_2, p_3) = (2, 0)$ this gives indeed $d_{\Lambda^{(1)}_{\text{hw},1}} = 6$.

It is convenient to encode the Dynkin labels, and, consequently, the index structure of a given representation module, in a Young tableau. We follow conventions where the first Dynkin label gives the number of columns with 1 box and the second Dynkin label gives the number of columns with 2 boxes[33]. For the representation $\mathbf{6}_1$ the first Dynkin label is 2 and the second is 0, hence the associated Young tableau is

$\mathbf{6}_1 \quad \Longleftrightarrow$  . (8.43)

At level $\ell = -1$ there is a corresponding negative generator F_{ij}. The generators E^{ij} and F_{ij} transform contravariantly and covariantly, respectively, under the level zero generators, i.e.,

$$\begin{aligned} [K^i{}_j, E^{kl}] &= \delta^k_j E^{il} + \delta^l_j E^{ki}, \\ [K^i{}_j, F_{kl}] &= -\delta^i_k F_{jl} - \delta^i_l F_{kj}. \end{aligned} \tag{8.44}$$

The internal commutator on level one can be obtained by first identifying

$$e_1 \equiv E^{11}, \qquad f_1 \equiv F_{11}, \tag{8.45}$$

and then by demanding $[e_1, f_1] = \alpha_1^\vee$ we find

$$[E^{ij}, F_{kl}] = 2\delta^{(i}_{(k} K^{j)}{}_{l)} - \delta^{(i}_k \delta^{k)}_l (K^1{}_1 + K^2{}_2 + K^3{}_3), \tag{8.46}$$

which is indeed compatible with the realisation of $\alpha_1^\vee$ given in Equation (8.33). The Killing form at level 1 takes the form

$$\left(F_{ij} | E^{kl}\right) = \delta^{(k}_i \delta^{l)}_j. \tag{8.47}$$

8.3.4 Constraints on Dynkin labels

As we go to higher and higher levels it is useful to employ a systematic method to investigate the representation content. It turns out that it is possible to derive a set of equations whose solutions give the Dynkin labels for the representations at each level [47].

We begin by relating the Dynkin labels to the expansion coefficients ℓ, m_2 and m_3 of a root $\gamma \in \mathfrak{h}^\star_{\mathfrak{g}}$, whose projection $\bar{\gamma}$ onto $\mathfrak{h}^\star_{\mathfrak{r}}$ is a lowest weight vector for some representation of $\mathfrak{r}$ at level ℓ. We let $a = 2, 3$ denote indices in the root space of the subalgebra $\mathfrak{sl}(3,\mathbb{R})$ and we let $i = 1, 2, 3$ denote indices in the full root space of AE_3. The formula for the Dynkin labels then gives

$$p_a = -(\alpha_a|\gamma) = -\ell A_{a1} - m_2 A_{a2} - m_3 A_{a3}, \tag{8.48}$$

where A_{ij} is the Cartan matrix for AE_3, given in Equation (8.30). Explicitly, we find the following relations between the coefficients m_2, m_3 and the Dynkin labels:

$$\begin{aligned} p_2 &= 2\ell - 2m_2 + m_3, \\ p_3 &= m_2 - 2m_3. \end{aligned} \tag{8.49}$$

[33] Since we are, in fact, using conjugate Dynkin labels, these conventions are equivalent to the standard ones if one replaces covariant indices by contravariant ones, and vice-versa.

These formulae restrict the possible Dynkin labels for each ℓ since the coefficients m_2 and m_3 must necessarily be non-negative integers. Therefore, by inverting Equation (8.49) we obtain two Diophantine equations that restrict the possible Dynkin labels,

$$\begin{aligned} m_2 &= \frac{4}{3}\ell - \frac{2}{3}p_2 - \frac{1}{3}p_3 \geq 0, \\ m_3 &= \frac{2}{3}\ell - \frac{1}{3}p_2 - \frac{2}{3}p_3 \geq 0. \end{aligned} \tag{8.50}$$

In addition to these constraints we can also make use of the fact that we are decomposing the adjoint representation of AE_3. Since the weights of the adjoint representation are the roots of the algebra we know that the lowest weight vector Λ must satisfy

$$(\Lambda|\Lambda) \leq 2. \tag{8.51}$$

Taking $\Lambda = \ell\alpha_1 + m_2\alpha_2 + m_3\alpha_3$ then gives the following constraint on the coefficients ℓ, m_2 and m_3:

$$(\Lambda|\Lambda) = 2\ell^2 + 2m_2^2 + 2m_3^2 - 4\ell m_2 - 2m_2m_3 \leq 2. \tag{8.52}$$

We are interested in finding an equation for the Dynkin labels, so we insert Equation (8.50) into Equation (8.52) to obtain the constraint

$$p_2^2 + p_3^2 + p_2p_3 - \ell^2 \leq 3. \tag{8.53}$$

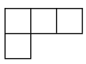
The inequalities in Equation (8.50) and Equation (8.53) are sufficient to determine the representation content at each level ℓ. However, this analysis does not take into account the outer multiplicities, which must be analyzed separately by comparing with the known root multiplicities of AE_3 as given in Table 38. We shall return to this issue later.

8.3.5 Level $\ell = 2$

Let us now use these results to analyze the case for which $\ell = 2$. The following equations must then be satisfied:

$$\begin{aligned} a8 - 2p_2 - p_3 &\geq 0, \\ 4 - p_2 - 2p_3 &\geq 0, \\ p_2^2 + p_3^2 + p_2p_3 &\leq 7. \end{aligned} \tag{8.54}$$

The only admissible solution is $p_2 = 2$ and $p_3 = 1$. This corresponds to a 15-dimensional representation $\mathbf{15}_2$ with the following Young tableau

$\mathbf{6}_1 \quad \Longleftrightarrow$  . (8.55)

Note that $p_2 = p_3 = 0$ is also a solution to Equation (8.54) but this violates the constraint that m_2 and m_3 be integers and so is not allowed.

Moreover, the representation $[p_2, p_3] = [0, 2]$ is also a solution to Equation (8.54) but has not been taken into account because it has vanishing outer multiplicity. This can be understood by examining Figure 48 a little closer. The representation $[0, 2]$ is six-dimensional and has highest weight $2\lambda_3$, corresponding to the middle node of the top horizontal line in Figure 48. This weight lies outside of the lightcone and so is a real root of AE_3. Therefore we know that it has root multiplicity one and may therefore only occur once in the level decomposition. Since the weight $2\lambda_3$ already appears in the larger representation $\mathbf{15}_2$ it cannot be a highest weight in another representation at this level. Hence, the representation $[0, 2]$ is not allowed within AE_3. A similar analysis reveals that also the representation $[p_2, p_3] = [1, 0]$, although allowed by Equation (8.54), has vanishing outer multiplicity.

The level two module is realized by the tensor ${E_i}^{jk}$ whose index structure matches the Young tableau above. Here we have used the $\mathfrak{sl}(3,\mathbb{R})$-invariant antisymmetric tensor ϵ^{abc} to lower the two upper antisymmetric indices leading to a tensor ${E_i}^{jk}$ with the properties

$$ {E_i}^{jk} = {E_i}^{(jk)}, \qquad {E_i}^{ik} = 0. \tag{8.56} $$

This corresponds to a positive root generator and by the Chevalley involution we have an associated negative root generator ${F^i}_{jk}$ at level $\ell = -2$. Because the level decomposition gives a gradation of AE_3 we know that all higher level generators can be obtained through commutators of the level one generators. More specifically, the level two tensor ${E_i}^{jk}$ corresponds to the commutator

$$ [E^{ij}, E^{kl}] = \epsilon^{mk(i}{E_m}^{j)l} + \epsilon^{ml(i}{E_m}^{j)k}, \tag{8.57} $$

where ϵ^{ijk} is the totally antisymmetric tensor in three dimensions. Inserting the result $p_2 = 2$ and $p_3 = 1$ into Equation (8.50) gives $m_2 = 1$ and $m_3 = 0$, thus providing the explicit form of the root taking us from the origin of the root diagram in Figure 47 to the lowest weight of $\mathbf{15}_2$ at level two:

$$ \Lambda^{(2)} = 2\alpha_1 + \alpha_2. \tag{8.58} $$

This is a real root of AE_3, $(\gamma|\gamma) = 2$, and hence the representation $\mathbf{15}_2$ has outer multiplicity one. We display the representation $\mathbf{15}_2$ of $\mathfrak{sl}(3,\mathbb{R})$ in Figure 48. The lower leftmost weight is the lowest weight $\Lambda^{(2)}$. The expansion of the lowest weight $\Lambda^{(2)}_{\mathrm{lw}}$ in terms of the fundamental weights λ_2 and λ_3 is given by the (conjugate) Dynkin labels

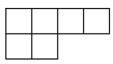
$$ -\Lambda^{(2)}_{\mathrm{hw}} = p_2\lambda_2 + p_3\lambda_3 = 2\lambda_2 + \lambda_3. \tag{8.59} $$

The three innermost weights all have multiplicity 2 as weights of $\mathfrak{sl}(3,\mathbb{R})$, as indicated by the black circles. These lie inside the lightcone of $\mathfrak{h}^\star_{\mathfrak{g}}$ and so are timelike roots of AE_3.

8.3.6 Level $\ell = 3$

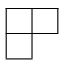
We proceed quickly past level three since the analysis does not involve any new ingredients. Solving Equation (8.50) and Equation (8.53) for $\ell = 3$ yields two admissible $\mathfrak{sl}(3,\mathbb{R})$ representations, $\mathbf{27}_3$ and $\mathbf{8}_3$, represented by the following Dynkin labels and Young tableaux:

$$ \mathbf{27}_3 : [p_2, p_3] = [2, 2] \quad \Longleftrightarrow $$



,

$$ \mathbf{8}_3 : [p_2, p_3] = [1, 1] \quad \Longleftrightarrow $$



. (8.60)

The lowest weight vectors for these representations are

$$ \begin{aligned} \Lambda^{(3)}_{\mathbf{15}} &= 3\alpha_1 + 2\alpha_2, \\ \Lambda^{(3)}_{\mathbf{8}} &= 3\alpha_1 + 3\alpha_2 + \alpha_3. \end{aligned} \tag{8.61} $$

The lowest weight vector for $\mathbf{27}_3$ is a real root of AE_3, $(\Lambda^{(3)}_{\mathbf{27}}|\Lambda^{(3)}_{\mathbf{27}}) = 2$, while the lowest weight vectors for $\mathbf{8}_3$ is timelike, $(\Lambda^{(3)}_{\mathbf{8}}|\Lambda^{(3)}_{\mathbf{8}}) = -4$. This implies that the entire representation $\mathbf{8}_3$ lies inside the lightcone of $\mathfrak{h}^\star_{\mathfrak{g}}$. Both representations have outer multiplicity one.

Note that $[0,3]$ and $[3,0]$ are also admissible solutions but have vanishing outer multiplicities by the same arguments as for the representation $[0,2]$ at level 2.

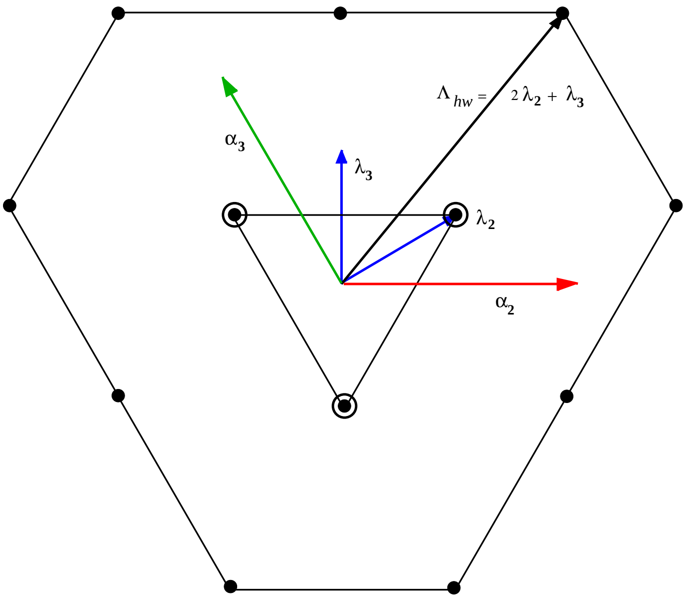


Figure 48: The representation $\mathbf{15}_2$ of $\mathfrak{sl}(3,\mathbb{R})$ appearing at level two in the decomposition of the adjoint representation of AE_3 into representations of $\mathfrak{sl}(3,\mathbb{R})$. The lowest leftmost node is the lowest weight of the representation, corresponding to the real root $\Lambda^{(2)} = 2\alpha_1 + \alpha_2$ of AE_3. This representation has outer multiplicity one.

8.3.7 Level $\ell = 4$

At this level we encounter for the first time a representation with non-trivial outer multiplicity. It is a 15-dimensional representation with the following Young tableau structure:

$$\overline{\mathbf{15}}_4 : [p_2, p_3] = [1, 2] \quad \Longleftrightarrow \quad \yng(3,2)\,. \tag{8.62}$$

The lowest weight vector is

$$\Lambda^{(4)}_{\overline{\mathbf{15}}} = 4\alpha_1 + 4\alpha_2 + \alpha_3, \tag{8.63}$$

which is an imaginary root of AE_3,

$$(\Lambda^{(4)}_{\overline{\mathbf{15}}}|\Lambda^{(4)}_{\overline{\mathbf{15}}}) = -6. \tag{8.64}$$

From Table 38 we find that this root has multiplicity 5 as a root of AE_3,

$$\text{mult}(\Lambda^{(4)}_{\overline{\mathbf{15}}}) = 5. \tag{8.65}$$

In order for Equation (8.26) to make sense, this multiplicity must be matched by the total multiplicity of $\Lambda^{(4)}_{\overline{\mathbf{15}}}$ as a weight of $\mathfrak{sl}(3,\mathbb{R})$ representations at level four. The remaining representations at this level are

$$\begin{aligned} \mathbf{24}_4 : [3, 1] &\quad \Longleftrightarrow \quad \yng(4,1)\,, \\ \overline{\mathbf{3}}_4 : [0, 1] &\quad \Longleftrightarrow \quad \yng(1,1)\,, \\ \mathbf{6}_4 : [2, 0] &\quad \Longleftrightarrow \quad \yng(2)\,, \\ \mathbf{42}_4 : [2, 3] &\quad \Longleftrightarrow \quad \yng(6,3)\,. \end{aligned} \tag{8.66}$$

By drawing these representations explicitly, one sees that the root $4\alpha_1 + 4\alpha_2 + \alpha_3$, representing the weight $\Lambda^{(4)}_{\bar{\mathbf{15}}}$, also appears as a weight (but not as a lowest weight) in the representations $\mathbf{42}_4$ and $\mathbf{24}_4$. It occurs with weight multiplicity 1 in the $\mathbf{24}_4$ but with weight multiplicity 2 in the $\mathbf{42}_4$. Taking also into account the representation $\bar{\mathbf{15}}_4$ in which it is the lowest weight we find a total weight multiplicity of 4. This implies that, since in AE_3

$$\mathrm{mult}(4\alpha_1 + 4\alpha_2 + \alpha_3) = 5, \tag{8.67}$$

the outer multiplicity of $\bar{\mathbf{15}}_4$ must be 2, i.e.,

$$\mu\left(\Lambda^{(2)}_{\bar{\mathbf{15}}}\right) = 2. \tag{8.68}$$

When we go to higher and higher levels, the outer multiplicities of the representations located entirely inside the lightcone in $\mathfrak{h}_{\mathfrak{g}}$ increase exponentially.

8.4 Level decomposition of E_{10}

As we have seen, the Kac–Moody algebra E_{10} is one of the four hyperbolic algebras of maximal rank, the others being BE_{10}, DE_{10} and CE_{10}. It can be constructed as an overextension of E_8 and is therefore often denoted by E_8^{++}. Similarly to E_8 in the rank 8 case, E_{10} is the unique indefinite rank 10 algebra with an even self-dual root lattice, namely the Lorentzian lattice $\Pi_{1,9}$.

Our first encounter with E_{10} in a physical application was in Section 5 where we have showed that the Weyl group of E_{10} describes the chaos that emerges when studying eleven-dimensional supergravity close to a spacelike singularity [45].

In Section 9.3, we will discuss how to construct a Lagrangian manifestly invariant under global E_{10}-transformations and compare its dynamics to that of eleven-dimensional supergravity. The level decomposition associated with the removal of the "exceptional node" labelled "10" in Figure 49 will be central to the analysis. It turns out that the low-level structure in this decomposition precisely reproduces the bosonic field content of eleven-dimensional supergravity [47].

Moreover, decomposing E_{10} with respect to different regular subalgebras reproduces also the bosonic field contents of the Type IIA and Type IIB supergravities. The fields of the IIA theory are obtained by decomposition in terms of representations of the $D_9 = \mathfrak{so}(9,9,\mathbb{R})$ subalgebra obtained by removing the first simple root α_1 [125]. Similarly the IIB-fields appear at low levels for a decomposition with respect to the $A_9 \oplus A_1 = \mathfrak{sl}(9,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ subalgebra found upon removal of the second simple root α_2 [126]. The extra A_1-factor in this decomposition ensures that the $SL(2,\mathbb{R})$-symmetry of IIB supergravity is recovered.

For these reasons, we investigate now these various level decompositions.

8.4.1 Decomposition with respect to $\mathfrak{sl}(10,\mathbb{R})$

Let $\alpha_1, \cdots, \alpha_{10}$ denote the simple roots of E_{10} and $\alpha_1^\vee, \cdots, \alpha_{10}^\vee$ the Cartan generators. These span the root space $\mathfrak{h}^\star$ and the Cartan subalgebra $\mathfrak{h}$, respectively. Since E_{10} is simply laced the Cartan

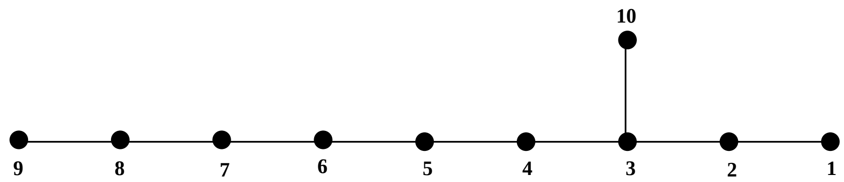


Figure 49: The Dynkin diagram of E_{10}. Labels $i = 1, \cdots, 9$ enumerate the nodes corresponding to simple roots α_i of the $\mathfrak{sl}(10, \mathbb{R})$ subalgebra and "10" labels the exceptional node.

matrix is given by the scalar products between the simple roots:

$$A_{ij}[E_{10}] = (\alpha_i|\alpha_j) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}. \tag{8.69}$$

The associated Dynkin diagram is displayed in Figure 49. We will perform the decomposition with respect to the $\mathfrak{sl}(10, \mathbb{R})$ subalgebra represented by the horizontal line in the Dynkin diagram so the level ℓ of an arbitrary root $\alpha \in \mathfrak{h}^\star$ is given by the coefficient in front of the exceptional simple root, i.e.,

$$\gamma = \sum_{i=1}^{9} m^i \alpha_i + \ell \alpha_{10}. \tag{8.70}$$

As before, the weight that is easiest to identify for each representation $\mathcal{R}(\Lambda^{(\ell)})$ at positive level ℓ is the lowest weight $\Lambda^{(\ell)}_{\text{lw}}$. We denote by $\bar{\Lambda}^{(\ell)}_{\text{lw}}$ the projection onto the spacelike slice of the root lattice defined by the level ℓ. The (conjugate) Dynkin labels $p_1, \cdots, p_9$ characterizing the representation $\mathcal{R}(\Lambda^{(\ell)})$ are defined as before as minus the coefficients in the expansion of $\bar{\Lambda}^{(\ell)}_{\text{lw}}$ in terms of the fundamental weights λ^i of $\mathfrak{sl}(10, \mathbb{R})$:

$$-\bar{\Lambda}^{(\ell)}_{\text{lw}} = \sum_{i=1}^{9} p_i \lambda^i. \tag{8.71}$$

The Killing form on each such slice is positive definite so the projected weight $\bar{\Lambda}^{(\ell)}_{\text{hw}}$ is of course real. The fundamental weights of $\mathfrak{sl}(10, \mathbb{R})$ can be computed explicitly from their definition as the duals of the simple roots:

$$\lambda^i = \sum_{j=1}^{9} B^{ij} \alpha_j, \tag{8.72}$$

where B^{ij} is the inverse of the Cartan matrix of A_9,

$$(B_{ij}[A_9])^{-1} = \frac{1}{10}\begin{pmatrix} 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 8 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ 7 & 14 & 21 & 18 & 15 & 12 & 9 & 6 & 3 \\ 6 & 12 & 18 & 24 & 20 & 16 & 12 & 8 & 4 \\ 5 & 10 & 15 & 20 & 25 & 20 & 15 & 10 & 5 \\ 4 & 8 & 12 & 16 & 20 & 24 & 18 & 12 & 6 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 14 & 7 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}. \tag{8.73}$$

Note that all the entries of B^{ij} are positive which will prove to be important later on. As we saw for the AE_3 case we want to find the possible allowed values for $(m_1, \cdots, m_9)$, or, equivalently, the possible Dynkin labels $[p_1, \cdots, p_9]$ for each level ℓ.

The corresponding diophantine equation, Equation (8.50), for E_{10} was found in [47] and reads

$$m^i = B^{i3}\ell - \sum_{j=1}^{9} B^{ij} p_j \geq 0. \tag{8.74}$$

Since the two sets $\{p_i\}$ and $\{m^i\}$ both consist of non-negative integers and all entries of B^{ij} are positive, these equations put strong constraints on the possible representations that can occur at each level. Moreover, each lowest weight vector $\Lambda^{(\ell)} = \gamma$ must be a root of E_{10}, so we have the additional requirement

$$(\Lambda^{(\ell)}|\Lambda^{(\ell)}) = \sum_{i,j=1}^{9} B^{ij} p_i p_j - \frac{1}{10}\ell^2 \leq 2. \tag{8.75}$$

The representation content at each level is represented by $\mathfrak{sl}(10,\mathbb{R})$-tensors whose index structure are encoded in the Dynkin labels $[p_1, \cdots, p_9]$. At level $\ell = 0$ we have the adjoint representation of $\mathfrak{sl}(10,\mathbb{R})$ represented by the generators $K^a{}_b$ obeying the same commutation relations as in Equation (8.32) but now with $\mathfrak{sl}(10,\mathbb{R})$-indices.

All higher (lower) level representations will then be tensors transforming contravariantly (covariantly) under the level $\ell = 0$ generators. The resulting representations are displayed up to level 3 in Table 39. We see that the level 1 and 2 representations have the index structures of a 3-form and a 6-form respectively. In the E_{10}-invariant sigma model, to be constructed in Section 9, these generators will become associated with the time-dependent physical "fields" $A_{abc}(t)$ and $A_{a_1\cdots a_6}(t)$ which are related to the electric and magnetic component of the 3-form in eleven-dimensional supergravity. Similarly, the level 3 generator $E^{a|b_1\cdots b_9}$ with mixed Young symmetry will be associated to the dual of the spatial part of the eleven-dimensional vielbein. This field is therefore sometimes referred to as the "dual graviton".

Algebraic structure at low levels

Let us now describe in a little more detail the commutation relations between the low-level generators in the level decomposition of E_{10} (see Table 39). We recover the Chevalley generators of A_9 through the following realisation:

$$e_i = K^{i+1}{}_i, \qquad f_i = K^i{}_{i+1}, \qquad h_i = K^{i+1}{}_{i+1} - K^i{}_i \qquad (i = 1, \cdots, 9), \tag{8.76}$$

where, as before, the $K^i{}_j$'s obey the commutation relations

$$[K^i{}_j, K^k{}_l] = \delta^k_j K^i{}_l - \delta^i_l K^k{}_j. \tag{8.77}$$

Table 39: The low-level representations in a decomposition of the adjoint representation of E_{10} into representations of its A_9 subalgebra obtained by removing the exceptional node in the Dynkin diagram in Figure 49.

ℓ	$\Lambda^{(\ell)} = [p_1, \cdots, p_9]$	$\Lambda^{(\ell)} = (m_1, \cdots, m_{10})$	A_9-representation	E_{10}-generator
1	$[0,0,1,0,0,0,0,0,0]$	$(0,0,0,0,0,0,0,0,0,1)$	$\mathbf{120}_1$	E^{abc}
2	$[0,0,0,0,0,1,0,0,0]$	$(1,2,3,2,1,0,0,0,0,2)$	$\mathbf{210}_2$	$E^{a_1\cdots a_6}$
3	$[1,0,0,0,0,0,0,1,0]$	$(1,3,5,4,3,2,1,0,0,3)$	$\mathbf{440}_3$	$E^{a\|b_1\cdots b_8}$

At levels ±1 we have the positive root generators E^{abc} and their negative counterparts $F_{abc} = -\tau(E^{abc})$, where τ denotes the Chevalley involution as defined in Section 4. Their transformation properties under the $\mathfrak{sl}(10,\mathbb{R})$-generators $K^a{}_b$ follow from the index structure and reads explicitly

$$\begin{aligned}
[K^a{}_b, E^{cde}] &= 3\delta^{[c}_b E^{de]a}, \\
[K^a{}_b, F_{cde}] &= -3\delta^a{}_{[c}F_{de]b}, \\
[E^{abc}, F_{def}] &= 18\delta^{[ab}_{[de}K^{c]}{}_{f]} - 2\delta^{abc}_{def}\sum_{a=1}^{10} K^a{}_a,
\end{aligned} \tag{8.78}$$

where we defined

$$\begin{aligned}
\delta^{ab}_{cd} &= \frac{1}{2}(\delta^a_c\delta^b_d - \delta^b_c\delta^a_d) \\
\delta^{abc}_{def} &= \frac{1}{3!}(\delta^a_d\delta^b_e\delta^c_f \pm 5 \text{ permutations}).
\end{aligned} \tag{8.79}$$

The "exceptional" generators e_{10} and f_{10} are fixed by Equation (8.76) to have the following realisation:

$$e_{10} = E^{123}, \qquad f_{10} = F_{123}. \tag{8.80}$$

The corresponding Cartan generator is obtained by requiring $[e_{10}, f_{10}] = h_{10}$ and upon inspection of the last equation in Equation (8.78) we find

$$h_{10} = -\frac{1}{3}\sum_{i\neq 1,2,3} K^a{}_a + \frac{2}{3}(K^1{}_1 + K^2{}_2 + K^3{}_3), \tag{8.81}$$

enlarging $\mathfrak{sl}(10,\mathbb{R})$ to $\mathfrak{gl}(10,\mathbb{R})$.

The bilinear form at level zero is

$$(K^i{}_j|K^k{}_l) = \delta^i_l\delta^k_j - \delta^i_j\delta^k_l \tag{8.82}$$

and can be extended level by level to the full algebra by using its invariance, $([x,y]|z) = (x|[y,z])$ for $x,y,z \in E_{10}$ (see Section 4). For level 1 this yields

$$\left(E^{abc}|F_{def}\right) = 3!\delta^{abc}_{def}, \tag{8.83}$$

where the normalization was chosen such that

$$(e_{10}|f_{10}) = \left(E^{123}|F_{123}\right) = 1. \tag{8.84}$$

Now, by using the graded structure of the level decomposition we can infer that the level 2 generators can be obtained by commuting the level 1 generators

$$[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_2. \tag{8.85}$$

Concretely, this means that the level 2 content should be found from the commutator

$$[E^{a_1a_2a_3}, E^{a_4a_5a_6}]. \tag{8.86}$$

We already know that the only representation at this level is $\mathbf{210}_2$, realized by an antisymmetric 6-form. Since the normalization of this generator is arbitrary we can choose it to have weight one and hence we find

$$E^{a_1\cdots a_6} = [E^{a_1a_2a_3}, E^{a_4a_5a_6}]. \tag{8.87}$$

The bilinear form is lifted to level 2 in a similar way as before with the result

$$(E^{a_1\cdots a_6}|F_{b_1\cdots b_6}) = 6!\delta^{a_1\cdots a_6}_{b_1\cdots b_6}. \tag{8.88}$$

Continuing these arguments, the level 3-generators can be obtained from

$$[[\mathfrak{g}_1, \mathfrak{g}_1], \mathfrak{g}_1] \subseteq \mathfrak{g}_3. \tag{8.89}$$

From the index structure one would expect to find a 9-form generator $E^{a_1\cdots a_9}$ corresponding to the Dynkin labels $[0,0,0,0,0,0,0,0,1]$. However, we see from Table 39 that only the representation $[1,0,0,0,0,0,0,1,0]$ appears at level 3. The reason for the disappearance of the representation $[0,0,0,0,0,0,0,0,1]$ is because the generator $E^{a_1\cdots a_9}$ is not allowed by the Jacobi identity. A detailed explanation for this can be found in [77]. The right hand side of Equation (8.89) therefore only contains the index structure compatible with the generators $E^{a|b_1\cdots b_8}$,

$$[[E^{ab_1b_2}, E^{b_3b_4b_5}], E^{b_6b_7b_8}] = -E^{[a|b_1b_2]b_3\cdots b_8}, \tag{8.90}$$

where the minus sign is purely conventional.

For later reference, we list here some additional commutators that are useful [53]:

$$\begin{aligned}
[E^{a_1\cdots a_6}, F_{b_1b_2b_3}] &= -5!\delta^{[a_1a_2a_3}_{b_1b_2b_3}E^{a_1a_2a_3]}, \\
[E^{a_1\cdots a_6}, F_{b_1\cdots b_6}] &= 6\cdot 6!\delta^{[a_1\cdots a_5}_{[b_1\cdots b_5}K^{a_6]}{}_{b_6]} - \frac{2}{3}\cdot 6!\delta^{a_1\cdots a_6}_{b_1\cdots b_6}\sum_{a=1}^{10}K^a{}_a, \\
[E^{a_1|a_2\cdots a_9}, F_{b_1b_2b_3}] &= -7\cdot 48\left(\delta^{a_1[a_2a_3}_{b_1b_2b_3}E^{a_4\cdots a_9]} - \delta^{[a_2a_3a_4}_{b_1b_2b_3}E^{a_5\cdots a_9]a_1}\right), \\
[E^{a_1|a_2\cdots a_9}, F_{b_1\cdots b_6}] &= -8!\left(\delta^{a_1[a_2\cdots a_6}_{b_1\cdots b_6}E^{a_7a_8a_9]} - \delta^{[a_2\cdots a_7}_{b_1\cdots b_6}E^{a_8a_9]a_1}\right).
\end{aligned} \tag{8.91}$$

8.4.2 "Gradient representations"

So far, we have only discussed the representations occurring at the first four levels in the E_{10} decomposition. This is due to the fact that a physical interpretation of higher level fields is yet to be found. There are, however, among the infinite number of representations, a subset of three (infinite) towers of representations with certain appealing properties. These are the "gradient representations", so named due to their conjectured relation to the emergence of space, through a Taylor-like expansion in spatial gradients [47]. We explain here how these representations arise and we emphasize some of their important properties, leaving a discussion of the physical interpretation to Section 9.

The gradient representations are obtained by searching for "affine representations", for which the coefficient m^9 in front of the overextended simple root of E_{10} vanishes, i.e., the lowest weights of the representations correspond to the following subset of E_{10} roots,

$$\gamma = \sum_{i=1}^{8} m^i\alpha_i + \ell\alpha_{10}. \tag{8.92}$$

The Dynkin labels allowed by this restricting are parametrized by an integer k which is related to the level at which a specific representation occurs in the following way:

$$\ell = 3k+1 \qquad [0,0,1,0,0,0,0,0,k], \tag{8.93}$$
$$\ell = 3k+2 \qquad [0,0,0,0,0,1,0,0,k], \tag{8.94}$$
$$\ell = 3k+3 \qquad [1,0,0,0,0,0,0,1,k]. \tag{8.95}$$

One easily verifies that these representations fulfill the diophantine constraints (8.74) and the lowest weight has length squared, Equation (8.75), equal to 2 and is thus indeed a real root of E_{10}. For $k=0$ these representations reduce to the ones for $\ell = 1,2$ and 3, and hence the gradient representations correspond to generalizations of these standard low-level structures. The corresponding generators have, respectively at levels $\ell = 3k+1, 3k+2$ or $3k+3$, additional sets of k "9-tuples" of antisymmetric indices,

$$\begin{array}{lcl} [0,0,1,0,0,0,0,0,k] & \Longrightarrow & E^{a_1\cdots a_9, b_1\cdots b_9, \cdots, c_1c_2c_3}, \\ [0,0,0,0,0,1,0,0,k] & \Longrightarrow & E^{a_1\cdots a_9, b_1\cdots b_9, \cdots, c_1\cdots c_6}, \\ [1,0,0,0,0,0,0,1,k] & \Longrightarrow & E^{a_1\cdots a_9, b_1\cdots b_9, \cdots, |c|d_1\cdots d_8} \end{array} \tag{8.96}$$

(with the irreducibility conditions expressing that antisymmetrizations involving one more index over the explicit antisymmetry are zero). Since these are $\mathfrak{sl}(10,\mathbb{R})$-representations we can use the rank 10 antisymmetric epsilon tensor $\epsilon_{a_1\cdots a_{10}}$ to dualize these representations, for instance for the $\ell = 3k+1$ tower we get

$$E^{a_1a_2a_3}{}_{b_1\cdots b_k} = \epsilon_{b_1c_1\cdots c_9}\epsilon_{b_2d_1\cdots d_9}\cdots\epsilon_{b_ke_1\cdots e_9}E^{c_1\cdots c_9, d_1\cdots d_9, \cdots, e_1\cdots e_9, a_1a_2a_3}, \tag{8.97}$$

where the lower indices $b_1\cdots b_k$ are now completely symmetric and furthermore, obey appropriate tracelessness conditions when contracted with an upper index.

Thus, in this way we obtain the three infinite towers of E_{10} generators

$$E^{a_1a_2a_3}{}_{b_1\cdots b_k}, \qquad E^{a_1\cdots a_6}{}_{b_1\cdots b_k}, \qquad E^{a_1|a_2\cdots a_9}{}_{b_1\cdots b_k}. \tag{8.98}$$

The lowest weight vectors of these representations are all spacelike and so these representations always come with outer multiplicity one.

The existence of these towers of representations is not special for E_{10} among the exceptional algebras, although the symmetric Young structure of the lower indices is actually a very special and important feature of E_{10}. In Section 9 we will discuss the tantalizing possibility that these representations encode an infinite set of spatial gradients that describe the emergence, or "unfolding", of space.

To illustrate the difference from other exceptional algebras, we consider, for instance, a similar search for affine representations within E_{11} (see, e.g. [141]). The same sets of 9-tuples appear, but now these should be dualized with the rank 11 epsilon tensor of $\mathfrak{sl}(11,\mathbb{R})$, leaving us with three towers of generators that have k *pairs of antisymmetric indices*, i.e.,

$$E^{\mu_1\mu_2\mu_3}{}_{[\nu_1\rho_1]\cdots[\nu_k\rho_k]}, \qquad E^{\mu_1\cdots\mu_6}{}_{[\nu_1\rho_1]\cdots[\nu_k\rho_k]}, \qquad E^{\mu_1|\sigma_2\cdots\sigma_9}{}_{[\nu_1\rho_1]\cdots[\nu_k\rho_k]}, \tag{8.99}$$

where all indices are $\mathfrak{sl}(11,\mathbb{R})$-indices and so run from 1 to 11. No interpretation in terms of spatial gradients exist for these generators. Note, however, that these representations have recently been interpreted as dual to scalars [149].

Finally, we note that because all these representations were found by setting $m^9 = 0$, we are really dealing with representations that also exist within E_9, in the sense that when restricting all indices to $\mathfrak{sl}(9,\mathbb{R})$-indices, these generators can be found in a level decomposition of E_9 with respect to its $\mathfrak{sl}(9,\mathbb{R})$-subalgebra. However, it is important to note that in E_{10} and E_{11} the affine representations constitute merely a small subset of all representations occurring in the level decomposition, while in E_9 they are actually the only ones and so they provide (together with their transposed partners) the full structure of the algebra. Moreover, in E_9 the epsilon tensor is of rank 9 so all the 9-tuples of antisymmetric indices are "swallowed" by the epsilon tensor. This reflects the fact that for affine algebras the level decomposition corresponds to an infinite repetition of the low-level representations.

8.4.3 Decomposition with respect to $\mathfrak{so}(9,9)$ and $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(9,\mathbb{R})$

A level decomposition can be performed with respect to any of the regular subalgebras encoded in the Dynkin diagram. We mention here two additional cases which are specifically interesting for our purposes, since they give rise to low-level field contents that coincide with the bosonic spectrum of Type IIA and IIB supergravity. The relevant decompositions are the following:

$$\begin{aligned} \text{IIA} \quad &\Longleftrightarrow \quad \mathfrak{so}(9,9) \subset E_{10}, \\ \text{IIB} \quad &\Longleftrightarrow \quad \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(9,\mathbb{R}) \subset E_{10}. \end{aligned} \tag{8.100}$$

The corresponding levels are defined as

$$\begin{aligned} \mathfrak{so}(9,9) : \gamma = \ell\alpha_1 + \sum_{i=2}^{10} m^i \alpha_i \in \mathfrak{h}^\star, \\ \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(9,\mathbb{R}) : \gamma = m^1\alpha_1 + \ell\alpha_2 + \sum_{j=3}^{10} m^j \alpha_j \in \mathfrak{h}^\star. \end{aligned} \tag{8.101}$$

It turns out that in the $\mathfrak{so}(9,9)$ decomposition the even levels correspond to vectorial representations of $\mathfrak{so}(9,9)$ while the odd levels give spinorial representations. This implies that the fields in the NS-NS sector arise at even levels and the R-R fields correspond to odd level representations of $\mathfrak{so}(9,9)$.

On the contrary, in the $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(9,\mathbb{R})$ decomposition the additional factor of $\mathfrak{sl}(2,\mathbb{R})$ causes mixing between the R-R and NS-NS fields at each level. This is to be expected since we know that for example the fundamental string (F1) and the D1-brane couples to the NS-NS 2-form B_2 and the R-R 2-form C_2, respectively, which transform as a doublet under the $SL(2,\mathbb{R})$-symmetry of Type IIB supergravity.

In the $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(9,\mathbb{R})$ the level $\ell = 0$ content is of course just the adjoint representation in the same way as in the $\mathfrak{sl}(10,\mathbb{R})$ decomposition considered above. In the other case instead we find the adjoint representation $M^{ab} = -M^{ba}$ of $\mathfrak{so}(9,9)$ with commutation relations

$$[M^{ab}, M^{cd}] = \eta^{ca}M^{bd} - \eta^{cb}M^{ad} - \eta^{da}M^{bc} + \eta^{db}M^{ac}, \tag{8.102}$$

where η^{ab} is the split diagonal metric with $(9,9)$-signature.

The procedure follows a similar structure as for the previous cases so we will not give the details here. We refer the interested reader to [125, 126] for a detailed account. The result of the decompositions up to level 3 for the two cases discussed here is displayed in Tables 40 and 41.

Table 40: The low-level representations in a decomposition of the adjoint representation of E_{10} into representations of its $\mathfrak{so}(9,9)$ subalgebra obtained by removing the first node in the Dynkin diagram in Figure 49. Note that the lower indices at levels 1 and 3 are spinor indices of $\mathfrak{so}(9,9)$.

ℓ	$\Lambda^{(\ell)} = [p_1, \cdots, p_9]$	E_{10}-generator
1	$[0,0,0,0,0,0,0,1,0]$	E_α
2	$[0,0,1,0,0,0,0,0,0]$	$E^{a_1a_2a_3}$
3	$[1,0,0,0,0,0,0,1,0]$	$E^a{}_\beta$

Table 41: The low-level representations in a decomposition of the adjoint representation of E_{10} into representations of its $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(9,\mathbb{R})$ subalgebra obtained by removing the second node in the Dynkin diagram in Figure 49. The index α at levels 1 and 3 corresponds to the fundamental representation of $\mathfrak{sl}(2,\mathbb{R})$.

ℓ	$\Lambda^{(\ell)} = [p_1, \cdots, p_9] \otimes \mathcal{R}[A_1]$	E_{10}-generator
1	$[0,0,0,0,0,0,0,1,0] \otimes \mathbf{2}$	$E^{ab}{}_\alpha$
2	$[0,0,0,0,0,1,0,0,0] \otimes \mathbf{1}$	$E^{a_1\cdots a_4}$
3	$[1,0,0,0,0,0,0,1,0] \otimes \mathbf{2}$	$E^{a_1\cdots a_6}{}_\alpha$

9 Hidden Symmetries Made Manifest – Infinite-Dimensional Coset Spaces

As we have indicated above, the emergence of hyperbolic Coxeter groups in the BKL-limit has been argued to be the tip of an iceberg signaling the existence of a huge number of symmetries underlying gravitational theories. However, while the appearance of hyperbolic Coxeter groups is a solid fact that will in our opinion survive future developments, the exact way in which the conjectured infinite-dimensional symmetry acts is still a matter of debate and research.

The aim of this section is to describe one line of investigation for making the infinite-dimensional symmetry manifest. This approach is directly inspired by the results obtained through toroidal dimensional reduction of gravitational theories, where the scalar fields form coset manifolds exhibiting explicitly larger and larger symmetries as one goes down in dimensions. In the case of eleven-dimensional supergravity, reduction on an n-torus T^n reveals a chain of exceptional U-duality symmetries $\mathcal{E}_{n(n)}$ [33, 34], culminating with $\mathcal{E}_{8(8)}$ in three dimensions [134]. This has lead to the conjecture [113] that the chain of enhanced symmetries should in fact remain unbroken and give rise to the infinite-dimensional duality groups $\mathcal{E}_{9(9)}, \mathcal{E}_{10(10)}$ and $\mathcal{E}_{11(11)}$, as one reduces the theory to two, one and zero dimensions, respectively.

The connection between the symmetry groups controlling the billiards in the BKL-limit, and the symmetry groups appearing in toroidal dimensional reduction to three dimensions, where coset spaces play a central role, has led to the attempt to reformulate eleven-dimensional supergravity as a $(1+0)$-dimensional nonlinear sigma model based on the infinite-dimensional coset manifold $\mathcal{E}_{10(10)}/\mathcal{K}(\mathcal{E}_{10(10)})$ [47]. This sigma model describes the geodesic flow of a particle moving on $\mathcal{E}_{10(10)}/\mathcal{K}(\mathcal{E}_{10(10)})$, whose dynamics can be seen to match the dynamics of the associated (suitably truncated) supergravity theory. Another, related, source of inspiration for the idea pushed forward in [47] has been the earlier proposal to reformulate eleven-dimensional supergravity as a nonlinear realisation of the even bigger symmetry $\mathcal{E}_{11(11)}$ [167], containing $\mathcal{E}_{10(10)}$ as a subgroup.

A central tool in the analysis of [47] is the level decomposition studied in Section 8. Although proposed some time ago and crowned with partial successes at low levels, the attempt to reformulate eleven-dimensional supergravity as an infinite-dimensional nonlinear sigma model, faces obstacles that have not yet been overcome at higher levels. This indicates that novel ideas are needed in order to make further progress towards a complete understanding of the role of infinite-dimensional symmetry groups in gravitational theories.

We begin by describing some general aspects of nonlinear sigma models for finite-dimensional coset spaces. We then explain how to generalize the construction to the infinite-dimensional case. We finally apply the construction in detail to the case of eleven-dimensional supergravity where the conjectured symmetry group is $\mathcal{E}_{10(10)}$. This is one of the most extensively investigated models in the literature in view of its connection with M-theory. The techniques presented, however, can be applied to all gravitational models exhibiting the $\mathcal{U}_3$-duality symmetries discussed in Sections 5 and 7.

9.1 Nonlinear sigma models on finite-dimensional coset spaces

A nonlinear sigma model describes maps ξ from one Riemannian space X, equipped with a metric γ, to another Riemannian space, the "target space" M, with metric g. Let x^m $(m = 1, \cdots, p = \dim X)$ be coordinates on X and ξ^α $(\alpha = 1, \cdots, q = \dim M)$ be coordinates on M. Then the standard action for this sigma model is

$$S = \int_X d^p x \sqrt{\gamma}\, \gamma^{mn}(x)\, \partial_m \xi^\alpha(x)\, \partial_n \xi^\beta(x)\, g_{\alpha\beta}\left(\xi(x)\right). \tag{9.1}$$

Solutions to the equations of motion resulting from this action will describe the maps ξ^α as functions of x^m.

A familiar example, of direct interest to the analysis below, is the case where X is one-dimensional, parametrized by the coordinate t. Then the action for the sigma model reduces to

$$S_{\text{geodesic}} = \int dt\, A \frac{d\xi^\alpha(t)}{dt}\frac{d\xi^\beta(t)}{dt} g_{\alpha\beta}\left(\xi(t)\right), \tag{9.2}$$

where A is $\gamma^{11}\sqrt{\gamma}$ and ensures reparametrization invariance in the variable t. Extremization with respect to A enforces the constraint

$$\frac{d\xi^\alpha(t)}{dt}\frac{d\xi^\beta(t)}{dt} g_{\alpha\beta}\left(\xi(t)\right) = 0, \tag{9.3}$$

ensuring that solutions to this model are null geodesics on M. We have already encountered such a sigma model before, namely as describing the free lightlike motion of the billiard ball in the $(\dim M - 1)$-dimensional scale-factor space. In that case A corresponds to the inverse "lapse-function" N^{-1} and the metric $g_{\alpha\beta}$ is a constant Lorentzian metric.

9.1.1 The Cartan involution and symmetric spaces

In what follows, we shall be concerned with sigma models on symmetric spaces $\mathcal{G}/\mathcal{K}(\mathcal{G})$ where $\mathcal{G}$ is a Lie group with semi-simple real Lie algebra $\mathfrak{g}$ and $\mathcal{K}(\mathcal{G})$ its maximal compact subgroup with real Lie algebra $\mathfrak{k}$, corresponding to the maximal compact subalgebra of $\mathfrak{g}$. Since elements of the coset are obtained by factoring out $\mathcal{K}(\mathcal{G})$, this subgroup is referred to as the "local gauge symmetry group" (see below). Our aim is to provide an algebraic construction of the metric on the coset and of the Lagrangian.

We have investigated real forms in Section 6 and have found that the Cartan involution θ induces a Cartan decomposition of $\mathfrak{g}$ into even and odd eigenspaces:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{9.4}$$

(direct sum of vector spaces), where

$$\begin{aligned} \mathfrak{k} &= \{x \in \mathfrak{g} \,|\, \theta(x) = x\}, \\ \mathfrak{p} &= \{y \in \mathfrak{g} \,|\, \theta(y) = -y\} \end{aligned} \tag{9.5}$$

play central roles. The decomposition (9.4) is orthogonal, in the sense that $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the invariant bilinear form $(\cdot|\cdot) \equiv B(\cdot,\cdot)$,

$$\mathfrak{p} = \{y \in \mathfrak{g} \,|\, \forall x \in \mathfrak{k} : (y|x) = 0\}. \tag{9.6}$$

The commutator relations split in a way characteristic for symmetric spaces,

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}, \qquad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}. \tag{9.7}$$

The subspace $\mathfrak{p}$ is not a subalgebra. Elements of $\mathfrak{p}$ transform in some representation of $\mathfrak{k}$, which depends on the Lie algebra $\mathfrak{g}$. We stress that if the commutator $[\mathfrak{p},\mathfrak{p}]$ had also contained elements in $\mathfrak{p}$ itself, this would not have been a symmetric space.

The coset space $\mathcal{G}/\mathcal{K}(\mathcal{G})$ is defined as the set of equivalence classes $[g]$ of $\mathcal{G}$ defined by the equivalence relation

$$g \sim g' \qquad \text{iff } gg'^{-1} \in \mathcal{K}(\mathcal{G}) \text{ and } g, g' \in \mathcal{G}, \tag{9.8}$$

i.e.,

$$[g] = \{kg \,|\, \forall k \in \mathcal{K}(\mathcal{G})\}. \tag{9.9}$$

Example: The coset space $SL(n,\mathbb{R})/SO(n)$

As an example to illustrate the Cartan involution we consider the coset space $SL(n,\mathbb{R})/SO(n)$. The group $SL(n,\mathbb{R})$ contains all $n\times n$ real matrices with determinant equal to one. The associated Lie algebra $\mathfrak{sl}(n,\mathbb{R})$ thus consists of real $n\times n$ traceless matrices. In this case the Cartan involution is simply (minus) the ordinary matrix transpose $(\,)^T$ on the Lie algebra elements:

$$\tau : a \quad\longmapsto\quad -a^T \qquad a\in\mathfrak{sl}(n,\mathbb{R}). \tag{9.10}$$

This implies that all antisymmetric traceless $n\times n$ matrices belong to $\mathfrak{k}=\mathfrak{so}(n)$. The Cartan involution θ is the differential at the identity of an involution Θ defined on the group itself, such that for real Lie groups (real or complex matrix groups), θ is just the inverse conjugate transpose. Defining

$$\mathcal{K}(\mathcal{G}) = \{g\in\mathcal{G}\,|\,\Theta g = g\} \tag{9.11}$$

then gives in this example the group $\mathcal{K}(\mathcal{G})=SO(n)$. The Cartan decomposition of $\mathfrak{sl}(n,\mathbb{R})$ thus splits all elements into symmetric and antisymmetric matrices, i.e., for $a\in\mathfrak{sl}(n,\mathbb{R})$ we have

$$\begin{aligned} &a-a^T\in\mathfrak{so}(n),\\ &a+a^T\in\mathfrak{p}. \end{aligned} \tag{9.12}$$

9.1.2 Nonlinear realisations

The group $\mathcal{G}$ naturally acts through (here, right) multiplication on the quotient space $\mathcal{G}/\mathcal{K}(\mathcal{G})$[34] as

$$[h]\mapsto[hg]. \tag{9.13}$$

This definition makes sense because if $h\sim h'$, i.e., $h'=kh$ for some $k\in\mathcal{K}(\mathcal{G})$, then $h'g\sim hg$ since $h'g=(kh)g=k(hg)$ (left and right multiplications commute).

In order to describe a dynamical theory on the quotient space $\mathcal{G}/\mathcal{K}(\mathcal{G})$, it is convenient to introduce as dynamical variable the group element $\mathrm{V}(x)\in\mathcal{G}$ and to construct the action for $\mathrm{V}(x)$ in such a way that the equivalence relation

$$\forall k(x)\in\mathcal{K}(\mathcal{G}) : \mathrm{V}(x)\sim k(x)\mathrm{V}(x) \tag{9.14}$$

corresponds to a gauge symmetry. The physical (gauge invariant) degrees of freedom are then parametrized indeed by points of the coset space. We also want to impose Equation (9.13) as a rigid symmetry. Thus, the action should be invariant under

$$\mathrm{V}(x)\quad\longmapsto\quad k(x)\mathrm{V}(x)g,\qquad k(x)\in\mathcal{K}(\mathcal{G}),\, g\in\mathcal{G}. \tag{9.15}$$

One may develop the formalism without fixing the $\mathcal{K}(\mathcal{G})$-gauge symmetry, or one may instead fix the gauge symmetry by choosing a specific coset representative $\mathrm{V}(x)\in\mathcal{G}/\mathcal{K}(\mathcal{G})$. When $\mathcal{K}(\mathcal{G})$ is a maximal compact subgroup of $\mathcal{G}$ there are no topological obstructions, and a standard choice, which is always available, is to take $\mathrm{V}(x)$ to be of upper triangular form as allowed by the Iwasawa decomposition. This is usually called the *Borel gauge* and will be discussed in more detail later. In this case an arbitrary global transformation,

$$\mathrm{V}(x)\quad\longmapsto\quad\mathrm{V}(x)'=\mathrm{V}(x)g,\qquad g\in\mathcal{G}, \tag{9.16}$$

[34] Strictly speaking, the coset space defined in this way should be written as $\mathcal{K}(\mathcal{G})\backslash\mathcal{G}$. However, we follow what has become common practice in the literature and denote it by $\mathcal{G}/\mathcal{K}(\mathcal{G})$.

will destroy the gauge choice because $\mathrm{V}'(x)$ will generically not be of upper triangular form. Then, a compensating local $\mathcal{K}(\mathcal{G})$-transformation is needed that restores the original gauge choice. The total transformation is thus

$$\mathrm{V}(x) \quad\longmapsto\quad \mathrm{V}(x)'' = k\left(\mathrm{V}(x), g\right)\mathrm{V}(x)g, \qquad k\left(\mathrm{V}(x), g\right) \in \mathcal{K}(\mathcal{G}),\ g \in \mathcal{G}, \tag{9.17}$$

where $\mathrm{V}''(x)$ is again in the upper triangular gauge. Because now $k\left(\mathrm{V}(x), g\right)$ depends nonlinearly on $\mathrm{V}(x)$, this is called a *nonlinear realisation* of $\mathcal{G}$.

9.1.3 Three ways of writing the quadratic $\mathcal{K}(\mathcal{G})_{\mathrm{local}} \times \mathcal{G}_{\mathrm{rigid}}$-invariant action

Given the field $\mathrm{V}(x)$, we can form the Lie algebra valued one-form (Maurer–Cartan form)

$$d\mathrm{V}(x)\,\mathrm{V}(x)^{-1} = dx^{\mu}\,\partial_{\mu}\mathrm{V}(x)\mathrm{V}(x)^{-1}. \tag{9.18}$$

Under the Cartan decomposition, this element splits according to Equation (9.4),

$$\partial_{\mu}\mathrm{V}(x)\,\mathrm{V}(x)^{-1} = \mathrm{Q}_{\mu}(x) + \mathrm{P}_{\mu}(x), \tag{9.19}$$

where $\mathrm{Q}_{\mu}(x) \in \mathfrak{k}$ and $\mathrm{P}_{\mu}(x) \in \mathfrak{p}$. We can use the Cartan involution θ to write these explicitly as projections onto the odd and even eigenspaces as follows:

$$\begin{aligned} \mathrm{Q}_{\mu}(x) &= \frac{1}{2}\left[\partial_{\mu}\mathrm{V}(x)\mathrm{V}(x)^{-1} + \theta\left(\partial_{\mu}\mathrm{V}(x)\mathrm{V}(x)^{-1}\right)\right] \in \mathfrak{k}, \\ \mathrm{P}_{\mu}(x) &= \frac{1}{2}\left[\partial_{\mu}\mathrm{V}(x)\mathrm{V}(x)^{-1} - \theta\left(\partial_{\mu}\mathrm{V}(x)\mathrm{V}(x)^{-1}\right)\right] \in \mathfrak{p}. \end{aligned} \tag{9.20}$$

If we define a *generalized transpose* $\mathcal{T}$ by

$$(\,)^{\mathcal{T}} \equiv -\theta(\,), \tag{9.21}$$

then $\mathrm{P}_{\mu}(x)$ and $\mathrm{Q}_{\mu}(x)$ correspond to symmetric and antisymmetric elements, respectively,

$$\mathrm{P}_{\mu}(x)^{\mathcal{T}} = \mathrm{P}_{\mu}(x), \qquad \mathrm{Q}_{\mu}(x)^{\mathcal{T}} = -\mathrm{Q}_{\mu}(x). \tag{9.22}$$

Of course, in the special case when $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{k} = \mathfrak{so}(n)$, the generalized transpose $(\,)^{\mathcal{T}}$ coincides with the ordinary matrix transpose $(\,)^{T}$. The Lie algebra valued one-forms with components $\partial_{\mu}\mathrm{V}(x)\mathrm{V}(x)^{-1}$, $\mathrm{Q}_{\mu}(x)$ and $\mathrm{P}_{\mu}(x)$ are invariant under rigid right multiplication, $\mathrm{V}(x) \mapsto \mathrm{V}(x)g$.

Being an element of the Lie algebra of the gauge group, $\mathrm{Q}_{\mu}(x)$ can be interpreted as a gauge connection for the local symmetry $\mathcal{K}(\mathcal{G})$. Under a local transformation $k(x) \in \mathcal{K}(\mathcal{G})$, $\mathrm{Q}_{\mu}(x)$ transforms as

$$\mathcal{K}(\mathcal{G}) : \mathrm{Q}_{\mu}(x) \quad\longmapsto\quad k(x)\mathrm{Q}_{\mu}(x)k(x)^{-1} + \partial_{\mu}k(x)k(x)^{-1}, \tag{9.23}$$

which indeed is the characteristic transformation property of a gauge connection. On the other hand, $\mathrm{P}_{\mu}(x)$ transforms covariantly,

$$\mathcal{K}(\mathcal{G}) : \mathrm{P}_{\mu}(x) \quad\longmapsto\quad k(x)\mathrm{P}_{\mu}(x)k(x)^{-1}, \tag{9.24}$$

because the element $\partial_{\mu}k(x)k(x)^{-1}$ is projected out due to the negative sign in the construction of $\mathrm{P}_{\mu}(x)$ in Equation (9.20).

Using the bilinear form $(\cdot|\cdot)$ we can now form a manifestly $\mathcal{K}(\mathcal{G})_{\mathrm{local}} \times \mathcal{G}_{\mathrm{rigid}}$-invariant expression by simply "squaring" $\mathrm{P}_{\mu}(x)$, i.e., the p-dimensional action takes the form (cf. Equation (9.1))

$$S_{\mathcal{G}/\mathcal{K}(\mathcal{G})} = \int_X d^p x\,\sqrt{\gamma}\gamma^{\mu\nu}\left(\mathrm{P}_{\mu}(x)\middle|\mathrm{P}_{\nu}(x)\right). \tag{9.25}$$

We can rewrite this action in a number of ways. First, we note that since $\mathrm{Q}_\mu(x)$ can be interpreted as a gauge connection we can form a "covariant derivative" D_μ in a standard way as

$$\mathrm{D}_\mu \mathrm{V}(x) \equiv \partial_\mu \mathrm{V}(x) - \mathrm{Q}_\mu(x)\mathrm{V}(x), \tag{9.26}$$

which, by virtue of Equation (9.20), can alternatively be written as

$$\mathrm{D}_\mu \mathrm{V}(x) = \mathrm{P}_\mu(x)\mathrm{V}(x). \tag{9.27}$$

We see now that the action can indeed be interpreted as a gauged nonlinear sigma model, in the sense that the local invariance is obtained by minimally coupling the globally $\mathcal{G}$-invariant expression $(\partial_\mu \mathrm{V}(x)\mathrm{V}(x)^{-1}|\partial^\mu \mathrm{V}(x)\mathrm{V}(x)^{-1})$ to the gauge field $\mathrm{Q}_\mu(x)$ through the "covariantization" $\partial_\mu \to \mathrm{D}_\mu$,

$$\left(\partial_\mu \mathrm{V}(x)\mathrm{V}(x)^{-1}\middle|\partial^\mu \mathrm{V}(x)\mathrm{V}(x)^{-1}\right) \quad \longrightarrow \quad \left(\mathrm{D}_\mu \mathrm{V}(x)\mathrm{V}(x)^{-1}\middle|\mathrm{D}^\mu \mathrm{V}(x)\mathrm{V}(x)^{-1}\right) = \left(\mathrm{P}_\mu(x)\middle|\mathrm{P}_\nu(x)\right). \tag{9.28}$$

Thus, the action then takes the form

$$S_{\mathcal{G}/\mathcal{K}(\mathcal{G})} = \int_X d^p x\, \sqrt{\gamma}\gamma^{\mu\nu} \left(\mathrm{D}_\mu \mathrm{V}(x)\mathrm{V}(x)^{-1}\middle|\mathrm{D}_\nu \mathrm{V}(x)\mathrm{V}(x)^{-1}\right). \tag{9.29}$$

We can also form a generalized "metric" $\mathrm{M}(x)$ that does not transform at all under the local symmetry, but only transforms under rigid $\mathcal{G}$-transformations. This is done, using the generalized transpose (extended from the algebra to the group through the exponential map [93]), in the following way,

$$\mathrm{M}(x) \equiv \mathrm{V}(x)^{\mathcal{T}}\mathrm{V}(x), \tag{9.30}$$

which is clearly invariant under local transformations

$$\mathcal{K}(\mathcal{G}) : \mathrm{M}(x) \quad \longmapsto \quad (k(x)\mathrm{V}(x))^{\mathcal{T}}(k(x)\mathrm{V}(x)) = \mathrm{V}(x)^{\mathcal{T}}\left(k(x)^{\mathcal{T}}k(x)\right)\mathrm{V}(x) = \mathrm{M}(x) \tag{9.31}$$

for $k(x) \in \mathcal{K}(\mathcal{G})$, and transforms as follows under global transformations on $\mathrm{V}(x)$ from the right,

$$\mathcal{G} : \mathrm{M}(x) \quad \longmapsto \quad g^{\mathcal{T}}\mathrm{M}(x)g, \qquad g \in \mathcal{G}. \tag{9.32}$$

A short calculation shows that the relation between $\mathrm{M}(x) \in \mathcal{G}$ and $\mathrm{P}(x) \in \mathfrak{p}$ is given by

$$\begin{aligned}\frac{1}{2}\mathrm{M}(x)^{-1}\partial_\mu \mathrm{M}(x) &= \frac{1}{2}\left(\mathrm{V}(x)^{\mathcal{T}}\mathrm{V}(x)\right)^{-1}\partial_\mu \mathrm{V}(x)^{\mathcal{T}}\mathrm{V}(x) + \left(\mathrm{V}(x)^{\mathcal{T}}\mathrm{V}(x)\right)^{-1}\mathrm{V}(x)^{\mathcal{T}}\partial_\mu \mathrm{V}(x)\\ &= \frac{1}{2}\mathrm{V}(x)^{-1}\left[\left(\partial_\mu \mathrm{V}(x)\mathrm{V}(x)^{-1}\right)^{\mathcal{T}} + \partial_\mu \mathrm{V}(x)\mathrm{V}(x)^{-1}\right]\mathrm{V}(x)\\ &= \mathrm{V}(x)^{-1}\mathrm{P}_\mu(x)\mathrm{V}(x).\end{aligned} \tag{9.33}$$

Since the factors of $\mathrm{V}(x)$ drop out in the squared expression,

$$\left(\mathrm{V}(x)^{-1}\mathrm{P}_\mu(x)\mathrm{V}(x)\middle|\mathrm{V}(x)^{-1}\mathrm{P}^\mu(x)\mathrm{V}(x)\right) = \left(\mathrm{P}_\mu(x)\middle|\mathrm{P}^\mu(x)\right), \tag{9.34}$$

Equation (9.33) provides a third way to write the $\mathcal{K}(\mathcal{G})_{\mathrm{local}} \times \mathcal{G}_{\mathrm{rigid}}$-invariant action, completely in terms of the generalized metric $\mathrm{M}(x)$,

$$S_{\mathcal{G}/\mathcal{K}(\mathcal{G})} = \frac{1}{4}\int_X d^p x\, \sqrt{\gamma}\gamma^{\mu\nu}\left(\mathrm{M}(x)^{-1}\partial_\mu \mathrm{M}(x)\middle|\mathrm{M}(x)^{-1}\partial_\nu \mathrm{M}(x)\right). \tag{9.35}$$

(We call M a "generalized metric" because in the $GL(n,\mathbb{R})/SO(n)$-case, it does correspond to the metric, the field V being the "vielbein"; see Section 9.3.2.)

All three forms of the action are manifestly gauge invariant under $\mathcal{K}(\mathcal{G})_{\mathrm{local}}$. If desired, one can fix the gauge, and thereby eliminating the redundant degrees of freedom.

9.1.4 Equations of motion and conserved currents

Let us now take a closer look at the equations of motion resulting from an arbitrary variation $\delta \mathrm{V}(x)$ of the action in Equation (9.25). The Lie algebra element $\delta \mathrm{V}(x)\mathrm{V}(x)^{-1} \in \mathfrak{g}$ can be decomposed according to the Cartan decomposition,

$$\delta \mathrm{V}(x)\mathrm{V}(x)^{-1} = \Sigma(x) + \Lambda(x), \qquad \Sigma(x) \in \mathfrak{k},\ \Lambda(x) \in \mathfrak{p}. \tag{9.36}$$

The variation $\Sigma(x)$ along the gauge orbit will be automatically projected out by gauge invariance of the action. Thus we can set $\Sigma(x) = 0$ for simplicity. Let us then compute $\delta \mathrm{P}_\mu(x)$. One easily gets

$$\delta \mathrm{P}_\mu(x) = \partial_\mu \Lambda(x) - [\mathrm{Q}_\mu(x), \Lambda(x)]. \tag{9.37}$$

Since $\Lambda(x)$ is a Lie algebra valued scalar we can freely set $\partial_\mu \Lambda(x) \to \nabla_\mu \Lambda(x)$ in the variation of the action below, where ∇^μ is a covariant derivative on X compatible with the Levi–Civita connection. Using the symmetry and the invariance of the bilinear form one then finds

$$\delta S_{\mathcal{G}/\mathcal{K}(\mathcal{G})} = \int_X d^p x \sqrt{\gamma}\gamma^{\mu\nu} 2 \left[\left(-\nabla_\nu \mathrm{P}_\mu(x) + [\mathrm{Q}_\nu(x), \mathrm{P}_\mu(x)] \middle| \Lambda(x) \right) \right]. \tag{9.38}$$

The equations of motion are therefore equivalent to

$$\mathrm{D}^\mu \mathrm{P}\mu(x) = 0, \tag{9.39}$$

with

$$\mathrm{D}_\mu \mathrm{P}_\nu(x) = \nabla_\mu \mathrm{P}_\nu(x) - [\mathrm{Q}_\mu(x), \mathrm{P}_\nu(x)], \tag{9.40}$$

and simply express the covariant conservation of $\mathrm{P}_\mu(x)$.

It is also interesting to examine the dynamics in terms of the generalized metric $\mathrm{M}(x)$. The equations of motion for $\mathrm{M}(x)$ are

$$\frac{1}{2}\nabla^\mu \left(\mathrm{M}(x)^{-1} \partial_\mu \mathrm{M}(x) \right) = 0. \tag{9.41}$$

These equations ensure the conservation of the current

$$\mathcal{J}_\mu \equiv \frac{1}{2} \mathrm{M}(x)^{-1} \partial_\mu \mathrm{M}(x) = \mathrm{V}(x)^{-1} \mathrm{P}_\mu(x) \mathrm{V}(x), \tag{9.42}$$

i.e.,

$$\nabla^\mu \mathcal{J}_\mu = 0. \tag{9.43}$$

This is the conserved Noether current associated with the rigid $\mathcal{G}$-invariance of the action.

9.1.5 Example: $SL(2,\mathbb{R})/SO(2)$ (hyperbolic space)

Let us consider the example of the coset space $SL(2, \mathbb{R}/SO(2)$, which, although very simple, is nevertheless quite illustrative. Recall from Section 6.2 that the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is constructed from the Chevalley triple (e, h, f),

$$\mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}f \oplus \mathbb{R}h \oplus \mathbb{R}e, \tag{9.44}$$

with the following standard commutation relations

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h \tag{9.45}$$

and matrix realisation

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{9.46}$$

In the Borel gauge, $\mathrm{V}(x)$ reads

$$\mathrm{V}(x) = \mathrm{Exp}\left[\frac{\phi(x)}{2}h\right] \mathrm{Exp}\left[\chi(x)e\right] = \begin{pmatrix} e^{\phi(x)/2} & \chi(x)e^{\phi(x)/2} \\ 0 & e^{-\phi(x)/2} \end{pmatrix}, \tag{9.47}$$

where $\phi(x)$ and $\chi(x)$ represent coordinates on the coset space, i.e., they describe the sigma model map

$$X \ni x \quad \longmapsto \quad (\phi(x), \chi(x)) \in SL(2,\mathbb{R})/SO(2). \tag{9.48}$$

An arbitrary transformation on $\mathrm{V}(x)$ reads

$$\mathrm{V}(x) \quad \longmapsto \quad k(x)\mathrm{V}(x)g, \qquad k(x) \in SO(2),\ g \in SL(2,\mathbb{R}), \tag{9.49}$$

which in infinitesimal form becomes

$$\delta_{\delta k(x),\delta g}\mathrm{V}(x) = \delta k(x)\mathrm{V}(x) + \mathrm{V}(x)\delta g, \qquad \delta k(x) \in \mathfrak{so}(2),\ \delta g \in \mathfrak{sl}(2,\mathbb{R}). \tag{9.50}$$

Let us then check how $\mathrm{V}(x)$ transforms under the generators $\delta g = e, f, h$. As expected, the Borel generators h and e preserve the upper triangular structure

$$\begin{aligned} \delta_e \mathrm{V}(x) &= \mathrm{V}(x)e = \begin{pmatrix} 0 & e^{\phi(x)/2} \\ 0 & 0 \end{pmatrix}, \\ \delta_h \mathrm{V}(x) &= \mathrm{V}(x)h = \begin{pmatrix} e^{\phi(x)/2} & -\chi(x)e^{\phi(x)/2} \\ 0 & -e^{\phi(x)/2} \end{pmatrix}, \end{aligned} \tag{9.51}$$

while the negative root generator f does not respect the form of $\mathrm{V}(x)$,

$$\delta_f \mathrm{V}(x) = \mathrm{V}(x)f = \begin{pmatrix} \chi(x)e^{\phi(x)/2} & 0 \\ e^{-\phi(x)/2} & 0 \end{pmatrix}. \tag{9.52}$$

Thus, in this case we need a compensating transformation to restore the upper triangular form. This transformation needs to cancel the factor $e^{-\phi(x)/2}$ in the lower left corner of the matrix $\delta_f \mathrm{V}(x)$ and so it must necessarily depend on $\phi(x)$. The transformation that does the job is

$$\delta k(x) = \begin{pmatrix} 0 & e^{-\phi(x)} \\ -e^{-\phi(x)} & 0 \end{pmatrix} \in \mathfrak{so}(2), \tag{9.53}$$

and we find

$$\begin{aligned} \delta_{\delta k(x),f}\mathrm{V}(x) &= \delta k(x)\mathrm{V}(x) + \mathrm{V}(x)f \\ &= \begin{pmatrix} \chi(x)e^{\phi(x)/2} & e^{-3\phi(x)/2} \\ 0 & -\chi(x)e^{-\phi(x)/2} \end{pmatrix} \in SL(2,\mathbb{R})/SO(2). \end{aligned} \tag{9.54}$$

Finally, since the generalized transpose $(\,)^{\mathcal{T}}$ in this case reduces to the ordinary matrix transpose, the "generalized" metric becomes

$$\mathrm{M}(x) = \mathrm{V}(x)^T\mathrm{V}(x) = \begin{pmatrix} e^{\phi(x)} & \chi(x)e^{\phi(x)} \\ \chi(x)e^{\phi(x)} & \chi(x)^2 e^{\phi(x)} + e^{-\phi(x)} \end{pmatrix}. \tag{9.55}$$

The Killing form $(\cdot|\cdot)$ corresponds to taking the trace in the adjoint representation of Equation (9.46) and the action (9.35) therefore takes the form

$$S_{SL(2,\mathbb{R})/SO(2)} = \frac{1}{2}\int_X d^p x\, \sqrt{\gamma}\gamma^{\mu\nu}\left[\partial_\mu\phi(x)\,\partial_\nu\phi(x) + e^{2\phi(x)}\partial_\mu\chi(x)\partial_\nu\chi(x)\right]. \tag{9.56}$$

9.1.6 Parametrization of $\mathcal{G}/\mathcal{K}(\mathcal{G})$

The Borel gauge choice is always accessible when the group $\mathcal{K}(\mathcal{G})$ is the maximal compact subgroup of $\mathcal{G}$. In the noncompact case this is no longer true since one cannot invoke the Iwasawa decomposition (see, e.g. [120] for a discussion of the subtleties involved when $\mathcal{K}(\mathcal{G})$ is noncompact). This point will, however, not be of concern to us in this paper. We shall now proceed to write down the sigma model action in the Borel gauge for the coset space $\mathcal{G}/\mathcal{K}(\mathcal{G})$, with $\mathcal{K}(\mathcal{G})$ being the maximal compact subgroup. Let $\Pi = \{\alpha_1^\vee, \cdots, \alpha_n^\vee\}$ be a basis of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and let $\Delta_+ \subset \mathfrak{h}^\star$ denote the set of positive roots. The Borel subalgebra of $\mathfrak{g}$ can then be written as

$$\mathfrak{b} = \sum_{i=1}^{n} \mathbb{R}\alpha_i^\vee + \sum_{\alpha\in\Delta_+} \mathbb{R}E_\alpha, \tag{9.57}$$

where E_α is the positive root generator spanning the one-dimensional root space $\mathfrak{g}_\alpha$ associated to the root α. The coset representative is then chosen to be

$$\mathrm{V}(x) = \mathrm{V}_1(x)\mathrm{V}_2(x) = \mathrm{Exp}\left[\sum_{i=1}^{n}\phi_i(x)\alpha_i^\vee\right]\,\mathrm{Exp}\left[\sum_{\alpha\in\Delta_+}\chi_\alpha(x)E_\alpha\right] \in \mathcal{G}/\mathcal{K}(\mathcal{G}). \tag{9.58}$$

Because $\mathfrak{g}$ is a finite Lie algebra, the sum over positive roots is finite and so this is a well-defined construction.

From Equation (9.58) we may compute the Lie algebra valued one-form $\partial_\mu \mathrm{V}(x)\mathrm{V}(x)^{-1}$ explicitly. Let us do this in some detail. First, we write the general expression in terms of $\mathrm{V}_1(x)$ and $\mathrm{V}_2(x)$,

$$\partial_\mu \mathrm{V}(x)\mathrm{V}(x)^{-1} = \partial_\mu \mathrm{V}_1(x)\mathrm{V}_1(x)^{-1} + \mathrm{V}_1(x)\left(\partial_\mu \mathrm{V}_2(x)\mathrm{V}_2(x)^{-1}\right)\mathrm{V}_1(x)^{-1}. \tag{9.59}$$

To compute the individual terms in this expression we need to make use of the Baker–Hausdorff formulas:

$$\begin{aligned} \partial_\mu e^A e^{-A} &= \partial_\mu A + \frac{1}{2!}[A, \partial_\mu A] + \frac{1}{3!}[A,[A,\partial_\mu A]] + \cdots, \\ e^A B e^{-A} &= B + [A,B] + \frac{1}{2!}[A,[A,B]] + \cdots. \end{aligned} \tag{9.60}$$

The first term in Equation (9.59) is easy to compute since all generators in the exponential commute. We find

$$\partial_\mu \mathrm{V}_1(x)\mathrm{V}_1(x)^{-1} = \sum_{i=1}^{n}\partial_\mu\phi_i(x)\alpha_i^\vee \in \mathfrak{h}. \tag{9.61}$$

Secondly, we compute the corresponding expression for $\mathrm{V}_2(x)$. Here we must take into account all commutators between the positive root generators $E_\alpha \in \mathfrak{n}_+$. Using the first of the Baker–Hausdorff formulas above, the first terms in the series become

$$\begin{aligned} \partial_\mu \mathrm{V}_2(x)\mathrm{V}_2(x)^{-1} &= \partial_\mu\,\mathrm{Exp}\left[\sum_{\alpha\in\Delta_+}\chi_\alpha(x)E_\alpha\right]\,\mathrm{Exp}\left[-\sum_{\alpha'\in\Delta_+}\chi_{\alpha'}(x)E_{\alpha'}\right] \\ &= \sum_{\alpha\in\Delta_+}\partial_\mu\chi_\alpha(x)E_\alpha + \frac{1}{2!}\sum_{\alpha,\alpha'\in\Delta_+}\chi_\alpha(x)\partial_\mu\chi_{\alpha'}(x)[E_\alpha, E_{\alpha'}] \\ &\quad + \frac{1}{3!}\sum_{\alpha,\alpha',\alpha''\in\Delta_+}\chi_\alpha(x)\chi_{\alpha'}(x)\partial_\mu\chi_{\alpha''}(x)[E_\alpha,[E_{\alpha'},E_{\alpha''}]] + \cdots. \end{aligned} \tag{9.62}$$

Each multi-commutator $[E_\alpha, [E_{\alpha'}, \cdots]\cdots, E_{\alpha'''}]$ corresponds to some new positive root generator, say $E_\gamma \in \mathfrak{n}_+$. However, since each term in the expansion (9.62) is a sum over all positive roots, the specific generator E_γ will get a contribution from all terms. We can therefore write the sum in "closed form" with the coefficient in front of an arbitrary generator E_γ taking the form

$$\mathcal{R}_{\gamma,\mu}(x) \equiv \partial_\mu \chi_\gamma(x) + \frac{1}{2!} \underbrace{\chi_\zeta(x)\partial_\mu\chi_{\zeta'}(x)}_{\zeta+\zeta'=\gamma} + \cdots + \frac{1}{k_\gamma!} \underbrace{\chi_\eta(x)\chi_{\eta'}(x)\cdots\chi_{\eta''}(x)\partial_\mu\chi_{\eta'''}(x)}_{\eta+\eta'+\cdots+\eta''+\eta'''=\gamma}, \tag{9.63}$$

where k_γ denotes the number corresponding to the last term in the Baker–Hausdorff expansion in which the generator E_γ appears. The explicit form of $\mathcal{R}_{\gamma,\mu}(x)$ must be computed individually for each root $\gamma \in \Delta_+$.

The sum in Equation (9.62) can now be written as

$$\partial_\mu \mathrm{V}_2(x)\mathrm{V}_2(x)^{-1} = \sum_{\alpha\in\Delta_+} \mathcal{R}_{\alpha,\mu}(x) E_\alpha. \tag{9.64}$$

To proceed, we must conjugate this expression with $\mathrm{V}_1(x)$ in order to compute the full form of Equation (9.59). This involves the use of the second Baker–Hausdorff formula in Equation (9.60) for each term in the sum, Equation (9.64). Let h denote an arbitrary element of the Cartan subalgebra,

$$h = \sum_{i=1}^{n} \phi_i(x)\alpha_i^\vee \in \mathfrak{h}. \tag{9.65}$$

Then the commutators we need are of the form

$$[h, E_\alpha] = \alpha(h)E_\alpha, \tag{9.66}$$

where $\alpha(h)$ denotes the value of the root $\alpha \in \mathfrak{h}^\star$ acting on the Cartan element $h \in \mathfrak{h}$,

$$\alpha(h) = \sum_{i=1}^{n} \phi_i(x)\alpha(\alpha_i^\vee) = \sum_{i=1}^{n} \phi_i(x)\left\langle \alpha, \alpha_i^\vee \right\rangle \equiv \sum_{i=1}^{n} \phi_i(x)\alpha_i. \tag{9.67}$$

So, for each term in the sum in Equation (9.64) we obtain

$$\begin{aligned}
\mathrm{V}_1(x)E_\alpha \mathrm{V}_1(x)^{-1} &= E_\alpha + \sum_i \phi_i(x)\alpha_i E_\alpha + \frac{1}{2}\sum_{i,j}\phi_i(x)\phi_j(x)\alpha_i\alpha_j E_\alpha + \cdots \\
&= \mathrm{Exp}\left[\sum_i \phi_i(x)\alpha_i\right] E_\alpha \\
&= e^{\alpha(h)}E_\alpha.
\end{aligned} \tag{9.68}$$

We can now write down the complete expression for the element $\partial_\mu \mathrm{V}(x)\mathrm{V}(x)^{-1}$,

$$\partial_\mu \mathrm{V}(x)\mathrm{V}(x)^{-1} = \sum_{i=1}^{n} \partial_\mu\phi_i(x)\alpha_i^\vee + \sum_{\alpha\in\Delta_+} e^{\alpha(h)}\mathcal{R}_{\alpha,\mu}(x)E_\alpha. \tag{9.69}$$

Projection onto the coset $\mathfrak{p}$ gives (see Equation (9.20) and Section 6.3)

$$\mathrm{P}_\mu(x) = \sum_{i=1}^{n} \partial_\mu\phi_i(x)\alpha_i^\vee + \frac{1}{2}\sum_{\alpha\in\Delta_+} e^{\alpha(h)}\mathcal{R}_{\alpha,\mu}(x)\left(E_\alpha + E_{-\alpha}\right), \tag{9.70}$$

where we have used that $E_\alpha^{\mathcal{T}} = E_{-\alpha}$ and $(\alpha_i^\vee)^{\mathcal{T}} = \alpha_i^\vee$.

Next we want to compute the explicit form of the action in Equation (9.25). Choosing the following normalization for the root generators,

$$(E_\alpha|E_{\alpha'}) = \delta_{\alpha,-\alpha'}, \qquad (\alpha_i^\vee|\alpha_j^\vee) = \delta_{ij}, \tag{9.71}$$

which implies

$$(E_\alpha|E_{\alpha'}^{\mathcal{T}}) = (E_\alpha|E_{-\alpha'}) = \delta_{\alpha,\alpha'} \tag{9.72}$$

one finds the form of the $\mathcal{K}(\mathcal{G})_{\text{local}} \times \mathcal{G}_{\text{rigid}}$-invariant action in the parametrization of Equation (9.58),

$$S_{\mathcal{G}/\mathcal{K}(\mathcal{G})} = \int_X d^p x \sqrt{\gamma}\gamma^{\mu\nu} \left[\sum_{i=1}^n \partial_\mu \phi_i(x) \partial_\nu \phi_i(x) + \frac{1}{2} \sum_{\alpha \in \Delta_+} e^{2\alpha(h)} \mathcal{R}_{\alpha,\mu}(x) \mathcal{R}_{\alpha,\nu}(x) \right]. \tag{9.73}$$

9.2 Geodesic sigma models on infinite-dimensional coset spaces

In the following we shall both "generalize and specialize" the construction from Section 9.1. The generalization amounts to relaxing the restriction that the algebra $\mathfrak{g}$ be finite-dimensional. Although in principle we could consider $\mathfrak{g}$ to be any indefinite Kac–Moody algebra, we shall be focusing on the case where it is of Lorentzian type. The analysis will also be a specialization, in the sense that we consider only *geodesic* sigma models, meaning that the Riemannian space X is the one-dimensional worldline of a particle, parametrized by one variable t. This restriction is of course motivated by the billiard description of gravity close to a spacelike singularity, where the dynamics at each spatial point is effectively described by a particle geodesic in the fundamental Weyl chamber of a Lorentzian Kac–Moody algebra.

The motivation is that the construction of a geodesic sigma model that exhibits this Kac–Moody symmetry in a manifest way, would provide a link to understanding the role of the full algebra $\mathfrak{g}$ beyond the BKL-limit.

9.2.1 Formal construction

For definiteness, we consider only the case when the Lorentzian algebra $\mathfrak{g}$ is a split real form, although this is not really necessary as the Iwasawa decomposition holds also in the non-split case.

A very important difference from the finite-dimensional case is that we now have nontrivial *multiplicities* of the imaginary roots (see Section 4). Recall that if a root $\alpha \in \Delta$ has multiplicity m_α, then the associated root space $\mathfrak{g}_\alpha$ is m_α-dimensional. Thus, it is spanned by m_α generators $E_\alpha^{[s]}$ $(s = 1, \cdots, m_\alpha)$,

$$\mathfrak{g}_\alpha = \mathbb{R}E_\alpha^{[1]} + \cdots + \mathbb{R}E_\alpha^{[m_\alpha]}. \tag{9.74}$$

The root multiplicities are not known in closed form for any indefinite Kac–Moody algebra, but must be computed recursively as described in Section 8.

Our main object of study is the coset representative $\mathcal{V}(t) \in \mathcal{G}/\mathcal{K}(\mathcal{G})$, which must now be seen as "formal" exponentiation of the infinite number of generators in $\mathfrak{p}$. We can then proceed as before and choose $\mathcal{V}(t)$ to be in the Borel gauge, i.e., of the form

$$\mathcal{V}(t) = \text{Exp}\left[\sum_{\mu=1}^{\dim \mathfrak{h}} \beta^\mu(t)\alpha_\mu^\vee\right] \text{Exp}\left[\sum_{\alpha\in\Delta_+}\sum_{s=1}^{m_\alpha} \xi_\alpha^{[s]}(t) E_\alpha^{[s]}\right] \in \mathcal{G}/\mathcal{K}(\mathcal{G}). \tag{9.75}$$

Here, the index μ does not correspond to "spacetime" but instead is an index in the Cartan subalgebra $\mathfrak{h}$, or, equivalently, in "scale-factor space" (see Section 2). In the following we shall dispose of writing the sum over μ explicitly. The second exponent in Equation (9.75) contains a formal infinite sum over all positive roots Δ_+. We will describe in detail in subsequent sections how it can be suitably truncated. The coset representative $\mathcal{V}(t)$ corresponds to a nonlinear realisation of $\mathcal{G}$ and transforms as

$$\mathcal{G} : \mathcal{V}(t) \quad \longmapsto \quad k\left(\mathcal{V}(t), g\right) \mathcal{V}(t) g, \qquad k\left(\mathcal{V}(t), g\right) \in \mathcal{K}(\mathcal{G}),\ g \in \mathcal{G}. \tag{9.76}$$

A $\mathfrak{g}$-valued “one-form” can be constructed analogously to the finite-dimensional case,

$$\partial \mathcal{V}(t)\mathcal{V}(t)^{-1} = \mathcal{Q}(t) + \mathcal{P}(t), \tag{9.77}$$

where $\partial \equiv \partial_t$. The first term on the right hand side represents a $\mathfrak{k}$-connection that is fixed under the Chevalley involution,

$$\tau(\mathcal{Q}) = \mathcal{Q}, \tag{9.78}$$

while $\mathcal{P}(t)$ lies in the orthogonal complement $\mathfrak{p}$ and so is anti-invariant,

$$\tau(\mathcal{P}) = -\mathcal{P} \tag{9.79}$$

(for the split form, the Cartan involution coincides with the Chevalley involution). Using the projections onto the coset $\mathfrak{p}$ and the compact subalgebra $\mathfrak{k}$, as defined in Equation (9.20), we can compute the forms of $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ in the Borel gauge, and we find

$$\begin{aligned} \mathcal{P}(t) &= \partial\beta^{\mu}(t)\alpha^{\vee}_{\mu} + \frac{1}{2}\sum_{\alpha\in\Delta_+}\sum_{s=1}^{m_\alpha} e^{\alpha(\beta)}\mathfrak{R}^{[s]}_{\alpha}(t)\left(E^{[s]}_{\alpha} + E^{[s]}_{-\alpha}\right), \\ \mathcal{Q}(t) &= \frac{1}{2}\sum_{\alpha\in\Delta_+}\sum_{s=1}^{m_\alpha} e^{\alpha(\beta)}\mathfrak{R}^{[s]}_{\alpha}(t)\left(E^{[s]}_{\alpha} - E^{[s]}_{-\alpha}\right), \end{aligned} \tag{9.80}$$

where $\mathfrak{R}^{[s]}_{\alpha}(t)$ is the analogue of $\mathcal{R}_{\alpha}(x)$ in the finite-dimensional case, i.e., it takes the form

$$\mathfrak{R}^{[s]}_{\alpha}(t) = \partial\xi^{[s]}_{\alpha}(t) + \frac{1}{2}\underbrace{\xi^{[s]}_{\zeta}(t)\partial\xi^{[s]}_{\zeta'}(t)}_{\zeta+\zeta'=\alpha} + \cdots, \tag{9.81}$$

which still contains a finite number of terms for each positive root α. The value of the root $\alpha \in \mathfrak{h}^{\star}$ acting on $\beta = \beta^{\mu}(t)\alpha^{\vee}_{\mu} \in \mathfrak{h}$ is

$$\alpha(\beta) = \alpha_{\mu}\beta^{\mu}. \tag{9.82}$$

Note that here the notation α_μ does not correspond to a simple root, but denotes the components of an arbitrary root vector $\alpha \in \mathfrak{h}^{\star}$.

The action for a particle moving on the infinite-dimensional coset space $\mathcal{G}/\mathcal{K}(\mathcal{G})$ can now be constructed using the invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$,

$$S_{\mathcal{G}/\mathcal{K}(\mathcal{G})} = \int dt\, n(t)^{-1}\left(\mathcal{P}(t)|\mathcal{P}(t)\right), \tag{9.83}$$

where $n(t)$ ensures invariance under reparametrizations of t. The variation of the action with respect to $n(t)$ constrains the motion to be a *null geodesic* on $\mathcal{G}/\mathcal{K}(\mathcal{G})$,

$$\left(\mathcal{P}(t)|\mathcal{P}(t)\right) = 0. \tag{9.84}$$

Defining, as before, a covariant derivative $\mathfrak{D}$ with respect to the local symmetry $\mathcal{K}(\mathcal{G})$ as

$$\mathfrak{D}\mathcal{P}(t) \equiv \partial\mathcal{P}(t) - \left[\mathcal{Q}(t), \mathcal{P}(t)\right], \tag{9.85}$$

the equations of motion read simply

$$\mathfrak{D}\left(n(t)^{-1}\mathcal{P}(t)\right) = 0. \tag{9.86}$$

The explicit form of the action in the parametrization of Equation (9.75) becomes

$$S_{\mathcal{G}/\mathcal{K}(\mathcal{G})} = \int dt\, n(t)^{-1}\left[G_{\mu\nu}\,\partial\beta^\mu(t)\,\partial\beta^\nu(t) + \frac{1}{2}\sum_{\alpha\in\Delta_+}\sum_{s=1}^{m_\alpha} e^{2\alpha(\beta)}\,\mathfrak{R}_\alpha^{[s]}(t)\,\mathfrak{R}_\alpha^{[s]}(t)\right], \tag{9.87}$$

where $G_{\mu\nu}$ is the flat Lorentzian metric, defined by the restriction of the bilinear form $(\cdot|\cdot)$ to the Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$. The metric $G_{\mu\nu}$ is exactly the same as the metric in scale-factor space (see Section 2), and the kinetic term for the Cartan parameters $\beta^\mu(t)$ reads explicitly

$$G_{\mu\nu}\,\partial\beta^\mu(t)\,\partial\beta^\nu(t) = \sum_{i=1}^{\dim\mathfrak{h}-1}\partial\beta^i(t)\,\partial\beta^i(t) - \left(\sum_{i=1}^{\dim\mathfrak{h}-1}\partial\beta^i(t)\right)\left(\sum_{j=1}^{\dim\mathfrak{h}-1}\partial\beta^j(t)\right) + \partial\phi(t)\,\partial\phi(t). \tag{9.88}$$

Although $\mathfrak{g}$ is infinite-dimensional we still have the notion of “formal integrability”, owing to the existence of an infinite number of conserved charges, defined by the equations of motion in Equation (9.86). We can define the generalized metric for any $\mathfrak{g}$ as

$$\mathcal{M}(t) \equiv \mathcal{V}(t)^{\mathcal{T}}\mathcal{V}(t), \tag{9.89}$$

where the transpose $(\,)^{\mathcal{T}}$ is defined as before in terms of the Chevalley involution,

$$(\,)^{\mathcal{T}} = -\tau(\,). \tag{9.90}$$

Then the equations of motion $\mathfrak{D}\mathcal{P}(t) = 0$ are equivalent to the conservation $\partial\mathfrak{J} = 0$ of the current

$$\mathfrak{J} \equiv \frac{1}{2}\mathcal{M}(t)^{-1}\partial\mathcal{M}(t). \tag{9.91}$$

This can be formally solved in closed form

$$\mathcal{M}(t) = e^{t\mathfrak{J}^{\mathcal{T}}}\mathcal{M}(0)e^{t\mathfrak{J}}, \tag{9.92}$$

and so an arbitrary group element $g\in\mathcal{G}$ evolves according to

$$g(t) = k(t)g(0)e^{t\mathfrak{J}}, \qquad k(t)\in\mathcal{K}(\mathcal{G}). \tag{9.93}$$

Although the explicit form of $\mathcal{P}(t)$ contains infinitely many terms, we have seen that each coefficient $\mathfrak{R}_\alpha^{[s]}(t)$ can, in principle, be computed exactly for each root α. This, however, is not the case for the current $\mathfrak{J}$. To find the form of $\mathfrak{J}$ one must conjugate $\mathcal{P}(t)$ with the coset representative $\mathcal{V}(t)$ and this requires an infinite number of commutators to get the correct coefficient in front of any generator in $\mathfrak{J}$.

9.2.2 Consistent truncations

One method for dealing with infinite expressions like Equation (9.80) consists in considering successive finite expansions allowing more and more terms, while still respecting the dynamics of the sigma model.

This leads us to the concept of a *consistent truncation* of the sigma model for $\mathcal{G}/\mathcal{K}(\mathcal{G})$. We take as definition of such a truncation any sub-model S' of $S_{\mathcal{G}/\mathcal{K}(\mathcal{G})}$ whose solutions are also solutions of the original model.

There are two main approaches to finding suitable truncations that fulfill this latter criterion. These are the so-called *subgroup truncations* and the *level truncations*, which will both prove to be useful for our purposes, and we consider them in turn below.

Subgroup truncation

The first consistent truncation we shall treat is the case when the dynamics of a sigma model for some global group $\mathcal{G}$ is restricted to that of an appropriately chosen subgroup $\bar{\mathcal{G}} \subset \mathcal{G}$. We consider here only subgroups $\bar{\mathcal{G}}$ which are obtained by exponentiation of regular subalgebras $\bar{\mathfrak{g}}$ of $\mathfrak{g}$. The concept of regular embeddings of Lorentzian Kac–Moody algebras was discussed in detail in Section 4.

To restrict the dynamics to that of a sigma model based on the coset space $\bar{\mathcal{G}}/\mathcal{K}(\bar{\mathcal{G}})$, we first assume that the initial conditions $g(t)\big|_{t=0} = g(0)$ and $\partial g(t)\big|_{t=0}$ are such that the following two conditions are satisfied:

1. The group element $g(0)$ belongs to $\bar{\mathcal{G}}$.

2. The conserved current $\mathfrak{J}$ belongs to $\bar{\mathfrak{g}}$.

When these conditions hold, then $g(0)e^{t\mathfrak{J}}$ belongs to $\bar{\mathcal{G}}$ for all t. Moreover, there always exists $\bar{k}(t) \in \mathcal{K}(\bar{\mathcal{G}})$ such that

$$\bar{g}(t) \equiv \bar{k}(t)g(0)e^{t\mathfrak{J}} \in \bar{\mathcal{G}}/\mathcal{K}(\bar{\mathcal{G}}), \tag{9.94}$$

i.e, $\bar{g}(t)$ belongs to the Borel subgroup of $\bar{\mathcal{G}}$. Because the embedding is regular, $\bar{k}(t)$ belongs to $\mathcal{K}(\mathcal{G})$ and we thus have that $\bar{g}(t)$ also belongs to the Borel subgroup of the full group $\mathcal{G}$.

Now recall that from Equation (9.93), we know that $\bar{g}(t) = \bar{k}(t)g(0)e^{t\mathfrak{J}}$ is a solution to the equations of motion for the sigma model on $\bar{\mathcal{G}}/\mathcal{K}(\bar{\mathcal{G}})$. But since we have found that $\bar{g}(t)$ preserves the Borel gauge for $\mathcal{G}/\mathcal{K}(\mathcal{G})$, it follows that $\bar{k}(t)g(0)e^{t\mathfrak{J}}$ is a solution to the equations of motion for the full sigma model. Thus, the dynamical evolution of the subsystem $S' = S_{\bar{\mathcal{G}}/\mathcal{K}(\bar{\mathcal{G}})}$ preserves the Borel gauge of $\mathcal{G}$. These arguments show that initial conditions in $\bar{\mathcal{G}}$ remain in $\bar{\mathcal{G}}$, and hence the dynamics of a sigma model on $\mathcal{G}/\mathcal{K}(\mathcal{G})$ can be consistently truncated to a sigma model on $\bar{\mathcal{G}}/\mathcal{K}(\bar{\mathcal{G}})$.

Finally, we recall that because the embedding $\bar{\mathfrak{g}} \subset \mathfrak{g}$ is regular, the restriction of the bilinear form on $\mathfrak{g}$ coincides with the bilinear form on $\bar{\mathfrak{g}}$. This implies that the Hamiltonian constraints for the two models, arising from time reparametrization invariance of the action, also coincide.

We shall make use of subgroup truncations in Section 10.

Level truncation and height truncation

Alternative ways of consistently truncating the infinite-dimensional sigma model rest on the use of *gradations* of $\mathfrak{g}$,

$$\mathfrak{g} = \cdots + \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 + \cdots, \tag{9.95}$$

where the sum is infinite but each subspace is finite-dimensional. One also has

$$[\mathfrak{g}_{\ell'}, \mathfrak{g}_{\ell''}] \subseteq \mathfrak{g}_{\ell'+\ell''}. \tag{9.96}$$

Such a gradation was for instance constructed in Section 8 and was based on a so-called *level decomposition* of the adjoint representation of $\mathfrak{g}$ into representations of a finite regular subalgebra $\mathfrak{r} \subset \mathfrak{g}$. We will now use this construction to truncate the sigma model based of $\mathcal{G}/\mathcal{K}(\mathcal{G})$, by "terminating" the gradation of $\mathfrak{g}$ at some finite level $\bar{\ell}$. More specifically, the truncation will involve setting to zero all coefficients $\mathfrak{R}^{[s]}_{\alpha}(t)$, in the expansion of $\mathcal{P}(t)$, corresponding to roots α whose generators $E^{[s]}_{\alpha}$ belong to subspaces $\mathfrak{g}_{\ell}$ with $\ell > \bar{\ell}$. Part of this section draws inspiration from the treatment in [47, 48, 124].

The level ℓ might be the height, or it might count the number of times a specified single simple root appears. In that latter case, the actual form of the level decomposition must of course be worked out separately for each choice of algebra $\mathfrak{g}$ and each choice of decomposition. We will do this in detail in Section 9.3 for a specific level decomposition of the hyperbolic algebra E_{10}. Here, we shall display the general construction in the case of the *height truncation*, which exists for any algebra.

Let α be a positive root, $\alpha \in \Delta_+$. It has the following expansion in terms of the simple roots

$$\alpha = \sum_i m_i \alpha_i \qquad (m_i \geq 0). \tag{9.97}$$

Then the *height* of α is defined as (see Section 4)

$$\mathrm{ht}(\alpha) = \sum_i m_i. \tag{9.98}$$

The height can thus be seen as a linear integral map $\mathrm{ht} : \Delta \to \mathbb{Z}$, and we shall sometimes use the notation $\mathrm{ht}(\alpha) = h_\alpha$ to denote the value of the map ht acting on a root $\alpha \in \Delta$.

To achieve the height truncation, we assume that the sum over all roots in the expansion of $\mathcal{P}(t)$, Equation (9.80), is ordered in terms of increasing height. Then we can consistently set to zero all coefficients $\mathfrak{R}_\alpha^{[s]}(t)$ corresponding to roots with greater height than some, suitably chosen, finite height $\bar{h}$. We thus find that the finitely truncated coset element $\mathcal{P}_0(t)$ is

$$\mathcal{P}_0(t) \equiv \mathcal{P}(t)\big|_{\mathrm{ht} \leq \bar{h}} = \partial\beta^\mu(t)\alpha_\mu^\vee + \frac{1}{2} \sum_{\substack{\alpha\in\Delta_+ \\ \mathrm{ht}(\alpha)\leq\bar{h}}} \sum_{s=1}^{m_\alpha} e^{\alpha(\beta)} \mathfrak{R}_\alpha^{[s]}(t) \left(E_\alpha^{[s]} + E_{-\alpha}^{[s]} \right), \tag{9.99}$$

which is equivalent to the statement

$$\mathfrak{R}_\gamma^{[s]}(t) = 0 \qquad \forall \gamma \in \Delta_+,\, \mathrm{ht}(\gamma) > \bar{h}. \tag{9.100}$$

For further use, we note here some properties of the coefficients $\mathfrak{R}_\alpha^{[s]}(t)$. By examining the structure of Equation (9.81), we see that $\mathfrak{R}_\alpha^{[s]}(t)$ takes the form of a temporal derivative acting on $\xi_\alpha^{[s]}(t)$, followed by a sequence of terms whose individual components, for example $\xi_\zeta^{[s]}(t)$, are all associated with roots of *lower* height than α, $\mathrm{ht}(\zeta) < \mathrm{ht}(\alpha)$. It will prove useful to think of $\mathfrak{R}_\alpha^{[s]}(t)$ as representing a kind of "generalized" derivative operator acting on the field $\xi_\alpha^{[s]}$. Thus we define the operator $\mathcal{D}$ by

$$\mathcal{D}\xi_\alpha^{[s]}(t) \equiv \partial\xi_\alpha^{[s]}(t) + \mathcal{F}_\alpha^{[s]}\left(\xi\partial\xi, \xi^2\partial\xi, \cdots\right), \tag{9.101}$$

where $\mathcal{F}_\alpha^{[s]}(t)$ is a polynomial function of the coordinates $\xi(t)$, whose explicit structure follows from Equation (9.81). It is common in the literature to refer to the level truncation as "setting all higher level covariant derivatives to zero", by which one simply means that all $\mathcal{D}\xi_\gamma^{[s]}(t)$ corresponding to roots γ above a given finite level $\bar{\ell}$ should vanish. Following [47] we shall call the operators $\mathcal{D}$ "covariant derivatives".

It is clear from the equations of motion $\mathfrak{D}\mathcal{P}(t) = 0$, that if all covariant derivatives $\mathcal{D}\xi_\gamma^{[s]}(t)$ above a given height are set to zero, this choice is preserved by the dynamical evolution. Hence, the height (and any level) truncation is indeed a consistent truncation. Let us here emphasize that it is *not* consistent by itself to merely put all fields $\xi_\gamma^{[s]}(t)$ above a certain level to zero, but one must take into account the fact that combinations of lower level fields may parametrize a higher level generator in the expansion of $\mathcal{P}(t)$, and therefore it is crucial to define the truncation using the derivative operator $\mathcal{D}\xi_\gamma^{[s]}(t)$.

9.3 Eleven-dimensional supergravity and $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$

We shall now illustrate the results of the previous sections by explicitly constructing an action for the coset space $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$. We employ the level decomposition of $E_{10} = \mathrm{Lie}\,\mathcal{E}_{10}$ in terms of its regular $\mathfrak{sl}(10,\mathbb{R})$-subalgebra (see Section 8), to write the coordinates on the coset space as (time-dependent) $\mathfrak{sl}(10,\mathbb{R})$-tensors. It is then shown that for a truncation of the sigma model at level $\ell = 3$, these fields can be interpreted as the physical fields of eleven-dimensional supergravity. This "dictionary" ensures that the equations of motion arising from the sigma model on $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$ are equivalent to the (suitably truncated) equations of motion of eleven-dimensional supergravity [47].

9.3.1 Low level fields

We perform the level decomposition of E_{10} with respect to the $\mathfrak{sl}(10,\mathbb{R})$-subalgebra obtained by removing the exceptional node in the Dynkin diagram in Figure 49. This procedure was described in Section 8. When using this decomposition, a sum over (positive) roots becomes a sum over all $\mathfrak{sl}(10,\mathbb{R})$-indices in each (positive) representation appearing in the decomposition.

We recall that up to level three the following representations appear

$$\begin{array}{ll} \ell = 0: & K^a{}_b, \\ \ell = 1: & E^{abc} = E^{[abc]}, \\ \ell = 2: & E^{a_1\cdots a_6} = E^{[a_1\cdots a_6]}, \\ \ell = 3: & E^{a|b_1\cdots b_8} = E^{a|[b_1\cdots b_8]}, \end{array} \tag{9.102}$$

where all indices are $\mathfrak{sl}(10,\mathbb{R})$-indices and so run from 1 to 10. The level zero generators $K^a{}_b$ correspond to the adjoint representation of $\mathfrak{sl}(10,\mathbb{R})$ and the higher level generators correspond to an infinite tower of raising operators of E_{10}. As indicated by the square brackets, the level one and two representations are completely antisymmetric in all indices, while the level three representation has a mixed Young tableau symmetry: It is antisymmetric in the eight indices $b_1\cdots b_8$ and is also subject to the constraint

$$E^{[a|b_1\cdots b_8]} = 0. \tag{9.103}$$

In the scale factor space (β-basis), the roots of E_{10} corresponding to the above generators act as follows on $\beta \in \mathfrak{h}$:

$$\begin{array}{rcl} K^a{}_b & \Longleftrightarrow & \alpha_{ab}(\beta) = \beta^a - \beta^b \qquad (a > b), \\ E^{abc} & \Longleftrightarrow & \alpha_{abc}(\beta) = \beta^a + \beta^b + \beta^c, \\ E^{a_1\cdots a_6} & \Longleftrightarrow & \alpha_{a_1\cdots a_6}(\beta) = \beta^{a_1} + \cdots + \beta^{a_6}, \\ E^{a|ab_1\cdots b_7} & \Longleftrightarrow & \alpha_{ab_1\cdots b_7}(\beta) = 2\beta^a + \beta^{b_1} + \cdots + \beta^{b_7}, \\ E^{a_1|a_2\cdots a_9} & \Longleftrightarrow & \alpha_{a_1\cdots a_9}(\beta) = \beta^{a_1} + \cdots + \beta^{a_9}. \end{array} \tag{9.104}$$

We can use the scalar product in root space, $\mathfrak{h}^\star_{E_{10}}$, to compute the norms of these roots. Recall from Section 5 that the metric on $\mathfrak{h}^\star_{E_{10}}$ is the inverse of the metric in Equation (9.88), and for E_{10} it takes the form

$$(\omega|\omega) = G^{ij}\omega_i\omega_j = \sum_{i=1}^{10}\omega_i\omega_i - \frac{1}{9}\left(\sum_{i=1}^{10}\omega_i\right)\left(\sum_{j=1}^{10}\omega_j\right), \qquad \omega \in \mathfrak{h}^\star_{E_{10}}. \tag{9.105}$$

The level zero, one and two generators correspond to real roots of E_{10},

$$(\alpha_{ab}|\alpha_{cd}) = 2, \qquad (\alpha_{abc}|\alpha_{def}) = 2, \qquad (\alpha_{a_1\cdots a_6}|\alpha_{b_1\cdots b_6}) = 2. \tag{9.106}$$

We have split the roots corresponding to the level three generators into two parts, depending on whether or not the special index a takes the same value as one of the other indices. The resulting two types of roots correspond to real and null roots, respectively,

$$(\alpha_{ab_1\cdots b_7}|\alpha_{cd_1\cdots d_7}) = 2, \quad (\alpha_{a_1\cdots a_9}|\alpha_{b_1\cdots b_9}) = 0. \tag{9.107}$$

Thus, the first time that generators corresponding to imaginary roots appear in the level decomposition is at level three. This will prove to be important later on in our analysis.

9.3.2 The $GL(10,\mathbb{R})/SO(10)$-sigma model

Because of the importance and geometric significance of level zero, we shall first develop the formalism for the $GL(10,\mathbb{R})/SO(10)$-sigma model. A general group element H in the subgroup $GL(10,\mathbb{R})$ reads

$$H = \text{Exp}\left[h_a{}^b K^a{}_b\right] \tag{9.108}$$

where $h_a{}^b$ is a 10×10 matrix (with a being the row index and b the column index). Although the $K^a{}_b$'s are generators of E_{10} and can, within this framework, at best be viewed as infinite matrices, it will prove convenient – for streamlining the calculations – to view them in the present section also as 10×10 matrices, since we confine our attention to the finite-dimensional subgroup $GL(10,\mathbb{R})$. Namely, $K^a{}_b$ is treated as a 10×10 matrix with 0's everywhere except 1 in position (a, b) (see Equation (6.83)). The final formulation in terms of the variables $h_a{}^b(t)$ – which are 10×10 matrices irrespectively as to whether one deals with $GL(10,\mathbb{R})$ *per se* or as a subgroup of E_{10} – does not depend on this interpretation.

It is also useful to describe $GL(10,\mathbb{R})$ as the set of linear combinations $m_i{}^j K^i{}_j$ where the 10×10 matrix $m_i{}^j$ is invertible. The product of the $K^i{}_j$'s is given by

$$K^i{}_j\, K^k{}_m = \delta^k{}_j\, K^i{}_m. \tag{9.109}$$

One easily verifies that if $M = m_i{}^j K^i{}_j$ and $N = n_i{}^j K^i{}_j$ belong to $GL(10,\mathbb{R})$, then $MN = (mn)_i{}^j K^i{}_j$ where mn is the standard product of the 10×10 matrices m and n. Furthermore, $\text{Exp}\left(h_i{}^j K^i{}_j\right) = \left(e^h\right)_i{}^j K^i{}_j$ where e^h is the standard matrix exponential.

Under a general transformation, the representative $H(t)$ is multiplied from the left by a time-dependent $SO(10)$ group element R and from the right by a constant linear $GL(10,\mathbb{R})$-group element L. Explicitly, the transformation takes the form (suppressing the time-dependence for notational convenience)

$$H \to H' = RHL. \tag{9.110}$$

In terms of components, with $H = e_a{}^b K^a{}_b$, $e_a{}^b = (e^h)_a{}^b$, $R = R_a{}^b K^a{}_b$ and $L = L_a{}^b K^a{}_b$, one finds

$$e'_a{}^b = R_a{}^c e_c{}^d L_d{}^b, \tag{9.111}$$

where we have set $H' = e'_a{}^b K^a{}_b$. The indices on the coset representative have different covariance properties. To emphasize this fact, we shall write a bar over the first index, $e_a{}^b \to e_{\bar{a}}{}^b$. Thus, barred indices transform under the local $SO(10)$ gauge group and are called "local", or also "flat", indices, while unbarred indices transform under the global $GL(10,\mathbb{R})$ and are called "world", or also "curved", indices. The gauge invariant matrix product $M = H^T H$ is equal to

$$M = g^{ab} K_{ab}, \tag{9.112}$$

with $K_{ab} \equiv K^c{}_b \delta_{ac}$ and

$$g^{ab} = \sum_{\bar{c}} e_{\bar{c}}{}^a e_{\bar{c}}{}^b. \tag{9.113}$$

The g^{ab} do not transform under local $SO(10)$-transformations and transform as a (symmetric) contravariant tensor under rigid $GL(10,\mathbb{R})$-transformations,

$$g'^{ab} = g^{cd} L_c{}^a L_d{}^b. \tag{9.114}$$

They are components of a nondegenerate symmetric matrix that can be identified with an inverse Euclidean metric.

Indeed, the coset space $GL(10,\mathbb{R})/SO(10)$ can be identified with the space of symmetric tensors of Euclidean signature, i.e., the space of metrics. This is because two symmetric tensors of Euclidean signature are equivalent under a change of frame, and the isotropy subgroup, say at the identity, is evidently $SO(10)$. In that view, the coset representative $e_a{}^b$ is the spatial vielbein.

The action for the coset space $GL(10,\mathbb{R})/SO(10)$ with the metric of Equation (8.82) is easily found to be

$$\mathcal{L}_0 = \frac{1}{4}\left(g^{ac}(t)g^{bd}(t) - g^{ab}(t)g^{cd}(t)\right)\partial g_{ab}(t)\,\partial g_{cd}(t). \tag{9.115}$$

Note that the quadratic form multiplying the time derivatives is just the "De Witt supermetric" [66]. Note also for future reference that the invariant form $\partial H\, H^{-1}$ reads explicitly

$$\partial H\, H^{-1} = \partial e_{\bar{a}}{}^{\;b}\, e_b{}^{\bar{c}}\, K^a{}_c, \tag{9.116}$$

where $e_b{}^{\bar{n}}$ is the inverse vielbein.

9.3.3 Sigma model fields and $SO(10)_{\mathrm{local}} \times GL(10,\mathbb{R})_{\mathrm{rigid}}$-covariance

We now turn to the full nonlinear sigma model for $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$. Rather than exponentiating the Cartan subalgebra separately as in Equation (9.75), it will here prove convenient to instead single out the level zero subspace $\mathfrak{g}_0 = \mathfrak{gl}(10,\mathbb{R})$. This permits one to control easily $SO(10)_{\mathrm{local}} \times GL(10,\mathbb{R})_{\mathrm{rigid}}$-covariance. To make this level zero covariance manifest, we shall furthermore assume that the Borel gauge has been fixed only for the non-zero levels, and we keep all level zero fields present. The residual gauge freedom is then just multiplication by an $SO(10)$ rotation from the left.

Thus, we take a coset representative of the form

$$\mathcal{V}(t) = \mathrm{Exp}\left[h_a{}^b(t)K^a{}_b\right]\ \mathrm{Exp}\left[\frac{1}{3!}\mathcal{A}_{abc}(t)E^{abc} + \frac{1}{6!}\mathcal{A}_{a_1\cdots a_6}(t)E^{a_1\cdots a_6} + \frac{1}{9!}\mathcal{A}_{a|b_1\cdots b_8}(t)E^{a|b_1\cdots b_8} + \cdots\right], \tag{9.117}$$

where the sum in the first exponent would be restricted to all $a \geq b$ if we had taken a full Borel gauge also at level zero. The parameters $\mathcal{A}_{abc}(t), \mathcal{A}_{a_1\cdots a_6}(t)$ and $\mathcal{A}_{a|b_1\cdots b_8}(t)$ are coordinates on the coset space $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$ and will eventually be interpreted as physical time-dependent fields of eleven-dimensional supergravity.

How do the fields transform under $SO(10)_{\mathrm{local}} \times GL(10,\mathbb{R})_{\mathrm{rigid}}$? Let $R \in SO(10)$, $L \in GL(10,\mathbb{R})$ and decompose $\mathcal{V}$ according to Equation (9.117) as the product

$$\mathcal{V} = HT, \tag{9.118}$$

with

$$\begin{aligned} H &= \mathrm{Exp}\left[h_a{}^b(t)K^a{}_b\right] \in GL(10,\mathbb{R}), \\ T &= \mathrm{Exp}\left[\frac{1}{3!}\mathcal{A}_{abc}(t)E^{abc} + \frac{1}{6!}\mathcal{A}_{a_1\cdots a_6}(t)E^{a_1\cdots a_6} + \frac{1}{9!}\mathcal{A}_{a|b_1\cdots b_8}(t)E^{a|b_1\cdots b_8} + \cdots\right]. \end{aligned} \tag{9.119}$$

One has

$$\mathcal{V} \to \mathcal{V}' = R(HT)L = (RHL)(L^{-1}TL). \tag{9.120}$$

Now, the first matrix $H' = RHL$ clearly belongs to $GL(10,\mathbb{R})$, since it is the product of a rotation matrix by two $GL(10,\mathbb{R})$-matrices. It has exactly the same transformation as in Equation (9.110) above in the context of the nonlinear sigma model for $GL(10,\mathbb{R})/SO(10)$. Hence, the geometric interpretation of $e_{\bar{a}}{}^b = (e^h)_{\bar{a}}{}^b$ as the vielbein remains.

Similarly, the matrix $T' \equiv L^{-1}TL$ has exactly the same form as T,

$$\begin{aligned} T' &= \mathrm{Exp}\left(L^{-1}\left[\frac{1}{3!}\mathcal{A}_{abc}(t)E^{abc} + \frac{1}{6!}\mathcal{A}_{a_1\cdots a_6}(t)E^{a_1\cdots a_6} + \frac{1}{9!}\mathcal{A}_{a|b_1\cdots b_8}(t)E^{a|b_1\cdots b_8} + \cdots\right]L\right) \\ &= \mathrm{Exp}\left[\frac{1}{3!}\mathcal{A}'_{abc}(t)E^{abc} + \frac{1}{6!}\mathcal{A}'_{a_1\cdots a_6}(t)E^{a_1\cdots a_6} + \frac{1}{9!}\mathcal{A}'_{a|b_1\cdots b_8}(t)E^{a|b_1\cdots b_8} + \cdots\right], \end{aligned} \tag{9.121}$$

where the variables $\mathcal{A}'_{abc}$, $\mathcal{A}'_{a_1\cdots a_6}$, ..., are obtained from the variables $\mathcal{A}_{abc}$, $\mathcal{A}_{a_1\cdots a_6}$, ..., by computing $L^{-1}E^{abc}L$, $L^{-1}E^{a_1\cdots a_6}L$, ..., using the commutation relations with $K^a{}_b$. Explicitly, one gets

$$\mathcal{A}'_{abc} = (L^{-1})_a{}^e(L^{-1})_b{}^f(L^{-1})_c{}^g\mathcal{A}_{efg}, \qquad \mathcal{A}'_{a_1\cdots a_6} = (L^{-1})_{a_1}{}^{b_1}\cdots(L^{-1})_{a_6}{}^{b_6}\mathcal{A}_{b_1\cdots b_6}, \qquad \text{etc.} \tag{9.122}$$

Hence, the fields $\mathcal{A}_{abc}$, $\mathcal{A}_{a_1\cdots a_6}$, ... do not transform under local $SO(10)$ transformations. However, they do transform under rigid $GL(10,\mathbb{R})$-transformations as tensors of the type indicated by their indices. Their indices are world indices and not flat indices.

9.3.4 "Covariant derivatives"

The invariant form $\partial\mathcal{V}\,\mathcal{V}^{-1}$ reads

$$\partial\mathcal{V}\,\mathcal{V}^{-1} = \partial H\,H^{-1} + H(\partial T\,T^{-1})H^{-1}. \tag{9.123}$$

The first term is the invariant form encountered above in the discussion of the level zero nonlinear sigma model for $GL(10,\mathbb{R})/SL(10)$. So let us focus on the second term. It is clear that $\partial T\,T^{-1}$ will contain only positive generators at level ≥ 1. So we set, in a manner similar to Equation (9.64),

$$\partial T\,T^{-1} = \sum_{\alpha\in\Delta_+}\sum_s \mathcal{D}\mathcal{A}_{\alpha,s}\,E_{\alpha,s}, \tag{9.124}$$

where the sum is over positive roots at levels one and higher and takes into account multiplicities (through the extra index s). The expressions $\mathcal{D}\mathcal{A}_{\alpha,s}$ are linear in the time derivatives $\partial\mathcal{A}$. As before, we call them "covariant derivatives". They are computed by making use of the Baker–Hausdorff formula, as in Section 9.1.6. Explicitly, up to level 3, one finds

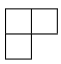
$$\dot{T}\,T^{-1} = \frac{1}{3!}\mathcal{D}\mathcal{A}_{abc}(t)E^{abc} + \frac{1}{6!}\mathcal{D}\mathcal{A}_{a_1\cdots a_6}(t)E^{a_1\cdots a_6} + \frac{1}{9!}\mathcal{D}\mathcal{A}_{a|b_1\cdots b_8}(t)E^{a|b_1\cdots b_8} + \cdots, \tag{9.125}$$

with

$$\begin{aligned} \mathcal{D}\mathcal{A}_{abc}(t) &= \partial\mathcal{A}_{abc}(t), \\ \mathcal{D}\mathcal{A}_{a_1\cdots a_6}(t) &= \partial\mathcal{A}_{a_1\cdots a_6}(t) + 10\mathcal{A}_{[a_1a_2a_3}(t)\partial\mathcal{A}_{a_4a_5a_6]}(t), \\ \mathcal{D}\mathcal{A}_{a|b_1\cdots b_8}(t) &= \partial\mathcal{A}_{a|b_1\cdots b_8}(t) + 42\mathcal{A}_{\langle ab_1b_2}(t)\partial\mathcal{A}_{b_3\cdots b_8\rangle}(t) - 42\partial\mathcal{A}_{\langle ab_1b_2}(t)\mathcal{A}_{b_3\cdots b_8\rangle}(t), \\ &\quad +280\mathcal{A}_{\langle ab_1b_2}(t)\mathcal{A}_{b_3b_4b_5}(t)\partial\mathcal{A}_{b_6b_7b_8\rangle}(t), \end{aligned} \tag{9.126}$$

as computed in [47]. The notation $\langle a_1\cdots a_k\rangle$ denotes projection onto the Young tableaux symmetry carried by the field upon which the covariant derivative acts[35]. It should be stressed that the

[35] As an example, consider the projection $P_{\alpha\beta\gamma} \equiv T_{\langle\alpha\beta\gamma\rangle}$ of a three index tensor $T_{\alpha\beta\gamma}$ onto the Young tableaux

.

This projection is given by

$$P_{\alpha\beta\gamma} = \frac{1}{3}(T_{\alpha\beta\gamma} + T_{\beta\alpha\gamma} - T_{\gamma\beta\alpha} - T_{\beta\gamma\alpha}),$$

which clearly satisfies

$$P_{\alpha\beta\gamma} = -P_{\gamma\beta\alpha}, \qquad P_{[\alpha\beta\gamma]} = 0.$$

Note also that $P_{\alpha\beta\gamma} \neq P_{\beta\alpha\gamma}$.

covariant derivatives $\mathcal{D}\mathcal{A}$ have the same transformation properties under $SO(10)$ (under which they are inert) and $GL(10,\mathbb{R})$ as $\mathcal{A}$ since the $GL(10,\mathbb{R})$ transformations do not depend on time.

9.3.5 The $\mathcal{K}(\mathcal{E}_{10}) \times \mathcal{E}_{10}$-invariant action at low levels

The action can now be computed using the bilinear form $(\cdot|\cdot)$ on E_{10},

$$S_{\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})} = \int dt\, n(t)^{-1} \left(\mathcal{P}(t)\big|\mathcal{P}(t)\right), \tag{9.127}$$

where $\mathcal{P}$ is obtained by projecting orthogonally onto the subalgebra $\mathfrak{k}_{E_{10}}$ by using the generalized transpose,

$$(K^a{}_b)^{\mathcal{T}} = K^b{}_a, \qquad (E^{abc})^{\mathcal{T}} = F_{abc}, \cdots \text{etc.}, \tag{9.128}$$

where as above $(\ )^{\mathcal{T}} = -\omega(\ )$ (with ω being the Chevalley involution). We shall compute the action up to, and including, level 3,

$$S_{\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})} = \int dt\, n(t)^{-1} \left(\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \cdots\right). \tag{9.129}$$

From Equation (9.123) and the fact that generators at level zero are orthogonal to generators at levels $\neq 0$, we see that $\mathcal{L}_0$ will be constructed from the level zero part $\dot{H}\, H^{-1}$ and will coincide with the Lagrangian (9.115) for the nonlinear sigma model $GL(10,\mathbb{R})/SO(10)$,

$$\mathcal{L}_0 = \frac{1}{4}\left(g^{ac}(t)g^{bd}(t) - g^{ab}(t)g^{cd}(t)\right)\partial g_{ab}(t)\,\partial g_{cd}(t). \tag{9.130}$$

To compute the other terms, we use the following trick. The Lagrangian must be a $GL(10,\mathbb{R})$ scalar. One can easily compute it in the frame where $H = 1$, i.e., where the metric g_{ab} is equal to δ_{ab}. One can then covariantize the resulting expression by replacing everywhere δ_{ab} by g_{ab}. To illustrate the procedure consider the level 1 term. One has, for $H = 1$ and at level 1, $\partial\mathcal{V}\,\mathcal{V}^{-1} = \frac{1}{3!}\mathcal{D}\mathcal{A}_{abc}(t)E^{abc}$ and thus, with the same gauge conditions, $\mathcal{P}(t) = \frac{1}{2\cdot 3!}\mathcal{D}\mathcal{A}_{abc}(t)\left(E^{abc} + F^{abc}\right)$ (where we have raised the indices of F_{abc} with δ^{ab}, $F_{123} \equiv F^{123}$ etc). Using $(E^{a_1a_2a_3}|F^{b_1b_2b_3}) = \delta^{a_1b_1}\delta^{a_2b_2}\delta^{a_3b_3}\, \pm$permutations that make the expression antisymmetric (3! terms; see Section 8.4), one then gets $\mathcal{L}_1 = \frac{1}{2\cdot 3!}\mathcal{D}\mathcal{A}_{abc}(t)\,\mathcal{D}\mathcal{A}_{def}(t)\,\delta^{ad}\delta^{be}\delta^{cf}$ in the frame where $g_{ab} = \delta_{ab}$. This yields the level 1 Lagrangian in a general frame,

$$\begin{aligned}\mathcal{L}_1 &= \frac{1}{2\cdot 3!} g^{a_1c_1}g^{a_2c_2}g^{a_3c_3}\mathcal{D}\mathcal{A}_{a_1a_2a_3}(t)\,\mathcal{D}\mathcal{A}_{c_1c_2c_3}(t)\\ &= \frac{1}{2\cdot 3!}\mathcal{D}\mathcal{A}_{a_1a_2a_3}(t)\,\mathcal{D}\mathcal{A}^{a_1a_2a_3}(t).\end{aligned} \tag{9.131}$$

By a similar analysis, the level 2 and 3 contributions are

$$\begin{aligned}\mathcal{L}_2 &= \frac{1}{2\cdot 6!}\,\mathcal{D}\mathcal{A}_{a_1\cdots a_6}(t)\,\mathcal{D}\mathcal{A}^{a_1\cdots a_6}(t),\\ \mathcal{L}_3 &= \frac{1}{2\cdot 9!}\,\mathcal{D}\mathcal{A}_{a|b_1\cdots b_8}(t)\,\mathcal{D}\mathcal{A}^{a|b_1\cdots b_8}(t).\end{aligned} \tag{9.132}$$

Collecting all terms, the final form of the action for $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$ up to and including level $\ell = 3$ is

$$\begin{aligned}S_{\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})} = \int dt\, n(t)^{-1}\Big[&\frac{1}{4}\left(g^{ac}(t)g^{bd}(t) - g^{ab}(t)g^{cd}(t)\right)\partial g_{ab}(t)\,\partial g_{cd}(t)\\ &+\frac{1}{2\cdot 3!}\mathcal{D}\mathcal{A}_{a_1a_2a_3}(t)\,\mathcal{D}\mathcal{A}^{a_1a_2a_3}(t) + \frac{1}{2\cdot 6!}\mathcal{D}\mathcal{A}_{a_1\cdots a_6}(t)\,\mathcal{D}\mathcal{A}^{a_1\cdots a_6}(t)\\ &+\frac{1}{2\cdot 9!}\mathcal{D}\mathcal{A}_{a|b_1\cdots b_8}(t)\,\mathcal{D}\mathcal{A}^{a|b_1\cdots b_8}(t) + \cdots\Big],\end{aligned} \tag{9.133}$$

which agrees with the action found in [47].

9.3.6 The correspondence

We shall now relate the equations of motion for the $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$ sigma model to the equations of motion of eleven-dimensional supergravity. As the precise correspondence is not yet known, we shall here only sketch the main ideas. These work remarkably well at low levels but need unknown ingredients at higher levels.

We have seen that the sigma model for $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$ can be consistently truncated level by level. More precisely, one can consistently set equal to zero all covariant derivatives of the fields above a given level and get a reduced system whose solutions are solutions of the full system. We shall show here that the consistent truncations of $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$ at levels 0, 1 and 2 yields equations of motion that coincide with the equations of motion of appropriate consistent truncations of eleven-dimensional supergravity, using a prescribed dictionary presented below. We will also show that the correspondence extends up to parts of level 3.

We recall that in the gauge $N^i = 0$ (vanishing shift) and $A_{0bc} = 0$ (temporal gauge), the bosonic fields of eleven-dimensional supergravity are the spatial metric $\mathrm{g}_{ab}(x^0, x^i)$, the lapse $N(x^0, x^i)$ and the spatial components $A_{abc}(x^0, x^i)$ of the vector potential 3-form. The physical field is $F = dA$ and its electric and magnetic components are, respectively, denoted F_{0abc} and F_{abcd}. The electric field involves only time derivatives of $A_{abc}(x^0, x^i)$, while the magnetic field involves spatial gradients.

Levels 0 and 1

If one keeps only levels zero and one, the sigma model action (9.133) reduces to

$$S[g_{ab}(t), \mathcal{A}_{abc}(t), n(t)] = \int dt\, n(t)^{-1} \left[\frac{1}{4}\left(g^{ac}(t)\, g^{bd}(t) - g^{ab}(t)\, g^{cd}(t)\right) \partial g_{ab}(t)\, \partial g_{cd}(t) + \frac{1}{2 \cdot 3!} \partial \mathcal{A}_{a_1a_2a_3}(t)\, \partial \mathcal{A}^{a_1a_2a_3}(t)\right]. \tag{9.134}$$

Consider now the consistent homogeneous truncation of eleven-dimensional supergravity in which the spatial metric, the lapse and the vector potential depend only on time (no spatial gradient). Then the reduced action for this truncation is precisely Equation (9.134) provided one makes the identification $t = x^0$ and

$$g_{ab}(t) = \mathrm{g}_{ab}(t), \tag{9.135}$$

$$\mathcal{A}_{abc}(t) = A_{abc}(t), \tag{9.136}$$

$$n(t) = \frac{N(t)}{\sqrt{\mathrm{g}(t)}} \tag{9.137}$$

(see, for instance, [61]). Also the Hamiltonian constraints (the only one left) coincide. Thus, there is a perfect match between the sigma model truncated at level one and supergravity "reduced on a 10-torus". If one were to drop level one, one would find perfect agreement with pure gravity. In the following, we shall make the gauge choice $N = \sqrt{\mathrm{g}}$, equivalent to $n = 1$.

Level 2

At levels 0 and 1, the supergravity fields g_{ab} and A_{abc} depend only on time. When going beyond this truncation, one needs to introduce some spatial gradients. Level 2 introduces spatial gradients of a very special type, namely allows for a homogeneous magnetic field. This means that A_{abc} acquires a space dependence, more precisely, a linear one (so that its gradient does not depend on x). However, because there is no room for x-dependence on the sigma model side, where the only independent variable is t, we shall use the trick to describe the magnetic field in terms of a dual potential $A_{a_1\cdots a_6}$. Thus, there is a close interplay between duality, the sigma model formulation, and the introduction of spatial gradients.

There is no tractable, fully satisfactory variational formulation of eleven-dimensional supergravity where both the 3-form potential and its dual appear as independent variables in the action, with a quadratic dependence on the time derivatives (this would be double-counting, unless an appropriate self-duality condition is imposed [35, 36]). This means that from now on, we shall not compare the actions of the sigma model and of supergravity but, rather, only their respective equations of motion. As these involve the electromagnetic field and not the potential, we rewrite the correspondence found above at levels 0 and 1 in terms of the metric and the electromagnetic field as

$$\begin{aligned} g_{ab}(t) &= \mathrm{g}_{ab}(t), \\ \mathcal{DA}_{abc}(t) &= F_{0abc}(t). \end{aligned} \tag{9.138}$$

The equations of motion for the nonlinear sigma model, obtained from the variation of the Lagrangian Equation (9.133), truncated at level two, read explicitly

$$\begin{aligned} \frac{1}{2}\partial\left(n(t)^{-1}g^{ac}(t)\,\partial g_{cb}(t)\right) &= \frac{n(t)^{-1}}{4}\left(\mathcal{DA}^{ac_1c_2}(t)\,\mathcal{DA}_{bc_1c_2}(t) - \frac{1}{9}\delta^a{}_b\mathcal{DA}^{c_1c_2c_3}(t)\,\mathcal{DA}_{c_1c_2c_3}(t)\right) \\ &\quad + \frac{n(t)^{-1}}{2\cdot 5!}\left(\mathcal{DA}^{ac_1\cdots c_5}(t)\,\mathcal{DA}_{bc_1\cdots c_5}(t) - \frac{1}{9}\delta^a{}_b\mathcal{DA}^{c_1\cdots c_6}(t)\,\mathcal{DA}_{c_1\cdots c_6}(t)\right), \\ \partial\left(n(t)^{-1}\mathcal{DA}^{a_1a_2a_3}(t)\right) &= -\frac{1}{3!}\,n(t)^{-1}\mathcal{DA}^{a_1\cdots a_6}(t)\,\mathcal{DA}_{a_4a_5a_6}(t), \\ \partial\left(n(t)^{-1}\mathcal{DA}^{a_1\cdots a_6}(t)\right) &= 0. \end{aligned} \tag{9.139}$$

In addition, we have the constraint obtained by varying n,

$$\begin{aligned} (\mathcal{P}(t)|\mathcal{P}(t)) &= \frac{1}{4}\left(g^{ac}(t)\,g^{bd}(t) - g^{ab}(t)\,g^{cd}(t)\right)\partial g_{ab}(t)\,\partial g_{cd}(t) \\ &\quad + \frac{1}{2\cdot 3!}\mathcal{DA}^{a_1a_2a_3}(t)\,\mathcal{DA}_{a_1a_2a_3}(t) + \frac{1}{2\cdot 6!}\mathcal{DA}^{a_1\cdots a_6}(t)\,\mathcal{DA}_{a_1\cdots a_6}(t) \\ &= 0. \end{aligned} \tag{9.140}$$

On the supergravity side, we truncate the equations to metrics $\mathrm{g}_{ab}(t)$ and electromagnetic fields $F_{0abc}(t)$, $F_{abcd}(t)$ that depend only on time. We take, as in Section 2, the spacetime metric to be of the form

$$ds^2 = -N^2(t)\,dt^2 + \mathrm{g}_{ab}(t)\,dx^a\,dx^b, \tag{9.141}$$

but now with $x^0 = t$. In the following we use Greek letters $\lambda, \sigma, \rho, \cdots$ to denote eleven-dimensional spacetime indices, and Latin letters $a, b, c, \cdots$ to denote ten-dimensional spatial indices.

The equations of motion and the Hamiltonian constraint for eleven-dimensional supergravity have been explicitly written in [61], so they can be expediently compared with the equations of motion of the sigma model. The dynamical equations for the metric read

$$\begin{aligned} \frac{1}{2}\partial\left(\sqrt{\mathrm{g}}N^{-1}\mathrm{g}^{ac}\partial\mathrm{g}_{cb}\right) &= \frac{1}{12}N\sqrt{\mathrm{g}}F^{a\rho\sigma\tau}F_{b\rho\sigma\tau} - \frac{1}{144}N\sqrt{\mathrm{g}}\,\delta^a{}_b\,F^{\lambda\rho\sigma\tau}F_{\lambda\rho\sigma\tau} \\ &= \frac{1}{4}N^{-1}\sqrt{\mathrm{g}}F^{0ac_1c_2}F_{0bc_1c_2} - \frac{1}{36}N^{-1}\sqrt{\mathrm{g}}\,\delta^a{}_b\,F^{0c_1c_2c_3}F_{0c_1c_2c_3} \\ &\quad + \frac{1}{12}N\sqrt{\mathrm{g}}F^{ac_1c_2c_3}F_{bc_1c_2c_3} - \frac{1}{144}N\sqrt{\mathrm{g}}\,\delta^a{}_b\,F^{c_1c_2c_3c_4}F_{c_1c_2c_3c_4}, \end{aligned} \tag{9.142}$$

and for the electric and magnetic fields we have, respectively, the equations of motion and the

Bianchi identity,

$$\begin{aligned} \partial\left(F^{0abc}N\sqrt{\mathrm{g}}\right) &= \frac{1}{144}\varepsilon^{0abcd_1d_2d_3e_1e_2e_3e_4}F_{0d_1d_2d_3}F_{e_1e_2e_3e_4}, \\ \partial F_{a_1a_2a_3a_4} &= 0. \end{aligned} \tag{9.143}$$

Furthermore we have the Hamiltonian constraint

$$\frac{1}{4}\left(\mathrm{g}^{ac}\mathrm{g}^{bd}-\mathrm{g}^{ab}\mathrm{g}^{cd}\right)\partial\mathrm{g}_{ab}\,\partial\mathrm{g}_{cd}+\frac{1}{12}F^{0abc}F_{0abc}+\frac{1}{48}N^2F^{abcd}F_{abcd}=0. \tag{9.144}$$

(We shall not consider the other constraints here; see remarks as at the end of this section.)

One finds again perfect agreement between the sigma model equations, Equation (9.139) and (9.140), and the equations of eleven-dimensional supergravity, Equation (9.142) and (9.144), provided one extends the above dictionary through [47]

$$\mathcal{DA}^{a_1\cdots a_6}(t)=-\frac{1}{4!}\varepsilon^{a_1\cdots a_6b_1b_2b_3b_4}F_{b_1b_2b_3b_4}(t). \tag{9.145}$$

This result appears to be quite remarkable, because the Chern–Simons term in Equation (9.143) is in particular reproduced with the correct coefficient, which in eleven-dimensional supergravity is fixed by invoking supersymmetry.

Level 3

Level 3 should correspond to the introduction of further controlled spatial gradients, this time for the metric. Because there is no room for spatial derivatives as such on the sigma model side, the trick is again to introduce a dual graviton field. When this dual graviton field is non-zero, the metric does depend on the spatial coordinates.

Satisfactory dual formulations of non-linearized gravity do not exist. At the linearized level, however, the problem is well understood since the pioneering work by Curtright [39] (see also [167, 14, 21]). In eleven spacetime dimensions, the dual graviton field is *described precisely by a tensor* $\mathcal{A}_{a|b_1\cdots b_8}$ *with the mixed symmetry of the Young tableau* $[1,0,0,0,0,0,0,1,0]$ *appearing at level 3 in the sigma model description*. Exciting this field, i.e., assuming $\mathcal{DA}_{a|b_1\cdots b_8}\neq 0$ amounts to introducing spatial gradients for the metric – and, for that matter, for the other fields as well – as follows. Instead of considering fields that are homogeneous on a torus, one considers fields that are homogeneous on non-Abelian group manifolds. This introduces spatial gradients (in coordinate frames) in a well controlled manner.

Let θ^a be the group invariant one-forms, with structure equations

$$d\theta^a=\frac{1}{2}C^a{}_{bc}d\theta^b\wedge d\theta^c. \tag{9.146}$$

We shall assume that $C^a{}_{ac}=0$ ("Bianchi class A"). Truncation at level 3 assumes that the metric and the electric and magnetic fields depend only on time in this frame and that the $C^a{}_{bc}$ are constant (corresponding to a group). The supergravity equations have been written in that case in [61] and can be compared with the sigma model equations. There is almost a complete match between both sets of equations provided one extends the dictionary at level 3 through

$$\mathcal{DA}^{a|b_1\cdots b_8}(t)=\frac{3}{2}\varepsilon^{b_1\cdots b_8cd}C^a{}_{cd} \tag{9.147}$$

(with the equations of motion of the sigma model implying indeed that $\mathcal{DA}^{a|b_1\cdots b_8}$ does not depend on time). Note that to define an invertible mapping between the level three fields and the $C^a{}_{bc}$, it is important that $C^a{}_{bc}$ be traceless; there is no "room" on level three on the sigma model side to incorporate the trace of $C^a{}_{bc}$.

With this correspondence, the match works perfectly for real roots up to, and including, level three. However, it fails for fields associated with imaginary roots (level 3 is the first time imaginary roots appear, at height 30) [47]. In fact, the terms that match correspond to "$SL(10,\mathbb{R})$-covariantized E_8", i.e., to fields associated with roots of E_8 and their images under the Weyl group of $SL(10,\mathbb{R})$.

Since the match between the sigma model equations and supergravity fails at level 3 under the present line of investigation, we shall not provide the details but refer instead to [47] for more information. The correspondence up to level 3 was also checked in [53] through a slightly different approach, making use of a formulation with local frames, i.e., using local flat indices rather than global indices as in the present treatment.

Let us note here that higher level fields of E_{10}, corresponding to imaginary roots, have been considered from a different point of view in [24], where they were associated with certain brane configurations (see also [23, 9]).

The dictionary

One may view the above failure at level 3 as a serious flaw to the sigma model approach to exhibiting the E_{10} symmetry[36]. Let us, however, be optimistic for a moment and assume that these problems will somehow get resolved, perhaps by changing the dictionary or by including higher order terms. So, let us proceed.

What would be the meaning of the higher level fields? As discussed in Section 9.3.7, there are indications that fields at higher levels contain higher order spatial gradients and therefore enable us to reconstruct completely, through something similar to a Taylor expansion, the most general field configuration from the fields at a given spatial point.

From this point of view, the relation between the supergravity degrees of freedom $\mathrm{g}_{ij}(t,x)$ and $F_{(4)}(t,x) = dA_{(3)}(t,x)$ would be given, at a specific spatial point $x = \mathbf{x}_0$ and in a suitable spatial frame $\theta^a(x)$ (that would also depend on x), by the following "dictionary":

$$
\begin{aligned}
g_{ab}(t) &= \mathrm{g}_{ab}(t,\mathbf{x}_0),\\
\mathcal{D}\mathcal{A}_{abc}(t) &= F_{tabc}(t,\mathbf{x}_0),\\
\mathcal{D}\mathcal{A}^{a_1\cdots a_6}(t) &= -\frac{1}{4!}\varepsilon^{a_1\cdots a_6 bcde}F_{bcde}(t,\mathbf{x}_0),\\
\mathcal{D}\mathcal{A}^{a|b_1\cdots b_8}(t) &= \frac{3}{2}\varepsilon^{b_1\cdots b_8 cd}C^a{}_{cd}(\mathbf{x}_0),
\end{aligned}
\tag{9.148}
$$

which reproduces in the homogeneous case what we have seen up to level 3.

This correspondence goes far beyond that of the algebraic description of the BKL-limit in terms of Weyl reflections in the simple roots of a Kac–Moody algebra. Indeed, the dynamics of the billiard is controlled entirely by the walls associated with *simple* roots and thus does not transcend height one. Here, we go to a much higher height and successfully extend (unfortunately incompletely) the intriguing connection between eleven-dimensional supergravity and E_{10}.

9.3.7 Higher levels and spatial gradients

We have seen that the correspondence between the $\mathcal{E}_{10}$-invariant sigma model and eleven-dimensional supergravity fails when we include spatial gradients beyond first order. It is nevertheless

[36] This does not exclude that other approaches would be successful. That E_{10}, or perhaps E_{11}, does encode a lot of information about M-theory is a fact, but that this should be translated into a sigma model reformulation of the theory appears to be questionable.

believed that the information about spatial gradients is somehow encoded within the algebraic description: One idea is that space is "smeared out" among the infinite number of fields contained in $\mathcal{E}_{10}$ and it is for this reason that a direct dictionary for the inclusion of spatial gradients is difficult to find. If true, this would imply that we can view the level expansion on the algebraic side as reflecting a kind of "Taylor expansion" in spatial gradients on the supergravity side. Below we discuss some speculative ideas about how such a correspondence could be realized in practice.

The "gradient conjecture"

One intriguing suggestion put forward in [47] was that fields associated to certain "affine representations" of E_{10} could be interpreted as spatial derivatives acting on the level one, two and three fields, thus providing a direct conjecture for how space "emerges" through the level decomposition of E_{10}. The representations in question are those for which the Dynkin label associated with the overextended root of E_{10} vanishes, and hence these representations are realized also in a level decomposition of the regular E_9-subalgebra obtained by removing the overextended node in the Dynkin diagram of E_{10}.

The affine representations were discussed in Section 8 and we recall that they are given in terms of three infinite towers of generators, with the following $\mathfrak{sl}(10,\mathbb{R})$-tensor structures,

$$E^{a_1a_2a_3}{}_{b_1\cdots b_k}, \qquad E^{a_1\cdots a_6}{}_{b_1\cdots b_k}, \qquad E^{a_1|a_2\cdots a_9}{}_{b_1\cdots b_k}, \tag{9.149}$$

where the upper indices have the same Young tableau symmetries as the $\ell = 1, 2$ and 3 representations, while the lower indices are all completely symmetric. In the sigma model these generators of $\mathcal{E}_{10}$ are parametrized by fields exhibiting the same index structure, i.e., $\mathcal{A}_{a_1a_2a_2}{}^{b_1\cdots b_k}(t)$, $\mathcal{A}_{a_1\cdots a_6}{}^{b_1\cdots b_k}(t)$ and $\mathcal{A}_{a_1|a_2\cdots a_9}{}^{b_1\cdots b_k}(t)$.

The idea is now that the three towers of fields have precisely the right index structure to be interpreted as spatial gradients of the low level fields

$$\begin{aligned}\mathcal{A}_{a_1a_2a_2}{}^{b_1\cdots b_k}(t) &= \partial^{b_1}\cdots\partial^{b_k}\mathcal{A}_{a_1a_2a_3}(t),\\ \mathcal{A}_{a_1\cdots a_6}{}^{b_1\cdots b_k}(t) &= \partial^{b_1}\cdots\partial^{b_k}\mathcal{A}_{a_1\cdots a_6}(t),\\ \mathcal{A}_{a_1|a_2\cdots a_9}{}^{b_1\cdots b_k}(t) &= \partial^{b_1}\cdots\partial^{b_k}\mathcal{A}_{a_1|a_2\cdots a_9}(t).\end{aligned} \tag{9.150}$$

Although appealing and intuitive as it is, this conjecture is difficult to prove or to check explicitly, and not much progress in this direction has been made since the original proposal. However, recently [73] this problem was attacked from a rather different point of view with some very interesting results, indicating that the gradient conjecture may need to be substantially modified. For completeness, we briefly review here some of the main features of [73].

U-duality and the Weyl Group of E_9

Recall from Section 4 that the infinite-dimensional Kac–Moody algebras E_9 and E_{10} can be obtained from E_8 through prescribed extensions of the E_8 Dynkin diagram: $E_9 = E_8^+$ is obtained by extending with one extra node, and $E_{10} = E_8^{++}$ by extending with two extra nodes. This procedure can be continued and after extending E_8 three times, one finds the Lorentzian Kac–Moody algebra $E_{11} = E_8^{+++}$, which is also believed to be relevant as a possible underlying symmetry of M-theory [167, 74].

These algebras are part of the chain of exceptional regular embeddings,

$$\cdots E_8 \subset E_9 \subset E_{10} \subset E_{11} \cdots, \tag{9.151}$$

which was used in [69] to show that a sigma model for the coset space $\mathcal{E}_{11}/\mathcal{K}(\mathcal{E}_{11})$ can be consistently truncated to a sigma model for the coset space $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$, which coincides with Equation (9.133). This result builds upon previous work devoted to general constructions of sigma models invariant under Lorentzian Kac–Moody algebras of $\mathfrak{g}^{+++}$-type [74, 71, 70, 72].

It was furthermore shown in [69] that by performing a suitable Weyl reflection before truncation, yet another sigma model based on $\mathcal{E}_{10}$ could be obtained. It differs from Equation (9.133) because the parameter along the geodesic is a *spacelike*, not timelike, variable in spacetime. This follows from the fact that the sigma model is constructed from the coset space $\mathcal{E}_{10}/\mathcal{K}^-(\mathcal{E}_{10})$, where $\mathcal{K}^-(\mathcal{E}_{10})$ coincides with the noncompact group $SO(1,9)$ at level zero, and not $SO(10)$ as is the case for Equation (9.133). The two sigma model actions were referred to in [69] as $S_{\text{cosmological}}$ and S_{brane}, since solutions to the first model translate to time-dependent (cosmological) solutions of eleven-dimensional supergravity, while the second model gives rise to stationary (brane) solutions, which are smeared in all but one spacelike direction. In particular, the $\ell = 1$ and $\ell = 2$ fields correspond to potentials for the $M2$- and $M5$-branes, respectively.

In [73], solutions associated to the infinite tower of affine representations for the brane sigma model based on $\mathcal{E}_{10}/\mathcal{K}^-(\mathcal{E}_{10})$ were investigated. The idea was that by restricting the indices to be $\mathfrak{sl}(9,\mathbb{R})$-indices, any such representation coincides with a generator of E_9, and so different fields in these affine towers must be related by Weyl reflections in E_9.

The Weyl group $\mathfrak{W}[E_9]$ is a subgroup of the U-duality group $\mathcal{E}_9(\mathbb{Z})$ of M-theory compactified on T^9 to two spacetime dimensions. Moreover, the continuous group $\mathcal{E}_9 = \mathcal{E}_9(\mathbb{R})$ is the M-theory analogue of the Geroch group, i.e., it is a symmetry of the space of solutions of $N = 16$ supergravity in two dimensions [140]. Under these considerations it is natural to expect that the fields associated with the affine representations should somehow be related to the infinite number of "dual potentials" appearing in connection with the Geroch group in two dimensions. Indeed, the authors of [73] were able to show, using the embedding $\mathfrak{W}[E_9] \subset \mathfrak{W}[E_{10}]$, that given, e.g., a representation in the $\ell = 1$ affine tower, there exists a $\mathfrak{W}[E_9] \subset \mathcal{E}_9(\mathbb{Z})$-transformation that relates the associated field to the lowest $\ell = 1$ generator $E^{a_1a_2a_3}$. The resulting solution, however, is different from the standard brane solution obtained from the $\ell = 1$-field because the new solution is smeared in all directions *except two spacelike directions*, i.e., the solution is an $M2$-brane solution which depends on two spacelike variables.

Thus, by taking advantage of the embedding $E_9 \subset E_{10}$, it was shown that the three towers of "gradient representations" encode a kind of "de-compactification" of one spacelike variable. In a way this therefore indicates that part of the gradient conjecture must be correct, in the sense that the towers of affine representations indeed contain information about the emergence of spacelike directions. On the other hand, it also seems that the correspondence is more complicated than was initially believed, perhaps deeply connected to U-duality in some, as of yet, unknown way.

9.4 Further comments

9.4.1 Massive type IIA supergravity

We have just seen that some of the higher level fields might have an interpretation in terms of spatial gradients. This would account for a subclass of representations at higher levels. The existence of other representations at each level besides the "gradient representations" shows that the sigma model contains further degrees of freedom besides the supergravity fields, conjectured in [47] to correspond to M-theoretic degrees of freedom and (quantum) corrections.

The gradient representations have the interesting properties that their highest $\mathfrak{sl}(10,\mathbb{R})$-weight is a real root. There are other representations with the same properties. An interesting interpretation of some of those has been put forward recently using dimensional reduction, as corresponding to the $(D-1)$-forms that generate the cosmological constant for maximal gauged supergravities in D spacetime dimensions [19, 150, 73]. (A cosmological constant that appears as a constant of integration can be described by a $(D-1)$-form [7, 99].) For definiteness, we shall consider here only the representations at level 4, related to the mass term of type IIA theory.

There are two representations at level 4, both of them with a highest weight which is a real root of E_{10}, namely $[0,0,1,0,0,0,0,0,1]$ and $[2,0,0,0,0,0,0,0,0]$ [141]. The lowest weight of the first one is, in terms of the scale factors, $2(\beta^1+\beta^2+\beta^3)+\beta^4+\beta^5+\beta^6+\beta^7+\beta^8+\beta^9$. The lowest weight of the second one is $3\beta^1+\beta^2+\beta^3+\beta^4+\beta^5+\beta^6+\beta^7+\beta^8+\beta^9+\beta^{10}$. Both weights are easily verified to have squared length equal to 2 and, since they are on the root lattice, they are indeed roots by the criterion for roots of hyperbolic algebras. The first representation is described by a tensor with mixed symmetry $\mathcal{A}_{a_1a_2a_3|b_1b_2\cdots b_9}$ corresponding, as we have seen, to the conjectured gradient representation (with one derivative) of the level 1 field $\mathcal{A}_{a_1a_2a_3}$. We shall thus focus on the second representation, described by a tensor $\mathcal{A}_{a_1|b_1|c_1c_2\cdots c_{10}}$.

By dimensional reduction along the first direction, the representation $[2,0,0,0,0,0,0,0,0]$ splits into various $\mathfrak{sl}(9,\mathbb{R})$ representations, one of which is described by the completely antisymmetric field $\mathcal{A}_{c_2\cdots c_{10}}$, i.e., a 9-form (in ten spacetime dimensions). It is obtained by taking $a_1=b_1=c_1=1$ in $\mathcal{A}_{a_1|b_1|c_1c_2\cdots c_{10}}$ and corresponds precisely to the lowest weight $\gamma=3\beta^1+\beta^2+\beta^3+\beta^4+\beta^5+\beta^6+\beta^7+\beta^8+\beta^9+\beta^{10}$ given above. If one rewrites the corresponding term $\sim \mathcal{D}\mathcal{A}^2_{1|1|1c_2\cdots c_{10}}e^{2\gamma}$ in the Lagrangian in terms of ten-dimensional scale factors and dilatons, one reproduces, using the field equations for $\mathcal{A}_{1|1|1c_2\cdots c_{10}}$, the mass term of massive Type IIA supergravity.

The fact that E_{10} contains information about the massive Type IIA theory is in our opinion quite profound because, contrary to the low level successes which are essentially a covariantization of known E_8 results, this is a true E_{10} test. The understanding of the massive Type IIA theory in the light of infinite Kac–Moody algebras was studied first in [157], where the embedding of the mass term in a nonlinear realisation of E_{11} was constructed. The precise connection between the mass term and an E_{10} positive real root was first explicitly made in Section 6.5 of [41]. It is interesting to note that even though the corresponding representation does not appear in E_9, it is present in E_{10} without having to go to E_{11}. The mass term of Type IIA was also studied from the point of view of the E_{10} coset model in [124].

This analysis suggests an interesting possibility for evading the no-go theorem of [13] on the impossibility to generate a cosmological constant in eleven-dimensional supergravity. This should be tried by introducing new degrees of freedom described by a mixed symmetry tensor $\mathcal{A}_{a_1|b_1|c_1c_2\cdots c_{10}}$. If this tensor can be consistently coupled to gravity (a challenge in the context of field theory with a finite number of fields!), it would provide the eleven-dimensional origin of the cosmological constant in massive Type IIA. There would be no contradiction with [13] since in eleven dimensions, the new term would not be a standard cosmological constant, but would involve dynamical degrees of freedom. This is, of course, quite speculative.

Finally, there are extra fields at higher levels besides spatial gradients and the massive Type IIA term. These might correspond to higher spin degrees of freedom [47, 21, 25, 169].

9.4.2 Including fermions

Another attractive aspect of the E_{10}-sigma model formulation is that it can easily account for the fermions of supergravity up to the levels that work in the bosonic sector. The fermions transform in representations of the compact subalgebra $\mathfrak{k}_{E_{10}}\subset E_{10}$. An interesting feature of the analysis is that E_{10}-covariance leads to $\mathfrak{k}_{E_{10}}$-covariant derivatives that coincides with the covariant derivatives dictated by supersymmetry. This has been investigated in detail in [56, 50, 57, 51, 128], to which we refer the interested reader.

9.4.3 Quantum corrections

If the gradient conjecture is correct (perhaps with a more sophisticated dictionary), then one sees that the sigma model action would contain spatial derivatives of higher order. It has been conjectured that these could perhaps correspond to higher quantum corrections [47]. This is supported by the fact that the known quantum corrections of M-theory do correspond to roots of E_{10} [54].

The idea is that with each correction curvature term of the form $R^N\sqrt{-{}^{(11)}\mathrm{g}}$, where R^N is a generic monomial of order N in the Riemann tensor, one can associate a linear form in the scale factors β^μ's in the BKL-limit. This linear form will be a root of E_{10} only for certain values of N. Hence compatibility of the corresponding quantum correction with the E_{10} structure constrains the power N.

The evaluation of the curvature components in the BKL-limit goes back to the paper by BKL themselves in four dimensions [16] and was extended to higher dimensions in [15, 63]. It was rederived in [54] for the purpose of evaluating quantum corrections. It is shown in these references that the leading terms in the curvature expressed in an orthonormal frame adapted to the slicing are, in the BKL-limit, $R_{\perp a\perp b}$ and R_{abab} $(a \neq b)$ which behave as

$$R_{\perp a\perp b} \sim e^{2\sigma}, \qquad R_{abab} \sim e^{2\sigma}, \tag{9.152}$$

where σ is the sum of all the scale factors

$$\sigma = \beta^1 + \beta^2 + \cdots + \beta^{10}, \tag{9.153}$$

and where we have set $R_{\perp a\perp b} = N^{-2}R_{0a0b}$. This implies that

$$R \sim e^{2\sigma}, \qquad R^N \sim e^{2N\sigma}, \qquad R^N\sqrt{-{}^{(11)}\mathrm{g}} \sim e^{2(N-1)\sigma}. \tag{9.154}$$

Now, σ is not on the root lattice. It is not an integer combination of the simple roots and it has length squared equal to $-10/9$. Integer combinations of the simple roots contains 3ℓ β^μ's, where ℓ is the level. Since 10 and 3 are relatively primes, the only multiples of σ that are on the root lattice are of the form $3k\sigma$, $k = 1, 2, 3, \cdots$. These are negative, imaginary roots. The smallest value is $k = 1$, corresponding to the imaginary root

$$\omega(\beta) = 3\sigma \tag{9.155}$$

at level -10, with squared length -10. It follows that the only quantum corrections compatible with the E_{10} structure must have $N - 1 = 3k$, i.e., $N = 3k + 1$ [54], since it it only in this case that $R^N\sqrt{-{}^{(11)}\mathrm{g}} \sim e^{-2\gamma}$ has $\gamma = -(N-1)\sigma$ on the root lattice. The first corrections are thus of the form R^4, R^7, R^{10} etc. This in in remarkable agreement with the quantum computations of [88] (see also [152]).

The analysis of [54] was completed in [42] where it was observed that the imaginary root (9.155) was actually one of the fundamental weights of E_{10}, namely, the fundamental weight conjugate to the exceptional root that defines the level. In the case of E_{10}, the root lattice and the weight lattice coincides, but this observation was useful in the analysis of the quantum corrections for other theories where the weight lattice is strictly larger than the root lattice. The compatibility conditions seem in those cases to be that quantum corrections should be associated with vectors on the weight lattice. (See also [130, 131, 12, 136].)

Finally, we note that recent work devoted to investigations of U-duality symmetries of compactified higher curvature corrections indicates that the results reported here in the context of E_{10} might require reconsideration [11].

9.4.4 Understanding duality

The previous analysis has revealed that the hyperbolic Kac–Moody algebra E_{10} contains a large amount of information about the structure and the properties of M-theory. How this should ultimately be incorporated in the final formulation of the theory is, however, not clear.

The sigma model approach exhibits some important drawbacks and therefore it does not appear to be the ultimate formulation of the theory. In addition to the absence of a complete dictionary enabling one to go satisfactorily beyond level 3 (the level where the first imaginary root appears), more basic difficulties already appear at low levels. These are:

The Hamiltonian constraint

There is an obvious discrepancy between the Hamiltonian constraint of the sigma model and the Hamiltonan constraint of supergravity. In the sigma model case, all terms are positive, except for the kinetic term of the scale factors, which contain a negative sign related to the conformal factor. On the supergravity side, the kinetic term of the scale factors matches correctly, but there are extra negative contributions coming from level 3 (something perhaps not too surprising if level 3 is to be thought as a dual formulation of gravity and hence contains in particular dual scale factors). How this problem can be cured by tractable redefinitions is far from obvious.

Gauge invariance

The sigma model formulation corresponds to a partially gauge-fixed formulation since there are no arbitrary functions of time in the solutions of the equations of motion (except for the lapse function $n(t)$). The only gauge freedom left corresponds to time-independent gauge transformation (this is the equivalent of the "temporal gauge" of electromagnetism). The constraints associated with the spatial diffeomorphisms and with the 3-form gauge symmetry have not been eliminated. How they are expressed in terms of the sigma model variables and how they fit with the E_{10}-symmetry is a question that should be answered. Progress along these lines may be found in recent work [52].

Electric-magnetic duality

The sigma model approach contains both the graviton and its dual, as well as both the 3-form and its dual 6-form. Since these obey second-order equations of motion, there is a double-counting of degrees of freedom. For instance, the magnetic field of the 3-form would also appear as a spatial gradient of the 3-form at level 4, but nothing in the formalism tells that this is the same magnetic field as the time derivative of the 6-form at level 2. A generalized self-duality condition should be imposed [35, 36], not just in the 3-form sector but also for the graviton. Better yet, one might search for a duality-invariant action without double-counting. Such actions have been studied both for p-forms [65, 64] and for gravity [21, 101, 114] and are not manifestly spacetime covariant (this is not an issue here since manifest spacetime covariance has been given up anyway in the $(1+0)$-dimensional $\mathcal{E}_{10}$-sigma model). One must pick a spacetime coordinate, which might be time, or one spatial direction [100, 159]. We feel that a better understanding of duality might yield an important clue [21, 25, 26, 148].

10 Cosmological Solutions from E_{10}

In this last main section we shall show that the low level equivalence between the $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$ sigma model and eleven-dimensional supergravity can be put to practical use for finding exact solutions of eleven-dimensional supergravity. This is a satisfactory result because even in the cosmological context of homogeneous fields $G_{\alpha\beta}(t)$, $F_{\alpha\beta\gamma\delta}(t)$ that depend only on time ("Bianchi I cosmological models" [61]), the equations of motion of eleven-dimensional supergravity remain notoriously complicated, while the corresponding sigma model is, at least formally, integrable.

We will remain in the strictly cosmological sector where it is assumed that all spatial gradients can be neglected so that all fields depend only on time. Moreover, we impose diagonality of the spatial metric. These conditions must of course be compatible with the equations of motion; if the conditions are imposed initially, they should be preserved by the time evolution.

A large class of solutions to eleven-dimensional supergravity preserving these conditions were found in [61]. These solutions have zero magnetic field but have a restricted number of electric field components turned on. Surprisingly, it was found that such solutions have an elegant interpretation in terms of so called *geometric configurations*, denoted (n_m, g_3), of n points and g lines (with $n \leq 10$) drawn on a plane with certain pre-determined rules. That is, for each geometric configuration (whose definition is recalled below) one can associate a diagonal solution with some non-zero electric field components F_{tijk}, determined by the configuration. In this section we re-examine this result from the point of view of the sigma model based on $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$.

We show, following [96], that each configuration (n_m, g_3) encodes information about a (regular) subalgebra $\bar{\mathfrak{g}}$ of E_{10}, and the supergravity solution associated to the configuration $(n_m g_3)$ can be obtained by restricting the $\mathcal{E}_{10}$-sigma model to the subgroup $\bar{\mathcal{G}}$ whose Lie algebra is $\bar{\mathfrak{g}}$. Therefore, we will here make use of both the level truncation and the subgroup truncation simultaneously; first by truncating to a certain level and then by restricting to the relevant $\bar{\mathfrak{g}}$-algebra generated by a subset of the representations at this level. Large parts of this section are based on [96].

10.1 Bianchi I models and eleven-dimensional supergravity

On the supergravity side, we will restrict the metric and the electromagnetic field to depend on time only,

$$\begin{aligned} ds^2 &= -N^2(t)\,dt^2 + \mathrm{g}_{ab}(t)\,dx^a\,dx^b, \\ F_{\lambda\rho\sigma\tau} &= F_{\lambda\rho\sigma\tau}(t). \end{aligned} \tag{10.1}$$

Recall from Section 9.3 that with these ansätze the dynamical equations of motion of eleven-dimensional supergravity reduce to [61]

$$\frac{1}{2}\partial\left(\sqrt{\mathrm{g}}N^{-1}\mathrm{g}^{ac}\partial\mathrm{g}_{cb}\right) = \frac{1}{12}N\sqrt{\mathrm{g}}F^{a\rho\sigma\tau}F_{b\rho\sigma\tau} - \frac{1}{144}N\sqrt{\mathrm{g}}\,\delta^a{}_b\,F^{\lambda\rho\sigma\tau}F_{\lambda\rho\sigma\tau}, \tag{10.2}$$

$$\partial\left(F^{tabc}N\sqrt{\mathrm{g}}\right) = \frac{1}{144}\varepsilon^{tabcd_1d_2d_3e_1e_2e_3e_4}F_{td_1d_2d_3}F_{e_1e_2e_3e_4}, \tag{10.3}$$

$$\partial F_{a_1a_2a_3a_4} = 0. \tag{10.4}$$

This corresponds to the truncation of the sigma model at level 2 which, as we have seen, completely matches the supergravity side. We also defined $\partial \equiv \partial_t$ as in Section 9.3. Furthermore we have the following constraints,

$$\frac{1}{4}\left(\mathrm{g}^{ac}\mathrm{g}^{bd} - \mathrm{g}^{ab}\mathrm{g}^{cd}\right)\partial\mathrm{g}_{ab}\,\partial\mathrm{g}_{cd} + \frac{1}{12}F^{tabc}F_{tabc} + \frac{1}{48}N^2F^{abcd}F_{abcd} = 0, \tag{10.5}$$

$$\frac{1}{6}NF^{tbcd}F_{abcd} = 0, \tag{10.6}$$

$$\varepsilon^{tabc_1c_2c_3c_4d_1d_2d_3d_4}F_{c_1c_2c_3c_4}F_{d_1d_2d_3d_4} = 0, \tag{10.7}$$

which are, respectively, the Hamiltonian constraint, momentum constraint and Gauss' law. Note that Greek indices $\alpha, \beta, \gamma, \cdots$ correspond to the full eleven-dimensional spacetime, while Latin indices $a, b, c, \cdots$ correspond to the ten-dimensional spatial part.

We will further take the metric to be purely time-dependent and diagonal,

$$ds^2 = -N^2(t)\, dt^2 + \sum_{i=1}^{10} a_i^2(t)(dx^i)^2. \tag{10.8}$$

This form of the metric has manifest invariance under the ten distinct spatial reflections

$$\begin{aligned} x^j &\to -x^j, \\ x^{i\neq j} &\to x^{i\neq j}, \end{aligned} \tag{10.9}$$

and in order to ensure compatibility with the Einstein equations, the energy-momentum tensor of the 4-form field strength must also be diagonal.

10.1.1 Diagonal metrics and geometric configurations

Assuming zero magnetic field (this restriction will be lifted below), one way to achieve diagonality of the energy-momentum tensor is to assume that the non-vanishing components of the electric field $F^{\perp abc} = N^{-1} F_{tabc}$ are determined by *geometric configurations* (n_m, g_3) with $n \leq 10$ [61].

A geometric configuration (n_m, g_3) is a set of n points and g lines with the following incidence rules [117, 105, 145]:

1. Each line contains three points.
2. Each point is on m lines.
3. Two points determine at most one line.

It follows that two lines have at most one point in common. It is an easy exercise to verify that $mn = 3g$. An interesting question is whether the lines can actually be realized as straight lines in the (real) plane, but, for our purposes, it is not necessary that it should be so; the lines can be bent.

Let (n_m, g_3) be a geometric configuration with $n \leq 10$ points. We number the points of the configuration $1, \cdots, n$. We associate to this geometric configuration a pattern of electric field components $F^{\perp abc}$ with the following property: $F^{\perp abc}$ can be non-zero only if the triple (a, b, c) is a line of the geometric configuration. If it is not, we take $F^{\perp abc} = 0$. It is clear that this property is preserved in time by the equations of motion (in the absence of magnetic field). Furthermore, because of Rule 3 above, the products $F^{\perp abc} F^{\perp a'b'c'} g_{bb'} g_{cc'}$ vanish when $a \neq a'$ so that the energy-momentum tensor is diagonal.

10.2 Geometric configurations and regular subalgebras of E_{10}

We prove here that the conditions on the electric field embodied in the geometric configurations (n_m, g_3) have a direct Kac–Moody algebraic interpretation. They simply correspond to a consistent truncation of the E_{10} nonlinear sigma model to a $\bar{\mathfrak{g}}$ nonlinear sigma model, where $\bar{\mathfrak{g}}$ is a rank g Kac–Moody subalgebra of E_{10} (or a quotient of such a Kac–Moody subalgebra by an appropriate ideal when the relevant Cartan matrix has vanishing determinant), with three crucial properties: (i) It is regularly embedded in E_{10} (see Section 4 for the definition of regular subalgebras), (ii) it is generated by electric roots only, and (iii) every node P in its Dynkin diagram $\mathbb{D}_{\bar{\mathfrak{g}}}$ is linked to a number k of nodes that is independent of P (but depend on the algebra). We find that the

Dynkin diagram $\mathbb{D}_{\bar{\mathfrak{g}}}$ of $\bar{\mathfrak{g}}$ is the *line incidence diagram* of the geometric configuration (n_m, g_3), in the sense that (i) each line of (n_m, g_3) defines a node of $\mathbb{D}_{\bar{\mathfrak{g}}}$, and (ii) two nodes of $\mathbb{D}_{\bar{\mathfrak{g}}}$ are connected by a single bond iff the corresponding lines of (n_m, g_3) have no point in common. This defines a geometric duality between a configuration (n_m, g_3) and its associated Dynkin diagram $\mathbb{D}_{\bar{\mathfrak{g}}}$. In the following we shall therefore refer to configurations and Dynkin diagrams related in this way as *dual*.

None of the algebras $\bar{\mathfrak{g}}$ relevant to the truncated models turn out to be hyperbolic: They can be finite, affine, or Lorentzian with infinite-volume Weyl chamber. Because of this, the solutions are non-chaotic. After a finite number of collisions, they settle asymptotically into a definite Kasner regime (both in the future and in the past).

10.2.1 General considerations

In order to match diagonal Bianchi I cosmologies with the sigma model, one must truncate the $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$ action in such a way that the sigma model metric g_{ab} is diagonal. This will be the case if the subalgebra $\bar{\mathfrak{g}}$ to which one truncates has no generator $K^i{}_j$ with $i \neq j$. Indeed, recall from Section 9 that the off-diagonal components of the metric are precisely the exponentials of the associated sigma model fields. The set of simple roots of $\bar{\mathfrak{g}}$ should therefore not contain any root at level zero.

Consider "electric" regular subalgebras of E_{10}, for which the simple roots are all at level one, where the 3-form electric field variables live. These roots can be parametrized by three indices corresponding to the indices of the electric field, with $i_1 < i_2 < i_3$. We denote them $\alpha_{i_1 i_2 i_3}$. For instance, $\alpha_{123} \equiv \alpha_{10}$. In terms of the β-parametrization of [45, 48], one has $\alpha_{i_1 i_2 i_3} = \beta^{i_1} + \beta^{i_2} + \beta^{i_3}$.

Now, for $\bar{\mathfrak{g}}$ to be a regular subalgebra, it must fulfill, as we have seen, the condition that the difference between any two of its simple roots is not a root of E_{10}: $\alpha_{i_1 i_2 i_3} - \alpha_{i'_1 i'_2 i'_3} \notin \Phi_{E_{10}}$ for any pair $\alpha_{i_1 i_2 i_3}$ and $\alpha_{i'_1 i'_2 i'_3}$ of simple roots of $\bar{\mathfrak{g}}$. But one sees by inspection of the commutator of $E^{i_1 i_2 i_3}$ with $F_{i'_1 i'_2 i'_3}$ in Equation (8.78) that $\alpha_{i_1 i_2 i_3} - \alpha_{i'_1 i'_2 i'_3}$ is a root of E_{10} if and only if the sets $\{i_1, i_2, i_3\}$ and $\{i'_1, i'_2, i'_3\}$ have exactly two points in common. For instance, if $i_1 = i'_1$, $i_2 = i'_2$ and $i_3 \neq i'_3$, the commutator of $E^{i_1 i_2 i_3}$ with $F_{i'_1 i'_2 i'_3}$ produces the off-diagonal generator $K^{i_3}{}_{i'_3}$ corresponding to a level zero root of E_{10}. In order to fulfill the required condition, one must avoid this case, i.e., one must choose the set of simple roots of the electric regular subalgebra $\bar{\mathfrak{g}}$ in such a way that given a pair of indices (i_1, i_2), there is at most one i_3 such that the root α_{ijk} is a simple root of $\bar{\mathfrak{g}}$, with (i, j, k) being the re-ordering of (i_1, i_2, i_3) such that $i < j < k$.

To each of the simple roots $\alpha_{i_1 i_2 i_3}$ of $\bar{\mathfrak{g}}$, one can associate the line (i_1, i_2, i_3) connecting the three points i_1, i_2 and i_3. If one does this, one sees that the above condition is equivalent to the following statement: *The set of points and lines associated with the simple roots of $\bar{\mathfrak{g}}$ must fulfill the third rule defining a geometric configuration, namely, that two points determine at most one line.* Thus, this geometric condition has a nice algebraic interpretation in terms of regular subalgebras of E_{10}.

The first rule, which states that each line contains 3 points, is a consequence of the fact that the E_{10}-generators at level one are the components of a 3-index antisymmetric tensor. The second rule, that each point is on m lines, is less fundamental from the algebraic point of view since it is not required to hold for $\bar{\mathfrak{g}}$ to be a regular subalgebra. It was imposed in [61] in order to allow for solutions isotropic in the directions that support the electric field. We keep it here as it yields interesting structure.

10.2.2 Incidence diagrams and Dynkin diagrams

We have just shown that each geometric configuration (n_m, g_3) with $n \leq 10$ defines a regular subalgebra $\bar{\mathfrak{g}}$ of E_{10}. In order to determine what this subalgebra $\bar{\mathfrak{g}}$ is, one needs, according to the theorem recalled in Section 4, to compute the Cartan matrix

$$C = [C_{i_1 i_2 i_3, i'_1 i'_2 i'_3}] = \left[\left(\alpha_{i_1 i_2 i_3} | \alpha_{i'_1 i'_2 i'_3} \right) \right] \tag{10.10}$$

(the real roots of E_{10} have length squared equal to 2). According to that same theorem, the algebra $\bar{\mathfrak{g}}$ is then just the rank g Kac–Moody algebra with Cartan matrix C, unless C has zero determinant, in which case $\bar{\mathfrak{g}}$ might be the quotient of that algebra by a nontrivial ideal.

Using for instance the root parametrization of [45, 48] and the expression of the scalar product in terms of this parametrization, one easily verifies that the scalar product $\left(\alpha_{i_1 i_2 i_3} | \alpha_{i'_1 i'_2 i'_3} \right)$ is equal to

$$\left(\alpha_{i_1 i_2 i_3} | \alpha_{i'_1 i'_2 i'_3} \right) = \begin{cases} 2 & \text{if all three indices coincide,} \\ 1 & \text{if two and only two indices coincide,} \\ 0 & \text{if one and only one index coincides,} \\ -1 & \text{if no indices coincide.} \end{cases} \tag{10.11}$$

The second possibility does not arise in our case since we deal with geometric configurations. For completeness, we also list the scalar products of the electric roots α_{ijk} $(i < j < k)$ with the symmetry roots $\alpha_{\ell m}$ $(\ell < m)$ associated with the raising operators $K^m{}_\ell$:

$$(\alpha_{ijk} | \alpha_{\ell m}) = \begin{cases} -1 & \text{if } \ell \in \{i,j,k\} \text{ and } m \notin \{i,j,k\}, \\ 0 & \text{if } \{\ell, m\} \subset \{i,j,k\} \text{ or } \{\ell, m\} \cap \{i,j,k\} = \emptyset, \\ 1 & \text{if } \ell \notin \{i,j,k\} \text{ and } m \in \{i,j,k\}, \end{cases} \tag{10.12}$$

as well as the scalar products of the symmetry roots among themselves,

$$(\alpha_{ij} | \alpha_{\ell m}) = \begin{cases} -1 & \text{if } j = \ell \text{ or } i = m, \\ 0 & \text{if } \{\ell, m\} \cap \{i,j\} = \emptyset, \\ 1 & \text{if } i = \ell \text{ or } j \neq m, \\ 2 & \text{if } \{\ell, m\} = \{i,j\}. \end{cases} \tag{10.13}$$

Given a geometric configuration (n_m, g_3), one can associate with it a "line incidence diagram" that encodes the incidence relations between its lines. To each line of (n_m, g_3) corresponds a node in the incidence diagram. Two nodes are connected by a single bond if and only if they correspond to lines with no common point ("parallel lines"). Otherwise, they are not connected[37]. By inspection of the above scalar products, we come to the important conclusion that *the Dynkin diagram of the regular, rank g, Kac–Moody subalgebra $\bar{\mathfrak{g}}$ associated with the geometric configuration (n_m, g_3) is just its line incidence diagram.* We shall call the Kac–Moody algebra $\bar{\mathfrak{g}}$ the algebra "dual" to the geometric configuration (n_m, g_3).

Because the geometric configurations have the property that the number of lines through any point is equal to a constant m, the number of lines parallel to any given line is equal to a number k that depends only on the configuration and not on the line. This is in fact true in general and not only for $n \leq 10$ as can be seen from the following argument. For a configuration with n points, g lines and m lines through each point, any given line Δ admits $3(m-1)$ true secants, namely, $(m-1)$ through each of its points[38]. By definition, these secants are all distinct since none of the lines that Δ intersects at one of its points, say P, can coincide with a line that it intersects at another of its points, say P', since the only line joining P to P' is Δ itself. It follows that the total

[37] One may also consider a point incidence diagram defined as follows: The nodes of the point incidence diagram are the points of the geometric configuration. Two nodes are joined by a single bond if and only if there is no straight line connecting the corresponding points. The point incidence diagrams of the configurations $(9_3, 9_3)$ are given in [105]. For these configurations, projective duality between lines and points lead to identical line and point incidence diagrams. Unless otherwise stated, the expression "incidence diagram" will mean "line incidence diagram".

[38] A true secant is here defined as a line, say Δ', distinct from Δ and with a non-empty intersection with Δ.

number of lines that Δ intersects is the number of true secants plus Δ itself, i.e., $3(m-1)+1$. As a consequence, each line in the configuration admits $k = g - [3(m-1)+1]$ parallel lines, which is then reflected by the fact that each node in the associated Dynkin diagram has the same number k of adjacent nodes.

10.3 Cosmological solutions with electric flux

Let us now make use of these considerations to construct some explicit supergravity solutions. We begin by analyzing the simplest configuration $(3_1, 1_3)$, of three points and one line. It is displayed in Figure 50. This case is the only possible configuration for $n = 3$.

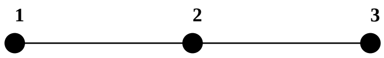


Figure 50: $(3_1, 1_3)$: The only allowed configuration for $n = 3$.

This example also exhibits some subtleties associated with the Hamiltonian constraint and the ensuing need to extend $\bar{\mathfrak{g}}$ when the algebra dual to the geometric configuration is finite-dimensional. We will come back to this issue below.

10.3.1 General discussion

In light of our discussion, considering the geometric configuration $(3_1, 1_3)$ is equivalent to turning on only the component $\mathcal{A}_{123}(t)$ of the 3-form that parametrizes the generator E^{123} in the coset representative $\mathcal{V}(t) \in \mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$. Moreover, in order to have the full coset description, we must also turn on the diagonal metric components corresponding to the Cartan generator $h = [E^{123}, F_{123}]$. The algebra has thus basis $\{e, f, h\}$ with

$$e \equiv E^{123}, \qquad f \equiv F_{123}, \qquad h = [e, f] = -\frac{1}{3} \sum_{a \neq 1,2,3} K^a{}_a + \frac{2}{3}(K^1{}_1 + K^2{}_2 + K^3{}_3), \tag{10.14}$$

where the form of h followed directly from the general commutator between E^{abc} and F_{def} in Section 8. The Cartan matrix is just (2) and is nondegenerate. It defines an $A_1 = \mathfrak{sl}(2, \mathbb{R})$ regular subalgebra. The Chevalley–Serre relations, which are guaranteed to hold according to the general argument, are easily verified. The configuration $(3_1, 1_3)$ is thus dual to A_1,

$$\mathfrak{g}_{(3_1,3_1)} = A_1. \tag{10.15}$$

This A_1 algebra is simply the $\mathfrak{sl}(2, \mathbb{R})$-algebra associated with the simple root α_{10}. Because the Killing form of A_1 restricted to the Cartan subalgebra $\mathfrak{h}_{A_1} = \mathbb{R}h$ is positive definite, one cannot find a solution of the Hamiltonian constraint if one turns on only the fields corresponding to A_1. One needs to enlarge A_1 (at least) by a one-dimensional subalgebra $\mathbb{R}l$ of $\mathfrak{h}_{E_{10}}$ that is timelike. As will be discussed further below, we take for l the Cartan element $K^4{}_4 + K^5{}_5 + K^6{}_6 + K^7{}_7 + K^8{}_8 + K^9{}_9 + K^{10}{}_{10}$, which ensures isotropy in the directions not supporting the electric field. Thus, the appropriate regular subalgebra of E_{10} in this case is $A_1 \oplus \mathbb{R}l$.

The need to enlarge the algebra A_1 was discussed in the paper [127] where a group theoretical interpretation of some cosmological solutions of eleven-dimensional supergravity was given. In that paper, it was also observed that $\mathbb{R}l$ can be viewed as the Cartan subalgebra of the (non-regularly embedded) subalgebra A_1 associated with an imaginary root at level 21, but since the corresponding field is not excited, the relevant subalgebra is really $\mathbb{R}l$.

10.3.2 The solution

In order to make the above discussion a little less abstract we now show how to obtain the relevant supergravity solution by solving the $\mathcal{E}_{10}$-sigma model equations of motion and then translating these, using the dictionary from Section 9, to supergravity solutions. For this particular example the analysis was done in [127].

In order to better understand the role of the timelike generator $l \in \mathfrak{h}_{E_{10}}$ we begin the analysis by omitting it. The truncation then amounts to considering the coset representative

$$\mathcal{V}(t) = e^{\phi(t)h}\, e^{\mathcal{A}_{123}(t)E^{123}} \in \mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10}). \tag{10.16}$$

The projection $\mathcal{P}(t)$ onto the coset becomes

$$\begin{aligned}\mathcal{P}(t) &= \frac{1}{2}\left[\partial\mathcal{V}(t)\,\mathcal{V}(t)^{-1} + \left(\partial\mathcal{V}(t)\,\mathcal{V}(t)^{-1}\right)^{\mathcal{T}}\right] \\ &= \partial\phi(t)h + \frac{1}{2}e^{2\phi(t)}\partial\mathcal{A}_{123}(t)\left(E^{123}+F_{123}\right),\end{aligned} \tag{10.17}$$

where the exponent is the linear form $\alpha(\phi) = 2\phi$ representing the exceptional simple root α_{123} of E_{10}. More precisely, it is the linear form α acting on the Cartan generator $\phi(t)h$, as follows:

$$\alpha(\phi h) = \phi\,\langle\alpha, h\rangle = \phi\,\langle\alpha,\alpha^{\vee}\rangle = \alpha^2\phi = 2\phi. \tag{10.18}$$

The Lagrangian becomes

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\mathcal{P}(t)|\mathcal{P}(t)\right) \\ &= \partial\phi(t)\partial\phi(t) + \frac{1}{4}e^{4\phi(t)}\,\partial\mathcal{A}_{123}(t)\,\partial\mathcal{A}_{123}(t).\end{aligned} \tag{10.19}$$

For convenience we have chosen the gauge $n = 1$ of the free parameter in the $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$-Lagrangian (see Section 9). Recall that for the level one fields we have $\mathcal{D}\mathcal{A}_{abc}(t) = \partial\mathcal{A}_{abc}(t)$, which is why only the partial derivative of $\mathcal{A}_{123}(t)$ appears in the Lagrangian.

The reason why this simple looking model contains information about eleven-dimensional supergravity is that the A_1 subalgebra represented by (e, f, h) is embedded in E_{10} through the level 1-generator E^{123}, and hence this Lagrangian corresponds to a consistent subgroup truncation of the $\mathcal{E}_{10}$- sigma model.

Let us now study the dynamics of the Lagrangian in Equation (10.19). The equations of motion for $\mathcal{A}_{123}(t)$ are

$$\partial\left(\frac{1}{2}e^{4\phi(t)}\partial\mathcal{A}_{123}(t)\right) = 0 \quad\Longrightarrow\quad \frac{1}{2}e^{4\phi(t)}\partial\mathcal{A}_{123}(t) = a, \tag{10.20}$$

where a is a constant. The equations for the $\ell = 0$ field ϕ may then be written as

$$\partial^2\phi(t) = 2a^2e^{-4\phi(t)}. \tag{10.21}$$

Integrating once yields

$$\partial\phi(t)\,\partial\phi(t) + a^2e^{-4\phi(t)} = E, \tag{10.22}$$

where E plays the role of the energy for the dynamics of $\phi(t)$. This equation can be solved exactly with the result [127]

$$\phi(t) = \frac{1}{2}\ln\left[\frac{2a}{\sqrt{E}}\cosh\sqrt{E}t\right] \equiv \frac{1}{2}\ln H(t). \tag{10.23}$$

We must also take into account the Hamiltonian constraint

$$\mathcal{H} = (\mathcal{P}|\mathcal{P}) = 0, \tag{10.24}$$

arising from the variation of $n(t)$ in the $\mathcal{E}_{10}$-sigma model. The Hamiltonian becomes

$$\begin{aligned}\mathcal{H} &= 2\partial\phi(t)\,\partial\phi(t) + \frac{1}{2}e^{4\phi(t)}\partial\mathcal{A}_{123}(t)\,\partial\mathcal{A}_{123}(t)\\ &= 2\left(\partial\phi(t)\partial\phi(t) + a^2e^{-4\phi(t)}\right)\\ &= 2E.\end{aligned} \tag{10.25}$$

It is therefore impossible to satisfy the Hamiltonian constraint unless $E = 0$. This is the problem which was discussed above, and the reason why we need to enlarge the choice of coset representative to include the timelike generator $l \in \mathfrak{h}_{E_{10}}$. We choose l such that it commutes with h and E^{123},

$$[l, h] = [l, E^{123}] = 0, \tag{10.26}$$

and such that isotropy in the directions not supported by the electric field is ensured. Most importantly, in order to solve the problem of the Hamiltonian constraint, l must be timelike,

$$l^2 = (l|l) < 0, \tag{10.27}$$

where $(\cdot|\cdot)$ is the scalar product in the Cartan subalgebra of E_{10}. The subalgebra to which we truncate the sigma model is thus given by

$$\bar{\mathfrak{g}} = A_1 \oplus \mathbb{R}l \subset E_{10}, \tag{10.28}$$

and the corresponding coset representative is

$$\tilde{\mathcal{V}}(t) = e^{\phi(t)h+\tilde{\phi}(t)l}e^{\mathcal{A}_{123}(t)E^{123}}. \tag{10.29}$$

The Lagrangian now splits into two disconnected parts, corresponding to the direct product $SL(2,\mathbb{R})/SO(2) \times \mathbb{R}$,

$$\tilde{\mathcal{L}} = \left(\partial\phi(t)\,\partial\phi(t) + \frac{1}{4}e^{4\phi(t)}\partial\mathcal{A}_{123}(t)\,\partial\mathcal{A}_{123}(t)\right) + \frac{l^2}{2}\partial\tilde{\phi}(t)\,\partial\tilde{\phi}(t). \tag{10.30}$$

The solution for $\tilde{\phi}$ is therefore simply linear in time,

$$\tilde{\phi} = |l^2|\sqrt{\tilde{E}}\,t. \tag{10.31}$$

The new Hamiltonian now gets a contribution also from the Cartan generator l,

$$\tilde{\mathcal{H}} = 2E - |l^2|\tilde{E}. \tag{10.32}$$

This contribution depends on the norm of l and since $l^2 < 0$, it is possible to satisfy the Hamiltonian constraint, provided that we set

$$\tilde{E} = \frac{2}{|l^2|}E. \tag{10.33}$$

We have now found a consistent truncation of the $\mathcal{K}(\mathcal{E}_{10}) \times \mathcal{E}_{10}$-invariant sigma model which exhibits $SL(2,\mathbb{R}) \times SO(2) \times \mathbb{R}$-invariance. We want to translate the solution to this model, Equation (10.23), to a solution of eleven-dimensional supergravity. The embedding of $\mathfrak{sl}(2,\mathbb{R}) \subset E_{10}$ in Equation (10.14) induces a natural "Freund–Rubin" type $(1+3+7)$ split of the coordinates in the physical metric, where the 3-form is supported in the three spatial directions x^1, x^2, x^3. We must also choose an embedding of the timelike generator l. In order to ensure isotropy in the directions

$x^4, \cdots, x^{10}$, where the electric field has no support, it is natural to let l be extended only in the "transverse" directions and we take [127]

$$l = K^4{}_4 + \cdots + K^{10}{}_{10}, \tag{10.34}$$

which has norm

$$(l|l) = \left(K^4{}_4 + \cdots + K^{10}{}_{10} | K^4{}_4 + \cdots + K^{10}{}_{10}\right) = -42. \tag{10.35}$$

To find the metric solution corresponding to our sigma model, we first analyze the coset representative at $\ell = 0$,

$$\tilde{\mathcal{V}}(t)\big|_{\ell=0} = \text{Exp}\left[\phi(t)h + \tilde{\phi}(t)l\right]. \tag{10.36}$$

In order to make use of the dictionary from Section 9.3.6 it is necessary to rewrite this in a way more suitable for comparison, i.e., to express the Cartan generators h and l in terms of the $\mathfrak{gl}(10,\mathbb{R})$-generators $K^a{}_b$. We thus introduce parameters $\xi^a{}_b(t)$ and $\tilde{\xi}^a{}_b(t)$ representing, respectively, ϕ and $\tilde{\phi}$ in the $\mathfrak{gl}(10,\mathbb{R})$-basis. The level zero coset representative may then be written as

$$\begin{aligned}\tilde{\mathcal{V}}(t)\big|_{\ell=0} &= \text{Exp}\left[\sum_{a=1}^{10}\left(\xi^a{}_a(t) + \tilde{\xi}^a{}_a(t)\right)K^a{}_a\right] \\ &= \text{Exp}\left[\sum_{a=4}^{10}\left(\xi^a{}_a(t) + \tilde{\xi}^a{}_a(t)\right)K^a{}_a + \left(\xi^1{}_1(t)K^1{}_1 + \xi^2{}_2(t)K^2{}_2 + \xi^3{}_3(t)K^3{}_3\right)\right]\end{aligned} \tag{10.37}$$

where in the second line we have split the sum in order to highlight the underlying spacetime structure, i.e., to emphasize that $\tilde{\xi}^a{}_b$ has no non-vanishing components in the directions x^1, x^2, x^3. Comparing this to Equation (10.14) and Equation (10.34) gives the diagonal components of $\xi^a{}_b$ and $\tilde{\xi}^a{}_b$,

$$\xi^1{}_1 = \xi^2{}_2 = \xi^3{}_3 = 2\phi/3, \qquad \xi^4{}_4 = \cdots = \xi^{10}{}_{10} = -\phi/3, \qquad \tilde{\xi}^4{}_4 = \cdots = \tilde{\xi}^{10}{}_{10} = \tilde{\phi}. \tag{10.38}$$

Now, the dictionary from Section 9 identifies the physical spatial metric as follows:

$$g_{ab}(t) = e_a{}^{\bar{a}}(t)e_b{}^{\bar{b}}(t)\delta_{\bar{a}\bar{b}} = \left(e^{\xi(t)+\tilde{\xi}(t)}\right)_a{}^{\bar{a}}\left(e^{\xi(t)+\tilde{\xi}(t)}\right)_b{}^{\bar{b}}\delta_{\bar{a}\bar{b}} \tag{10.39}$$

By observation of Equation (10.38) we find the components of the metric to be

$$\begin{aligned} g_{11} = g_{22} = g_{33} &= e^{4\phi/3}, \\ g_{44} = \cdots = g_{(10)(10)} &= e^{-2\phi/3+2\tilde{\phi}}. \end{aligned} \tag{10.40}$$

This result shows clearly how the embedding of h and l into E_{10} is reflected in the coordinate split of the metric. The gauge fixing $N = \sqrt{g}$ (or $n = 1$) gives the g_{tt}-component of the metric,

$$g_{tt} = N^2 = e^{14\tilde{\phi}-2\phi/3}. \tag{10.41}$$

Next we consider the generator E^{123}. The dictionary tells us that the field strength of the 3-form in eleven-dimensional supergravity at some fixed spatial point $\mathbf{x}_0$ should be identified as

$$F_{t123}(t, \mathbf{x}_0) = \mathcal{D}\mathcal{A}_{123}(t) = \partial\mathcal{A}_{123}(t). \tag{10.42}$$

It is possible to eliminate the $\mathcal{A}_{123}(t)$ in favor of the Cartan field $\phi(t)$ using the first integral of its equations of motion, Equation (10.20),

$$\frac{1}{2}e^{-4\phi(t)}\partial\mathcal{A}_{123}(t) = a. \tag{10.43}$$

In this way we may write the field strength in terms of a and the solution for ϕ,

$$F_{t123}(t,\mathbf{x}_0) = 2ae^{4\phi(t)} = 2aH^{-2}(t). \tag{10.44}$$

Finally, we write down the solution for the spacetime metric explicitly:

$$\begin{aligned} ds^2 &= -e^{14\tilde{\phi}+2\phi/3}\,dt^2 + e^{4\phi/3}\left[(dx^1)^2+(dx^2)^2+(dx^3)^2\right] + e^{2\tilde{\phi}-2\phi/3}\sum_{\bar{a}=4}^{10}(dx^{\bar{a}})^2 \\ &= -H^{1/3}(t)e^{\frac{1}{3}\sqrt{\tilde{E}t}}\,dt^2 + H^{-2/3}(t)\left[(dx^1)^2+(dx^2)^2+(dx^3)^2\right] + H^{1/3}(t)e^{\frac{\sqrt{\tilde{E}}}{21}t}\sum_{\bar{a}=4}^{10}(dx^{\bar{a}})^2, \end{aligned} \tag{10.45}$$

where

$$H(t) = \frac{2a}{\sqrt{E}}\cosh\sqrt{E}t. \tag{10.46}$$

This solution coincides with the cosmological solution first found in [61] for the geometric configuration $(3_1,3_1)$, and it is intriguing that it can be exactly reproduced from a manifestly $\mathcal{E}_{10}\times\mathcal{K}(\mathcal{E}_{10})$-invariant action, a priori unrelated to any physical model.

Note that in modern terminology, this solution is an $SM2$-brane solution (see, e.g., [143] for a review) since it can be interpreted as a spacelike (i.e., time-dependent) version of the $M2$-brane solution. From this point of view the world volume of the $SM2$-brane is extended in the directions x^1, x^2 and x^3, and so is Euclidean.

In the BKL-limit this solution describes two asymptotic Kasner regimes, at $t\to\infty$ and at $t\to-\infty$. These are separated by a collision against an electric wall, corresponding to the blow-up of the electric field $F_{t123}(t)\sim H^{-2}(t)$ at $t=0$. In the billiard picture the dynamics in the BKL-limit is thus given by free-flight motion interrupted by one geometric reflection against the electric wall,

$$e_{123}(\beta) = \beta^1+\beta^2+\beta^3, \tag{10.47}$$

which is the exceptional simple root of E_{10}. This indicates that in the strict BKL-limit, electric walls and $SM2$-branes are actually equivalent.

10.3.3 Intersecting spacelike branes from geometric configurations

Let us now examine a slightly more complicated example. We consider the configuration $(6_2,4_3)$, shown in Figure 51. This configuration has four lines and six points. As such the associated supergravity model describes a cosmological solution with four components of the electric field turned on, or, equivalently, it describes a set of four intersecting $SM2$-branes [96].

From the configuration we read off the Chevalley–Serre generators associated to the simple roots of the dual algebra:

$$e_1 = E^{123}, \qquad e_2 = E^{145}, \qquad e_3 = E^{246}, \qquad e_4 = E^{356}. \tag{10.48}$$

The first thing to note is that all generators have one index in common since in the graph any two lines share one node. This implies that the four lines in $(6_2,4_3)$ define four *commuting* A_1 subalgebras,

$$(6_2,4_3) \quad\Longleftrightarrow\quad \mathfrak{g}_{(6_2,4_3)} = A_1\oplus A_1\oplus A_1\oplus A_1. \tag{10.49}$$

One can make sure that the Chevalley–Serre relations are indeed fulfilled for this embedding. For instance, the Cartan element $h=[E^{b_1b_2b_3},F_{b_1b_2b_3}]$ (no summation on the fixed, distinct indices b_1,b_2,b_3) reads

$$h = -\frac{1}{3}\sum_{a\neq b_1,b_2,b_3}K^a{}_a + \frac{2}{3}(K^{b_1}{}_{b_1}+K^{b_2}{}_{b_2}+K^{b_3}{}_{b_3}). \tag{10.50}$$

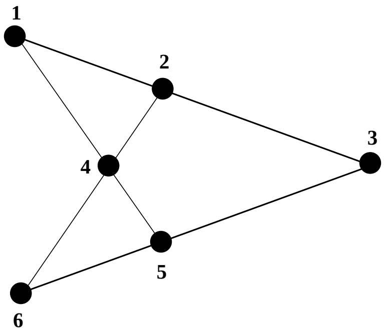


Figure 51: The configuration $(6_2, 4_3)$, dual to the Lie algebra $A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

Hence, the commutator $[h, E^{b_i cd}]$ vanishes whenever $E^{b_i cd}$ has only one b-index,

$$\begin{aligned}[h, E^{b_i cd}] &= -\frac{1}{3}[(K^c{}_c + K^d{}_d), E^{b_i cd}] + \frac{2}{3}[(K^{b_1}{}_{b_1} + K^{b_2}{}_{b_2} + K^{b_3}{}_{b_3}), E^{b_i cd}] \\ &= \left(-\frac{1}{3} - \frac{1}{3} + \frac{2}{3}\right) E^{b_i cd} = 0 \qquad (i = 1,2,3).\end{aligned} \tag{10.51}$$

Furthermore, multiple commutators of the step operators are immediately killed at level 2 whenever they have one index or more in common, e.g.,

$$[E^{123}, E^{145}] = E^{123145} = 0. \tag{10.52}$$

To fulfill the Hamiltonian constraint, one must extend the algebra by taking a direct sum with $\mathbb{R}l$, $l = K^7{}_7 + K^8{}_8 + K^9{}_9 + K^{10}{}_{10}$. Accordingly, the final algebra is $A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus \mathbb{R}l$. Because there is no magnetic field, the momentum constraint and Gauss' law are identically satisfied.

By investigating the sigma model solution corresponding to the algebra $\mathfrak{g}_{(6_2,4_3)}$, augmented with the timelike generator l,

$$\bar{\mathfrak{g}} = A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus \mathbb{R}l, \tag{10.53}$$

we find a supergravity solution which generalizes the one found in [61]. The solution describes a set of four intersecting $SM2$-branes, with a five-dimensional transverse spacetime in the directions t, x^7, x^8, x^9, x^{10}.

Let us write down also this solution explicitly. The full set of generators for $\mathfrak{g}_{(6_2,4_3)}$ is

$$\begin{aligned}
&e_1 = E^{123}, \quad e_2 = E^{145}, \quad e_3 = E^{246}, \quad e_4 = E^{356} \\
&f_1 = F_{123}, \quad f_2 = F_{145}, \quad f_3 = F_{246}, \quad f_4 = F_{356} \\
&h_1 = -\frac{1}{3}\sum_{a\neq 1,2,3} K^a{}_a + \frac{2}{3}(K^1{}_1 + K^2{}_2 + K^3{}_3), \\
&h_2 = -\frac{1}{3}\sum_{a\neq 1,4,5} K^a{}_a + \frac{2}{3}(K^1{}_1 + K^4{}_4 + K^5{}_5), \\
&h_3 = -\frac{1}{3}\sum_{a\neq 2,4,6} K^a{}_a + \frac{2}{3}(K^2{}_2 + K^4{}_4 + K^6{}_6), \\
&h_4 = -\frac{1}{3}\sum_{a\neq 3,5,6} K^a{}_a + \frac{2}{3}(K^3{}_3 + K^5{}_5 + K^6{}_6).
\end{aligned} \tag{10.54}$$

The coset element for this configuration then becomes

$$\mathcal{V}(t) = e^{\phi_1(t)h_1+\phi_2(t)h_2+\phi_3(t)h_3+\phi_4(t)h_4+\tilde{\phi}(t)l}\, e^{\mathcal{A}_{123}(t)E^{123}+\mathcal{A}_{145}(t)E^{145}+\mathcal{A}_{246}(t)E^{246}+\mathcal{A}_{356}(t)E^{356}}. \quad (10.55)$$

We must further choose the timelike Cartan generator, $l \in \mathfrak{h}_{E_{10}}$, appropriately. Examination of Equation (10.54) reveals that the four electric fields are supported only in the spatial directions $x^1, \cdots, x^6$ so, again, in order to ensure isotropy in the directions transverse to the S-branes, we choose the timelike Cartan generator as follows:

$$l = K^7{}_7 + K^8{}_8 + K^9{}_9 + K^{10}{}_{10}, \quad (10.56)$$

which implies

$$l^2 = (l|l) = \left(K^7{}_7 + K^8{}_8 + K^9{}_9 + K^{10}{}_{10}|K^7{}_7 + K^8{}_8 + K^9{}_9 + K^{10}{}_{10}\right) = -12. \quad (10.57)$$

The Lagrangian for this system becomes

$$\mathcal{L}_{(6_2,4_3)} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \frac{l^2}{2}\partial\tilde{\phi}(t)\,\partial\tilde{\phi}(t), \quad (10.58)$$

where $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ and $\mathcal{L}_4$ represent the $SL(2,\mathbb{R}) \times SO(2)$-invariant Lagrangians corresponding to the four A_1-algebras. The solutions for $\phi_1(t), \cdots, \phi_4(t)$ and $\tilde{\phi}(t)$ are separately identical to the ones for $\phi(t)$ and $\tilde{\phi}(t)$, respectively, in Section 10.3.2. From the embedding into E_{10}, provided in Equation (10.54), we may read off the solution for the spacetime metric,

$$\begin{aligned} ds^2_{(6_2,4_3)} = &-(H_1H_2H_3H_4)^{1/3}e^{\frac{2}{3}\sqrt{E_-}t}\,dt^2 + (H_1H_4)^{-2/3}(H_2H_3)^{1/3}(dx^1)^2 \\ &+(H_1H_3)^{-2/3}(H_2H_4)^{1/3}(dx^2)^2 + (H_1H_2)^{-2/3}(H_3H_4)^{1/3}(dx^3)^2 \\ &+(H_3H_4)^{-2/3}(H_1H_2)^{1/3}(dx^4)^2 + (H_2H_4)^{-2/3}(H_1H_3)^{1/3}(dx^5)^2 \\ &+(H_2H_3)^{-2/3}(H_1H_4)^{1/3}(dx^6)^2 + (H_1H_2H_3H_4)^{1/3}e^{\frac{1}{6}\sqrt{E_-}t}\sum_{\bar{a}=7}^{10}(dx^{\bar{a}})^2. \end{aligned} \quad (10.59)$$

As announced, this describes four intersecting $SM2$-branes with a $1+4$-dimensional transverse spacetime. For example the brane that couples to the field associated with the first Cartan generator is extended in the directions x^1, x^2, x^3. By restricting to the case $\phi_1 = \phi_2 = \phi_3 = \phi_4 \equiv \phi$ the metric simplifies to

$$\begin{aligned} ds^2_{(6_2,4_3)} = &-\left(\frac{2a}{\sqrt{E}}\right)^{4/3}\cosh^{4/3}\sqrt{E}t\, e^{\frac{2}{3}\sqrt{\tilde{E}}t}dt^2 + \left(\frac{2a}{\sqrt{E}}\right)^{-2/3}\cosh^{-2/3}\sqrt{E}t\sum_{a'=1}^{6}(dx^{a'})^2 \\ &+\left(\frac{2a}{\sqrt{E}}\right)^{4/3}\cosh^{4/3}\sqrt{E}t\, e^{\frac{1}{6}\sqrt{\tilde{E}}t}\sum_{\bar{a}=7}^{10}(dx^{\bar{a}})^2, \end{aligned} \quad (10.60)$$

which coincides with the cosmological solution found in [61] for the configuration $(6_2, 4_3)$. We can therefore conclude that the algebraic interpretation of the geometric configurations found in this paper generalizes the solutions given in the aforementioned reference.

In a more general setting where we excite more roots of E_{10}, the solutions of course become more complex. However, as long as we consider *commuting* subalgebras there will naturally be no coupling in the Lagrangian between fields parametrizing different subalgebras. This implies that if we excite a direct sum of m A_1-algebras the total Lagrangian will split according to

$$\mathcal{L} = \sum_{k=1}^{m}\mathcal{L}_k + \tilde{\mathcal{L}}, \quad (10.61)$$

where $\mathcal{L}_k$ is of the same form as Equation (10.19), and $\tilde{\mathcal{L}}$ is the Lagrangian for the timelike Cartan element, needed in order to satisfy the Hamiltonian constraint. It follows that the associated solutions are

$$\begin{aligned} \phi_k(t) &= \frac{1}{2}\ln\left[\frac{a_k}{E_k}\cosh\sqrt{E_k}t\right] \qquad (k=1,\cdots,m), \\ \tilde{\phi}(t) &= |l^2|\sqrt{\tilde{E}}t. \end{aligned} \tag{10.62}$$

Furthermore, the resulting structure of the metric depends on the embedding of the A_1-algebras into E_{10}, i.e., which level 1-generators we choose to realize the step-operators and hence which Cartan elements that are associated to the ϕ_k's. Each excited A_1-subalgebra will turn on an electric 3-form that couples to an $SM2$-brane and hence the solution for the metric will describe a set of m intersecting $SM2$-branes.

As an additional nice example, we mention here the configuration $(7_3, 7_3)$, also known as the *Fano plane*, which consists of 7 lines and 7 points (see Figure 52). This configuration is well known for its relation to the octonionic multiplication table [8]. For our purposes, it is interesting because none of the lines in the configuration are parallell. Thus, the algebra dual to the Fano plane is a direct sum of seven A_1-algebras and the supergravity solution derived from the sigma model describes a set of seven intersecting $SM2$-branes.

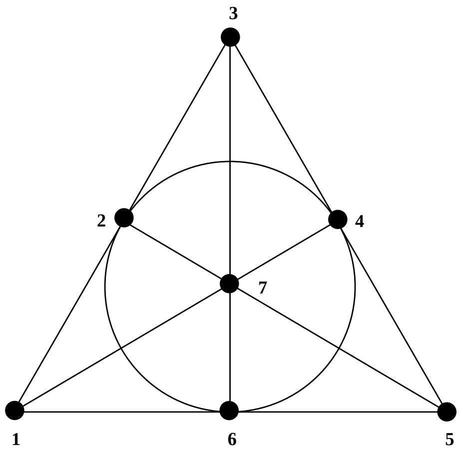


Figure 52: The Fano Plane, $(7_3, 7_3)$, dual to the Lie algebra $A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

10.3.4 Intersection rules for spacelike branes

For multiple brane solutions, there are rules for how these branes may intersect in order to describe allowed BPS-solutions [6]. These intersection rules also apply to spacelike branes [144] and hence they apply to the solutions considered here. In this section we will show that the intersection rules for multiple S-brane solutions are encoded in the associated geometric configurations [96].

For two spacelike q-branes, A and B, in M-theory the rules are

$$SMq_A \cap SMq_B = \frac{(q_A+1)(q_B+1)}{9} - 1. \tag{10.63}$$

So, for example, if we have two $SM2$-branes the result is

$$SM2 \cap SM2 = 0, \tag{10.64}$$

which means that they are allowed to intersect on a 0-brane. Note that since we are dealing with spacelike branes, a zero-brane is extended in one spatial direction, so the two $SM2$-branes may therefore intersect in one spatial direction only. We see from Equation (10.59) that these rules are indeed fulfilled for the configuration $(6_2, 4_3)$.

In [72] it was found in the context of $\mathfrak{g}^{+++}$-algebras that the intersection rules for extremal branes are encoded in orthogonality conditions between the various roots from which the branes arise. This is equivalent to saying that the subalgebras that we excite are commuting, and hence the same result applies to $\mathfrak{g}^{++}$-algebras in the cosmological context[39]. From this point of view, the intersection rules can also be read off from the geometric configurations in the sense that the configurations encode information about whether or not the algebras commute.

The next case of interest is the Fano plane, $(7_3, 7_3)$. As mentioned above, this configuration corresponds to the direct sum of 7 commuting A_1 algebras and so the gravitational solution describes a set of 7 intersecting $SM2$-branes. The intersection rules are guaranteed to be satisfied for the same reason as before.

10.4 Cosmological solutions with magnetic flux

We will now briefly sketch how one can also obtain the $SM5$-brane solutions from geometric configurations and regular subalgebras of E_{10}. In order to do this we consider "magnetic" subalgebras of E_{10}, constructed only from simple root generators at level two in the level decomposition of E_{10}. To the best of our knowldege, there is no theory of geometric configurations developed for the case of having 6 points on each line, which would be needed here. However, we may nevertheless continue to investigate the simplest example of such a configuration, namely $(6_1, 1_6)$, displayed in Figure 53.

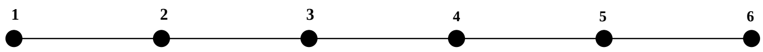


Figure 53: The simplest "magnetic configuration" $(6_1, 1_6)$, dual to the algebra A_1.The associated supergravity solution describes an $SM5$-brane, whose world volume is extended in the directions $x^1, \cdots, x^6$.

The algebra dual to this configuration is an A_1-subalgebra of E_{10} with the following generators:

$$\begin{aligned} e &= E^{123456} = F_{123456}, \\ h &\equiv [E^{123456}, F_{123456}] = -\frac{1}{6}\sum_{a\neq 1,\cdots,6} K^a{}_a + \frac{1}{3}(K^1{}_1 + \cdots + K^6{}_6). \end{aligned} \tag{10.65}$$

Although the embedding of this algebra is different from the electric cases considered previously, the sigma model solution is still associated to an $SL(2,\mathbb{R})/SO(2)$ coset space and therefore the solutions for $\phi(t)$ and $\tilde{\phi}(t)$ are the same as before. Because of the embedding, however, the sigma model translates to a different type of supergravity solution, namely a spacelike five-brane whose world volume is extended in the directions $x^1, \cdots, x^6$. The metric is given by

$$ds^2 = -H^{-4/3}(t)\, e^{\frac{2}{3}\sqrt{E_-}t}\, dt^2 + H^{-1/3}(t)\sum_{a'=1}^{6}(dx^{a'})^2 + H^{1/6}(t)e^{\frac{1}{6}\sqrt{E_-}t}\sum_{\bar{a}=7}^{10}(dx^{\bar{a}})^2. \tag{10.66}$$

This solution coincides with the $SM5$-brane found by Strominger and Gutperle in [90][40]. Note that the correct power of $H(t)$ for the five-brane arises here entirely due to the embedding of h into E_{10} through Equation (10.65).

[39] This was also pointed out in [127].

[40] In [90] they were dealing with a hyperbolic internal space so there was an additional sinh-function in the transverse spacetime.

Because of the existence of electric-magnetic duality on the supergravity side, it is suggestive to expect the existence of a duality between the two types of configurations (n_m, g_3) and (n_m, g_6), of which we have here seen the simplest realisation for the configurations $(3_1, 1_3)$ and $(6_1, 1_6)$.

10.5 The Petersen algebra and the Desargues configuration

We want to end this section by considering an example which is more complicated, but very interesting from the algebraic point of view. There exist ten geometric configurations of the form $(10_3, 10_3)$, i.e., with exactly ten points and ten lines. In [61], these were associated to supergravity solutions with ten components of the electric field turned on. This result was re-analyzed by some of the present authors in [96] where it was found that many of these configurations have a dual description in terms of Dynkin diagrams of rank 10 Lorentzian Kac–Moody subalgebras of E_{10}. One would therefore expect that solutions of the sigma models for these algebras should correspond to new solutions of eleven-dimensional supergravity. However, since these algebras are infinite-dimensional, the corresponding sigma models are difficult to solve without further truncation. Nevertheless, one may argue that explicit solutions should exist, since the algebras in question are all non-hyperbolic, so we know that the supergravity dynamics is non-chaotic.

We shall here consider one of the $(10_3, 10_3)$-configurations in some detail, referring the reader to [96] for a discussion of the other cases. The configuration we will treat is the well known *Desargues configuration*, displayed in Figure 54. The Desargues configuration is associated with the 17th century French mathematician *Gérard Desargues* to illustrate the following "Desargues theorem" (adapted from [145]):

> *Let the three lines defined by* $\{4, 1\}, \{5, 2\}$ *and* $\{6, 3\}$ *be concurrent, i.e., be intersecting at one point, say* $\{7\}$*. Then the three intersection points* $8 \equiv \{1, 2\} \cap \{4, 5\}, 9 \equiv \{2, 3\} \cap \{5, 6\}$ *and* $10 \equiv \{1, 3\} \cap \{4, 6\}$ *are colinear.*

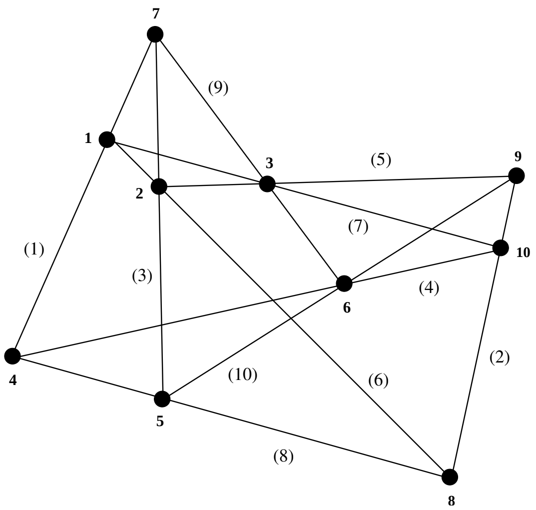


Figure 54: $(10_3, 10_3)_3$: The Desargues configuration, dual to the Petersen graph.

Another way to say this is that the two triangles $\{1, 2, 3\}$ and $\{4, 5, 6\}$ in Figure 54 are in perspective from the point $\{7\}$ and in perspective from the line $\{8, 10, 9\}$.

As we will see, a new fascinating feature emerges for this case, namely that the Dynkin diagram dual to this configuration *also* corresponds in itself to a geometric configuration. In fact, the Dynkin diagram dual to the Desargues configuration turns out to be the famous *Petersen graph*, denoted $(10_3, 15_2)$, which is displayed in Figure 55.

To construct the Dynkin diagram we first observe that each line in the configuration is disconnected from three other lines, e.g., $\{4,1,7\}$ have no nodes in common with the lines $\{2,3,9\}$, $\{5,6,9\}$, $\{8,10,9\}$. This implies that all nodes in the Dynkin diagram will be connected to three other nodes. Proceeding as in Section 10.2.2 leads to the Dynkin diagram in Figure 55, which we identify as the Petersen graph. The corresponding Cartan matrix is

$$A(\mathfrak{g}_{\text{Petersen}}) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}, \tag{10.67}$$

which is of Lorentzian signature with

$$\det A(\mathfrak{g}_{\text{Petersen}}) = -256. \tag{10.68}$$

The Petersen graph was invented by the Danish mathematician *Julius Petersen* in the end of the 19th century. It has several embeddings on the plane, but perhaps the most famous one is as a star inside a pentagon as depicted in Figure 55. One of its distinguishing features from the point of view of graph theory is that it contains a *Hamiltonian path* but no *Hamiltonian cycle*[41].

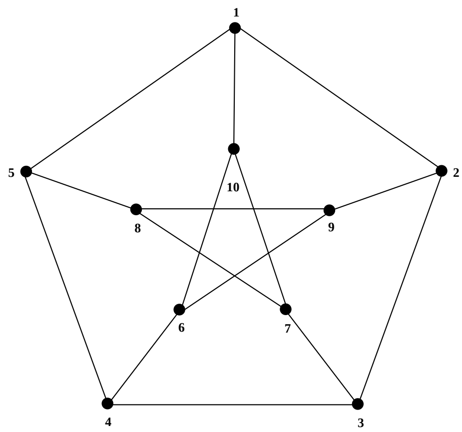


Figure 55: This is the so-called Petersen graph. It is the Dynkin diagram dual to the Desargues configuration, and is in fact a geometric configuration itself, denoted $(10_3, 15_2)$.

Because the algebra is Lorentzian (with a metric that coincides with the metric induced from the embedding in E_{10}), it does not need to be enlarged by any further generator to be compatible with the Hamiltonian constraint.

It is interesting to examine the symmetries of the various embeddings of the Petersen graph

[41] We recall that a Hamiltonian path is defined as a path in an undirected graph which intersects each node once and only once. A Hamiltonian cycle is then a Hamiltonian path which also returns to its initial node.

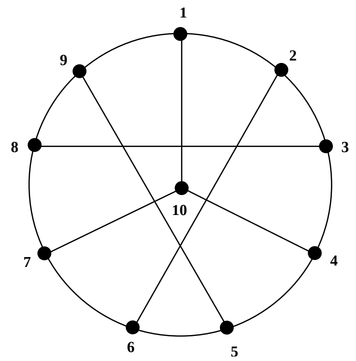


Figure 56: An alternative drawing of the Petersen graph in the plane. This embedding reveals an S_3 permutation symmetry about the central point.

in the plane and the connection to the Desargues configurations. The embedding in Figure 55 clearly exhibits a $\mathbb{Z}_5 \times \mathbb{Z}_2$-symmetry, while the Desargues configuration in Figure 54 has only a $\mathbb{Z}_2$-symmetry. Moreover, the embedding of the Petersen graph shown in Figure 56 reveals yet another symmetry, namely an S_3 permutation symmetry about the central point, labeled "10". In fact, the external automorphism group of the Petersen graph is S_5, so what we see in the various embeddings are simply subgroups of S_5 made manifest. It is not clear how these symmetries are realized in the Desargues configuration that seems to exhibit much less symmetry.

10.6 Further comments

- The analysis of the present section exhibits subgroups of the Coxeter group E_{10}, with the property that their Coxeter exponents m_{ij} (see Section 3) are either 2 or 3, but never infinity[42]. Furthermore, the associated Coxeter graphs all have incidence index $\mathfrak{I} = 3$, meaning that each node in the graph is connected to three and only three other nodes. A classification of all rank 10 and 11 Coxeter groups with these properties has been given in [97].
- Integrability of sigma models for "cosmological billiards" in relation to dimensional reduction to three dimensions has been extensively investigated in [81, 82, 79, 80, 83].

[42]When no Coxeter exponent m_{ij} is equal to infinity, the Coxeter group is called 2-*spherical*. 2-spherical Coxeter subgroups of E_{10} are rare [27].

11 Conclusions

In this review, we have investigated the remarkable structures that emerge when studying gravitational theories in the BKL-limit, i.e., close to a spacelike singularity. Although it has been known for a long time that in this limit the dynamics can be described in terms of billiard motion in hyperbolic space, it is only recently that the connection between the billiards and Coxeter groups have been uncovered. Furthermore, the relevant Coxeter groups turn out to be the Weyl groups of the Lorentzian Kac–Moody algebra obtained by double extension (sometimes twisted) of the U-duality algebra appearing upon dimensional reduction to three dimensions.

These results, which in our opinion are solid and here to stay, necessitate some mathematical background which is not part of the average physicist's working knowledge. For this reason, we have also devoted a few sections to the development of the necessary mathematical concepts.

We have then embarked on the exploration of more speculative territory. A natural question that arises is whether or not the emergence of Weyl groups of Kac–Moody algebras in the BKL-limit has a profound meaning independently of the BKL-limit (which would serve only as a "revelator") and could indicate that the gravitational theories under investigation – possibly supplemented by additional degrees of freedom – possess these infinite Kac–Moody algebras as "hidden symmetries" (in any regime). The existence of these infinite-dimensional symmetries was also advocated in the pioneering work [113] and more recently [167, 156, 157, 102, 103, 74, 104, 158, 168] from a somewhat different point of view. It is also argued in those references that even bigger symmetries (E_{11} that contains E_{10}, or Borcherds subalgebras) might actually be relevant. In order to make the conjectured E_{10}-symmetry manifest (which is perhaps itself part of a bigger symmetry), we have investigated a nonlinear sigma model for the coset space $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$ using the level decomposition techniques introduced in [47]. Although very suggestive and partially successful, this approach exhibits limitations which, in spite of many efforts, have not yet been overcome. It is likely that new ideas are needed, or that the implementation of the symmetry must be made in a more subtle fashion, where duality will perhaps play a more central role.

Independently of the way they are actually implemented, it appears that infinite-dimensional Kac–Moody algebras (e.g, E_{10} or, perhaps, E_{11}) do encode important features of gravitational theories, and the idea that they constitute essential elements of the final formulation will surely play an important role in future developments.

12 Acknowledgements

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A Proof of Some Important Properties of the Bilinear Form

We demonstrate in this appendix the properties of the bilinear form B associated with the geometric realisation of a Coxeter group $\mathfrak{C}$. Recall that the matrix

$$(B_{ij}) = \left(-\cos\left(\frac{\pi}{m_{ij}}\right)\right)$$

has only 1's on the diagonal and non-positive numbers off the diagonal. Recall also that a vector v is said to be positive if and only if *all* its components v_i are strictly positive, $v_i > 0$; this is denoted $v > 0$. Similarly, a vector v is non-negative, $v \geq 0$, if and only if *all* its components v_i are non-negative, $v_i \geq 0$. Finally, a vector is non-zero if and only if at least one of its component is non-zero, which is denoted $v \neq 0$. Our analysis is based on reference [116]. We shall assume throughout that B is indecomposable.

Main theorem:

1. The Coxeter group $\mathfrak{C}$ is of finite type if and only if there exists a positive vector $v_i > 0$ such that $\sum_j B_{ij} v_j > 0$.
2. The Coxeter group $\mathfrak{C}$ is of affine type if and only if there exists a positive vector $v_i > 0$ such that $\sum_j B_{ij} v_j = 0$.
3. The Coxeter group $\mathfrak{C}$ is of indefinite type if and only if there exists a positive vector $v_i > 0$ such that $\sum_j B_{ij} v_j < 0$.

These cases are mutually exclusive and exhaust all possibilities.

Proof: The proof follows from a series of lemmata. The inequalities $v \geq 0$ define a convex cone, namely the first quadrant Q. Similarly, the inequalities $Bv \geq 0$ define also a convex cone K_B. One has indeed:

$$u, v \in K_B \quad \Rightarrow \quad \lambda u + (1-\lambda) v \in K_B \qquad \forall \lambda \in [0,1].$$

Note that one has also

$$v \in K_B \quad \Rightarrow \quad \lambda v \in K_B \qquad \forall \lambda \geq 0$$

and $\ker B = \{v | Bv = 0\} \subset K_B$. There are three distinct cases for the intersection $K_B \cap Q$:

1. Case 1: $K_B \cap Q \neq \{0\}$, $K_B \subset Q$.
2. Case 2: $K_B \cap Q \neq \{0\}$, $K_B \not\subset Q$.
3. Case 3: $K_B \cap Q = \{0\}$.

These three distinct cases correspond, as we shall now show, to the three distinct cases of the theorem. To investigate these distinct cases, we need the following lemmata:

Lemma 1: The conditions $Bv \geq 0$ and $v \geq 0$ imply either $v > 0$ or $v = 0$. In other words

$$K_B \cap Q \equiv \{v | Bv \geq 0\} \cap \{v | v \geq 0\} \subset \{v | v > 0\} \cup \{v = 0\}.$$

Proof: Assume that $v \geq 0$ fulfills $Bv \geq 0$ and has at least one component equal to zero. We shall show that all its components are then zero. Assume $v_i = 0$ for $i = 1, \cdots, s$ and $v_i > 0$ for $i > s$. One has $1 \leq s \leq n$ (with no non-vanishing component v_i if $s = n$). From $Bv \geq 0$ one gets $(Bv)_i = \sum_{j=1}^n B_{ij} v_j = \sum_{j=s+1}^n B_{ij} v_j \geq 0$. Take $i \leq s$. As $j > s$ in the previous sum, one has

$B_{ij} < 0$ and thus $\sum_{j=s+1}^{n} B_{ij}v_j \leq 0$, which implies $\sum_{j=s+1}^{n} B_{ij}v_j = 0$. As $v_j > 0$ $(j > s)$, this leads to $B_{ij} = 0$ for $i \leq s$ and $j > s$. The matrix B would be decomposable, unless $s = n$, i.e. when all components v_i vanish.

Lemma 2: Consider the system of linear homogeneous inequalities

$$\lambda_\alpha \equiv \sum_i a_{\alpha i} v_i > 0$$

on the vector v. This system possesses a solution if and only if there is no set of numbers $\mu_\alpha \geq 0$ that are not all zero such that $\sum_\alpha \mu_\alpha a_{\alpha i} = 0$.

Proof: This is a classical result in the theory of linear inequalities (see [116], page 47).

We can now study more thoroughly the three cases listed above.

Case 1: $K_B \cap Q \neq \{0\}$, $K_B \subset Q$

In that case, one has

$$Bv \geq 0 \quad \Rightarrow \quad v > 0 \text{ or } v = 0$$

by Lemma 1. Furthermore, K_B cannot contain a nontrivial subspace W since $w \in W$ implies $-w \in W$, but only one of the two can be in Q when $w \neq 0$. Hence $\ker B = 0$, i.e., $\det B \neq 0$ and

$$Bv = 0 \quad \Rightarrow \quad v = 0.$$

This excludes in particular the existence of a vector $u > 0$ such that $Bu < 0$ or $Bu = 0$.

Finally, the interior of K_B is non-empty since B is nondegenerate. Taking a non-zero vector v such that $Bv > 0$, one concludes that there exists a vector $v > 0$ such that $Bv > 0$. This shows that Case 1 corresponds to the first case in the theorem. We shall verify below that B_{ij} is indeed positive definite.

Case 2: $K_B \cap Q \neq \{0\}$, $K_B \not\subset Q$

K_B reduces in that case to a straight line. Indeed, let $v \neq 0$ be an element of $K_B \cap Q$ and let $w \neq 0$ be in K_B but not in Q. Let ℓ be the straight line joining w and v. Consider the line segment from w to v. This line segment is contained in K_B and crosses the boundary ∂Q of Q at some point r. But by Lemma 1, this point r must be the origin. Thus, $w = \mu v$, for some real number $\mu < 0$. This implies that the entire line ℓ is in K_B since $v \in K_B \Rightarrow \lambda v \in K_B$ for all $\lambda > 0$, and also for all $\lambda < 0$ since $w \in K_B$.

Let q be any other point in K_B. If $q \notin Q$, the segment joining q to v intersects ∂Q and this can only be at the origin by Lemma 1. Hence $q \in \ell$. If $q \in Q$, the segment joining q to w intersects ∂Q and this can only be at the origin by Lemma 1. Hence, we find again that $q \in \ell$. This shows that K_B reduces to the straight line ℓ.

Since $v \in K_B \Rightarrow -v \in K_B$, one has $Bv = 0 \quad \forall v \in K_B$. Hence, $Bv \geq 0 \Rightarrow Bv = 0$, which excludes the existence of a vector $v > 0$ such that $Bv < 0$ (one would have $B(-v) > 0$ and hence $Bv = 0$). Furthermore, there exists $v > 0$ such that $Bv = 0$. This shows that Case 2 corresponds to the second case in the theorem. We shall verify below that B_{ij} is indeed positive semi-definite. Note that $\det B = 0$ and that the corank of B is one.

Case 3: $K_B \cap Q = \{0\}$

In that case, there is a vector $v > 0$ such that $Bv < 0$, which corresponds to the third case in the theorem. Indeed, consider the system of homogeneous linear inequalities

$$-\sum_j B_{uj} v_j > 0, \qquad v_j > 0.$$

By Lemma 2, this system possesses a solution if and only if there is no non-trivial $\mu_\alpha \equiv (\mu_i, \bar{\mu}_i) \geq 0$ such that $\sum_i \mu_i(-B_{ij}) + \bar{\mu}_j = 0$.

Consider thus the equations $\sum_i \mu_i(-B_{ij}) + \bar{\mu}_j = 0$ for $\mu_\alpha \geq 0$, or, as B_{ij} is symmetric, $\sum_j B_{ij}\mu_j = \bar{\mu}_i$. Since $\bar{\mu}_i \geq 0$, these conditions are equivalent to $\sum_j B_{ij}\mu_j \geq 0$ (if $\sum_j B_{ij}\mu_j \geq 0$, one defines $\bar{\mu}_i$ through $\sum_j B_{ij}\mu_j = \bar{\mu}_i$), i.e., $\mu \in K_B$. But $\mu_i \geq 0$, i.e., $\mu \in Q$, which implies $\mu_i = 0$ and hence also $\bar{\mu}_i = 0$. The μ_α all vanish and the general solution $\mu \geq 0$ to the equations $\sum_i \mu_i(-B_{ij}) + \bar{\mu}_j = 0$ is accordingly trivial.

To conclude the proof of the main theorem, we prove the following proposition:

Proposition: The Coxeter group $\mathfrak{C}$ belongs to Case 1 if and only if B is positive definite; it belongs to Case 2 if and only if B is positive semi-definite with $\det B = 0$.

Proof: If B is positive semi-definite, then it belongs to Case 1 or Case 2 since otherwise there would be a vector $w > 0$ such that $Bw < 0$ and thus $B_{ij}w_iw_j < 0$, leading to a contradiction. In the finite case, B is positive definite and hence, $\det B \neq 0$: This corresponds to Case 1. In the affine case, there are zero eigenvectors and $\det B = 0$: This corresponds to Case 2.

Conversely, assume that the Coxeter group $\mathfrak{C}$ belongs to Case 1 or Case 2. Then there exists a vector w such that $Bw \geq 0$. This yields $(B - \lambda I)w > 0$ for $\lambda < 0$ and therefore $B - \lambda I$ belongs to Case 1 $\forall \lambda < 0$. In particular, $\det(B - \lambda I) \neq 0 \quad \forall \lambda < 0$, which shows that the eigenvalues of B are all non-negative: B is positive semi-definite. We have seen furthermore that it has the eigenvalue zero only in Case 2.

This completes the proof of the main theorem.

B Existence and "Uniqueness" of the Aligned Compact Real Form

We prove in this appendix the crucial result that for any real form of a complex semi-simple Lie algebra, one can always find a compact real form aligned with it [93, 129].

Let $\mathfrak{g}_0$ be a specific real form of the semi-simple, complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Let $\mathfrak{c}_0$ be a compact real form of $\mathfrak{g}^{\mathbb{C}}$. We may introduce on $\mathfrak{g}^{\mathbb{C}}$ two conjugations. A first one (denoted by σ) with respect to $\mathfrak{g}_0$ and another one (denoted by τ) with respect to the compact real form $\mathfrak{c}_0$. The product of these two conjugations constitutes an automorphism $\lambda = \sigma\tau$ of $\mathfrak{g}^{\mathbb{C}}$. For any automorphism φ we have the identity

$$\mathrm{ad}(\varphi Z) = \varphi \;\mathrm{ad}\, Z\, \varphi^{-1}, \tag{B.1}$$

and, as a consequence, the invariance of the Killing form with respect to the automorphisms of the Lie algebra:

$$B(\varphi Z,\, \varphi Z') = \mathrm{Tr}(\mathrm{ad}\,(\varphi\, Z)\,\mathrm{ad}\,(\varphi\, Z')) = \mathrm{Tr}(\varphi \;\mathrm{ad}\, Z\, \varphi^{-1}\, \varphi \;\mathrm{ad}\, Z'\, \varphi^{-1}) = B(Z,\, Z'). \tag{B.2}$$

The automorphism $\lambda = \sigma\tau$ is symmetric with respect to the Hermitian product B^τ defined by $B^\tau(X, Y) = -B(X, \tau(Y))$. Indeed $(\sigma\tau)^{-1}\tau = \tau(\sigma\tau)$ implies that $B^\tau(\sigma\tau[Z],\, Z') = B^\tau(Z,\, \sigma\tau[Z'])$. Thus its square $\rho = (\sigma\tau)^2$ is positive definite. It can be proved that ρ^t $(t \in \mathbb{R})$ is a one-parameter group of internal automorphisms of $\mathfrak{g}_0$ such that[43] $\rho^t\, \tau = \tau\, \rho^{-t}$. It follows that

$$\rho^{\frac{1}{4}}\tau\rho^{-\frac{1}{4}}\sigma = \rho^{\frac{1}{2}}\tau\,\sigma = \rho^{-\frac{1}{2}}\rho\,\tau\,\sigma = \rho^{-\frac{1}{2}}\sigma\,\tau = \sigma\,\tau\,\rho^{-\frac{1}{2}} = \sigma\,\rho^{\frac{1}{4}}\tau\rho^{-\frac{1}{4}}. \tag{B.3}$$

In other words, the conjugation σ always commutes with the conjugation $\tilde{\tau} = \rho^{\frac{1}{4}}\tau\rho^{-\frac{1}{4}}$, which is the conjugation with respect to the compact real algebra $\rho^{\frac{1}{4}}[\mathfrak{c}_0]$. This shows that the compact real form $\rho^{\frac{1}{4}}[\mathfrak{c}_0]$ is aligned with the given real form $\mathfrak{g}_0$.

Note also that if there are two Cartan involutions, θ and θ', defined on a real semi-simple Lie algebra, they are conjugated by an internal automorphism. Indeed, as we just mentioned, then an automorphism $\phi = ((\theta\theta')^2)^{\frac{1}{4}}$ exists, such that θ and $\psi = \phi\theta'\phi^{-1}$ commute. If $\psi \neq \theta$, the eigensubspaces of eigenvalues $+1$ and -1 of these two involutions are disitnct but, because they commute, a vector X exists, such that $\theta[X] = X$ and $\psi[X] = -X$. For this vector we obtain

$$\begin{aligned} 0 < B^\theta(X, X) &= -B(X, \theta[X]) = -B(X, X), \\ 0 < B^\psi(X, X) &= -B(X, \psi[X]) = +B(X, X), \end{aligned} \tag{B.4}$$

which constitutes a contradiction, and thus implies $\theta = \psi$. An important consequence of this is that any real semi-simple Lie algebra possesses a "unique" Cartan involution[44]. In the same way, if $\mathfrak{g}$ is a complex semi-simple Lie algebra, the only Cartan involutions of $\mathfrak{g}^{\mathbb{R}}$ are obtained from the conjugation with respect to a compact real form of $\mathfrak{g}$; all compact real forms being conjugated to each other by internal automorphisms.

[43] To convince oneself of the validity of this commutation relation, it suffices to check it in a basis where the (finite-dimensional) matrix ρ is diagonal, using the symmetry of the matrix $\sigma\tau$.

[44] The uniqueness derives from the fact that the internal automorphism groups of $\mathfrak{g}^{\mathbb{R}}$ and $\mathfrak{g}$ are identical.

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