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SPACES FOR WHICH THE GENERALIZED CANTOR SPACE 2^J IS A REMAINDER

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ABSTRACT. Let X be a locally compact noncompact space, m be an infinite cardinal and $|J| = m$. Let $F(X)$ be the algebra of continuous functions from X into $\mathbf{R}$ which have finite range outside of an open set with compact closure and let $I(X) = \{g \in F(X)$: g vanishes outside of an open set with compact closure$\}$. Conditions on $R(X) = F(X)/I(X)$ and internal conditions are obtained which characterize when X has 2^J as a remainder.

1. Introduction. Throughout this paper all spaces are assumed to be completely regular and Hausdorff. We let LC denote the class of all locally compact and noncompact spaces. A compactification of a space X is a compact space which contains X as a dense subspace and a remainder of X is any $aX \setminus X$ where aX is a compactification of X. If aX and bX are two compactifications of X, then $aX \leqslant bX$ if there is a continuous function g: $aX \to bX$ such that $g(x) = x$ for each $x \in X$. For a set A let $|A|$ denote the cardinality of A.

Recently Hatzenbuhler and Mattson [**HM**] have obtained an internal characterization which characterizes when a given space $X \in$ LC has every compact metric space as a remainder. The condition given by them assures that if X satisfies this condition the Cantor space $2^{\mathbf{N}}$ is a remainder of X, where $\mathbf{N}$ is the set of natural numbers and 2 is the discrete space $\{0, 1\}$. Their result then follows from the fact that every compact metric space is a continuous image of $2^{\mathbf{N}}$. It is thus natural to ask when for a given cardinal m and a space $X \in$ LC, 2^J is a remainder of X where $|J| = m$. In this connection we briefly recall the construction of the Freudenthal compactification.

DEFINITION 1.1. Let X be a space. An ordered pair (G, H) is called an *f-pair* in X if G and H are disjoint open subsets of X and $X \setminus (G \cup H)$ is compact.

Let $X \in$ LC. For subsets A and B of X, let us define the relation δ by $A\ \delta\ B$ if and only if there is an f-pair (G, H) in X such that $\mathrm{cl}_X A \subseteq G$ and $\mathrm{cl}_X B \subseteq H$. It is well known that δ is a compatible proximity relation on X and the Samuel compactification fX corresponding to this proximity relation is the Freudenthal compactification of X [**W**, 41.2, 41B]. It is known [**R**] that $fX \setminus X$ is zero dimensional and if aX is any compactification of X such that $aX \setminus X$ is zero dimensional, then $aX \leqslant fX$. By a zero dimensional space, we mean a space which has a basis consisting of clopen, i.e., both

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closed and open sets. It follows in particular that if 2^J is a remainder of X, then 2^J is a continuous image of $fX \setminus X$.

In **[E]** Efimov introduces the concept of a dyadic family of power m and proves that $[0, 1]^J$, where $|J| = m$, is a continuous image of a compact space Y if and only if Y has a dyadic family of power m. In the view of these observations, one naturally expects to obtain an internal characterization, via the Freudenthal compactification which characterizes when a space $X \in \mathrm{LC}$ has 2^J as a remainder. For this purpose we slightly modify Efimov's definition of a dyadic family.

DEFINITION 1.2. Let X be a space, m be a cardinal and J be a set with $|J| = m$. A family $\{(U_j^{-1}, U_j^1): j \in J\}$ consisting of f-pairs in X is called a *dyadic family of power* m in X if for every finite collection of distinct elements $j_1, \ldots, j_n$ of J and any finite sequence $i_1, \ldots, i_n$ in $\{-1, 1\}$, $\mathrm{cl}_X(U_{j_1}^{i_1} \cap \cdots \cap U_{j_n}^{i_n})$ is not compact.

We prove that if $X \in \mathrm{LC}$, m is an infinite cardinal and $|J| = m$, then 2^J is a remainder of X if and only if X has a dyadic family of power m. We also give an algebraic characterization which is equivalent to the one given above.

For a space X let $\kappa(X)$ denote the set of all open subsets of X with compact closure in X. Let $C(X)$ be the algebra of all continuous functions from X into the set of real numbers $\mathbf{R}$.

DEFINITION 1.3. Let $X \in \mathrm{LC}$. We set

$$F(X) = \{g \in C(X): g(X \setminus V) \text{ is finite for some } V \in \kappa(X)\},$$
$$I(X) = \{g \in C(X): X \setminus g^{-1}(0) \in \kappa(X)\}.$$

Clearly $F(X)$ is a subalgebra of $C(X)$ and $I(X)$ is an ideal in $F(X)$. We set $R(X) = F(X)/I(X)$.

We prove that if $X \in \mathrm{LC}$, then the structure space $\mathrm{Max}\, F(X)$ of $F(X)$ can be identified with fX. We also prove that if m is an infinite cardinal and $|J| = m$, then 2^J is a remainder of X if and only if the group of units of $R(X)$ has a subgroup G of cardinality m such that $g^2 = 1$ for every $g \in G$ and G is linearly independent over $\mathbf{R}$.

2. Structure space of $F(X)$. If g is a function from a set A into $\mathbf{R}$, $Z(g) = \{a \in A: g(a) = 0\}$ is the zero set of g. If $\alpha \in \mathbf{R}$, then we will use the same notation α to denote the constant function from A into $\mathbf{R}$ whose value is α. Let $X \in \mathrm{LC}$. It is easy to verify that $\{X \setminus Z(g): g \in I(X)\} = B_X$ forms a base for open sets in X. For $x \in X$ let us define $M_x = \{g \in F(X): g(x) = 0\}$. Then M_x is a maximal ideal in $F(X)$ and if x and y are distinct elements of X, then $M_x \neq M_y$ since B_X forms a base for open sets in X. A maximal ideal M of $F(X)$ is, by definition, *fixed* if $M = M_x$ for some $x \in X$, otherwise M is *free*.

Let S be any commutative ring with identity. The structure space of S is the set of all maximal ideals $\mathrm{Max}\, S$ of S topologized by taking the sets of the form $E(s) = \{M \in \mathrm{Max}\, S: s \in M\}$ as a base for closed sets **[GJ**, 7M**]**. $\mathrm{Max}\, S$ with this topology is compact but not necessarily Hausdorff. If S is von Neumann regular, then $\mathrm{Max}\, S$ is Hausdorff. Recall that S is von Neumann regular if for each $a \in S$, there exists $b \in S$ such that $a^2 b = a$.

PROPOSITION 2.1. *Let* $X \in \mathrm{LC}$ *and* $M \in \operatorname{Max} F(X)$.

(a) $R(X)$ *is von Neumann regular and hence* $\operatorname{Max} R(X)$ *is a compact Hausdorff space.*

(b) M *is free if and only if* $I(X) \subseteq M$.

PROOF. (a) Let $g \in F(X)$ and $V \in \kappa(X)$ be such that $g(X \setminus V) = \{\alpha_1, \ldots, \alpha_n\}$. Note that $X \setminus V \neq \varnothing$ since X is not compact. Let $K = Z(g)$ and $L = g^{-1}(\{\alpha_i\colon \alpha_i \neq 0\})$. Then K and L are disjoint zero sets in X. Thus there is a continuous function $h\colon X \to [0,1]$ such that $K \subseteq \operatorname{int}_X Z(h) \subseteq Z(h) = A$ and $L \subseteq h^{-1}(1) \subseteq X \setminus \operatorname{int}_X Z(h) = B$. A and B are closed sets in X and $A \cup B = X$. Define $w\colon X \to \mathbf{R}$ by $w(x) = 0$ if $x \in A$ and $w(x) = h(x)/g(x)$ if $x \in B$. w is well defined and continuous. Note that $w(X \setminus V) \subseteq \{0\} \cup \{1/\alpha_i\colon \alpha_i \neq 0\}$. It follows that $w \in F(X)$ and $g^2 w - g \in I(X)$. Thus $R(X)$ is von Neumann regular.

(b) Let $x \in X$ and $V \in \kappa(X)$ be a neighbourhood of x. Then there is a continuous function $g\colon X \to [0,1]$ such that $g(x) = 1$ and $g(X \setminus V) = \{0\}$. Thus $g \in I(X) \setminus M_x$. Consequently a fixed maximal ideal cannot contain $I(X)$. Now, let M be a free maximal ideal. Suppose that there exists $g \in I(X) \setminus M$. Since M is maximal, then $gk - 1 \in M$ for some $k \in F(X)$. Let $V = X \setminus Z(g) \in \kappa(X)$. For each $x \in \operatorname{cl}_X V$, $M \setminus M_x \neq \varnothing$. Thus $\operatorname{cl}_X V \subseteq \bigcup\{X \setminus Z(t)\colon t \in M\}$. Since $\operatorname{cl}_X V$ is compact, then there are $t_1, \ldots, t_n \in M$ such that $\operatorname{cl}_X V \subseteq \bigcup\{X \setminus Z(t_i)\colon i = 1, \ldots, n\}$. Let $t = t_1^2 + \cdots + t_n^2 \in M$. Then $\operatorname{cl}_X V \subseteq X \setminus Z(t)$. There is an $0 < \varepsilon < 1$ such that $t(x) \geqslant \varepsilon$ for each $x \in \operatorname{cl}_X V$. If $r = (gk - 1)^2 + t$, then $r \in M$ and $r(x) \geqslant \varepsilon$ for all $x \in X$. Thus M contains an invertible element, a contradiction. So $M \supseteq I(X)$.

We have already seen that the function $x \to M_x$ sets up a one-to-one correspondence between X and the fixed maximal ideals in $F(X)$. Hence X already constitutes an index set for the fixed maximal ideals in $F(X)$. We enlarge it to an index set fX for $\operatorname{Max} F(X)$, so that $\operatorname{Max} F(X) = \{M_y\colon y \in fX\}$ and for distinct $y, z \in fX$, $M_y \neq M_z$. For $g \in F(X)$ let $F(g) = \{y \in fX\colon g \in M_y\}$. If $\theta\colon fX \to \operatorname{Max} F(X)$ is the function defined by $\theta(y) = M_y$, then $\theta^{-1}(E(g)) = F(g)$ for $g \in F(X)$. Thus $\{F(g)\colon g \in F(X)\}$ forms a base for closed sets of a topology on fX and with this topology fX is compact and homeomorphic to $\operatorname{Max} F(X)$.

THEOREM 2.2. *Let* $X \in \mathrm{LC}$. *For a subset* A *of* X *let* $A' = \operatorname{cl}_{fX} A \setminus X$.

(a) fX *is a compactification of* X *and* $fX \setminus X$ *is homeomorphic to* $\operatorname{Max} R(X)$.

(b) *Each function* $g \in F(X)$ *has a unique continuous extension* $g^e\colon fX \to \mathbf{R}$ *and* $F(g) = Z(g^e)$.

(c) *Let* (G, H) *be an f-pair in* X. *Then there exist* $g \in F(X)$ *and* $W \in \kappa(X)$ *such that* $X \setminus (G \cup H) \subseteq W$, $\operatorname{cl}_X G \setminus W \subseteq g^{-1}(-1)$ *and* $\operatorname{cl}_X H \setminus W \subseteq g^{-1}(1)$. *If* U *is any open subset of* X, *then* $(G \cap U)' = G' \cap U'$. *In particular* G' *and* H' *are disjoint clopen subsets of* X' *whose union is* X'.

(d) fX *is the Freudenthal compactification of* X.

PROOF. (a) We have already observed that fX is compact. By 2.1(b) if $g \in I(X)$, then $fX \setminus F(g) = X \setminus Z(g)$. It follows that the topology on X coincides with the subspace topology inherited from fX. Also, $X = \bigcup\{X \setminus Z(g)\colon\ g \in I(X)\} = \bigcup\{fX \setminus F(g)\colon\ g \in I(X)\}$. So X is an open subspace of fX. If $h \in F(X)$ and $fX \setminus F(h) \neq \varnothing$, then $h \neq 0$. So $x \in fX \setminus F(h)$ for some $x \in X$. Thus X is dense in fX. We now proceed to show that fX is Hausdorff. Let $y, z \in fX$ and $y \neq z$. If $y, z \in X$, then y and z can be separated by open sets in X and hence in fX since X is open in fX. Thus suppose without loss of generality that $z \notin X$. Let $g \in M_y \setminus M_z$. By 2.1(a), $g^2h - g \in I(X) \subseteq M_z$ for some $h \in F(X)$. Let $b = gh - 1$. Since M_z is prime, then $b \in M_z$. Note also that $b \notin M_y$. Let $V = X \setminus Z(gb) \in \kappa(X)$. Let $W \in \kappa(X)$ be such that $\mathrm{cl}_X V \subseteq W$. There is a continuous function $u\colon X \to [0, 1]$ such that $\mathrm{cl}_X V \subseteq Z(u)$ and $X \setminus W \subseteq Z(u - 1)$. Note that $u - 1 \in I(X)$ and $ugb = 0$. So $z \in fX \setminus F(ug)$, $y \in fX \setminus F(b)$ and $F(ug) \cup F(b) = fX$. This proves that fX is Hausdorff. To see that X' and $P = \operatorname{Max} R(X)$ are homeomorphic, consider the natural homomorphism ϕ: $F(X) \to R(X)$. ϕ induces a bijection $\phi'\colon P \to X'$ defined by $\phi'(N) = z$ if and only if $\phi^{-1}(N) = M_z$. ϕ' is continuous since $(\phi')^{-1}(F(g) \setminus X) = E(g + I(X))$ for every $g \in F(X)$. Both X' and P are compact Hausdorff, thus ϕ' is a homeomorphism.

(b) Let $g \in F(X)$ and $V \in \kappa(X)$ be such that $g(X \setminus V) = \{\alpha_1, \ldots, \alpha_n\}$ where $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $h = (g - \alpha_1) \cdots (g - \alpha_n)$. Since $X \setminus Z(h) \subseteq V$, then $h \in I(X)$. So $fX = \mathrm{cl}_X V \cup F(h) = \mathrm{cl}_X V \cup F(g - \alpha_1) \cup \cdots \cup F(g - \alpha_n)$. If $i \neq j$, then $0 \neq \alpha_i - \alpha_j \notin M_y$, for any $y \in fX$. Thus $F(g - \alpha_i) \cap F(g - \alpha_j) = \varnothing$ for $i \neq j$. Moreover if $x \in \mathrm{cl}_X V \cap F(g - \alpha_i)$, then $g - \alpha_i \in M_x$, i.e., $g(x) = \alpha_i$. We define g^e: $fX \to \mathbf{R}$ by $g^e(y) = \alpha_i$ if $y \in F(g - \alpha_i)$ and $g^e(y) = g(y)$ if $y \in \mathrm{cl}_X V$. Then g^e is well defined and continuous. It is routine to verify that $F(g) = Z(g^e)$ and g^e extends g.

(c) Let $L = X \setminus (G \cup H)$. Let $V, W \in \kappa(X)$ be such that $L \subseteq W \subseteq \mathrm{cl}_X W \subseteq V$. Then $C = \mathrm{cl}_X G \cap \mathrm{cl}_X V \setminus W$ and $D = \mathrm{cl}_X H \cap \mathrm{cl}_X V \setminus W$ are disjoint closed subsets of the compact space $\mathrm{cl}_X V$. Thus there is a continuous function $h\colon \mathrm{cl}_X V \to [-1, 1]$ such that $C \subseteq h^{-1}(-1)$ and $D \subseteq h^{-1}(1)$. Let $g\colon X \to [-1, 1]$ be defined by $g(x) = -1$ if $x \in \mathrm{cl}_X G \setminus W$, $g(x) = 1$ if $x \in \mathrm{cl}_X H \setminus W$ and $g(x) = h(x)$ if $x \in \mathrm{cl}_X V$. It is easy to see that g satisfies the required properties. Since $W \in \kappa(X)$ and L is compact, then $g^e(G) \subseteq \{-1\}$, $g^e(H) \subseteq \{1\}$ and $G' \cup H' = X'$. So G' and H' are disjoint clopen subsets of X' whose union is X'. Now let U be any open subset of X. Suppose that $y \in U' \cap G' \setminus (U \cap G)'$ for some $y \in X'$. There is an open neighbourhood S of y in fX such that $S \cap (L \cup H' \cup (U \cap G)) = \varnothing$. Then $S \cap U \subseteq H$ and consequently $y \in (S \cap U)' \subseteq H'$, a contradiction. So $(U \cap G)' = U' \cap G'$.

(d) We must show that if $A, B \subseteq X$, then $\mathrm{cl}_{fX} A \cap \mathrm{cl}_{fX} B = \varnothing$ if and only if there is an f-pair (G, H) in X such that $\mathrm{cl}_X A \subseteq G$ and $\mathrm{cl}_X B \subseteq H$. "if" part is clear from (c). Thus suppose that $A, B \subseteq X$ and $\mathrm{cl}_{fX} A \cap \mathrm{cl}_{fX} B = \varnothing$. $\{F(g)\colon\ g \in F(X)\} = \{Z(g^e)\colon\ g \in F(X)\}$ is a basis for closed sets in fX and it is closed under finite intersections. Hence there exist $g, h \in F(X)$ such that $\mathrm{cl}_{fX} A \subseteq F(g)$, $\mathrm{cl}_{fX} B \subseteq F(h)$ and $F(g) \cap F(h) = Z(g^e) \cap Z(h^e) = Z((g^2 + h^2)^e) = \varnothing$. This implies that $g^2 + h^2$ is a unit in $F(X)$. Let $w = g^2/(g^2 + h^2)$. Then $0 \leqslant w(x) \leqslant 1$ for all $x \in X$. Let $V \in \kappa(X)$ be such that $w(X \setminus V) = \{\alpha_1, \ldots, \alpha_n\}$. Pick a real number $0 < \alpha < 1$ such that $\alpha \neq \alpha_i$ for $i = 1, \ldots, n$. Let $G = \{x \in X\colon w(x) < \alpha\}$ and $H = \{x \in X\colon w(x) > \alpha\}$. Then $\mathrm{cl}_X A \subseteq G$, $\mathrm{cl}_X B \subseteq H$, $G \cap H = \varnothing$ and $X \setminus (G \cup H) \subseteq (w^e)^{-1}(\alpha) \subseteq X$. Thus (G, H) is an f-pair with the required properties.

3. 2^J as a remainder. Let D be the discrete space $\{-1, 1\}$. Then 2^J is homeomorphic to D^J. In what follows, it will be more convenient to work with D than 2 and we will do so. We first state a lemma which follows easily from Theorem 2.2(c) by induction.

LEMMA 3.1. *Let $X \in$ LC and $(G_1, H_1), \ldots, (G_n, H_n)$ be a finite sequence of f-pairs in X. Then $(G_1 \cap \cdots \cap G_n)' = G_1' \cap \cdots \cap G_n'$, where for a subset A of X, $A' = \mathrm{cl}_{fX} A \setminus X$.*

We now state our main result.

THEOREM 3.2. *Let $X \in$ LC, m be an infinite cardinal and J be a set of cardinality m. Then the following are equivalent.*

(a) *X has a dyadic family of power m.*

(b) *The group of units of $R(X)$ has a subgroup G of cardinality m such that $g^2 = 1$ for all $g \in G$ and G is linearly independent over* $\mathbf{R}$.

(c) *D^J is a remainder of X.*

PROOF. (a) implies (b). Let $\Delta = \{(U_j^{-1}, U_j^1): j \in J\}$ be a dyadic family of power m in X. For each $j \in J$ we pick a function $g_j \in F(X)$ and a member V_j of $\kappa(X)$ such that for $i \in D$ $\mathrm{cl}_X U_j^i \setminus V_j \subseteq g_j^{-1}(i)$. The existence of g_j is guaranteed by 2.2(c). Let $r_j = g_j + I(X), j \in J$. Since

$$X \setminus Z\left(g_j^2 - 1\right) \subseteq L_j \cup \mathrm{cl}_X V_j$$

where $L_j = X \setminus (U_j^{-1} \cup U_j^1)$, then $r_j^2 = 1$ for all $j \in J$. Let G be the group generated by $\{r_j: j \in J\}$. Then clearly $r^2 = 1$ for all $r \in G$. Let $j_1, \ldots, j_n$ be distinct elements of J and H be the subgroup of G generated by $A = \{r_{j_k}: k = 1, \ldots, n\}$. Let T be the linear subspace of $R(X)$ spanned by H. Since $|A| \leq n$ and $r_j^2 = 1$ for all $j \in J$, then $|H| \leq 2^n$. It follows that $\dim_{\mathbf{R}} T \leq 2^n$. For an n-tuple $\eta = (\eta_1, \ldots, \eta_n) \in D^n$ let $e_\eta \in F(X)$ be defined by

$$e_\eta = 2^{-n}\left(1 + \eta_1 g_{j_1}\right) \cdots \left(1 + \eta_n g_{j_n}\right).$$

Let $P_k = \mathrm{cl}_X V_{j_k} \cup L_{j_k}, 1 \leq k \leq n$, and $P = P_1 \cup \cdots \cup P_n$. We claim that $e_\eta \notin I(X)$. For suppose that $e_\eta \in I(X)$ and $V = X \setminus Z(e_\eta)$. Then $P \cup \mathrm{cl}_X V$ is compact. Thus if $Q = U_{j_1}^{\eta_1} \cap \cdots \cap U_{j_n}^{\eta_n}$, then $\hat{Q} = Q \setminus P \cup \mathrm{cl}_X V \neq \varnothing$ since Δ is a dyadic family. Let $x \in \hat{Q}$. Then for $1 \leq k \leq n$, $x \in U_{j_k}^k \setminus V_{j_k}$ which implies that $2^{-1}(1 + \eta_k g_{j_k}(x)) = 2^{-1}(1 + \eta_k^2) = 1$. Thus $0 = e(x) = 1$, a contradiction. Let $\hat{e}_\eta = e_\eta + I(X)$. Then $\hat{e}_\eta$ is a nonzero element of T. If η and ρ are distinct n-tuples in D^n, then it is easy to see that $\hat{e}_\eta \hat{e}_\rho = 0$ and $\hat{e}_\eta$ is an idempotent in T. Thus $\{\hat{e}_\eta: \eta \in D^n\}$ is a linearly independent subset of T containing exactly 2^n elements. This shows that $\dim_{\mathbf{R}} T = 2^n$. Thus $|H| = 2^n$ and H is linearly independent over $\mathbf{R}$. Since every finite subset B of G is also a subset of a subgroup H described as above, then B is linearly independent. Also the argument given above shows that $|\{r_j: j \in J\}| = m$. Thus $|G| = m$.

(b) implies (c). G is a 2-group and so it has a basis. This means that there is a subset B of G such that if $b_1, \ldots, b_n$ are distinct elements of B, then $b_1 \cdots b_n \neq 1$ and each element of G can be written as a finite product of elements in B. Since $|G| = m$ and m is an infinite cardinal, then $|B| = m$. We define a function ψ: Max $R(X) \to D^B$ as follows: Let $P = \text{Max } R(X)$ and $M \in P$. If $b \in B$ then $b^2 - 1 \in M$. Thus either

$b-1 \in M$ or $b+1 \in M$. But $2 \notin M$ and so only one of $b-1$ or $b+1$ may be in M. We define $\psi(M)(b) = -1$ if $b+1 \in M$ and $\psi(M)(b) = 1$ if $b-1 \in M$. Let π_b: $D^B \to D$ denote the bth projection. If $b_1, \ldots, b_n$ are distinct elements of B and $i_1, \ldots, i_n \in D$, then $\psi^{-1}(\pi_{b_1}^{-1}(i_1) \cap \cdots \cap \pi_{b_n}^{-1}(i_n)) = P \setminus E((b_1+i_1)\cdots(b_n+i_n))$. Hence ψ is continuous. Moreover ψ is onto, for let $x \in D^B$. The ideal T of $R(X)$ generated by $\{b - x(b)\colon b \in B\}$ is distinct from $R(X)$. For otherwise there are elements $r_1, \ldots, r_n \in R(X)$ and $b_1, \ldots, b_n \in B$ such that b_i's are distinct and

$$\text{(i)} \qquad r_1(b_1 - x(b_1)) + \cdots + r_n(b_n - x(b_n)) = 1.$$

Let r be the product of the elements $b_k + x(b_k)$, $1 \leq k \leq n$. Then multiplying both sides of (i) by r we obtain $r = 0$. Since B is a basis for G, then r is a linear combination of the pairwise distinct elements $1, b_1, \ldots, b_n, b_1b_2, \ldots, b_1b_2\cdots b_n$ of G with 1 having the coefficient ∓ 1. This is a contradiction as G is linearly independent over $\mathbf{R}$. So $T \neq R(X)$. If M is a maximal ideal containing T, then $\psi(M) = x$. So ψ is onto. It follows that D^B is a continuous image of P and so of $fX \setminus X$ by 2.2(a). Now, utilizing upper semicontinuous decompositions as in **[M]** we can construct a compactification aX of X with $aX \setminus X = D^B$.

(c) implies (a). Let aX be a compactification of X such that $aX \setminus X = D^J$. Then $fX \geq aX$ since D^J is zero dimensional. Let $\phi\colon fX \setminus X \to D^J$ be a continuous surjection. For $j \in J$ and $i \in D$, let us set $W_j^i = \phi^{-1}(\pi_j^{-1}(i))$. Then W_j^{-1} and W_j^1 are disjoint clopen sets in X' whose union is X'. Let V_j^{-1} and V_j^1 be disjoint open neighbourhoods of W_j^{-1} and W_j^1, respectively, in fX. Let $U_j^i = X \cap V_j^i$ for $i \in D$ and $j \in J$. If $T = V_j^{-1} \cup V_j^1$, then $(X \setminus T \cap X) \cap T = \varnothing$ and T is an open neighbourhood of X'. Thus $\mathrm{cl}_{fX}(X \setminus X \cap T) \subseteq X$, i.e., $X \setminus X \cap T$ is compact. So (U_j^{-1}, U_j^1) is an f-pair in X for each $j \in J$. Let Δ be the set of all these pairs. If $j_1, \ldots, j_n$ are distinct elements of J and $i_1, \ldots, i_n \in D$, then by Lemma 3.1, $(\bigcap U_{j_k}^{i_k})' = \bigcap (U_{j_k}^{i_k})' = \bigcap W_{j_k}^{i_k} = \phi^{-1}(\bigcap \pi_{j_k}^{-1}(i_k)) \neq \varnothing$ where the intersections are taken over k, $1 \leq k \leq n$. Thus Δ is a dyadic family of power m in X.

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