BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 1, January 1973

SPACES OF EQUIVARIANT SELF-EQUIVALENCES OF SPHERES

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Communicated by William Browder, July 13, 1972

ABSTRACT. Let $F(S^m)$ denote the identity component of the space of homotopy self-equivalences of S^m and let $F = \text{inj lim}_m F(S^m)$. This paper studies the homotopy properties of certain equivariant analogs of the infinite loop space F.

1. **Introduction.** Let G be a compact Lie group and let W be a free, finite dimensional, real G-module equipped with a G-invariant metric. Let $S(W)$ be the unit sphere of W and denote by $F(W)$ the identity component of the space of equivariant self-equivalences of $S(W)$ with the compact-open topology.

If V and W are free G-modules as above, then $V \oplus W$ is also a free G-module. Since $S(V \oplus W)$ is equivariantly homeomorphic to the join of $S(V)$ and $S(W)$, there is a continuous inclusion of $F(V)$ into $F(V \oplus W)$ defined by taking joins with the identity on $S(W)$. In particular, if kW denotes the direct sum of k copies of W, there is an inclusion of $F(kW)$ in $F((k+1)W)$. Define

$$F_G = \text{inj lim}_k F(kW). \tag{1.1}$$

If G is the trivial group then $F_G = F$ is a familiar and widely studied object. An important aspect of this space is the existence of two infinite loop space structures, one induced by composition multiplication, the other induced by a canonical homotopy equivalence from F to the identity component of inj $\lim_m \Omega^m(S^m)$. One can show that F_G also has an infinite loop space structure induced by composition multiplication. Our results generalize to F_G the second infinite loop space structure on F.

Let BG denote a classifying space for G, let $\mathfrak{g}$ be the Lie algebra of G and let G act on $\mathfrak{g}$ via the adjoint representation. The balanced product of EG and $\mathfrak{g}$ is a vector bundle over BG that we shall call ζ. Let BG^ζ denote its Thom space.

THEOREM 1. *On the category of connected finite* CW-*complexes there is a natural equivalence of homotopy functors*

AMS (MOS) subject classifications (1970). Primary 55D10, 55D35; Secondary 55E45.

Key words and phrases. Free linear representation, equivariant self-equivalence, infinite loop space, transfer map, ex-space.

[1] Partially supported by NSF Grant GP-34197.

[2] Partially supported by NSF Grant GP-19530.

$$\lambda_G : [\ ; F_G] \to \{\ ; BG^\zeta\}$$

Here $\{A; B\}$ denotes the homotopy classes of pointed S-maps from A to B.

If Y is a pointed space, let $Q_0(Y)$ be the identity component of inj $\lim_k \Omega^k S^k(Y)$. The exponential law provides a natural equivalence from $\{\ ; Y\}$ to $[\ ; Q_0(Y)]$ on the category of connected finite CW-complexes. Combining this with Theorem 1, we obtain the following result.

THEOREM 2.[3] *The space F_G is homotopy equivalent to $Q_0(BG^\zeta)$.*

If G is the trivial group, Theorem 1 reduces to the usual equivalence

(1.2) $$\lambda[\ ; F] \to \{\ ; S^0\}$$

given by sending $f : X \to F(S^n)$ to the map $h(f) : X * S^n \to S^{n+1}$ obtained by applying the Hopf construction to the adjoint of f. If G is a finite group, then ζ is 0-dimensional and $BG^\zeta = BG^+$, the disjoint union of BG with a point. In this case F_G has the homotopy type of $Q_0(BG) \times Q_0(S^0)$. The only compact Lie groups of positive dimension that act freely on spheres are S^1, S^3, and $N(S^1)$, the normalizer of S^1 in S^3 [**5**]. If $G = S^1$ then ζ is a trivial line bundle and in this case F_G has the homotopy type of $Q_0(CP^\infty) \times Q_0(S^1)$. If $G = S^3$, then BG^ζ is an infinite dimensional quasi-projective space as defined by James (see [**2**, Proposition (5.3)]).

2. **Naturality properties.** Let G be as above and let H be a closed subgroup of G. Then we may take BH to be EG/H, and the canonical map from BH to BG to be the projection. Techniques of J. M. Boardman [**4**] imply the existence of a "wrong way" map (in the stable homotopy category)

(2.1) $$\tau : BG^{\zeta(G)} \to BH^{\zeta(H)}.$$

If G and H are finite, τ agrees with the transfer defined in [**9**]. Let

(2.2) $$\rho : F_G \to F_H$$

denote the natural forgetful map.

THEOREM 3. *The following diagram is commutative*

$$\begin{array}{ccc} [\ ; F_G] & \xrightarrow{\rho_*} & [\ ; F_H] \\ \downarrow \lambda_G & & \downarrow \lambda_H \\ \{\ ; BG^{\zeta(G)}\} & \xrightarrow{\tau_*} & \{\ ; BH^{\zeta(H)}\}. \end{array}$$

If G and H are finite, the map

[3] ADDED IN PROOF. Spaces related to F_G have been studied by G. Segal [**12**] using bordism techniques. Theorem 2 is similar to Proposition 4 of [**12**].

(2.3) $$p_*^+ : \{\ ; BH^+\} \to \{\ ; BG^+\}$$

has a geometrical interpretation in terms of a transfer map $t : F_H \to F_G$; details will appear elsewhere.

3. **Applications.** The above results are useful in describing the image of

$$\rho_* : \pi_*(F_G) \to \pi_*(F_H).$$

For example, a theorem of D. S. Kahn and S. B. Priddy [**8**] implies that the transfer

$$\tau : \Sigma_n(RP^{\infty +}) \to \Sigma_n(S^0), \qquad n > 0,$$

is surjective. Hence we have the following.

THEOREM 4. *The forgetful map* $\rho_* : \pi_*(F_{Z_2}) \to \pi_*(F)$ *is surjective.*

On the other hand we have the following result.

THEOREM 5. *Let k be a positive integer, let* $\sigma_k \in \pi_{8k-1}(F)$ *generate the image of J, and let* $\mu_k \in \pi_{8k+1}(F)$ *be an Adams-Barratt element* [**1**]. *Then neither* σ_k *nor* μ_k *is in the image of* $\rho_* : \pi_*(F_{S^1}) \to \pi_*(F)$.

Geometrical applications of the result on μ_k will be given in [**10**].

4. **Spaces over *B*.** Fix a CW-complex B and let $\mathscr{C}(B)$ denote the category having objects $\xi = (E_\xi, B, p_\xi, \Delta_\xi)$ where $p_\xi : E_\xi \to B$ is a fiber bundle and Δ_ξ is a cross section to p_ξ. We assume that ξ is admissible in the sense of [**3**]. In the terminology of James [**7**], ξ is an ex-space of B. The set $[\xi, \xi']$ of maps in $\mathscr{C}(B)$ is the set of homotopy classes of fiber and cross section preserving maps $E_\xi \to E_{\xi'}$. The category $\mathscr{C}(B)$ is a natural extension of the category of pointed spaces, and much of the homotopy theory of pointed spaces can be extended to $\mathscr{C}(B)$. For detailed accounts see [**3**], [**6**], [**7**].

Let $\xi \wedge \alpha$ denote the fiberwise reduced join of ξ and α and define

(4.1) $$\sigma : [\xi ; \xi'] \to [\xi \wedge \alpha ; \xi' \wedge \alpha]$$

by $f \to f \wedge 1$. We then have the following suspension theorem (compare [**6**, Theorem (7.4)]).

THEOREM 6. *Assume that* α *is a sphere bundle and the fiber of* ξ' *is* $(n-1)$-*connected. Then* σ *is injective if* E_ξ *is* $(2n-1)$-*coconnected and surjective if* E_ξ *is* $2n$-*coconnected.*

Let $T(\xi) = E_\xi / \Delta_\xi(B)$. If X is a space with base point x_0 let $\dot{X}$ denote the object $(B \times X, B, p, \Delta)$ where $p(b, x) = b$ and $\Delta(b) = (b, x)$. Note that $T(X \wedge \xi) = X \wedge T(\xi)$. Observe also that the projection map $B \times X$

$\to X$ induces a one-one correspondence $[\xi;X] \to [T(\xi);X]$.

If β is a vector bundle over B let $\bar{\beta}$ denote the object of $\mathscr{C}(B)$ obtained by taking the fiberwise one point compactification of E_β and letting $\Delta_{\bar{\beta}}$ be the cross section at infinity.

5. **Proof of Theorem 1.** Let W be a free G-module of dimension n, let $M(W) = S(W)/G$ and let $\xi = (S(W) \times S(W)/G, M(W), p, \Delta)$ where $p[w, w'] = [w]$ and $\Delta[w] = [w, w]$. Suppose that X is a finite connected complex and $\dim(X) < n - 2$. We have a bijection

$$\theta:[X;F(W)] \to [\dot{X};\xi] \tag{5.1}$$

defined as follows: given $f:X \to F(W)$ define

$$\theta(f):M(W) \times X \to S(W) \times S(W)/G$$

by

$$\theta(f)([w], x) = [w, f(x)(w)].$$

If M is a smooth manifold let $\tau(M)$ denote its tangent bundle. Let ζ denote the bundle with fiber $\mathfrak{g}$ associated with the principal bundle $S(W) \to M(W)$. We then have **[11]**

$$\xi \simeq \overline{\tau(S(W))/G} \simeq \overline{\tau(M(W)) \oplus \zeta}.$$

Making this identification (and abbreviating $\tau(M(W))$ to τ) we have

$$\theta:[X;F(W)] \to [\dot{X};\overline{\tau \oplus \zeta}]. \tag{5.2}$$

Now choose (a) an embedding $h:M(W) \subset R^s$ and (b) a monomorphism $\phi:\zeta \to B \times R^t$. Let ν denote the normal bundle determined by h and ζ' the complementary bundle determined by ϕ. From this data we obtain (a′) an equivalence $\psi:\overline{(\tau \oplus \zeta) \oplus (\nu \oplus \zeta')} \to S^{s+t}$ and (b′) a duality map $\mu:S^{s+t} \to T(\zeta) \wedge T(\nu \oplus \zeta')$.

Define

$$\kappa:[\dot{X};\overline{\tau \oplus \zeta}] \to [X \wedge T(\nu \oplus \zeta');S^{s+t}] \tag{5.3}$$

to be composition

$$[\dot{X};\overline{\tau \oplus \zeta}] \xrightarrow{\sigma} [\dot{X} \wedge \overline{\nu \oplus \zeta'};\overline{\tau \oplus \zeta \oplus \nu \oplus \zeta'}]$$

$$\xrightarrow{\psi_*} [\dot{X} \wedge \overline{\nu \oplus \zeta'};\dot{S}^{s+t}] \to [X \wedge T(\nu \oplus \zeta');S^{s+t}].$$

Since $\dim(X) < n - 2$, σ and hence κ is bijective.

The duality map μ defines a bijection

$$D_\mu:\{X \wedge T(\nu \oplus \zeta');S^{s+t}\} \to \{X;T(\zeta)\}. \tag{5.4}$$

Since we are in the stable range we may define

$$\lambda_W : [X, F(W)] \to \{X; T(\zeta)\} \tag{5.5}$$

by $\lambda_W = D_\mu \kappa\theta$. It is easily seen that λ_W is independent of the choice of h and ϕ. Moreover, if V is a second free G-module, it is compatible with the inclusion $F(V) \to F(V \oplus W)$ in the obvious sense. Now λ_G in Theorem 1 is defined to be inj $\lim_k \lambda_{kW}$.

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