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Spacelike Surfaces with Harmonic Inverse Mean Curvature

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Abstract. We study fundamental properties of spacelike surfaces with harmonic inverse mean curvature in Lorentzian space forms. Further we classify spacelike Bonnet surfaces with constant curvature in Lorentzian space forms.

Introduction

Surfaces with nonzero constant mean curvature (CMC surfaces) in Riemannian and Lorentzian 3-space forms have been studied extensively. (See [1]-[3], [5]-[8], [13], [18], [28], [30]-[31] and references therein.)

As is well known, since the Gauss-Codazzi equations of an umbilic free CMC surface can be transformed to Sinh-Laplace equation, we can apply methods in the theory of integrable systems to the study of CMC surfaces. The starting point of such methods is a *zero curvature representation* (ZCR), more precisely, a representation of the Gauss-Codazzi equations in a form of compatibility condition for the Lax equations with *spectral parameter*.

From the view point of integrability theory, it is natural to consider the following problem:

What kinds of surfaces can be considered as natural generalisations of CMC surfaces?

In 1994, Bobenko [8] has introduced the notion of surfaces with harmonic inverse mean curvature in Euclidean 3-space $\mathbf{E}^3$ (HIMC surfaces). A

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surface in $\mathbf{E}^3$ is said to be a *surface with harmonic inverse mean curvature* if the reciprocal of the mean curvature is a harmonic function. He showed that HIMC surfaces are natural generalisation of CMC surfaces in terms of integrability theory. In fact, he showed HIMC surfaces admit ZCR with *variable* spectral parameter. The first named author [19] generalised the notion of HIMC surface to surfaces in Riemannian 3-space forms. He obtained immersion formulae for HIMC surfaces. Furthermore he established Lawson type correspondences between HIMC surfaces in Riemannian 3-space forms.

Fokas and Gelfand [17] found another characterisation of HIMC surfaces. They showed that the class of HIMC surfaces (including CMC surfaces) is the only class of surfaces which admit a Lie-point group of symmetries.

Further, Bobenko, Eitner and Kitaev [12] have developed detailed study on HIMC surfaces. In the general class of HIMC surfaces they distinguished a subclass of θ-isothermic HIMC surfaces. They showed that Gauss equation of θ-isothermic HIMC surface reduce to the ordinary differential equation:

$$(*) \qquad \left(\frac{q''(t)}{q'(t)}\right)' - q'(t) = \mathcal{S}(t)\left(2 - \frac{q^2(t)+c}{q'(t)}\right), \quad q'(t) < 0, \quad c > 0.$$

This ordinary differential equation is called the *generalised Hazzidakis equation.* Bobenko, Eitner and Kitaev solved the generalised Hazzidakis equation $(*)$ in terms of Painlevé transcendents P_V and P_{VI}.

The authors are interested in the study of following generalisation of $(*)$:

$$(\bigstar_c^{\pm}) \qquad \left(\frac{q''(t)}{q'(t)}\right)' - q'(t) = \mathcal{S}(t)\left(2 - \frac{q^2(t)+c}{q'(t)}\right), \quad \pm q'(t) > 0, \quad c \in \mathbf{R}.$$

We shall also call this equation the generalised Hazzidakis equation. In this paper we shall show that if $c = \theta^2 > 0$ then a solution $q(t)$ to $(\bigstar_c^+)$ describes a spacelike surface with harmonic inverse mean curvature in Minkowski 3-space $\mathbf{E}^3_1$. Furthermore a solution $q(t)$ describes a spacelike Bonnet surface in pseudo hyperbolic space of constant curvature $-1/\theta^2$ if $\theta \neq 0$ and spacelike Bonnet surface in 3-space $\mathbf{E}^3_1$ if $\theta = 0$.

We would like to remark that solutions $q(t)$ to $(\bigstar_c^-)$ with $c = -\theta^2$ describes timelike HIMC surfaces in $\mathbf{E}^3_1$, timelike Bonnet surfaces in $\mathbf{E}^3_1$ if

$\theta = 0$ and Bonnet surfaces in hyperbolic 3-space if $\theta \neq 0$. For more details we refer to [22] and [23].

On the other hand it is also interesting to generalise CMC surfaces in terms of *variational problem*. So called *H-surface equations* are examples of such generalisations. The first named author introduced gauge-theoretic equations whose solutions are considered as a generalisation of surfaces with prescribed mean curvature in Riemannian 3-space forms [18]. Further we obtained a characterisation of HIMC surfaces in terms of a reduction condition for H-surfaces equations [21].

In this paper we shall introduce the notion of spacelike surfaces with harmonic inverse mean curvature (SHIMC surfaces) in Lorentzian 3-space forms and study their fundamental properties. Furthermore we shall apply our results on SHIMC surfaces to the classification of spacelike Bonnet surfaces with constant curvature.

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1. Models for Lorentzian Space Forms

First of all, we shall describe complete connected Lorentzian manifolds of constant curvature explicitly for later use.

Let $\mathbf{E}^n_\nu$ be a *semi-Euclidean n-space* of index ν, that is, Cartesian n-space $\mathbf{R}^n(\xi_1, ..., \xi_n)$ with scalar product $\langle\cdot,\cdot\rangle$ of index ν. The scalar product $\langle\cdot,\cdot\rangle$ is expressed as $\langle\cdot,\cdot\rangle = -\sum_{i=1}^{\nu} d\xi_i^2 + \sum_{i=\nu+1}^{n} d\xi_i^2$ in terms of natural coordinates.

A semi-Euclidean n-space $\mathbf{E}^n_\nu$ is a complete connected and simply connected flat semi- Riemannian manifold. For $n \geq 2$, $\mathbf{E}^n_1$ is called a *Minkowski n-space*. Semi-Euclidean space $\mathbf{E}^n_\nu$ contains two kinds of central hyperquadrics:

$$S^n_\nu(r) = \{\xi \in \mathbf{E}^n_\nu \mid \langle\xi,\xi\rangle = r^2\},$$

$$H^n_{\nu-1}(r) = \{\xi \in \mathbf{E}^n_\nu \mid \langle\xi,\xi\rangle = -r^2\}.$$

The hyperquadrics $S^n_\nu(r)$ and $H^n_{\nu-1}(r)$ are called *pseudosphere* of radius r and *pseudohyperbolic space* of radius r respectively. The pseudosphere $S^n_\nu(r)$ is a semi-Riemannian manifold of index ν and of constant curvature $1/r^2$. The pseudohyperbolic space is a semi-Riemannian manifolds of index

$\nu - 1$ and of constant curvature $-1/r^2$. Note that these hyperquadrics are not necessarily simply connected. For $n \geq 2$, $S_1^n = S_1^n(1)$ is called a *de Sitter n-space* and $H_1^n = H_1^n(1)$ is called an *anti de Sitter n-space*.

In this paper we shall denote by $\mathfrak{M}_1^3(c)$ the following model spaces. ($c = 0$ or ± 1.)

$c = 0$: $\mathfrak{M}_1^3(0) = \mathbf{E}_1^3$, the Minkowski 3-space,

$c = 1$: $\mathfrak{M}_1^3(1) = S_1^3$, the de Sitter 3-space,

$c = -1$: $\mathfrak{M}_1^3(-1) = H_1^3$, the anti de Sitter 3-space.

We shall call Lorentzian 3-manifolds $\mathfrak{M}_1^3(c)$, *Lorentzian 3-space forms.*

Note that Minkowski 3-space and de Sitter 3-space are simply connected but anti de Sitter 3-space is not.

For more details on Semi-Riemannian geometry, we refer to O'Neill's textbook [29].

2. Split-quaternion Formalism

To study spacelike surfaces in Minkowski 3-space $\mathbf{E}_1^3$ by the methods of integrability theory, it is convenient to use 2 by 2 matrix-formalism. Our idea for this purpose is to identify the Minkowski 3–space with the imaginary part Im $\mathbf{H}'$ of the split-quaternion algebra $\mathbf{H}'$.

In this section we shall summarise the fundamental equations of spacelike surfaces for later use.

Let M be a connected 2-manifold and $F : M \to \mathbf{E}_1^3$ an immersion. The immersion F is said to be *spacelike* if the induced metric of M is positive definite. Hereafter we may assume that M is an orientable spacelike surface in $\mathbf{E}_1^3$ immersed by F.

The induced Riemannian metric I (the first fundamental form) of a spacelike surface M determines a conformal structure on M. We treat M as a Riemann surface with respect to this conformal structure and F as a conformal immersion. Let $z = x + \sqrt{-1}y$ be a local complex coordinate of M. The induced metric I of M can be written as

$$I = e^u dz d\bar{z} = e^u(dx^2 + dy^2). \tag{2.1}$$

Since the immersion F is conformal, partial derivatives of F satisfy the following formulae.

$$\langle F_z, F_z \rangle = \langle F_{\bar{z}}, F_{\bar{z}} \rangle = 0, \;\; \langle F_z, F_{\bar{z}} \rangle = \frac{1}{2} e^u. \tag{2.2}$$

Here we denote the complex bilinear extension of the Minkowski metric by the same letter. Let N be a unit normal vector field to M. Then the vector fields N, F_z and $F_{\bar{z}}$ define a moving frame along F.

The compatibility conditions (Gauss-Codazzi equations) of the moving frame equations have the following form:

$$u_{z\bar{z}} - \frac{1}{2}H^2 e^u + 2|Q|^2 e^{-u} = 0, \tag{G$_0$}$$

$$\bar{Q}_z = \frac{e^u}{2}H_{\bar{z}}, \quad Q_{\bar{z}} = \frac{e^u}{2}H_z. \tag{C$_0$}$$

Here H is the mean curvature of M defined by $H = -2e^{-u}\langle F_{z\bar{z}}, N\rangle$. The function $Q = -\langle F_{zz}, N\rangle$ defines a global 2-differential $Q^{\#} = Qdz^2$ on M. The differential $Q^{\#}$ is called the *Hopf differential of* M.

Now, let us denote the algebra of split-quaternions by $\mathbf{H}'$ and its natural basis by $\{\mathbf{1},\ \mathbf{i},\ \mathbf{j}',\ \mathbf{k}'\}$. The multiplication of $\mathbf{H}'$ is defined as follows:

$$\mathbf{i}\mathbf{j}' = -\mathbf{j}'\mathbf{i} = \mathbf{k}',\ \mathbf{j}'\mathbf{k}' = -\mathbf{k}'\mathbf{j}' = -\mathbf{i},\ \mathbf{k}'\mathbf{i} = -\mathbf{i}\mathbf{k}' = \mathbf{j}', \tag{2.3}$$

$$\mathbf{i}^2 = -\mathbf{1},\ \mathbf{j}'^2 = \mathbf{k}'^2 = \mathbf{1}.$$

An element $\xi = \xi_0\mathbf{1} + \xi_1\mathbf{i} + \xi_2\mathbf{j}' + \xi_3\mathbf{k}' \in \mathbf{H}'$ is called a *split-quaternion*. For a split-quaternion ξ, the *conjugate* $\bar{\xi}$ *of* ξ is defined by $\bar{\xi} = \xi_0\mathbf{1} - \xi_1\mathbf{i} - \xi_2\mathbf{j}' - \xi_3\mathbf{k}'$. It is easy to see that $-\xi\bar{\xi} = -\xi_0^2 - \xi_1^2 + \xi_2^2 + \xi_3^2$. The algebra $\mathbf{H}'$ is naturally identified with a semi-Euclidean 4-space $\mathbf{E}^4_2$

$$\mathbf{E}^4_2 = (\mathbf{R}^4(\xi_0, \xi_1, \xi_2, \xi_3),\ -d\xi_0^2 - d\xi_1^2 + d\xi_2^2 + d\xi_3^2).$$

Let $G = \{\xi \in \mathbf{H}' \mid \xi\bar{\xi} = 1\}$ be the multiplicative group of timelike unit split-quaternions. The Lie algebra $\mathfrak{g}$ of G is the imaginary part of $\mathbf{H}'$, that is,

$$\mathfrak{g} = \mathrm{Im}\ \mathbf{H}' = \{\xi_1\mathbf{i} + \xi_2\mathbf{j}' + \xi_3\mathbf{k}' \mid \xi_1,\ \xi_2,\ \xi_3 \in \mathbf{R}\}.$$

The Lie algebra $\mathfrak{g}$ is naturally identified with a Minkowski 3-space

$$\mathbf{E}^3_1 = (\mathbf{R}^3(\xi_1, \xi_2, \xi_3),\ -d\xi_1^2 + d\xi_2^2 + d\xi_3^2)$$

as a metric linear space.

We shall use the standard matricial expression of $\mathbf{H}'$ in $\mathrm{M}_2\mathbf{C}$:

$$\mathbf{1} \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} \longleftrightarrow \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \tag{2.4}$$

$$\mathbf{j}' \longleftrightarrow \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \mathbf{k}' \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Under the identification (2.4), the group G of timelike unit split-quaternions corresponds to an indefinite special unitary group (see [29, p.324].)

$$\mathrm{SU}_1(2) = \left\{ \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \;\middle|\; -|\alpha|^2 + |\beta|^2 = -1,\ \alpha,\ \beta \in \mathbf{C} \right\}.$$

The semi-Euclidean metric of $\mathbf{H}'$ corresponds to the following scalar product on $\mathrm{M}_2\mathbf{C}$.

$$\langle X, Y \rangle = \frac{1}{2}\{\mathrm{tr}(XY) - \mathrm{tr}(X)\mathrm{tr}(Y)\} \tag{2.5}$$

for all $X,\ Y \in \mathrm{M}_2\mathbf{C}$. The metric of G induced by (2.5) is a bi-invariant Lorentz metric of constant curvature -1. Hence the Lie group G is identified with an anti de Sitter 3-space H^3_1 of constant curvature -1. (See [29].)

Now, we shall rewrite the Gauss-Codazzi equations (G_0) and (C_0) in 2 by 2 matrix-form. Let $F : M \to \mathbf{E}^3_1$ be a spacelike surface with a moving frame $(N,\ F_z,\ F_{\bar{z}})$ as before. Recall that $\mathrm{SU}_1(2)$ is the double covering group of $\mathrm{O}^{++}_1(3)$. Let us take an $\mathrm{SU}_1(2)$-valued framing Φ. We shall define a framing Φ by

$$\mathrm{Ad}(\Phi)(\mathbf{i},\ \mathbf{j}',\ \mathbf{k}') = (N,\ e^{-\frac{u}{2}}F_x,\ e^{-\frac{u}{2}}F_y). \tag{2.6}$$

The $\mathbf{H}'$-valued function Φ satisfies the following system of linear differential equations:

$$\frac{\partial}{\partial z}\Phi = \Phi U,\ \frac{\partial}{\partial \bar{z}}\Phi = \Phi V, \tag{2.7}$$

$$U = \begin{pmatrix} \frac{1}{4}u_z & \frac{H}{2}e^{\frac{u}{2}} \\ Qe^{-\frac{u}{2}} & -\frac{1}{4}u_z \end{pmatrix},\ V = \begin{pmatrix} -\frac{1}{4}u_{\bar{z}} & \bar{Q}e^{-\frac{u}{2}} \\ \frac{H}{2}e^{\frac{u}{2}} & \frac{1}{4}u_{\bar{z}} \end{pmatrix}. \tag{2.8}$$

3. Spacelike Surfaces in H^3_1

In this section, we shall describe the fundamental equations of spacelike surfaces in anti de Sitter 3-space H^3_1.

As we saw in the preceding section, H^3_1 is naturally identified with the indefinite special unitary group $G = \mathrm{SU}_1(2)$ with biinvariant Lorentz metric $\langle\cdot,\cdot\rangle$ defined by (2.5). The Lie group G can be characterised by $\mathbf{i}'$ as follows:

$$G = \left\{ g \in \mathrm{M}_2\mathbf{C} \mid g\ \mathbf{i}'\ g^* = \mathbf{i}' \right\}.$$

It is easy to see that

$$\langle X, X\rangle = -\det X = -\frac{1}{2}\mathrm{tr}(\mathbf{i}'X^*\mathbf{i}'X)$$

for all $X \in \mathbf{H}'$ under the identification (2.4). The Lie group $G \times G$ acts transitively and isometrically on H^3_1 as follows:

$$\mu_H : (G \times G) \times H^3_1 \longrightarrow H^3_1, \quad \mu_H(g_1, g_2)X = g_1\ X\ g_2^{-1}$$

for $(g_1, g_2) \in G \times G,\ X \in H^3_1$. The isotropy subgroup Δ of $G \times G$ at $\mathbf{1}$ is the diagonal subgroup of $G \times G$, that is, $\Delta = \{(g_1, g_1) \mid g_1 \in G\}$. Hence the anti de Sitter 3-space H^3_1 is represented as a Lorentzian symmetric space:

$$H^3_1 = (G \times G)/\Delta = \left\{ g_1 g_2^{-1} \mid (g_1, g_2) \in G \times G \right\}.$$

Note that the Lie group $G \times G$ is a double covering group of the identity component $\mathrm{O}_2^{++}(4)$ of the full isometry group $\mathrm{O}_2(4)$ of H^3_1 [29, p. 238]. The natural projection $p_H : G \times G \to H^3_1$ is given explicitly by $p_H(g_1, g_2) = \mu_H(g_1, g_2)\mathbf{1} = g_1 g_2^{-1},\ (g_1, g_2) \in G \times G$.

Let $F : M \to H^3_1 \subset \mathbf{H}'$ be a spacelike surface and $z = x + \sqrt{-1}y$ a local complex coordinate. Then the induced metric I can be written as

$$I = e^u dz d\bar{z}.$$

We can take a moving frame σ along F defined by $\sigma = (F, N, F_z, F_{\bar{z}})$. The moving frame σ satisfies the following Frenet (Gauss-Weingarten) equations:

$$\sigma_z = \sigma\ \mathcal{U}, \quad \sigma_{\bar{z}} = \sigma\ \mathcal{V} \tag{3.1}$$

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & 0 & \frac{e^u}{2} \\ 0 & 0 & Q & \frac{H}{2}e^u \\ 1 & H & u_z & 0 \\ 0 & 2Qe^{-u} & 0 & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} 0 & 0 & \frac{e^u}{2} & 0 \\ 0 & 0 & \frac{H}{2}e^u & \bar{Q} \\ 0 & 2\bar{Q}e^{-u} & 0 & 0 \\ 1 & H & 0 & u_{\bar{z}} \end{pmatrix},$$

where $H = -2e^u\langle F_{z\bar{z}}, N\rangle$ is the mean curvature of M and $Q = -\langle F_{zz}, N\rangle$. The function Q defines a global 2-differential on M as like in Section 2. The Gauss-Codazzi equations for (M, F) have the following form:

$$(\mathrm{G}_{-1}) \qquad u_{z\bar{z}} - \frac{1}{2}(H^2+1)e^u + 2|Q|^2e^{-u} = 0,$$

$$(\mathrm{C}_{-1}) \qquad \bar{Q}_z = \frac{e^u}{2}H_{\bar{z}}, \quad Q_{\bar{z}} = \frac{e^u}{2}H_z.$$

4. Spacelike Surfaces in S_1^3

In this section we shall discuss the 2 by 2 matrix-model of the de Sitter 3-space.

Let $\{\mathbf{1}, \mathbf{i}, \mathbf{j}', \mathbf{k}'\}$ be the natural basis of split-quaternion algebra $\mathbf{H}'$ as before. In this section, instead of the complex structure $\mathbf{i}$, we use the following split-complex structure $\mathbf{i}'$:

$$(4.1) \qquad \mathbf{i}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is easy to see that $\mathbf{i}'^2 = \mathbf{1}$. The linear space $\mathbb{H}$ spanned by $\mathbf{1}$, $\mathbf{i}'$, $\mathbf{j}'$ and $\mathbf{k}'$ is the linear space of all Hermitian 2-matrices:

$$\mathbb{H} = \left\{ \begin{pmatrix} \xi_0 + \xi_1 & \xi_3 - \sqrt{-1}\xi_2 \\ \xi_3 + \sqrt{-1}\xi_2 & \xi_0 - \xi_1 \end{pmatrix} \Bigg| \ \xi_0,\ \xi_1,\ \xi_2,\ \xi_3 \in \mathbf{R} \right\}.$$

On the linear space $\mathbb{H}$, The following scalar product $(\cdot, \cdot)$

$$(4.2) \qquad (X, Y) = -\frac{1}{2}\mathrm{tr}(\mathbf{j}'X\ \mathbf{j}'Y^t)$$

gives a Minkowski metric. In fact under the identification:

$$(4.3) \quad \xi = \xi_0\mathbf{1} + \xi_1\mathbf{i}' + \xi_2\mathbf{j}' + \xi_3\mathbf{k}' \longleftrightarrow \Xi = \begin{pmatrix} \xi_0 + \xi_1 & \xi_3 - \sqrt{-1}\xi_2 \\ \xi_3 + \sqrt{-1}\xi_2 & \xi_0 - \xi_1 \end{pmatrix},$$

the scalar product $(\cdot,\cdot)$ corresponds to the Minkowski metric

$$(\xi,\eta) = -\xi_0\eta_0 + \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3.$$

Hence the metric linear space $(\mathbb{H},\ (\cdot,\cdot)\ )$ is identified with Minkowski 4-space $\mathbf{E}^4_1(\xi_0,\xi_1,\xi_2,\xi_3)$. See Aiyama and Akutagawa [1] and Bryant [13]. However our description is slightly different from [1] and [13]. This matrix model is suitable for our approach. Note that $\det\Xi = -(\Xi,\Xi)$ under (4.3). The de Sitter 3-space S^3_1 is represented by

$$S^3_1 = \{\Xi \in \mathbb{H} \mid \det\Xi = -1\}.$$

The special linear group $\mathrm{SL}_2\mathbf{C}$ acts transitively and isometrically on S^3_1 as follows

$$\mu_S : \mathrm{SL}_2\mathbf{C} \times S^3_1 \to S^3_1, \quad \mu_S(g) = g\ X\ g^*.$$

Here g^* denotes the transposed complex conjugate of g. The de Sitter space S^3_1 is a homogeneous manifold of $\mathrm{SL}_2\mathbf{C}$. The isotropy subgroup of $\mathrm{SL}_2\mathbf{C}$ at $\mathbf{i}'$ is $G = \mathrm{SU}_1(2)$. The special linear group $\mathrm{SL}_2\mathbf{C}$ is a complexification of G. Hence the de Sitter 3-space S^3_1 is represented as $S^3_1 = G^{\mathbf{C}}/G$.

It is well known that $\mathrm{SL}_2\mathbf{C}$ is a double covering group of the full isometry group $\mathrm{O}_1(4)$ of S^3_1. In particular $G^{\mathbf{C}}/G$ is a Lorentzian symmetric space. The natural projection $p_S : G^{\mathbf{C}} \to G^{\mathbf{C}}/G$ is given explicitly $p_S(g) = \mu_S(g)\ \mathbf{i}' = g\ \mathbf{i}'\ g^*,\ g \in G^{\mathbf{C}}$.

Let $F : M \to S^3_1$ be a spacelike surface and $z = x + \sqrt{-1}y$ a local complex coordinate. The induced metric I of M can be written as

$$I = e^u dz d\bar{z}.$$

Let N be a unit normal vector field to M in S^3_1. Since F is conformal, we get

$$(F,F) = 1,\ (F_z,F_z) = (F_{\bar z},F_{\bar z}) = 0,\ (F_z,F_{\bar z}) = \frac{1}{2}e^u. \tag{4.4}$$

As in the geometry of spacelike surfaces in $\mathbf{E}^3_1$, the function

$$Q = -(F_{z\bar z},N)$$

defines a global 2-differential $Q^{\#} = Qdz^2$ on M. The differential $Q^{\#}$ is also called the *Hopf differential* of M.

The vector fields $N,\ F,\ F_z,\ F_{\bar{z}}$ define a moving frame $\sigma = (N, F, F_z, F_{\bar{z}})$. The Frenet equations for σ are slightly different from (3.1).

$$\sigma_z = \sigma\,\mathcal{U}, \quad \sigma_{\bar{z}} = \sigma\,\mathcal{V}, \tag{4.5}$$

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & Q & \frac{H}{2}e^u \\ 0 & 0 & 0 & -\frac{1}{2}e^u \\ H & 1 & u_z & 0 \\ 2Qe^{-u} & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} 0 & 0 & \frac{H}{2}e^u & \bar{Q} \\ 0 & 0 & -\frac{1}{2}e^u & 0 \\ 2\bar{Q}e^{-u} & 0 & 0 & 0 \\ H & 1 & 0 & u_{\bar{z}} \end{pmatrix}.$$

Here $H = -2e^{-u}(F_{z\bar{z}}, N)$ is the mean curvature of M. The compatibility condition (Gauss–Codazzi equations) of (4.5) has the following form:

$$u_{z\bar{z}} - \frac{1}{2}(H^2 - 1)e^u + 2|Q|^2 e^{-u} = 0, \tag{G$_1$}$$

$$\bar{Q}_z = \frac{e^u}{2}H_{\bar{z}}, \quad Q_{\bar{z}} = \frac{e^u}{2}H_z. \tag{C$_1$}$$

Comparing Gauss-Codazzi equations $(\mathrm{G}_c) - (\mathrm{C}_c)$, $c = \pm 1, 0$ we can deduce that spacelike CMC surfaces in S^3_1 with mean curvature $H^2 > 1$ locally corresponds to spacelike CMC surfaces in H^3_1 and $\mathbf{E}^3_1$.

In hyperbolic 3-space H^3, global properties of CMC surfaces are influenced by the range of mean curvature. For example, there are no compact CMC surfaces in H^3 with mean curvature $0 < H^2 < 1$. On the other hand there are many compact CMC surfaces with $H^2 > 1$. In particular Bobenko [6] classified all CMC tori in H^3. In addition CMC surfaces with $H^2 > 1$ locally corresponds to CMC surfaces in S^3 and $\mathbf{E}^3$. (so-called Lawson correspondences.) However CMC surfaces in H^3 with $H^2 < 1$ have no Lawson correspondents in S^3 and $\mathbf{E}^3$.

In S^3_1 compactness is very strong restriction for spacelike CMC surfaces. In fact Akutagawa and Ramanathan obtained the following fundamental result. (See also M. Dajczer and K. Nomizu [16] and H. Mori [28].)

THEOREM 4.1 ([3], [31]).

Let $F : M \to S^3_1$ be a complete spacelike surface with nonzero constant mean curvature. Then
(1) *M is compact if and only if $H^2 < 1$. In this case M is congruent to a totally umbilic 2-sphere of constant curvature $1 - H^2$:*

$$F : S^2(1/\sqrt{1-H^2}) \subset \mathbf{E}^3(x_1, x_2, x_3) \to S^3_1 \subset \mathbb{H} \ ;$$

$$F(x_1, x_2, x_3) = \begin{pmatrix} \frac{H}{\sqrt{1-H^2}} + x_1 & x_3 - \sqrt{-1}x_2 \\ x_3 + \sqrt{-1}x_2 & \frac{H}{\sqrt{1-H^2}} - x_1 \end{pmatrix}.$$

(2) *If $H^2 \equiv 1$ then M is a flat totally umbilic surface. More precisely, M is congruent to a parabolic type spacelike surface of revolution*:

$$F : \mathbf{E}^2(x, y) \to S^3_1 \subset \mathbb{H} \ ; \quad F(x, y) = \begin{pmatrix} 1 & y - \sqrt{-1}x \\ y + \sqrt{-1}x & x^2 + y^2 \end{pmatrix}.$$

Remark. In the case $H^2 > 1$, complete spacelike surfaces with mean curvature $H^2 > 1$ are not unique.

It is easy to find totally umbilic spacelike surface with constant mean curvature such that $H^2 > 1$. In fact for any $H \in \mathbf{R}$ such that $H^2 > 1$, we can construct an isometric and totally umbilical immersion of hyperbolic 2-space $H^2(1/\sqrt{H^2-1})$ of constant curvature $1 - H^2 < 0$ into S^3_1 by

$$F : H^2(1/\sqrt{H^2-1}) \subset \mathbf{E}^3_1(x_0, x_1, x_2) \to S^3_1 \subset \mathbb{H} \ ;$$

$$F(x_0, x_1, x_2) = \begin{pmatrix} x_0 + x_1 & \frac{H}{\sqrt{H^2-1}} - \sqrt{-1}x_2 \\ \frac{H}{\sqrt{H^2-1}} + \sqrt{-1}x_2 & x_0 - x_1 \end{pmatrix}.$$

However there exist complete nontotally umbilic spacelike surfaces of constant mean curvature $H^2 > 1$ in S^3_1. For example, spacelike surfaces of revolution provide such examples. See Akutagawa [3] and Mori [28]. More generally Mori [28] classified constant mean curvature spacelike surfaces of revolution in S^3_1. See also Ramanathan [31, Example 10, 11] and [16].

Tribuzy [32] and Umehara [33] proved the nonexistence of compact Bonnet surfaces with nonconstant mean curvature in Riemannian 3-space forms.

In Lorentzian 3-space forms compactness is still a strong restriction for spacelike Bonnet surface. It is well known that only Lorentzian 3-space form which admits compact spacelike surfaces is S^3_1. We saw as above only compact spacelike CMC surfaces are totally umbilic 2-spheres. Recently Alías proved the nonexistence of compact spacelike Bonnet surfaces.

Theorem 4.2 ([4]). *There are no compact spacelike Bonnet surfaces in Lorentzian 3-space forms. Moreover there are no compact spacelike surfaces which admit Bonnet pairs.*

These facts tell us that geometry of spacelike surfaces in S^3_1 is very complicated and not a cheap analogue of geometry of surfaces in H^3. For more details on Lawson correspondences between spacelike CMC surfaces in $\mathfrak{M}^3_1(c)$ we refer to [1]-[2].

Remark. The definitions of H and Q for surfaces in $\mathfrak{M}^3_1(c)$ have opposite sign of those in Chen and Li [15].

5. Spacelike HIMC Surfaces in Minkowski 3-space

In this section we shall insert a variable spectral parameter λ in the sense of Butsev, Zakharov and Mikhairov [14]— *i.e.*, an additional complex parameter λ depends on the coordinates z and $\bar{z}$— in the Lax pair (2.8). Further we shall introduce the notion of spacelike HIMC surfaces.

For the Lax pair $\{U,\ V\}$ in (2.8) we shall introduce an additional complex parameter λ in the following way:

$$U_\lambda = \begin{pmatrix} \frac{1}{4}u_z & \frac{H}{2}\lambda e^{\frac{u}{2}} \\ Qe^{-\frac{u}{2}} & -\frac{1}{4}u_z \end{pmatrix},\ V_\lambda = \begin{pmatrix} -\frac{1}{4}u_{\bar{z}} & \bar{Q}e^{-\frac{u}{2}} \\ \frac{H}{2}\lambda^{-1}e^{\frac{u}{2}} & \frac{1}{4}u_{\bar{z}} \end{pmatrix}. \tag{5.1}$$

Then the compatibility condition

$$\frac{\partial}{\partial z}V_\lambda - \frac{\partial}{\partial \bar{z}}U_\lambda + [\ U_\lambda, V_\lambda\ ] = 0 \tag{5.2}$$

for the deformed Lax pair U_λ, V_λ yields

$$u_{z\bar{z}} - \frac{1}{2}H^2e^u + 2|Q|^2e^{-u} = 0, \tag{G$_0$}$$

$$Q_{\bar{z}} = \frac{e^u}{2}(H\lambda^{-1})_z,\quad \bar{Q}_z = \frac{e^u}{2}(H\lambda)_{\bar{z}}. \tag{5.3}$$

The Lax pair U_λ, V_λ describe a spacelike surface in $\mathbf{E}^3_1$ if and only if the equations (5.3) are consistent with Codazzi equation (C_0). Hence the mean curvature H and the spectral parameter λ should satisfy the following equations

$$\frac{\partial}{\partial \bar{z}}\{H(1-\lambda)\} = 0, \quad \frac{\partial}{\partial z}\{H(1-\lambda^{-1})\} = 0. \tag{5.4}$$

These equations (5.4) can be easily solved as follows:

$$H = \frac{1}{h+\bar{h}}, \quad \lambda = -\frac{\bar{h}}{h}, \tag{5.5}$$

where $h(z)$ is a holomorphic function. It is easy to see that the mean curvature H is invariant under the one parametric deformation

$$h \longmapsto h + \frac{1}{2\sqrt{-1}\tau}, \quad \tau \in \mathbf{R}.$$

Under this deformation, spectral parameter λ is transformed as

$$\lambda \longmapsto \frac{1-2\sqrt{-1}\bar{h}\tau}{1+2\sqrt{-1}h\tau}, \quad \tau \in \mathbf{R}.$$

The form (5.5) of H is equivalent to the harmonicity of $1/H$. As in the Euclidean surface geometry [8], we shall call a spacelike surface M in $\mathbf{E}^3_1$, a *spacelike surface with harmonic inverse mean curvature* (SHIMC surface) if $1/H$ is a harmonic function.

Example 5.1 (SHIMC cylinders). Let $a(x) = (a_1(x), a_2(x))$ be a spacelike curve in Minkowski plane $\mathbf{E}^2_1(\xi_1, \xi_2)$ parametrised by the arclength parameter $x \in I$. Here I is an interval. *A cylinder over the curve* a is a spacelike surface in $\mathbf{E}^3_1$ defined by the immersion $F : I \times \mathbf{R} \longrightarrow \mathbf{E}^3_1$; $F(x,y) = (a_1(x), a_2(x), y)$. The fundamental quantities of F is given as follows

$$I = dx^2 + dy^2, \; II = (a_1' a_2'' - a_1'' a_2')dx^2 = -\kappa^2 dx^2.$$

Here the prime denotes the differentiation with respect to x and κ is the curvature of a. (See Appendix.) In particular the mean curvature H of F is

$H = \kappa/2$. Since the mean curvature H depends only on x, the harmonicity of $1/H$ implies that the reciprocal ρ of the curvature κ is a linear function of x, namely ρ has the form $\rho = C_1 x + C_2$, $C_1,\ C_2 \in \mathbf{R}$.

We can see in Appendix that spacelike curves with reciprocal curvature $\frac{1}{C_1 x + C_2}$ are logarithmic pseudo-spirals or spacelike hyperbolas. Hence all the SHIMC cylinders are congruent to cylinders over a logarithmic pseudo-spiral or a spacelike hyperbola.

In [25], we have obtained a one-parameter isometric deformation of spacelike surfaces with constant mean curvature (SCMC surfaces). For SHIMC surfaces in $\mathbf{E}^3_1$, we get the following one-parameter family of *conformal* deformation.

PROPOSITION 5.2 (*Sym formula*). *Let $F : M \to \mathbf{E}^3_1$ be a spacelike surface with harmonic inverse mean curvature*

$$H = \frac{1}{h(z) + \bar{h}(z)},$$

where $h(z)$ is a holomorphic function. Then F has the following zero curvature representation with variable spectral parameter λ:

$$\frac{\partial}{\partial z}\Phi_\lambda = \Phi_\lambda U_\lambda,\ \frac{\partial}{\partial \bar{z}}\Phi_\lambda = \Phi_\lambda V_\lambda, \tag{5.6}$$

$$U_\lambda = \begin{pmatrix} \frac{1}{4}u_z & \frac{H}{2}\lambda e^{\frac{u}{2}} \\ Q e^{-\frac{u}{2}} & -\frac{1}{4}u_z \end{pmatrix},\quad V_\lambda = \begin{pmatrix} -\frac{1}{4}u_{\bar{z}} & \bar{Q}e^{-\frac{u}{2}} \\ \frac{H}{2}\lambda^{-1}e^{\frac{u}{2}} & \frac{1}{4}u_{\bar{z}} \end{pmatrix},$$

where $\lambda = (1-2\sqrt{-1}\bar{h}\tau)/(1+2\sqrt{-1}h\tau)$, $\tau \in \mathbf{R}$. Let $\Phi_\lambda(z,\bar{z})$ be a solution of (5.6)*. Then*

$$F_\lambda = \frac{\partial}{\partial \tau}\Phi_\lambda \cdot \Phi_\lambda^{-1},\quad N_\lambda = \mathrm{Ad}(\Phi_\lambda)\ \mathbf{i} \tag{5.7}$$

describes a real loop of SHIMC surfaces through $F = F_1$ with Gauss mapping N_λ. The fundamental associated quantities of F_λ are given as follows:

$$I_\lambda = e^{u_\lambda} dz d\bar{z},\quad e^{u_\lambda} = \frac{e^u}{(1+2\sqrt{-1}h\tau)^2(1-2\sqrt{-1}\bar{h}\tau)^2}, \tag{5.8}$$

$$\frac{1}{H_\lambda} = h_\lambda + \bar{h}_\lambda, \quad h_\lambda = \frac{h}{(1 + 2\sqrt{-1}h\tau)}, \tag{5.9}$$

$$Q_\lambda^{\#} = \frac{Q^{\#}}{(1 + 2\sqrt{-1}h\tau)^2}, \tag{5.10}$$

$$K_\lambda = (1 + 2\sqrt{-1}h\tau)(1 - 2\sqrt{-1}\bar{h}\tau)K, \tag{5.11}$$

$$H_\lambda^2/K_\lambda \equiv H^2/K. \tag{5.12}$$

Proof. Differentiating (5.7),

$$\frac{\partial}{\partial z}F_\lambda = \mathrm{Ad}(\Phi_\lambda)\frac{\partial}{\partial \tau}U_\lambda, \quad \frac{\partial}{\partial \bar{z}}F_\lambda = \mathrm{Ad}(\Phi_\lambda)\frac{\partial}{\partial \tau}V_\lambda.$$

Direct calculations show

$$e^{u_\lambda} = 2\langle \frac{\partial}{\partial z}F_\lambda, \frac{\partial}{\partial \bar{z}}F_\lambda \rangle = \frac{e^u}{(1 + 2\sqrt{-1}h\tau)^2(1 - 2\sqrt{-1}\bar{h}\tau)^2},$$

$$\begin{aligned} H_\lambda =& 2e^{-u_\lambda}\langle \frac{\partial}{\partial z}F_\lambda, \frac{\partial}{\partial \bar{z}}N_\lambda \rangle = -e^{-u_\lambda} \ \mathrm{tr}\left\{\frac{\partial}{\partial t}U_\lambda[V_\lambda, \mathbf{i}]\right\} \\ =& \frac{1}{h_\lambda + \bar{h}_\lambda}, \end{aligned}$$

$$Q_\lambda = \langle \frac{\partial}{\partial z}F_\lambda, \frac{\partial}{\partial z}N_\lambda \rangle = \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial \tau}U_\lambda[U_\lambda, \mathbf{i}]\right\} = \frac{Q}{(1 + 2\sqrt{-1}h\tau)^2}. \ \square$$

The formula (5.12) implies that the members of the one parameter family F_λ have same ratio of the principal curvatures.

6. Christoffel Transformations of SHIMC Surfaces

It is well known that every umbilic free SCMC surfaces in $\mathbf{E}^3_1$ possess isometric and principal curvature preserving deformations [25], [30]. Thus spacelike Bonnet surfaces may be thought as another generalisation of SCMC surfaces.

Wang [34] studied spacelike and timelike Bonnet surfaces in $\mathbf{E}^3_1$. More generally, Chen and Li [15] studied spacelike Bonnet surfaces in Lorentzian space forms. In this section we shall study the relation between SHIMC surfaces and spacelike Bonnet surfaces.

We shall start with the following definition.

DEFINITION 6.1. Let $F : M \to \mathfrak{M}^3_1(c)$ be a spacelike immersion. Then (M, F) is said to be *isothermic* if there exists a local isothermic coordinate system around any point of M.

Here an isothermic coordinate system is a local isothermal coordinate system such that both of parameter curves are curvature lines. (*i.e.,* principal curves.)

The isothermic property for spacelike surfaces in $\mathfrak{M}^3_1(c)$ can be reformulated as follows. (*cf.* [8, p. 93].)

PROPOSITION 6.2. *A spacelike surface (M, F) in $\mathfrak{M}^3_1(c)$ is isothermic if and only if there exists a local complex coordinate z around any point of M such that the Hopf differential Q takes the following form*:

$$Q(z, \bar{z}) = \frac{1}{2}\mathfrak{q}(z, \bar{z})f(z). \tag{6.1}$$

Here $\mathfrak{q}$ is a real smooth function and f is a holomorphic function.

Typical examples of isothermic spacelike surfaces are spacelike surfaces of revolution in $\mathbf{E}^3_1$. We shall recall the notion of spacelike surface of revolution in $\mathbf{E}^3_1$. A *revolution of* $\mathbf{E}^3_1$ is a linear isometry which lies in the identity component of the Lorentz group $\mathrm{O}_1(3)$. Every revolution fixes a line pointwisely. Such a fixed line of a revolution is called the *axis of* revolution. Hence revolutions of $\mathbf{E}^3_1$ can be characterised by the causal character of the axes.

By a *spacelike surfaces of revolution* in $\mathbf{E}_1^3$ we mean a spacelike surface obtained by revolving about an axis a regular curve lying in some plane containing the axis. We refer to McNertney [27] and Hano and Nomizu [24] for more details on surfaces of revolution in $\mathbf{E}_1^3$. One can easily show that spacelike surfaces of revolution are isothermic in much the same way in [12].

Example 6.3 (**Spacelike axis**). Let $F : M \longrightarrow \mathbf{E}_1^3$ be a spacelike surface of revolution with spacelike axis. Then there exists an isothermic parametrisation

$$F(x, y) = \frac{1}{a}\left(e^{\frac{u(x)}{2}} \cosh(ay), e^{\frac{u(x)}{2}} \sinh(ay), c(x)\right), \ a \in \mathbf{R}^*$$

so that

$$c'(x)^2 e^{-u(x)} - \left(\frac{u'(x)}{2}\right)^2 = a^2.$$

With respect to this isothermic coordinates, the mean curvature is given by

$$H(x) = -\frac{1}{8c'(x)}\{2u''(x) + u'(x)^2 + 4a^2\}, \ c''(x) = -e^{u(x)}u'(x)H(x).$$

Example 6.4 (**Timelike axis**). Let $F : M \longrightarrow \mathbf{E}_1^3$ be a spacelike surface of revolution with timelike axis. Then there exists an isothermic parametrisation

$$F(x, y) = \frac{1}{a}\left(c(x), e^{\frac{u(x)}{2}} \cos(ay), e^{\frac{u(x)}{2}} \sin(ay),\right), \ a \in \mathbf{R}^*$$

so that

$$-c'(x)^2 e^{-u(x)} + \left(\frac{u'(x)}{2}\right)^2 = a^2.$$

With respect to this isothermic coordinates, the mean curvature is given by

$$H(x) = -\frac{1}{8c'(x)}\{-2u''(x) - u'(x)^2 + 4a^2\}, \ c''(x) = -e^{u(x)}u'(x)H(x).$$

Example 6.5 (**Null axis**). Let $F : M \longrightarrow \mathbf{E}^3_1$ be a spacelike surface of revolution with null axis. Then there exists a null frame $\{L_1, L_2, L_3\}$ and an isothermic parametrisation

$$F(x, y) = \left(a(x), b(x) - \frac{y^2}{2} a(x), y a(x) \right), \ a \in \mathbf{R}^*$$

relative to the null frame $\{L_1, L_2, L_3\}$ so that

$$2a'(x)b'(x) = a^2(x).$$

Here a null frame means a basis of $\mathbf{E}^3_1$ such that

$$\langle L_1, L_1 \rangle = \langle L_2, L_2 \rangle = 0, \quad \langle L_1, L_2 \rangle = 1,$$

$$\langle L_3, L_3 \rangle = 1, \quad \langle L_1, L_3 \rangle = \langle L_2, L_3 \rangle = 1.$$

With respect to this isothermic parametrisation, the mean curvature of F is given by

$$H(x) = \frac{a''(x)a(x) + a'(x)^2}{4a(x)^2 a'(x)}.$$

The following propositions 6.6–6.8 can be easily checked. (*cf.* [12, Section 4].)

PROPOSITION 6.6. *For any SHIMC surface of revolution with nonconstant mean curvature, there exists an isothermic coordinates* (x, y) *such that* $H(x) = 1/x$.

PROPOSITION 6.7. *Let* $F : M \longrightarrow \mathbf{E}^3_1$ *be a spacelike surface of revolution with spacelike axis parametrised as in Example* 6.3 *with harmonic inverse mean curvature* $1/H = x$ *and* $a = 2$. *Then there exists a real valued function* ϕ *such that*

$$e^{u(x)} = \frac{x^2}{4} \left(\phi'(x) + 2\cosh \phi(x) \right)^2,$$

$$c(x) = \frac{x^2}{4} \{ \phi'(x)^2 - 4\cosh^2 \phi(x) \}.$$

Furthermore ϕ is a solution to the ordinary differential equation:

$$x\left\{\phi''(x) - 2\sinh(2\phi(x))\right\} - \phi'(x) - 2\cosh\phi(x) = 0. \tag{6.2}$$

Remark. Every solution ϕ to

$$\phi'(x) + 2\cosh\phi(x) = 0$$

solves (6.2). In this case $\phi(x)$ can be solved by quadratures:

$$x + C = -\frac{1}{2}\int\frac{d\phi}{\cosh\phi} = -\tan^{-1}e^{\phi},\ C \in \mathbf{R}.$$

Proposition 6.8. *Let $F : M \longrightarrow \mathbf{E}^3_1$ be a spacelike surface of revolution with timelike axis parametrised as in Example* 6.4 *with harmonic inverse mean curvature $1/H = x$ and $a = 2$. Then there exists a real valued function ϕ such that*

$$e^{u(x)} = \frac{x^2}{4}\left(\phi'(x) \pm 2\sinh\phi(x)\right)^2,$$

$$c(x) = -\frac{x^2}{4}\{\phi'(x)^2 - 4\sinh^2\phi(x)\}.$$

Furthermore ϕ is a solution to the ordinary differential equation:

$$x\left\{\phi''(x) - 2\sinh(2\phi(x))\right\} - \phi'(x) \mp 2\sinh\phi(x) = 0. \tag{6.3}$$

Remark. Every solution ϕ to

$$\phi'(x) \pm 2\sinh\phi(x) = 0$$

solves (6.3). In this case $\phi(x)$ can be solved by quadratures:

$$x + C = \mp\frac{1}{2}\int\frac{d\phi}{\sinh\phi} = \mp\log\tanh\frac{\phi}{2},\ C \in \mathbf{R}.$$

REMARK. The ordinary differential equations (6.2) and (6.3) have similar forms to the third Painlevé equation. More precisely let $\omega=\omega(x)$ be a solution to the third Painlevé equation:

$$(P_{III}) \qquad \omega''-\frac{1}{\omega}(\omega')^2+\frac{\omega'}{x}-\frac{\alpha\omega^2-\alpha}{x}-\frac{\gamma}{\omega^3}-\frac{\gamma}{\omega}=0$$

with unit modulus, *i.e.*, $\omega(x)=e^{\sqrt{-1}\psi(x)}$ for some real valued function $\psi(x)$. Then (P_{III}) is equivalent to the following ordinary differential equation (trigonometric form of P_{III}):

$$x\left\{\psi''(x)+2\gamma\sin(2\psi(x))\right\}+\psi'(x)+2\alpha\sin\psi(x)=0.$$

In addition, if we complexified the above third Painlevé equation in trigonometric form as above and put $\psi=\sqrt{-1}\phi$ then ϕ satisfies

$$x\left\{\phi''(x)+2\gamma\sinh(2\phi(x))\right\}+\phi'(x)+2\alpha\sinh\phi(x)=0.$$

On the other hand SHIMC surfaces of revolution with null axis have restrictive shape. In fact such surfaces can be classified as follows:

PROPOSITION 6.9. *Let $F: M \longrightarrow \mathbf{E}^3_1$ be a spacelike surface of revolution with null axis parametrised as in Example* 6.5 *with harmonic inverse mean curvature $1/H=4x$. Then the function $a(x)$ is a solution to the following ordinary differential equation*:

$$x\left\{a''(x)a(x)+a'(x)^2\right\}=a^2(x)a'(x). \tag{6.4}$$

This ordinary differential equation is explicitly solved by quadratures. In fact the solution $a(x)$ is given as follows.

$$12\int\frac{a}{2a^3+3a^2+c_1}da=2\log|x|+c_2, \quad c_1,\ c_2\in\mathbf{R}. \tag{6.5}$$

PROOF. The ordinary differential equation (6.4) can be written as

$$\frac{1}{2}x(a^2(x))''=a^2(x)a'(x).$$

Adding $\frac{1}{2}(a^2(x))'$ to both hand sides of this equation, we get

$$\frac{1}{2}\{x(a^2(x))'\}' = \frac{1}{3}(a^3(x))' + \frac{1}{2}(a^2(x))'.$$

Hence we have

$$\frac{1}{2}x(a^2(x))' = \frac{a^3(x)}{3} + \frac{a^2(x)}{2} + C, \quad C \in \mathbf{R}.$$

This differential equation is solved by quadratures. In fact

$$\int \frac{2a}{\frac{a^3}{3} + \frac{a^2}{2} + C} da = \int \frac{2}{x} dx.$$

Thus we obtain (6.5). □

As in the geometry of surfaces in $\mathbf{E}^3$, we can formulate the Christoffel transformations for isothermic spacelike surfaces. (*cf.* [8, Proposition 2].)

PROPOSITION 6.10. *Let (M, F) be an isothermic spacelike surface in $\mathbf{E}^3_1$ and $(\mathfrak{D}, z)$ a simply connected local isothermic coordinate region as in* (6.1). *Then the formulae*:

$$F^*_z = e^{-u} f F_{\bar{z}}, \quad F^*_{\bar{z}} = e^{-u} \bar{f}\ F_z, \quad N^* = N \tag{6.6}$$

define a spacelike immersion $F^* : \mathfrak{D} \to \mathbf{E}^3_1$. The conformal structure of $\mathfrak{D}$ induced by F^* is anti-conformal to the original conformal structure determined by I. The fundamental quantities of F^* are given as follows:

$$e^{u^*} = e^{-u}|f|^2, \quad H^* = \mathfrak{q}, \quad Q^* = \frac{f}{2}H. \tag{6.7}$$

Here $I = e^u dz d\bar{z}$ is the first fundamental form of the original surface (M, F).

The immersion F^* is called a *Christoffel transform* (or *dual surface*) *of* F.

The following two propositions are immediate consequences of (6.6) and Theorem 6.1 in [15] (see also Lemma 6 in [12]).

PROPOSITION 6.11. *Let (M, F) be a spacelike Bonnet surface in $\mathbf{E}^3_1$. Then (M, F) is isothermic and its Christoffel transform is a SHIMC surface.*

PROPOSITION 6.12. *For any isothermic SHIMC surface in $\mathbf{E}^3_1$, its Christoffel transform is a spacelike Bonnet surface.*

The notion of isothermic surfaces can be generalised to the notion of θ-isothermic surfaces. (*cf.* [10, Definition 2].)

DEFINITION 6.13. A spacelike surface (M, F) is said to be *ϑ-isothermic* if there exists a local complex coordinate z around any point of M such that the Hopf differential Q has the following form:

$$Q(z, \bar{z}) = \frac{1}{2}(\mathfrak{q}(z, \bar{z}) + \sqrt{-1}\vartheta) f(z). \tag{6.8}$$

Here $\mathfrak{q}$ is a real smooth function, f is a holomorphic function and ϑ is a real constant.

Note that the constant ϑ has no global meaning, in fact, ϑ depends on the choice of z and f.

For any ϑ-isothermic SHIMC surface in $\mathbf{E}^3_1$ we can consider *dual* Bonnet surface in H^3_1. (*cf.* [12, Lemma 10].)

PROPOSITION 6.14. *Let (M, F) be a ϑ-isothermic spacelike surface in $\mathbf{E}^3_1$ and $(\mathfrak{D}, z)$ a simply connected ϑ-isothermic coordinate region such that the Hopf differential Q takes the following form:*

$$Q = \frac{1}{2}(\mathfrak{q} + \sqrt{-1}\vartheta).$$

Then there exists a spacelike immersion

$$F^* : \mathfrak{D} \longrightarrow \begin{cases} H^3_1(\frac{1}{|\vartheta|}), & \vartheta \neq 0, \\ \mathbf{E}^3_1, & \vartheta = 0 \end{cases}$$

with fundamental quantities

$$e^{u^*} = e^{-u}, \quad H^* = \mathfrak{q}, \quad Q^* = \frac{1}{2}(H \pm \sqrt{-1}\vartheta).$$

The spacelike immersion F^* is called a dual surface of F. In particular if F is a SHIMC surface then F^* is a spacelike Bonnet surface and vice versa.

Remark. In section 8, we shall prove a Lawson correspondence between SHIMC surfaces in Lorentzian space forms. Combining the duality in the preceding proposition and Lawson correspondence, we get a duality between SHIMC surfaces and spacelike Bonnet surfaces in H^3_1.

The following proposition is a direct consequence of Definition 6.13. See [12, Proposition 3].

Proposition 6.15. *A ϑ-isothermic spacelike surface with $\vartheta \neq 0$ is isothermic if and only if it is a spacelike Bonnet surface (including spacelike CMC surface.)*

7. Generalised Hazzidakis Equation

In this section we shall study normal forms of Gauss-Codazzi equations for SHIMC surfaces in $\mathbf{E}^3_1$. We assume that the mean curvature H is *non constant* in this section.

Let $F : M \to \mathbf{E}^3_1$ be a spacelike surface and ζ a local complex coordinate. Differentiating the Codazzi equation:

$$e^u = \frac{2Q_{\bar\zeta}}{H_\zeta} \tag{C$_0$}$$

with respect to ζ, we have

$$u_\zeta = \frac{Q_{\zeta\bar\zeta}}{Q_{\bar\zeta}} - \frac{H_{\zeta\zeta}}{H_\zeta}.$$

Inserting this equation into Gauss equation (G_0), we get

$$\left(\frac{Q_{\zeta\bar\zeta}}{Q_{\bar\zeta}}\right)_{\bar\zeta} - \left(\frac{H_{\zeta\zeta}}{H_\zeta}\right)_{\bar\zeta} = \frac{H^2}{H_\zeta}Q_{\bar\zeta} - \frac{|Q|^2 H_\zeta}{Q_{\bar\zeta}}. \tag{7.1}$$

Now we shall consider a SHIMC surface (M, F) with harmonic inverse mean curvature

$$H(\zeta,\bar\zeta) = \frac{1}{h(\zeta) + \bar h(\bar\zeta)}. \tag{7.2}$$

Inserting (7.2) into (C_0) we get

$$h_\zeta(\zeta)\bar{Q}_\zeta(\zeta,\bar{\zeta}) = \bar{h}_{\bar{\zeta}}(\bar{\zeta})Q_{\bar{\zeta}}(\zeta,\bar{\zeta}). \tag{7.3}$$

By using (7.3), the Codazzi equation (C_0) can be rewritten as

$$h_\zeta(\zeta)\left(\frac{Q_{\zeta\bar{\zeta}}}{Q_{\bar{\zeta}}}\right)_{\bar{\zeta}} + Q_{\bar{\zeta}} = \frac{|h_\zeta(\zeta)|^2}{(h(\zeta)+\bar{h}(\bar{\zeta}))^2}\left(2h_\zeta(\zeta) + \frac{|Q|^2}{\bar{Q}_\zeta}\right). \tag{7.4}$$

As long as $h'(\zeta) \neq 0$, we may assume $w := h(\zeta)$ as a local complex coordinate. With respect to the local complex coordinate w, Gauss equation (G_0) become:

$$\left(\frac{Q_{w\bar{w}}}{Q_{\bar{w}}}\right)_{\bar{w}} + Q_{\bar{w}} = \frac{1}{(w+\bar{w})^2}\left(2 + \frac{|Q|^2}{\bar{Q}_w}\right), \quad Q_{\bar{w}} = \bar{Q}_w. \tag{7.5}$$

We should remark that solutions $Q(w,\bar{w})$ to

$$2 + \frac{|Q|^2}{\bar{Q}_w} = 0 \tag{7.6}$$

solve (7.5).

By the Codazzi equtions and the formula $1/H = w + \bar{w}$, we get

$$e^{u(w,\bar{w})} = -2(w+\bar{w})^2\bar{Q}_w = (w+\bar{w})^2|Q(w,\bar{w})|^2.$$

Hence the solution $Q(w,\bar{w})$ to (7.6) defines a SHIMC surface with meric:

$$I = e^{u(w,\bar{w})}dwd\bar{w} = |Q(w,\bar{w})|^2(w+\bar{w})^2dwd\bar{w}. \tag{7.7}$$

(Compare with Euclidean case [12, p. 203].)

Hereafter we restrict our attention to ϑ-isothermic SHIMC surfaces. Namely we assume

$$Q(w,\bar{w}) = \frac{1}{2}\left(\mathfrak{q}(w,\bar{w}) + \sqrt{-1}\vartheta\right)f(w). \tag{7.8}$$

Throughout this section, to adapt our computation to [12] and avoid a plethora of unnecessary 1/2's in the description, we shall use the following convention:

$$q = \frac{1}{2}\mathfrak{q}, \ \theta := \frac{1}{2}\vartheta. \tag{7.9}$$

And we call a coordinate w simply a θ-*isothermic coordinate* in this section.

Inserting (7.8) to (7.3), we get

$$\bar{f}(\bar{w})q_w(w, \bar{w}) = f(w)q_{\bar{w}}(w, \bar{w}). \tag{7.10}$$

Now we shall introduce a new complex coordinate z by $z = \int f(w)dw$. Then the formula (7.8) implies that q depends only on $t = -(z + \bar{z})$.

We should separate our consideration to the following two cases:

(1) $2 + |Q|^2/\bar{Q}_w = 0$, (2) $2 + |Q|^2/\bar{Q}_w \neq 0$.

Case 1: $2 + |Q|^2/\bar{Q}_w = 0$.

In this case, the Hopf differential is given by

$$Q(w, \bar{w}) = f(w)(q(t) + \sqrt{-1}\theta), \quad q(t) = \begin{cases} \theta \tan(\frac{\theta t}{2}), & \theta \neq 0 \\ -2/t, & \theta = 0. \end{cases} \tag{7.11}$$

Next we shall compute the Riemannian metric I. Inserting (7.9) into (7.10), we have

$$I = e^{u(z,\bar{z})}dzd\bar{z} = \frac{\theta^2(w(z) + \bar{w}(\bar{z}))^2}{\cos^2(\theta t/2)}dzd\bar{z},$$

for $\theta \neq 0$ and

$$I = e^{u(z,\bar{z})}dzd\bar{z} = \frac{4(w(z) + \bar{w}(\bar{z}))^2}{t^2}dzd\bar{z}$$

for $\theta = 0$. The dual surface F^* of F with $\theta \neq 0$ in pseudo hyperbolic space of radius $1/(2|\theta|)$ is described by

$$I^* = \frac{\cos^2(\frac{\theta t}{2})}{\theta^2(w(z) + \bar{w}(\bar{z}))^2}dzd\bar{z},$$

$$(7.12)\qquad Q^*(w,\bar{w}) = \frac{1}{2(w(z)+\bar{w}(\bar{z}))^2} \pm \sqrt{-1}, \quad H^*(w,\bar{w}) = 2\theta\tan(\frac{\theta t}{2}).$$

The formulae (7.12) show that F^* is a spacelike Bonnet surface in $H_1^3(1/(2|\theta|))$ of type D in the classification table due to Chen and Li [15, Theorem 6.3]. Similary we can compute the dual surface F^* of isothermic SHIMC surface F.

PROPOSITION 7.1. *Let (M,F) be a θ-isothermic SHIMC surface in $\mathbf{E}_1^3$ with θ-isothermic coordinate w of the form (7.8) – (7.9). If $2\bar{Q}_w + |Q|^2 = 0$. Then the fundamental quantities of (M,F) are given by*

$$Q(w,\bar{w}) = \frac{q(t)+\sqrt{-1}\theta}{f(w(\zeta))}, \quad q(t) = \begin{cases} \theta\tan(\frac{\theta t}{2}), & \theta \neq 0 \\ -2/t, & \theta = 0. \end{cases}$$

$$H(w,\bar{w}) = \frac{1}{w(z)+\bar{w}(\bar{z})},$$

$$e^{u(z,\bar{z})} = \begin{cases} \frac{\theta^2(w(z)+\bar{w}(\bar{z}))^2}{\cos^2(\theta t/2)}, & \theta \neq 0 \\ \frac{4}{t^2}, & \theta = 0 \end{cases}$$

The dual surface of (M,F) is given by the following formulae :
If $\theta \neq 0$ then the dual surface F^* in $H_1^3(1/(2|\theta|))$ is defined by the folloing formulae :

$$e^{u^*(z,\bar{z})} = \frac{\cos^2(\theta t/2)}{\theta^2(w(z)+\bar{w}(\bar{z}))^2},$$

$$Q^*(w,\bar{w}) = \frac{1}{2(w+\bar{w})} \pm \sqrt{-1}, \quad H^*(w,\bar{w}) = 2\theta\tan(\frac{\theta t}{2}).$$

The dual surface F^* in $H_1^3(\frac{1}{2|\theta|})$ is a spacelike Bonnet surface which belongs to D-family of Chen-Li.
If $\theta = 0$ then the dual surface F^* in $\mathbf{E}_1^3$ is defined by

$$e^{u^*(z,\bar{z})} = \frac{t^2}{4},$$

$$Q^*(w,\bar{w}) = \frac{1}{2(w+\bar{w})}, \quad H^*(w,\bar{w}) = \frac{-4}{t}.$$

The dual surface F^* in $\mathbf{E}^3_1$ is a spacelike Bonnet surface which belongs to D-family of Chen-Li.

We shall call SHIMC surface (M, F) *generic* if (M, F) is not dual to a spacelike Bonnet surface which belongs to D-family.

Case 2: $2 + |Q|^2/\bar{Q}_w \neq 0$.
In this case, inserting

$$Q(w, \bar{w}) = f(w)(q(t) + \sqrt{-1}\theta)$$

into (7.3) and the assumption $2 + |Q|^2/\bar{Q}_w \neq 0$, we can define the following function

$$\mathcal{S}(t) = \frac{1}{|f(w(z))|^2 (w(z) + \bar{w}(\bar{z}))^2}. \tag{7.13}$$

Lemma 7.2. *Let (M, F) be a generic θ-isothermic SHIMC surface in $\mathbf{E}^3_1$. Namely a θ-isothermic SHIMC surface such that $2 + |Q|^2/\bar{Q}_w \neq 0$ Then with respect to the complex coordinate $z = \int f(w)dw$, the Gauss equation reduces to the following ordinary differential equation*:

$$\left(\frac{q''(t)}{q'(t)}\right)' - q'(t) = \mathcal{S}(t)\left(2 - \frac{q^2(t) + \theta^2}{q'(t)}\right), \quad q'(t) > 0. \tag{$\bigstar^+_{\theta^2}$}$$

Here $\mathcal{S}(t)$ is a real analytic function of $t = -(z + \bar{z})$ away from a discrete set. With respect to z, fundamental quantities of M are described as

$$e^{u(z,\bar{z})} = 2q'(t)(w(z) + \bar{w}(\bar{z}))^2, \tag{7.14}$$

$$Q(z, \bar{z}) = \frac{q(t) + \sqrt{-1}\theta}{f(w(z))}, \quad H(z, \bar{z}) = \frac{1}{w(z) + \bar{w}(\bar{z})}. \tag{7.15}$$

The following three lemmata are proved by much the same way in [12].

LEMMA 7.3. *Let $f(w)$ be the function in Lemma 7.2. Then $f(w)$ has the following form*:

$$f(w) = \frac{1}{aw^2 + \sqrt{-1}bw + c}, \quad a, b, c \in \mathbf{R}.$$

LEMMA 7.4 ([12]). *Let (M, F) be a generic θ-isothermic SHIMC surface in $\mathbf{E}^3_1$. Then up to scaling and reparametrisation, its fundamental quantities (u, Q, H) are given by (7.14) – (7.15) where $f(w)$ and $w(z)$ have following forms*:

$$\text{(A)} \qquad f(w) = \frac{1}{4\sqrt{-1}w}, \quad w(z) = -\sqrt{-1}e^{4\sqrt{-1}z},$$

$$\text{(B)} \qquad f(w) = \frac{1}{-w^2 + 4}, \quad w(z) = 2\coth(2z),$$

$$\text{(C)} \qquad f(w) = 2, \quad w(z) = \frac{z}{2},$$

$$\text{(D)} \qquad f(w) = \frac{1}{2w^2}, \quad w(z) = -\frac{1}{2z},$$

$$\text{(E)} \qquad f(w) = \frac{1}{w^2 + 4}, \quad w(z) = -2\cot(2z).$$

LEMMA 7.5. *In the preceding lemma, there are following isomorphisms*:

$$A \cong E \text{ and } C \cong D.$$

THEOREM 7.6. *There exit three classes- A, B and C- of associated families of generic θ-isothermic SHIMC surfaces in $\mathbf{E}^3_1$. The immersion function of each family is given by the Sym formula (5.6) – (5.7) in Proposition 5.2, where the data (u, Q, H) in (5.6) are determined by*

$$e^{u(z,\bar{z})} = 2q'(t)(w(z) + \bar{w}(\bar{z}))^2,$$

$$Q(z, \bar{z}) = \frac{q(t) + \sqrt{-1}\theta}{f(w(z))}, \quad H(z, \bar{z}) = \frac{1}{w(z) + \bar{w}(\bar{z})}.$$

Here $q(t)$ is a solution to the generalised Hazzidakis equation:

$$(\bigstar^{+}_{\theta^2}) \qquad \left(\frac{q''(t)}{q'(t)}\right)' - q'(t) = \mathcal{S}(t)\left(2 - \frac{q^2(t)+\theta^2}{q'(t)}\right), \quad q'(t) > 0.$$

Here coefficient function $\mathcal{S}(t)$ in the generalised Hazzidakis equation is given by

Family	*Coefficient*
A-family	$\mathcal{S}(t) = 1/\sin^2(2t)$
B-family	$\mathcal{S}(t) = 1/\sinh^2(2t)$
C-family	$\mathcal{S}(t) = 1/t^2$

Any generic θ-isothermic SHIMC surface belongs to one of these families A, B or C.

8. Spacelike HIMC Surfaces in $\mathfrak{M}^3_1(c)$

In this section we shall introduce the notion of spacelike surfaces with harmonic inverse mean curvature in $\mathfrak{M}^3_1(c)$. First of all, we shall recall the following two results due to the first-named author [19].

Proposition 8.1. *Let $I(c)$ be a 1-dimensional Riemannian manifold defined by*

$$I(c) = \begin{cases} (\mathbf{R}, g(c)\ ) & c = 0,\ 1, \\ (\mathbf{R}\setminus\{\pm 1\}, g(c)\ ) & c = -1, \end{cases}$$

$$g(c) = \frac{dt^2}{(1+ct^2)^2}.$$

Let $\varphi : (M, z) \to I(c)$ be a smooth mapping. Then φ is a harmonic map if and only if

$$(8.1) \qquad \frac{\partial^2\varphi}{\partial z\partial\bar{z}} - \frac{2c\varphi}{1+c\varphi^2}\left|\frac{\partial\varphi}{\partial z}\right|^2 = 0.$$

Note that the Riemannian manifolds $I(c)$ may be considered as 1-dimensional Riemannian space forms. The distance function derived from the metric $g(-1)$ is an example of *Hilbert distance* on the interval $(-1, 1)$.

PROPOSITION 8.2. *The equation* (8.1) *can be solved as follows*:

$$\varphi = \begin{cases} h + \bar{h}, & c = 0, \\ \frac{h+\bar{h}}{1-c|h|^2} \text{ or } \frac{1-c|h|^2}{h+\bar{h}}, & c = \pm 1, \end{cases}$$

where h is a holomorphic function.

The following definition may be considered as a generalisation of that in Section 5.

DEFINITION 8.3. Let $F : M \to \mathfrak{M}^3_1(c)$ be a spacelike surface in Lorentzian 3-space form. Then M is said to be a *spacelike surface with harmonic inverse mean curvature* (SHIMC surface) if $1/H$ is a harmonic map into $I(-c)$.

Note that surfaces in Riemannian space forms $\mathfrak{M}^3_1(c)$ is HIMC provided the reciprocal of mean curvature is a harmonic map into $I(c)$.

Hereafter we assume that M is *simply connected*. We denote $\mathcal{C}_H$ the *moduli space* of conformal immersions of M into $\mathfrak{M}^3_1(c)$ with prescribed H, namely $\mathcal{C}_H = \{F : M \to \mathfrak{M}^3_1(c) |$ conformal spacelike immersion with mean curvature H $\}/\mathfrak{I}_0(c)$.
Here $\mathfrak{I}_0(c)$ is the identity component of the full isometry group of $\mathfrak{M}^3_1(c)$. Then we can easily deduce (by the fundamental theorem of surface theory) that

$$\mathcal{C}_H \cong \{(u, Q) | \text{ solution to } (\mathrm{GC})_c\}.$$

Here $(\mathrm{GC})_c$ denotes the Gauss-Codazzi equations (G_c) and (C_c).

Lawson correspondences between spacelike CMC surfaces in Lorentzian space forms $\mathfrak{M}^3_1(c)$ can be generalised for SHIMC surfaces.

THEOREM 8.4 (Generalised Lawson correspondences).
Let M be a simply connected Riemann surface and h a holomorphic function on M. We define a function H by $H_c := (1 + c|h|^2)/(h + \bar{h})$. Then the following three spaces are mutually isomorphic.

$$\mathcal{C}_{H_0} \cong \mathcal{C}_{H_1} \cong \mathcal{C}_{H_{-1}}.$$

PROOF. First we should remark that if the ambient space is S_1^3 then $H_1^2 > 1$.

Let (u, Q) be a solution of (GC_c) for $c = \pm 1$. Then $(\tilde{u}, \tilde{Q})$ defined by

$$e^{\tilde{u}} := |1 - ch^2\ |^2\ e^u, \quad \tilde{Q} := (1 - ch^2)Q$$

is a solution to (GC_0). □

In the case of spacelike CMC surfaces, Lawson correspondents preserves Riemannian metrics. However Lawson correspondences for SHIMC surfaces are not metric preserving correspondence but conformal one. This property is consistent with the fact that SHIMC surfaces in $\mathbf{E}_1^3$ admits conformal one-parameter deformations described in Proposition 5.2.

Using the Lawson correspondences described above, we can give immersion formulae for SHIMC surfaces in $\mathfrak{M}_1^3(c)$, $c = \pm 1$.

Let Φ_λ be a solution of the zero curvature equation (5.6) with variable spectral parameter λ. To describe immersion formulae we shall use the following notational convention.

$$\Phi[\tau] := \Phi_\lambda,\ \lambda = (1 - 2\sqrt{-1}\bar{h}\tau)/(1 + 2\sqrt{-1}h\tau), \quad \tau \in \mathbf{R}.$$

Since the zero curvature equation (5.6) is completely integrable, (5.6) have also solutions for all $t \in \mathbf{C}$.

THEOREM 8.5 (Immersion formulae).
Let $\Phi[\tau] : M \times \mathbf{C} \to G^{\mathbf{C}}$ be a complexified solution to (5.2). *Then the followings hold.*

$(c = 0)$ *For every $\tau \in \mathbf{R}$,*

$$F_{\mathbf{E}}(\tau) := \frac{\partial}{\partial \tau}\Phi[\tau] \cdot (\Phi[\tau])^{-1},\ \tau \in \mathbf{R}$$

describes a SHIMC surface in $\mathbf{E}_1^3$ given in Proposition 5.2.

$(c = -1)$ *For any distinct $\tau_1, \tau_2 \in \mathbf{R}$,*

$$F_H(\tau_1, \tau_2) := \mu_H(\Phi[\tau_1], \Phi[\tau_2]) = \Phi[\tau_1]\Phi[\tau_2]^{-1}$$

is a SHIMC surface in H_1^3 *with unit normal vector field*

$$N = \mu_H(\Phi[\tau_1], \Phi[\tau_2])\mathbf{i}.$$

$(c = 1)$ *Let* $\Phi[\sqrt{-1}\tau]$, $\tau \in \mathbf{R}$ *be a complexified solution to* (5.6). *Then for every* $\tau \in \mathbf{R}$*:*

$$F_S(\tau) := p_S(\Phi[\sqrt{-1}t]) = \Phi[\sqrt{-1}t]\ \mathbf{i}'\ \Phi[\sqrt{-1}t]^*$$

is a SHIMC surface in S_1^3 *of mean curvature* $H^2 > 1$. *Its unit normal vector field is*

$$N = -\mu_S(\Phi_{[0]})\mathbf{1}.$$

In particular $F_H(\frac{1}{2}, -\frac{1}{2})$ *and* $F_S(\frac{1}{2})$ *are Lawson correspondents of* $F_{\mathbf{E}}(0)$.

PROOF. Direct computations similar to the proof of Proposition 5.2 give the required result. Here we shall give a proof of the case $c = -1$.

Differentiating $F_H(\tau_1, \tau_2)$, we have

$$\frac{\partial}{\partial z}F_H(\tau_1, \tau_2) = \mu_H(U_1 - U_2), \quad \frac{\partial}{\partial \bar{z}}F_H(\tau_1, \tau_2) = \mu_H(V_1 - V_2),$$

$$U_i = \Phi[\tau_i]^{-1}\frac{\partial}{\partial z}\Phi[\tau_i], \quad V_i = \Phi[\tau_i]^{-1}\frac{\partial}{\partial \bar{z}}\Phi[\tau_i], \quad i = 1, 2.$$

Direct calculation show

$$e^{u(z,\bar{z};\tau_1,\tau_2)} = \frac{(t_1 - t_2)^2}{|(1 + 2\sqrt{-1}f(z)\tau_1)(1 + 2\sqrt{-1}f(z)\tau_2)|^2},$$

$$H(z, \bar{z}; \tau_1, \tau_2) = \frac{1 + 4|f(z)|^2\tau_1\tau_2 + \sqrt{-1}(\tau_1 + \tau_2)(f(z) - \bar{f}(\bar{z}))}{(f(z) + \bar{f}(\bar{z}))(\tau_1 - \tau_2)}.$$

In particular if we choose $(\tau_1, \tau_2) = (1/2, -1/2)$ then

$$e^{u(z,\bar{z};\frac{1}{2},\frac{-1}{2})} = \frac{e^{u(z,\bar{z})}}{|1 + f(z)^2|}, \quad H(z, \bar{z}; \frac{1}{2}, \frac{-1}{2}) = \frac{1 - |f(z)|^2}{f(z) + \bar{f}(\bar{z})}.$$

These formulae imply that $F_H(\frac{1}{2}, \frac{-1}{2})$ is the Lawson correspondent of $F_{\mathbf{E}}(0)$. □

Finally we shall consider SHIMC surfaces in $\mathfrak{M}^3_1(\pm 1)$ with mean curvature $H = (h+\bar{h})/(1+c|h|^2)$.

By similar computations to the preceding theorem, one can prove the following theorem.

THEOREM 8.6. *Let $\Psi[\nu]$ be a solution to*

$$\frac{\partial}{\partial z}\Psi[\nu] = \Psi[\nu]U[\nu],\ \frac{\partial}{\partial \bar{z}}\Psi[\nu] = \Psi[\nu]V[\nu], \tag{8.2}$$

$$\begin{aligned} U[\nu] &= \begin{pmatrix} \frac{1}{4}u_z & \frac{1}{2}\lambda(H_\nu - c)e^{\frac{u}{2}} \\ Qe^{-\frac{u}{2}} & -\frac{1}{4}u_z \end{pmatrix}, \\ V[\nu] &= \begin{pmatrix} -\frac{1}{4}u_{\bar{z}} & \bar{Q}e^{-\frac{u}{2}} \\ \frac{1}{2}\bar{\lambda}(H_\nu + c)e^{\frac{u}{2}} & \frac{1}{4}u_{\bar{z}} \end{pmatrix}, \end{aligned} \tag{8.3}$$

$$\lambda = \frac{1}{\bar{\nu}}\frac{1+ch^2}{\nu^2+ch^2},\quad H_\nu = \frac{\nu\bar{h}+\bar{\nu}h}{1+c|h|^2},\quad \nu \in \mathrm{U}(1).$$

Then the followings hold.

($c=-1$) *For any distinct $\nu_1,\ \nu_2 \in \mathrm{U}(1)$, $F_H := p_H(\Psi[\nu_1], \Psi[\nu_2])$ is a SHIMC surface in H^3_1 with unit normal vector field $N = \mu_H(\Psi_{[1]}, \Psi_{[2]})\mathbf{i}$.*

($c=1$) *For any $\nu \in \mathrm{U}(1)$, $F[\nu] := p_S(\Psi[\nu])$ is a SHIMC surface in S^3_1 with mean curvature $0 < H^2_\nu < 1$ and unit normal vector field $N = \mu_S(\Psi_\lambda)\mathbf{1}$.*

9. Spacelike Bonnet Surfaces with Constant Curvature c

In our previous paper [20], we have classified Bonnet surfaces with constant curvature in Riemannian space forms $\mathfrak{M}^3(c)$. We shall classify spacelike Bonnet surfaces in Lorentzian space forms $\mathfrak{M}^3_1(c)$ in this section and next section. (For timelike case, see [22].) Hereafter we assume that (M,F) is *umbilic free* and $H \neq 0$.

First of all we shall recall the following result. (*cf.* Theorem 6.11 and [15, Theorem 6.1].)

PROPOSITION 9.1. *Let $F : M \longrightarrow \mathfrak{M}^3_1(c)$ be a spacelike surface. Then F is a spacelike Bonnet surface if and only if*

(1) *F is isothermic,*
(2) $1/Q$ *is harmonic with respect to an isothermic coordinate system.*

As in the case of Bonnet surfaces in Riemannian space forms, in order to classify spacelike Bonnet surfaces with constant curvature in $\mathfrak{M}^3_1(c)$ we have only to consider the following two cases: (*cf.* [32].)

(1) spacelike Bonnet surfaces with extrinsic curvature 0 (*i.e.*, $K = c$ for $c = \pm 1$.);
(2) flat spacelike Bonnet surfaces (*i.e.*, $K = 0$.)

In this section we shall consider spacelike Bonnet surfaces with constant curvature c in $\mathfrak{M}^3_1(c)$. Flat surfaces will be treated in the next section.

Let $F : M \longrightarrow \mathfrak{M}^3_1(c)$ be a spacelike Bonnet surface with constant curvature $c = \pm 1$. We take an isothermic coordinate $z = x + \sqrt{-1}y$ and put $Q = \mathfrak{q}/2$. Here $\mathfrak{q}$ is a real smooth function. Then the Gauss-Codazzi equations $(\mathrm{GC})_c$ become:

$$\mathfrak{q}^2 = e^{2u} H^2, \tag{9.1}$$

$$\mathfrak{q}_{\bar{z}} = e^u H_z. \tag{9.2}$$

By using similar arguments in our previous paper [20], we get the following.

THEOREM 9.2. *Let $F : M \longrightarrow \mathfrak{M}^3_1(c)$ be a spacelike Bonnet surface with constant curvature c for $c = \pm 1$ and $z = x + \sqrt{-1}y$ its isothermic coordinate. Then the first fundamental form, the Hopf differential and the mean curvature function are given as follows*:

$$(u, \mathfrak{q}, H) = \left(u(\eta), \frac{e^{\frac{u}{2}}}{f(\xi)}, \frac{\epsilon e^{\frac{u}{2}}}{f(\xi)} \right),$$

where $(\eta, \xi, \epsilon) = (x, y, -1)$ or $(y, x, 1)$.

$$u(\eta) = \begin{cases} \log \frac{\alpha^2}{\cosh^2(\alpha\eta+\beta)} & c = 1, \\ \log \frac{\alpha^2}{\sinh^2(\alpha\eta+\beta)}, \ \log \frac{\alpha^2}{\cosh^2(\alpha\eta+\beta)} \text{ or } \log \frac{1}{(\eta+\beta)^2} & c = -1, \end{cases}$$

$$f(\xi) = \begin{cases} C_1 \cos\alpha\xi + C_2 \sin\alpha\xi, & c = 1, \\ C_1 \cos\alpha\xi + C_2 \sin\alpha\xi,\ C_1 e^{\alpha\xi} + C_2 e^{-\alpha\xi} & \\ \qquad \text{or } C_1\xi + C_2 \ \textit{respectively}, & c = -1. \end{cases}$$

for $\alpha > 0,\ \beta \in \mathbf{R}, (C_1, C_2) \in \mathbf{R}^2 \setminus \{\mathbf{0}\}$.

In general H^3_1 [resp. S^3_1] is regarded as a Lorentzian analogue of S^3 [resp. H^3]. For example, both S^3 and H^3_1 are identified with 3-dimensional simple Lie groups with biinvariant metrics. Both H^3 and S^3_1 are realised as quadrics in Minkowski 4-space. This theorem 9.2 implies a strange fact that every spacelike Bonnet surface with constant curvature 1 [resp. -1] in S^3_1 [resp. H^3_1] has same data $(u, \mathfrak{q}, H)$ with those in S^3 [resp. H^3]. This result also tell us that geometry of surfaces in Lorentzian 3-space forms is not necessarily an easy modification of geometry of surfaces in Riemannian 3-space forms.

10. Flat Spacelike Bonnet Surfaces

In this section we shall investigate flat spacelike Bonnet surfaces in $\mathfrak{M}^3_1(c)$.

Let $F : M \longrightarrow \mathfrak{M}^3_1(c)$ be a flat spacelike Bonnet surface. As in the preceding section we take an isothermic coordinate $z = x + \sqrt{-1}y$ and put $Q = q/2$. Then we have

$$\mathfrak{q}^2 = e^{2u}(H^2 - c), \tag{10.1}$$

$$\mathfrak{q}_{\bar{z}} = e^u H_z. \tag{10.2}$$

Differentiating (10.1) by z and using (10.1) and (10.2), we have

$$\frac{\mathfrak{q}_z}{\mathfrak{q}} = u_z + \frac{HH_z}{H^2 - c}. \tag{10.3}$$

Since (M, F) is a flat spacelike Bonnet surface, both $1/\mathfrak{q}$ and u are harmonic with respect to z. Hence we get

$$\frac{|H_z|^2}{H^2 - c} = \left(\frac{HH_z}{H^2 - c}\right)_{\bar{z}} \tag{10.4}$$

If $H \neq 0$ then (10.4) is equivalent to the harmonicity of $\varphi = 1/H$ with respect to the metric $g(-c)$. In fact one can deduce the following.

$$\frac{\partial^2 \varphi}{\partial z \partial \bar{z}} - \frac{2c\varphi}{1 - c\varphi^2} \left| \frac{\partial \varphi}{\partial z} \right|^2 = 0.$$

Namely (M, F) is a spacelike HIMC surface in $\mathfrak{M}^3_1(c)$.

Theorem 10.1. *Every flat spacelike Bonnet surface in* $\mathbf{E}^3_1$ *is a flat spacelike HIMC surface. Using the same notations in Theorem* 9.2, *we get*

$$(u(\eta), f(\xi)) = (\beta, C_1\xi + C_2) \text{ or } \left(\alpha\eta + \beta, C_1 \cos \frac{\alpha}{2}\xi + C_2 \sin \frac{\alpha}{2}\xi\right).$$

The former case with $C_1 = 0$ corresponds to a spacelike flat and constant mean curvature surface. Hence (M, F) is an open portion of a hyperbolic cylinder. The former case with $C_1 \neq 0$ corresponds to certain spacelike cone or generalised cone.

The equation (10.1) implies that $H^2 - c > 0$. As we saw in Section 8, such SHIMC surfaces in one Lorentzian space form correspond to those in another locally. By using the result on generalised Lawson correspondences described in Theorem 8.4, we get

Theorem 10.2. *All flat spacelike Bonnet surface in* $\mathfrak{M}^3_1(c)$ *for* $c = \pm 1$ *are obtained by generalised Lawson correspondences from flat spacelike Bonnet surfaces in* $\mathbf{E}^3_1$.

Remark. Wang [34] proved that spacelike or timelike Bonnet surfaces with constant curvature in $\mathbf{E}^3_1$ are flat. However she did not classify flat spacelike or timelike Bonnet surfaces in $\mathbf{E}^3_1$.

11. A Certain Reduction of H-surface Equations

In our previous paper [18], the first-named author introduced equations (called H-surface equations) whose solutions are considered as a generalisation of surfaces with prescribed mean curvature in Riemannian space forms $\mathfrak{M}^3(c)$. Here we shall introduce the following equations:

$$dA + \frac{1}{2}[A \wedge A] - c[\psi \wedge \bar{\psi}] = 0, \tag{11.1}$$

$$d\psi + [A \wedge \psi] = \sqrt{-1}H[\psi \wedge \bar{\psi}], \tag{11.2}$$

where $c = 0, \pm 1$, H is a function on a Riemann surface M, A is a Lie algebra $\mathfrak{o}_1^{++}(3)$-valued 1-form on M and ψ is a $(1,0)$-type $\mathfrak{o}_1^{++}(3)^{\mathbf{C}}$-valued 1-form on M. If M is simply connected, the set of solutions (A, ψ) to (11.1) and (11.2) which satisfy the conditions $\mathrm{tr}(\psi \otimes \psi) = 0$ and $\mathrm{tr}(\psi \otimes \bar{\psi}) \neq 0$ is identified with the set of conformal immersions of M into $\mathfrak{M}_1^3(c)$ with mean curvature function H. We call the equations (11.1)–(11.2) *the (gauge theoretic) spacelike H-surface equation*. In this section we shall consider the following reduction of spacelike H-surface equations:

$$A = f\psi + \bar{f}\bar{\psi}. \tag{11.3}$$

Here f is a holomorphic function on M. We shall determine the necessary condition of the reduction. Substituting (11.3) into (11.1) we get

$$f d\psi + \bar{f} d\bar{\psi} + |f|^2 \ [\psi \wedge \bar{\psi}] - c[\psi \wedge \bar{\psi}] = 0.$$

Using (11.2), we obtain

$$f\left(-[A \wedge \psi] + \sqrt{-1}H[\psi \wedge \bar{\psi}]\right) + \bar{f}\left(-[A \wedge \bar{\psi} - \sqrt{-1}H[\psi \wedge \bar{\psi}]\right) + \left(|f|^2 - c\right)[\psi \wedge \bar{\psi}] = 0.$$

Using (11.3) again, we have

$$\left\{\sqrt{-1}H(f - \bar{f}) - (|f|^2 + c)\right\}[\psi \wedge \bar{\psi}] = 0. \tag{11.4}$$

Thus if we assume that

$$H = \frac{|f|^2 + c}{\sqrt{-1}(f - \bar{f})} \tag{11.5}$$

then (11.4) is always satisfied. It is easy to see that the reciprocal of the function H given by the formula (11.5) is a harmonic map into $I(-c)$. Thus we obtained a characterisation of SHIMC surfaces. Namely SHIMC surfaces in $\mathfrak{M}_1^3(c)$ are spacelike H-surfaces in $\mathfrak{M}_1^3(c)$ which satisfy the reduction condition (11.3).

Appendix

In this appendix we shall give a brief story of Frenet-Serret formulae for spacelike plane curves.

Let $a(s) = (a_1(s), a_2(s))$ be a spacelike curve in Minkowski plane $\mathbf{E}^2_1(\xi_1, \xi_2)$ parametrised by the arclength parameter s. We shall define an orthonormal frame field $\mathcal{E} = (e_1, e_2)$ along a by

$$e_2(s) = \frac{d}{ds}a(s), \quad e_1(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e_2(s)$$

and call it the *Frenet frame field of* a. The Frenet frame field $\mathcal{E}$ satisfies the following Frenet-Serret equation.

$$\text{(A.1)} \qquad \frac{d}{ds}\mathcal{E} = \mathcal{E}\begin{pmatrix} 0 & \kappa \\ \kappa & 0 \end{pmatrix}$$

for some function $\kappa(s)$ defined along a. The function κ is called the *curvature of* a. It is easy to see that a spacelike curve with curvature $\kappa = 0$ is congruent to a spacelike line. Further a spacelike curve with nonzero constant curvature is congruent to a spacelike hyperbola. Since the matrix valued function $\mathcal{E}^{-1}d\mathcal{E}/ds$ takes the value in the Lie algebra $\mathfrak{o}_1(2)$ of the Lorentz group $\mathrm{O}_1(2)$, we can easily get the fundamental theorem of spacelike curve theory.

Next we shall consider spacelike curves with prescribed curvature. Let $a(s)$ be a spacelike curve with Frenet frame e as before. We may assume that e_2 takes value in the upper half component $(S^1_1)_+ = \{(\xi_1, \xi_2) \in \mathbf{E}^2_1 \mid -\xi_1^2 + \xi_2^2 = 1, \ \xi_1 > 0\}$ of a unit spacelike hyperbola S^1_1. We can write $e_2(s) = (\cosh\phi(s), \sinh\phi(s))$. Here ϕ is the hyperbolic angle function. The curvature κ of a satisfies

$$\text{(A.2)} \qquad \kappa = \frac{d}{ds}\phi.$$

The spacelike curve $a(s)$ with curvature $\kappa = \frac{1}{C_1 s + C_2}$ is given by the following formula.

$$\text{(A.3)} \qquad (a_1, a_2) = (\int \sinh\phi(s)ds, \int \cosh\phi(s)ds).$$

We shall determine spacelike curves with curvature $\kappa = \frac{1}{C_1 s + C_2}$, $C_1, C_2 \in \mathbf{R}$. Hereafter we shall treat the case $C_1 \neq 0$. (If $C_1 = 0$ then, as we saw above, the spacelike curve is congruent to a spacelike line or spacelike hyperbola.)

Let us denote the reciprocal of κ by ρ. Then we have

$$\phi(s) = \int_0^s \frac{1}{\rho} ds + \phi_0 = \int_0^s \frac{1}{C_1} \frac{d\rho}{\rho} = \frac{1}{C_1} \log \frac{\rho}{\rho_0} + \phi_0.$$

Without loss of generality we can choose $\phi_0 = 0$. Thus we get $\rho = C_2 e^{C_1 \phi}$.

Under a linear isometry

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} C_1 & -1 \\ -1 & C_1 \end{pmatrix} \begin{pmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{pmatrix} + \frac{C_2 e^{C_1}}{C_1^2 - 1} \begin{pmatrix} 1 \\ -C_1 \end{pmatrix}$$

of $\mathbf{E}_1^2$, the required spacelike curve $a(s)$ is transformed as

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} = \frac{C_2 e^{C_1 \phi}}{\sqrt{C_1^2 - 1}} \begin{pmatrix} \sinh \phi \\ \cosh \phi \end{pmatrix}.$$

Thus we get the following Lorentzian polar representation of the required curve.

$$-\tilde{a}_1^2 + \tilde{a}_2^2 = \frac{C_2^2}{C_1^2 - 1} e^{2C_1 \phi}.$$

This curve $\tilde{a} = (\tilde{a}_1, \tilde{a}_2)$ is a Lorentzian analogue of a logarithmic spiral in Euclidean plane. We shall call this curve a *logarithmic pseudo-spairal.*

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