SPACES OF POLYTOPES AND COBORDISM OF TORIC MANIFOLDS

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Abstract. Our aim is to bring the theory of analogous polytopes to bear on the study of omnioriented toric manifolds. By way of application, we simplify and correct certain proofs of the first and third author on the representability of complex cobordism classes. These proofs concern quotient polytopes; the first involves framed embeddings in the positive cone, and the second considers the role of orientations in forming connected sums. Analogous polytopes provide an illuminating context within which to deal with several of the details. Our modified connected sum incorporates an oriented cube of appropriate dimension, and is directly relevant to the proof that any complex cobordism class may be represented by an omnioriented toric manifold. We illustrate the results by means of 4–dimensional examples.

1. Introduction

The theory of analogous polytopes was initiated by Alexandrov [1] in the 1930s, and extended more recently by Khovanskii and Puhlikov [10]. Our aim is to apply this theory to the algebraic topology of torus actions, in the context of Davis and Januszkiewicz's work [5] on toric geometry.

Davis and Januszkiewicz explain how to construct a $2n$–dimensional toric manifold M from a characteristic pair (P,λ), where P is a simple convex polytope of dimension n, and λ is a function with certain special properties which assigns a subcircle of the torus T^n to each facet of P. By construction M admits a locally standard T^n action, whose quotient space is homeomorphic to P.

Every such polytope is equivalent to an arrangement $\mathcal{H}$ of m closed half-spaces in $\mathbb{R}^n$, whose bounding hyperplanes meet only in general position. The intersection of the half-spaces is assumed to be bounded, and defines P. The $(n-1)$ dimensional faces form the facets of P, and general position ensures that any face of codimension k is the intersection of precisely k facets. In particular, every vertex is the intersection of n facets, and lies in an open neighbourhood isomorphic to the positive cone $\mathbb{R}^n_{\geqslant}$. For any characteristic pair (P,λ), it is possible to vary P within its combinatorial equivalence class without affecting the θ-equivariant diffeomorphism type [5] of the toric manifold M.

For a fixed arrangement, we consider the vector $d_{\mathcal{H}}$ of signed distances from the origin O to the bounding hyperplanes in $\mathbb{R}^n$; a coordinate is positive when

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O lies in the interior of the corresponding half-space, and negative in the complement. We then identify the m-dimensional vector space $\mathbb{R}^{\mathcal{H}}$ with the space of arrangements analogous to $\mathcal{H}$. Under this identification, $d_{\mathcal{H}}$ corresponds to $\mathcal{H}$ itself, and every other vector corresponds to the arrangement obtained by the appropriate parallel displacement of half-spaces. For small displacements, the intersections of the half-spaces are polytopes similar to P. For larger displacements the intersections may be degenerate, or empty; in either case, they are known as virtual polytopes, analogous to P.

In [4], the first and third authors consider dicharacteristic pairs (P, ℓ), where λ is replaced by a homomorphism $\ell\colon T^{\mathcal{H}} \to T^n$. This has the effect of orienting each of the subcircles $\lambda(F_j)$ of T^n, and leads to the construction of an omnioriented toric manifold M; [4, Theorem 3.8] claims that a canonical stably complex structure may then be chosen for M. The proof, however, has two flaws. Firstly, it fails to provide a sufficiently detailed explanation of how a certain complexified neighbourhood of P may be framed, and secondly, it requires an orientation of M (and hence of P) for the stably complex stucture to be uniquely defined. The latter issue has already been raised in [2, §5.3], but amended proofs have not been given. One of our aims is to show that analogous polytopes offer a natural setting for some of the details.

The main application of [4, Theorem 3.8] is as follows.

Theorem 6.11 [4]. *In dimensions > 2, every complex cobordism class contains a toric manifold, necessarily connected, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the action of the torus.*

By [3], this result already holds for additive generators of the complex cobordism groups Ω^U_n. So the proof proceeds by considering $2n$–dimensional omnioriented toric manifolds M_1 and M_2, with quotient polytopes P_1 and P_2 respectively, and constructs a third such manifold M, which is complex cobordant to the connected sum $M_1 \# M_2$. For the quotient polytope of M, the authors use the connected sum $P_1 \# P_2$, over which the dicharacteristics naturally extend.

In the light of the preceding observations, we must amend the proof so as to incorporate orientations of P_1 and P_2. However, it is not always possible to form $P_1 \# P_2$ in the oriented sense, and simultaneously extend the dicharacteristics. Instead, we replace M_2 with a complex cobordant toric manifold M_2', whose quotient polytope is $I^n \# P_2$, where I^n denotes an appropriately oriented n–cube. It turns out that the resulting gain in geometrical freedom allows us to extend both orientations and dicharacteristics; the result is the omnioriented toric manifold $M_1 \# M_2'$ over the polytope

$$P_1 \,\Box\, P_2 \;=\; P_1 \# I^n \# P_2,$$

which we call the *box sum* of P_1 and P_2. We may then complete the proof of Theorem 6.11 as described in Section 5 below.

In dimension 2, $P_1 \,\Box\, P_2$ is combinatorially equivalent to the Minkowski sum $P_1 + P_2$, which is central to the theory of analogous polytopes.

In [4], the authors compare Theorem 6.11 with a famous question of Hirzebruch, who asks for a description of those complex cobordism classes which may be represented by connected algebraic varieties. This is a difficult problem, and remains unsolved; nevertheless, our modification to the proof of Theorem 6.11 adds some value to the comparison, in the following sense.

Given complex cobordism classes $[N_1]$ and $[N_2]$ of the same dimension, suppose that N_1 and N_2 are connected. Then we may form the connected sum $N_1 \# N_2$ in the standard fashion, so that it is also a connected, stably complex manifold, and represents $[N_1] + [N_2]$. If, on the other hand, N_1 and N_2 are algebraic varieties, then $N_1 \# N_2$ is not usually algebraic. In these circumstances we might proceed by analogy with the toric case, and look for an alternative representative N_2' such that $N_1 \# N_2'$ is also algebraic.

For the reader's convenience we retain most of the notation and conventions of [4], with two obvious exceptions concerning the half-spaces (2.1). Firstly, we assume that the set of half-spaces defining any polytope is ordered, and discuss the effects of choosing alternative orderings as required. Secondly, we reverse the direction of the inequalitites, so that the normals to the bounding hyperplanes point outwards, in agreement with Khovanskii.

The contents of our sections are as follows.

In section 2 we recall various definitions and notation concerning simple convex polytopes with ordered facets. We also introduce the space $\mathbb{R}(P)$ of polytopes analogous to a fixed example P, and re-interpret the cokernel of an associated transformation. In section 3 we summarise Davis and Januszkiewicz's construction of a toric manifold M over a polytope P having m facets. We also offer a quadratic description of the auxiliary T^m-space $\mathcal{Z}_P$, and define M as its quotient by the kernel of a dicharacteristic homomorphism.

In Section 4 we amend the definition of omniorientation so as to include an orientation of M, and recall the stably complex structure which results. In so doing, we utilise a framing of P as a submanifold with corners of the positive cone in $\mathbb{R}(P)$. We review the construction of connected sum for omnioriented toric manifolds in Section 5, by encoding the additional orientations as signs attached to the fixed points. By way of application, we correct the proof of [4, Theorem 3.8].

Finally, in Section 6, we exemplify the realisation of 4–dimensional complex cobordism classes by omnioriented toric manifolds, and comment on analogous situations in higher dimensions.

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2. Analogous polytopes

We work in a real vector space V of dimension n, equipped with a euclidean inner product $\langle\ ,\ \rangle$. An *arrangement* $\mathcal{H}$ of closed half-spaces in V is a collection of subsets

$$H_i = \{x \in V : \langle a_i, x\rangle \leqslant b_i\} \quad \text{for} \quad 1 \leqslant i \leqslant m, \tag{2.1}$$

where a_i lies in V and b_i is a real scalar. Unless stated otherwise, we shall assume that $\mathcal{H}$ has cardinality $m \geqslant n$, and that a_i has unit length for every

$1 \leqslant i \leqslant m$. We consider H_i as a smooth manifold, whose boundary ∂H_i is its *bounding hyperplane*

$$Y_i = \{x \in V : \langle a_i, x\rangle = b\} \quad \text{for} \quad 1 \leqslant i \leqslant m, \tag{2.2}$$

with outward pointing normal vector a_i.

If the intersection $\cap_i H_i$ is bounded, it forms a *convex polytope* P; otherwise, it is a *polyhedron*. We assume that P has maximal dimension n, and that none of the half-spaces is redundant, in the sense that no H_i may be deleted without enlarging P. In these circumstances, $\mathcal{H}$ and P are interchangeable. We may also specify P by a matrix inequality $A_P x \leqslant b_P$, where A_P is the $m \times n$ matrix of row vectors a_i, and b_P is the column vector of scalars b_i in $\mathbb{R}^m$.

The *positive cone* $\mathbb{R}^n_{\geqslant}$ is an important polyhedron in $\mathbb{R}^n$. It is determined by the half-spaces

$$\{x \in \mathbb{R}^n : \langle -e_j, x_j\rangle \leqslant 0\} \quad \text{for} \quad 1 \leqslant j \leqslant n,$$

and consists of the vectors $\{x : x_j \geqslant 0,\ 1 \leqslant j \leqslant n\}$.

A *supporting hyperplane* is characterised by the property that P lies within one of its two associated half-spaces. A *proper face* of P is defined by its intersection with any supporting hyperplane, and forms a convex polytope of lower dimension. We regard P as an n–dimensional face of itself; the faces of dimension 0, 1, and $n-1$ are known as *vertices*, *edges*, and *facets* respectively. There is one facet $F_i = P \cap Y_i$ for every bounding hyperplane (2.2), so the facets corresponds bijectively to the half-spaces (2.1). We deem a vertex v and facet F_i to be *opposite* whenever v lies in the interior of H_i. If the bounding hyperplanes are in general position, then every vertex of P is the intersection of exactly n facets, and has $m-n$ opposite half-spaces. In these circumstances, P is *simple*.

From this point on, we deal only with simple polytopes, and reserve the notation $q = q(P)$ and $m = m(P)$ for the number of vertices and the number of facets respectively. We assume that the ordering o of the half-spaces H_i is such that the intersection $F_1 \cap \cdots \cap F_n$ of the first n facets is a vertex of P, and describe P as *strongly ordered* by o. We call v the *initial vertex* of P. The faces of codimension k may then be labelled with their list of defining facets and ordered lexicographically, for every $1 \leqslant k \leqslant n$. In particular, the vertices of P are ordered by this procedure.

With respect to inclusion, the faces form a poset $\mathfrak{L}_F(P)$, with unique maximal element P. This poset fails to be a lattice only because we usually omit the empty face, which would otherwise form a unique minimal element. Two polytopes are *combinatorially equivalent* whenever their face posets are isomorphic; this occurs precisely when the polytopes are diffeomorphic as smooth n–dimensional manifolds with corners. A combinatorial equivalence class of polytopes is known as a *combinatorial polytope*, and most of our constructions are defined on such classes. Nevertheless, it is usually helpful to keep a representative polytope in mind, rather than the underlying poset. Examples include the *vertex figures* P_v, which are formed by intersecting P with any closed half-space whose interior contains a single vertex v. Because P is simple, P_v is an n–simplex for any v.

For computational purposes, it is sometimes convenient to locate the initial vertex of P at the origin, and use the first n normal vectors $a_1, \ldots, a_n$ as an orthonormal basis for V. This may be achieved by an appropriate affine transformation, and does not affect the combinatorial equivalence class of P.

Fixing $\mathcal{H}$, we consider the vector $d_{\mathcal{H}} \in \mathbb{R}^m$, whose ith coordinate is the signed distance from the origin O to Y_i in V, for $1 \leqslant i \leqslant m$. The sign is positive when O lies in the interior of H_i, and negative in the exterior. So long as we maintain our convention that the normal vectors a_i have unit length, $d_{\mathcal{H}}$ coincides with b_P; otherwise, the distances have to be scaled accordingly. Every vector $d_{\mathcal{H}} + h$ in $\mathbb{R}^m$ may then be identified with an *analogous arrangement* of half-spaces, defined by translating each of the half-spaces H_i by h_i, for $1 \leqslant i \leqslant m$. Some such arrangements determine convex polytopes $P(h)$, and others, dubbed *virtual polytopes*, do not. In either case, they are described as being analogous to P. We note that $P(h)$ is given by

$$\{x \in V : A_P x \leqslant b_P + h\}, \tag{2.3}$$

and is combinatorially equivalent to P when h is small, because P is simple. In particular, we have that $P(0) = P$.

Examples 2.4.

(1) The zero vector $0 \in \mathbb{R}^m$ is identified with the central arrangement $\mathcal{H}_0$, whose bounding hyperplanes contain the origin; the corresponding polytope $P(-b_P) = \{0\}$ is virtual. The basis vector $e_i \in \mathbb{R}^m$ is identified with the arrangement obtained from $\mathcal{H}_0$ by translating H_i by 1; the corresponding polytope $P(-b_P + e_i) = P_i$ may be virtual, or a simplex.

(2) Any $x \in V$ defines a vector $A_P x \in \mathbb{R}^{\mathcal{H}}$. Then $b_P - A_P x$ is identified with the arrangement given by translating $\mathcal{H}$ by $-x$; the corresponding polytope $P(-A_P x)$ is the translate $P - x$, and is congruent to P. As x varies, we obtain an n–parameter family of analogous polytopes, each of which is congruent to P.

The *Minkowski sum* of subsets P, $Q \subseteq V$ is given by

$$P + Q = \{x + y : x \in P,\, y \in Q\} \;\subseteq\; V.$$

If P and Q are convex polytopes, so is $P + Q$; moreover, when P is analogous to Q, so is $P + Q$. Under the identification of $b_P + h$ with $P(h)$, vector addition corresponds to Minkowski sum, and scalar multiplication to rescaling. In this context, we denote the m-dimensional vector space of polytopes analogous to P by $\mathbb{R}(P)$, and consider the identification as an isomorphism

$$k\colon \mathbb{R}^m \longrightarrow \mathbb{R}(P), \quad \text{where} \quad k(b_P + h) = P(h). \tag{2.5}$$

We may interpret the matrix A_P as a linear transformation $V \to \mathbb{R}^m$. Since the points of P are specified by the constraint $A_P x \leqslant b_P$, the intersection of the affine subspace $b_P - A_P(V)$ with the positive cone $\mathbb{R}^m_{\geqslant}$ is a copy of P in $\mathbb{R}^m$. In other words, the formula $i_P(x) = b_P - A_P x$ defines an affine injection

$$i_P\colon V \longrightarrow \mathbb{R}^m, \tag{2.6}$$

which embeds P as a submanifold with corners of the positive cone. Since i_P maps the half-space H_i to the half-space $\{y : y_i \geqslant 0\}$, this embedding respects the codimension of faces.

The composition $\chi_P = k \circ i_P$ restricts to an affine injection $P \to \mathbb{R}(P)$, and Example (2.4)(2) identifies $\chi_P(x)$ as the polytope congruent to P, obtained by translating the origin to x, for all x in P. Of course, $\chi_P(P)$ is a submanifold with corners of the positive cone $\mathbb{R}(P)_{\geqslant}$.

Given any shift vector h in $\mathbb{R}^m$, the half-spaces $H_i + h_i$ are also ordered by o, and determine the initial vertex $v(h)$ of $P(h)$. For every $1 \leqslant i \leqslant m$, we write $d_i(h)$ for the signed distance between $v(h)$ and the supporting hyperplane $Y_i + h_i$; in other words,

$$d_i(h) = b_i + h_i - \langle a_i, v(h)\rangle \quad \text{for all} \quad 1 \leqslant i \leqslant m,$$

and $d_1(h) = \cdots = d_n(h) = 0$ by construction. We then define the linear transformation $C_o \colon \mathbb{R}^m \to \mathbb{R}^{m-n}$ by the formula

$$C_o(b_P + h) \;=\; (d_{m-n+1}(h), \ldots, d_m(h)). \tag{2.7}$$

Using (2.5), we may interpret C_o as a transformation $\mathbb{R}(P) \to \mathbb{R}^{m-n}$, which acts by $P(h) \mapsto (d_{m-n+1}(h), \ldots, d_m(h))$. Clearly, C_o is epimorphic.

Proposition 2.8. *As a transformation $V \to \mathbb{R}^{m-n}$, the composition $C_o \cdot A_P$ is zero.*

Proof. The $d_i(h)$ are metric invariants of the polytope $P(h)$, so C_o takes identical values on congruent polytopes. In particular, it is constant on the translates $P - x$ for all values $x \in V$, and therefore on the affine plane $b_P - A_P(V)$. So $C_o(A_P(V)) = 0$, as required. $\square$

Proposition 2.8 shows that o determines a short exact sequence

$$0 \longrightarrow V \xrightarrow{A_P} \mathbb{R}^m \xrightarrow{C_o} \mathbb{R}^{m-n} \longrightarrow 0,$$

or equivalently, a choice of basis for $\operatorname{coker} A_P$. A matrix $(c_{i,j})$ for C_o is most easily computed by assuming that the normal vectors $a_1, \ldots, a_n$ form an orthonormal basis for V, as described in Section 2. Then the basis polytopes P_j of (2.4)(1) satisfy

$$d_i(P_j) = \begin{cases} -a_{i,j} & \text{if } 1 \leqslant j \leqslant n \\ \delta_{i,j} & \text{if } n+1 \leqslant j \leqslant m, \end{cases}$$

for all $n+1 \leqslant i \leqslant m$, and we deduce that

$$(c_{i,j}) \;=\; \begin{pmatrix} -a_{n+1,1} & \ldots & -a_{n+1,n} & 1 & 0 & \ldots & 0 \\ -a_{n+2,1} & \ldots & -a_{n+2,n} & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{m,1} & \ldots & -a_{m,n} & 0 & 0 & \ldots & 1 \end{pmatrix}. \tag{2.9}$$

An alternative strong ordering o' provides an alternative basis for $\operatorname{coker} A_P$, and the corresponding matrix $(c'_{i,j})$ is obtained by substituting $a'_{n+1}, \ldots, a'_m$ into (2.9). This procedure sets up a bijection between strong orderings on P and matrices of the form (2.9).

Of course, any $(m-n)\times m$ matrix C of full rank for which $CA_P=0$ also yields a basis for $\operatorname{coker} A_P$, and satisfies the following property.

Lemma 2.10. *Let C' be the $(m-n)\times(m-k)$ matrix obtained from C by deleting columns c_{j_1}, ..., c_{j_k}, for some $1\leqslant k\leqslant n$; if the intersection $F_{j_1}\cap\dots\cap F_{j_k}$ is a face of P of codimension k, then C' has rank $m-n$.*

Proof. Let $\iota\colon\mathbb{R}^{m-k}\to\mathbb{R}^m$ be the inclusion of the subspace

$$\{x : x_{j_1}=\dots=x_{j_k}=0\}$$

and $\kappa\colon\mathbb{R}^m\to\mathbb{R}^k$ the associated quotient map. Then C' is the matrix of the composition $C\cdot\iota$, and the $k\times n$ matrix A' of the composition $\kappa\cdot A_P$ consists of the rows a_{j_1}, ..., a_{j_k} of A_P. The data implies that A' has rank k, and therefore that $\kappa\cdot A_P$ is an epimorphism; so $C\cdot\iota$ is also an epimorphism, and its matrix has rank $m-n$. □

3. Toric manifolds

In this section we include a summary of Davis and Januszkiewicz's construction of toric manifolds over a simple polytope P. We appeal to their auxiliary space $\mathcal{Z}_P$, for which we provide an alternative description in terms of quadratic hypersurfaces. Throughout, we use the methods and notation of [4], under the additional assumption that P is strongly ordered by o. In particular, we denote the ith coordinate subcircle of the standard m–torus T^m by T_i, for every $1\leqslant i\leqslant m$.

For each point p in P, we define the subgroup $T(p)$ by

$$\prod_{H_i\,\ni\,p} T_i \;<\; T^m.$$

If p is a vertex, then $T(p)$ has maximal dimension n; if p is an interior point of P, then $T(p)$ consists of the trivial subgroup $\{1\}$. Davis and Januszkiewicz introduce the identification space $\mathcal{Z}_P$ as

$$T^m\times P/\!\sim\,, \tag{3.1}$$

where $(t_1,p)\sim(t_2,p)$ if and only if $t_1^{-1}t_2\in T(p)$. So $\mathcal{Z}_P$ is an $(m+n)$-dimensional manifold with a canonical left T^m-action, whose isotropy subgroups are precisely the subgroups $T(p)$.

Construction (3.1) may equally well be applied to the positive cone $\mathbb{R}^m_{\geqslant}$, in which case the result is the complex vector space $\mathbb{C}^m$. Since the embedding i_P of (2.6) respects facial codimensions, there is a pullback diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{\;i_Z\;} & \mathbb{C}^m \\ {\scriptstyle\varrho_P}\big\downarrow & & \big\downarrow{\scriptstyle\varrho} \\ P & \xrightarrow{\;i_P\;} & \mathbb{R}^m \end{array} \tag{3.2}$$

of identification spaces. Here $\varrho(z_1,\dots,z_m)$ is given by $(|z_1|^2,\dots,|z_m|^2)$, the vertical maps are projections onto the quotients by the T^m-actions, and i_Z is a T^m-equivariant embedding. It is sometimes convenient to rewrite $\mathbb{C}^m$ as $\mathbb{R}^{2m}$, in which case we substitute $q_j + ir_j$ for the jth coordinate z_j, and let T act by rotation.

Then Proposition 2.8 and Diagram (3.2) imply that i_Z embeds $\mathcal{Z}_P$ in $\mathbb{R}^{2m}$ as the space of solutions of the $m-n$ quadratic equations

$$\sum_{k=1}^{m} c_{j,k}\left(q_k^2 + r_k^2 - b_k\right) = 0, \quad \text{for} \quad 1 \leqslant j \leqslant m-n. \tag{3.3}$$

In Lemma 4.1, we will confirm that $\mathcal{Z}_P$ is a framed submanifold of $\mathbb{R}^{2m}$.

The spaces $\mathcal{Z}_P$ are of considerable independent interest in toric topology, having originated in [5]. They arise in homotopy theory as homotopy colimits [9], in symplectic topology as level surfaces for the moment maps of Hamiltonian torus actions, and in the theory of arrangements as complements of coordinate subspace arrangements. Details are given in [2], where they play a central rôle as *moment-angle complexes*; relationships with combinatorial geometry and commutative algebra are also developed, and topological invariants described.

In order to construct toric manifolds over P, we need one further set of data. This consists of a homomorphism $\ell\colon T^m \to T^n$, satisfying Davis and Januszkiewicz's independence condition, namely

$$F_{j_1} \cap \cdots \cap F_{j_k} \text{ is a face of codimension } k \quad \Longrightarrow \quad \ell \text{ is monic on } \prod_s T_{j_s}. \tag{3.4}$$

Any such ℓ is called a *dicharacteristic* in [4]; the condition (3.4) ensures that the kernel $K(\ell)$ of ℓ is isomorphic to an $(m-n)$–dimensional subtorus of T^m. Wherever possible we abbreviate $K(\ell)$ to K. We write the subcircle $\ell(T_i) < T^n$ as $T(F_i)$, and denote the subgroup

$$\prod_{H_i \ni p} T(F_i) \;\leqslant\; T^n$$

by $S(p)$; it is, of course, the image of $T(p)$ under ℓ. So (3.4) implies, for example, that $S(w) = T^n$ for any vertex w of P.

When applied to the initial vertex v, (3.4) ensures that the restriction of ℓ to $T_1 \times \cdots \times T_n$ is an isomorphism. The homomorphism of Lie algebras induced by ℓ may therefore be represented by an integral matrix of the form

$$L \;=\; \begin{pmatrix} 1 & 0 & \dots & 0 & \lambda_{1,n+1} & \dots & \lambda_{1,m} \\ 0 & 1 & \dots & 0 & \lambda_{2,n+1} & \dots & \lambda_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \lambda_{n,n+1} & \dots & \lambda_{n,m} \end{pmatrix}, \tag{3.5}$$

in terms of appropriate coordinates for T^n.

Since K acts freely on $\mathcal{Z}_P$ there is a principal K-bundle $\pi_\ell\colon \mathcal{Z}_P \to M$, whose base space is a $2n$–dimensional manifold. By construction, M may be expressed as the identification space

$$T^n \times P/{\approx} \tag{3.6}$$

where $(s_1, p) \approx (s_2, p)$ if and only if $s_1^{-1}s_2 \in S(p)$. Furthermore, M admits a canonical T^n-action α, which is locally isomorphic to the standard action on $\mathbb{C}^n$, and has quotient map $\pi\colon M \to P$. The fixed points of α lie over the vertices of

P. The construction identifies a neighbourhood of the fixed point $\pi^{-1}(v)$ with $\mathbb{C}^n$, on which the action of T^n is precisely standard. Note that $\pi \cdot \pi_\ell = \varrho_P$ as maps $\mathcal{Z}_P \to P$.

The quadruple (M, α, π, P) is an example of a *toric manifold*, as defined by Davis and Januszkiewicz. Any manifold with a similarly well-behaved torus action over P is equivariantly diffeomorphic to one of the form (3.6). In this sense, M is typical, and we follow the lead of [4] in working with (3.6) as our arbitrary toric manifold.

Additional structure on M is associated to the *facial submanifolds* M_i, defined as the inverse images of the facets F_i under π. It is clear that each M_i has codimension 2, and we may check that its isotropy subgroup is $T(F_i) < T^n$. Furthermore, the quotient map

$$\mathcal{Z}_P \times_K \mathbb{C}_i \longrightarrow M \tag{3.7}$$

defines a canonical complex line-bundle ρ_i, whose restriction to M_i is isomorphic to the normal bundle ν_i of its embedding in M.

As explained in [5], the bundles ρ_i play an important part in understanding the integral cohomology ring of M. If u_i denotes the first Chern class $c_1(\rho_i)$ in $H^2(M)$, then $H^*(M)$ is generated multiplicatively by $u_1, \ldots, u_m$, with two sets of relations. The first are monomial, and arise from the Stanley-Reisner ideal of P; the second are linear, and arise from the matrix form (3.5) of the dicharacteristic. The latter may be expressed as

$$u_i = -\lambda_{i,n+1}u_{n+1} - \ldots - \lambda_{i,m}u_m \quad \text{for} \quad 1 \leqslant i \leqslant n. \tag{3.8}$$

The following example is straightforward, but will be used in later sections.

Example 3.9. Let the polytope P be the n–cube I^n, where I denotes the unit interval $[0,1] \subset \mathbb{R}$. It has defining half-spaces

$$H_i = \begin{cases} \{x : -x_i \leqslant 0\} & \text{for } 1 \leqslant i \leqslant n \\ \{x : x_i \leqslant 1\} & \text{for } n+1 \leqslant i \leqslant 2n \end{cases}, \tag{3.10}$$

so the vertices are the binary sequences $(\epsilon_1, \ldots, \epsilon_n)$, where $\epsilon_i = 0$ or 1. The cube is strongly ordered by (3.10), with initial vertex the origin. Then i_P embeds I^n in $\mathbb{R}^{2n}$ by $i_P(x) = (x_1, \ldots, x_n, 1-x_1, \ldots, 1-x_n)$, and $\mathcal{Z}_P$ is the product of unit 3–spheres $(S^3)^n \subset (\mathbb{C}^2)^n$. The dicharacteristic is specified by the $n \times 2n$ matrix $(I_n : I_n)$, and its kernel K is the n–torus

$$\{(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1})\} \;<\; T^{2n}.$$

So M is the product of 2–spheres $(S^2)^n$, and $(e^{i\theta_1}, \ldots, e^{i\theta_n}) \in T^n$ acts on the ith factor S_i^2 as rotation by θ_i. The facial bundles are $\overline{\zeta}_1, \ldots, \overline{\zeta}_n, \zeta_1, \ldots, \zeta_n$, where ζ_i denotes the Hopf line bundle over S_i^2. The integral cohomology ring of M is generated by the 2–dimensional elements u_i for $1 \leqslant i \leqslant m$, and the relations (3.8) give $u_i = -u_{n+i}$. The Stanley-Reisner relations then reduce to $u_i^2 = 0$ for all i.

4. Stably complex structures, orientations, and framings

On a smooth manifold N of dimension d, a *stably complex structure* is an equivalence class of real $2k$–plane bundle isomorphisms $\tau(N) \oplus \mathbb{R}^{2k-d} \cong \zeta$, where ζ denotes a fixed $GL(k, \mathbb{C})$-bundle and k is suitably large. Two such isomorphisms are equivalent when they agree up to stabilisation; or, alternatively, when the corresponding lifts to BU of the classifying map of the stable tangent bundle of N are homotopic through lifts. In this section we identify the geometric data required to induce such structures on toric manifolds.

According to [4], an *omniorientation* of a toric manifpld M consists of a choice of orientation for each ν_i; since the dicharacteristic ℓ determines a complex structure on each ρ_i, it encodes equivalent information. In [2], a choice of orientation for M is added to the definition, since no such choice is implied by ℓ. We adopt the latter convention henceforth, and refer to the dicharacteristic and the orientation of the omniorientation as necessary.

An interior point of P admits an open neighborhood U, whose inverse image under the projection π is canonically diffeomorphic to $T^n \times U$ as a subspace of M. Since T^n is oriented by the standard choice of basis, the orientations of P correspond to the orientations of M. We shall therefore take the view that an omnioriented toric manifold over P is equivalent to a dicharacteristic ℓ and an orientation of P. A strong ordering o also induces an orientation on P, by determining an affine isomorphism between a neighbourhood of the initial vertex and the positive cone $\mathbb{R}^n_{\geqslant}$. This is independent of the orientation of any omniorientation on M.

In order to explain the stably complex structure induced on M, it is convenient to study the embedding i_Z of (3.2) in more detail.

Lemma 4.1. *The embedding $i_Z\colon \mathcal{Z}_P \to \mathbb{R}^{2m}$ is T^m-equivariently framed by any choice of matrix $(c_{i,j})$ for the transformation C_o of (2.7).*

Proof. We describe i_Z by the $m-n$ quadratic equations (3.3) over $P \subset \mathbb{R}^m_{\geqslant}$. At each point $(q_1, r_1, \ldots, q_m, r_m) \in \mathcal{Z}_P$, the $m-n$ associated gradient vectors are given by

$$(4.2) \qquad 2\,(c_{j,1}q_1,\, c_{j,1}r_1,\, \ldots,\, c_{j,m}q_m,\, c_{j,m}r_m) \quad \text{for} \quad 1 \leqslant j \leqslant m-n,$$

and so form the rows of the $(m-n) \times 2m$ matrix $2(c_{i,j})R$, where

$$R \;=\; \begin{pmatrix} q_1 & r_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & q_m & r_m \end{pmatrix}$$

is $m \times 2m$. By definition of i_P, the set of integers $j_1, \ldots, j_k$ with the property that $q_{j_1} = r_{j_1} = \cdots = q_{j_k} = r_{j_k} = 0$ at some point $z \in \mathcal{Z}_P$ corresponds to an intersection $F_{j_1} \cap \cdots \cap F_{j_k}$ of facets forming a face of P of codimension k. Lemma 2.10 then applies to show that the matrix obtained by deleting the columns $c_{j_1}, \ldots, c_{j_k}$ of $(c_{i,j})$ has rank $m-n$. It follows that $2(c_{i,j})R$ has rank $m-n$, and therefore that the gradient vectors (4.2) are linearly independent at z, and so frame i_Z.

Furthermore, each of the gradient vectors (4.2) frames the corresponding quadratic hypersurface in $\mathbb{R}^{2m}$, and is T^m-invariant. □

Remark 4.3. Lemma 4.1 provides an alternative to [4, Proposition 3.4], which gives insufficient detail for readers to complete the proof.

Combining the details of Lemma 4.1 with [4, Theorem 6.11] yields an interesting extension of the latter.

Theorem 4.4. *Every complex cobordism class may be represented by the quotient of a free torus action on a real quadratic complete intersection.*

It is particularly illuminating to describe the framing of i_Z in terms of analogous polytopes, as follows.

Factoring out by the action of T^m yields a framing of the embedding i_P, and therefore of P as a submanifold with corners of $\mathbb{R}^m_{\geqslant}$. Under the identification (2.5), the framing vectors may be represented by $m-n$ independent 1–parameter families of polytopes analogous to P. These families are made explicit by applying the differential $d\varrho_P$ to the rows of the matrix $2(c_{i,j})R$. At the point $(q_1, r_1, \dots, q_m, r_m)$ in $\mathcal{Z}_P$, the matrix of $d\varrho_P$ is given by $2R$, so the framing vectors are the rows of the $(m-n)\times m$ matrix $4(c_{i,j})RR^t$. When $(c_{i,j})$ takes the form (2.9), we may take the jth framing vector to be

$$f_j \;=\; (-a_{n+j,1}y_1, \dots, -a_{n+j,n}y_n, 0, \dots, 0, y_{n+j}, 0, \dots, 0)$$

at $y = i_P(x)$, for $1 \leqslant j \leqslant m-n$. Applying (2.5), we conclude that the corresponding 1–parameter family of polytopes $P(f_j, t)$ (for $-1 \leqslant t \leqslant 1$) is obtained from P by: retaining the origin at x, rescaling H_k by $-a_{n+j,k}t$ for $1 \leqslant k \leqslant n$, fixing every facet opposite the initial vertex except H_{n+j}, and rescaling the latter by t.

It is possible to reverse this procedure, and begin with a framing of i_P. The corresponding T^m-equivariant framing of i_Z is then recovered by applying the contruction (3.1). Since P is contractible, all framings of i_P are equivalent, and their lifts to i_Z are equivariantly equivalent. In particular, the equivalence class of the framings described in Lemma 4.1 does not depend on the choice of strong ordering on P.

We may now return to the tangent bundle $\tau(M)$ of M. Our analysis is nothing more than a special case of a proof of Szczarba [11, (1.1)], and replaces that given in [4, (3.9)] which takes no account of the orientation on M.

Proposition 4.5. *Any omnioriented toric manifold admits a canonical stably complex structure, which is invariant under the T^n-action.*

Proof. Following Szczarba, there is a K-equivariant decomposition

$$\tau(\mathcal{Z}_P) \oplus \nu(i_Z) \cong \mathcal{Z}_P \times \mathbb{C}^m,$$

obtained by restricting the tangent bundle $\tau(\mathbb{C}^m)$ to $\mathcal{Z}_P$. Factoring out K yields

$$\tau(M) \oplus (\xi/K) \oplus (\nu(i_Z)/K) \cong \mathcal{Z}_P \times_K \mathbb{C}^m, \tag{4.6}$$

where ξ denotes the $(m-n)$–plane bundle of tangents along the fibres of π_ℓ. The right-hand side of (4.6) is isomorphic to $\bigoplus_{i=1}^m \rho_i$ as $GL(m,\mathbb{C})$-bundles.

Szczarba [11, 6.2)] identifies ξ/K with the adjoint bundle of π_ℓ, which is trivial because K is abelian; $\nu(i_Z)/K$ is trivial by Lemma 4.1. So (4.6) reduces to an isomorphism

$$\tau(M)\oplus\mathbb{R}^{2(m-n)}\cong\rho_1\oplus\ldots\oplus\rho_m,$$

although different choices of trivialisations may lead to different isomorphisms. Since M is connected and $GL(2(m-n),\mathbb{R})$ has two connected components, such isomorphisms are equivalent when and only when the induced orientations agree on $\mathbb{R}^{2(m-n)}$. We choose the orientation which is compatible with those on $\tau(M)$ and $\rho_1\oplus\ldots\oplus\rho_m$, as given by the omniorientation.

The induced structure is invariant under the action of T^n, because i_Z is T^m-equivariant. □

The complex cobordism classes represented by the two choices of orientation in Proposition 4.5 differ by sign. The underlying smooth structure is also T^n-invariant, and is identical to that induced by Lemma 4.1.

5. Connected Sums

In this section we review the construction of the connected sum of omnioriented toric manifolds M' and M'', bearing in mind from Proposition 4.5 that we must incorporate the orientations. We omitted this requirement in [4], and we deal with it here in terms of certain signs associated to the vertices of P.

We assume throughout that the omniorientations on M' and M'' consist of dicharacteristics ℓ' and ℓ'', corrresponding to matrices L' and L'' of the form (3.5); and of orientations on the manifolds themselves, or equivalently, on P' and P''. We let P' and P'' be strongly ordered by o' and o'', with initial vertices v' and v'' respectively.

The *connected sum* $P'\#_{v',v''}P''$ may be described informally as follows. First construct the polytope Q' by deleting the interior of the vertex figure $P'_{v'}$ from P'; so Q' has one new facet $\varDelta(v')$ (which is an $(n-1)$–simplex), whose incident facets are ordered by o'. Then construct the polytope Q'' from P'' by the same procedure. Finally, glue Q' to Q'' by identifying $\varDelta(v')$ with $\varDelta(v'')$, in such a way that the jth facet of Q' combines with the jth facet of Q'' to give a single new facet for each $1\leqslant j\leqslant n$. The gluing is carried out by applying appropriate projective transformations to Q' and Q''. More precise details are given in [4, §6]. Note that $P'\#_{v',v''}P''$ inherits no strong ordering, because v' and v'' disappear during its formation.

The combinatorial type of the connected sum may be altered, for example, by choosing alternative strong orderings on P' and P''. When the choices are clear, or their effect on the result is irrelevant, we use the abbreviation $P'\#P''$. The face lattice $\mathfrak{L}_F(P'\#P'')$ is obtained from $\mathfrak{L}_F(P')\cup\mathfrak{L}_F(P'')$ by identifying the jth facets of each, for $1\leqslant j\leqslant n$. In particular,

$$q(P'\#P'')=q(P')+q(P'')-2\quad\text{and}\quad m(P'\#P'')=m(P')+m(P'')-n. \tag{5.1}$$

By definition, the *connected sum* $M' \#_{v',v''} M''$ is the toric manifold constructed over $P' \#_{v',v''} P''$ using the dicharacteristic $\ell_\# \colon T^{m'+m''-n} \to T^n$ associated to the matrix

$$L_\# = \begin{pmatrix} 1 & 0 & \dots & 0 & \lambda'_{1,n+1} & \dots & \lambda'_{1,m'} & \lambda''_{1,n+1} & \dots & \lambda''_{1,m''} \\ 0 & 1 & \dots & 0 & \lambda'_{2,n+1} & \dots & \lambda'_{2,m'} & \lambda''_{2,n+1} & \dots & \lambda''_{2,m''} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \lambda'_{n,n+1} & \dots & \lambda'_{n,m'} & \lambda''_{n,n+1} & \dots & \lambda''_{n,m''} \end{pmatrix}. \tag{5.2}$$

It is diffeomorphic to the equivariant connected sum of M' and M'' at their initial fixed points. If M' and M'' are omnioriented, the construction shows that the only obstruction to defining a compatible omniorientation on $M' \# M''$ is given by the orientations. We deal with this issue in Proposition 5.4 below.

We write $p' \colon M' \# M'' \to M'$ and $p'' \colon M' \# M'' \to M''$ for the maps collapsing the connected sum onto its constituent manifolds.

We recall from [8] that an omniorientation associates a *sign* $\sigma(w)$ to every vertex w of the quotient polytope P (or, equivalently, to every fixed point of M). If w is the intersection of facets $F_{i_1} \cap \cdots \cap F_{i_n}$, then $\sigma(w) = \pm 1$ measures the difference between the orientations induced on the tangent space at w by the dicharacteristic and the orientation of M respectively. The former is determined by the sum of line bundles $\rho_{i_1} \oplus \ldots \oplus \rho_{i_n}$, and $\sigma(w)$ is given by the Chern number

$$\sigma(w) \;=\; \langle u_{i_1} \cdots u_{i_n}, \mu_M \rangle\,, \tag{5.3}$$

where μ_M denotes the fundamental class in $H_{2n}(M)$ corresponding to the orientation of M.

Proposition 5.4. *The connected sum $M' \#_{v',v''} M''$ admits an orientation compatible with those of M' and M'' if and only if $\sigma(v') = -\sigma(v'')$.*

Proof. The facets of $P' \# P''$ give rise to complex line bundles ξ_i, ξ'_j and ξ''_k over $M' \# M''$, corresponding to the columns of (5.2). We denote their first Chern classes by

$$c_1(\xi_i) = w_i, \quad c_1(\xi'_j) = w'_j, \quad \text{and} \quad c_1(\xi''_k) = w''_k$$

in $H^2(M' \# M'')$, for

$$1 \leqslant i \leqslant n, \quad n+1 \leqslant j \leqslant m', \quad \text{and} \quad n+1 \leqslant k \leqslant m''$$

respectively. The relations (3.8) become

$$w_i = -\lambda'_{i,n+1} w'_{n+1} - \ldots - \lambda'_{i,m'} w'_{m'} - \lambda''_{i,n+1} w''_{n+1} - \ldots - \lambda''_{i,m''} w''_{m''}, \tag{5.5}$$

which imply that

$$w_i = p'^* u'_i + p''^* u''_i \quad \text{for} \quad 1 \leqslant i \leqslant n.$$

Since the first n facets of $P' \# P''$ do not define a vertex, it follows that $w_1 \cdots w_n = 0$ in $H^{2n}(M' \# M'')$, and

$$(p'^* u'_1 + p''^* u''_1) \cdots (p'^* u'_n + p''^* u''_n) = p'^*(u'_1 \cdots u'_n) + p''^*(u''_1 \cdots u''_n) = 0.$$

For any choice of fundamental class in $H_{2n}(M' \# M'')$, we deduce that

$$\big\langle u'_1 \cdots u'_n \,,\; p'_* \mu_{M' \# M''} \big\rangle + \big\langle u''_1 \cdots u''_n \,,\; p''_* \mu_{M' \# M''} \big\rangle = 0.$$

But the corresponding orientation of $M' \# M''$ is compatible with those of M' and M'' if and only if $p'_* \mu_{M' \# M''} = \mu_{M'}$ and $p''_* \mu_{M' \# M''} = \mu_{M''}$; that is, if and only if

$$\sigma(v') + \sigma(v'') = 0,$$

as required. □

Corollary 5.6. *Let M' and M'' be omnioriented toric manifolds over strongly ordered polytopes P' and P'' respectively, with $\sigma(v')+\sigma(v'')=0$; then the stably complex structure induced on $M' \#_{v',v''} M''$ by Proposition 4.5 and Proposition 5.4 is equivalent to the connected sum of those induced on M' and M''. Moreover, the associated complex cobordism classes satisfy*

$$[M' \#_{v',v''} M''] = [M'] + [M''].$$

Proof. The stably complex structures on M' and M'' combine to give an isomorphism

$$\begin{aligned} \tau(M' \# M'') \oplus \mathbb{R}^{2(m'+m''-n)} \;\cong\; & \xi_1 \oplus \ldots \oplus \xi_n \oplus \xi'_{n+1} \oplus \ldots \oplus \xi'_{m'} \\ & \oplus \xi''_{n+1} \oplus \ldots \oplus \xi''_{m''}. \end{aligned} \tag{5.7}$$

As explained in [4, Theorem 6.9], the isomorphism (5.7) belongs to one of the two equivalence classes specified by Proposition 4.5 over $M' \# M''$. The choice of orientation is then provided by Proposition 5.4.

The equation of cobordism classes follows immediately, because the connected sum is cobordant to the disjoint union. □

Proposition 5.4 implies that we cannot always form the connected sum of two omniorented toric manifolds. If the sign of every vertex of P is positive, for example, then it is impossible to construct $M \# M$ directly; we illustrate this situation in Example 6.1. Such restrictions are vital in making applications to complex cobordism theory. Corollary 5.6 confirms that the complex cobordism class $[M' \#_{v',v''} M'']$ is independent of the strong orderings o' and o'', and therefore of the initial vertices.

Example 5.8. In example 3.9, an omniorientation is defined on $S = (S^2)^n$ by investing the cube I^n with its natural orientation as a submanifold of $\mathbb{R}^n$. The stably complex structure induced by Proposition 4.5 takes the form

$$\tau(S) \oplus \mathbb{R}^{2n} \;\cong\; \overline{\zeta}_1 \oplus \zeta_1 \oplus \cdots \oplus \overline{\zeta}_n \oplus \zeta_n$$

This structure bounds because $\overline{\zeta}_i \oplus \zeta_i$ is trivial over S^2_i for every i, and extends over the disc D^3_i. The signs of the vertices are given by

$$\sigma(\epsilon_1, \ldots, \epsilon_n) = (-1)^{\epsilon_1} \ldots (-1)^{\epsilon_n},$$

so those of adjacent vertices are opposite.

We are now in a position to illustrate our principal philosophical point; that good alternative representatives can be chosen for a complex cobordism class $[M]$, when M itself is an omnioriented toric manifold which is not amenable to forming connected sums.

Lemma 5.9. *Let M be an omnioriented toric manifold of dimension $\geqslant 4$, over a strongly ordered polytope P; then there exists an omnioriented M' over a polytope P' such that $[M'] = [M]$ and P' has vertices of opposite sign.*

Proof. Suppose that v is the initial vertex of P. Let S be the omnioriented product of 2–spheres given by example (3.9), with initial vertex w.

If $\sigma(v) = -1$, define M' to be $S \#_{v,w} M$ over the polytope $P' = I^n \#_{v,w} P$. Then $[M'] = [M]$, because S bounds; moreover, adjacent pairs of non-initial vertices of I^n have opposites signs, which survive under the formation of P', as required. If $\sigma(v) = +1$, we make the same construction using the opposite orientation on I^n (and therefore on S). Since $-S$ also bounds, the same conclusions hold. □

We may now complete the proof of our amended [4, Theorem 6.11].

Theorem 5.10. *In dimensions > 2, every complex cobordism class contains a toric manifold, necessarily connected, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the action of the torus.*

Proof. Following [4], we consider cobordism classes $[M_1]$ and $M_2]$, represented by $2n$-dimensional omnioriented toric manifolds over quotient polytopes P_1 and P_2 respectively, It suffices to construct a third such manifold M, over a quotiemt polytope P, whose cobordism class is $[M_1] + [M_2]$.

To achieve this aim, we replace M_2 by M_2' over P_2' following Lemma 5.9. We are then guaranteed to be able to construct $M_1 \# M_2'$ over $P_1 \# P_2'$, using appropriate strong orderings on P_1 and P_2'. The omniorientation on $M_1 \# M_2'$ defines the required cobordism class, by Corollary 5.6 and Lemma 5.9. □

We refer to the polytope P of Theorem 5.10 as the *box sum* $P_1 \square P_2$ of P_1 and P_2, because it is constructed by connecting them with an itermediate cube. The following observation of [4] is unaffected: for any complex cobordism class, the representing toric manifold may be chosen so that its quotient polytope is a connected sum of products of simplices.

6. Examples and concluding remarks

We were taught the importance of adding an orientation to the original definition of omniorientation by certain 4–dimensional examples of Feldman [6]. In this section we describe and develop his examples (noting that 4 is the smallest dimension to which Proposition 5.4 is relevant). They lead to our concluding remarks concerning higher dimensions.

Example 6.1. The complex projective plane $\mathbb{C}P^2$ admits a standard omniorientation, arising from its structure as a complex projective toric variety. The polytope P is the standard 2–simplex $\varDelta(2)$, strongly ordered by the standard basis for $\mathbb{R}^2$, with initial vertex at the origin. Then $\mathcal{Z}_P$ is the unit sphere $S^5 \subset \mathbb{C}^3$. The dicharacteristic is specified by the 2×3 matrix $\left(\begin{smallmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{smallmatrix}\right)$, and its kernel K is the diagonal subcircle

$$T_\delta \;=\; \{(t,t,t)\} \;<\; T^3.$$

So T^3/T_δ is isomorphic to T^2, and (t_1, t_2) acts on $[z_1, z_2, z_3] \in \mathbb{C}P^2$ to give $[t_1 z_1, t_2 z_2, z_3]$. Every facial bundle is isomorphic to $\overline{\zeta}$. The integral cohomology ring of M is generated by 2–dimensional elements u_1, u_2, u_3, and the relations (3.8) give $u_1 = u_2 = u_3$; the Stanley-Reisner relations reduce to $u_1^3 = 0$. Every vertex of $\varDelta(2)$ has sign $+1$.

The complex cobordism class $[\mathbb{C}P^2]$ is an additive generator of the cobordism group $\varOmega_4^U \cong \mathbb{Z}^2$, which immediately raises the question of representing $2[\mathbb{C}P^2]$. This is not, however, possible by omniorienting $\mathbb{C}P^2 \# \mathbb{C}P^2$, because no vertices of opposite sign are available in $\varDelta(2)$, as demanded by Proposition 5.4. Instead, we appeal to Lemma 5.9, and replace the second $\mathbb{C}P^2$ by the omnioriented toric manifold $(-S) \# \mathbb{C}P^2$ over $P' = I^2 \# \varDelta(2)$. Of course $(-S) \# \mathbb{C}P^2$ is cobordant to $\mathbb{C}P^2$, and P' is a pentagon. These observations lead naturally to our second example.

Example 6.2. The omnioriented toric manifold $\mathbb{C}P^2 \# (-S) \# \mathbb{C}P^2$ represents $2[\mathbb{C}P^2]$, and lies over the box sum $\varDelta(2) \,\square\, \varDelta(2)$, which is a hexagon. Figure 1 illustrates the procedure diagramatically, in terms of dicharacteristics and orientations. Every vertex of the hexagon has sign 1.

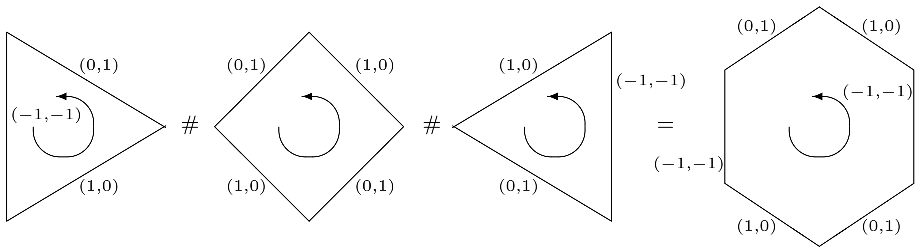

FIGURE 1. The omnioriented connected sum $\mathbb{C}P^2 \# (-S) \# \mathbb{C}P^2$.

Our analysis is supported by a result of [8], which identifies the top Chern number of any $2n$–dimensional omnioriented toric manifold as

$$c_n(M) \;=\; \sum_w \sigma(w). \tag{6.3}$$

Given the quotient polytope P, it is convenient to refine the notation of (5.1) by writing

$$q(P) = q_+(M) + q_-(M),$$

where $q_\pm(M)$ denotes the number of vertices with sign ± 1 respectively. Then (6.3) shows that $q_(M) - q_-(M)$ is a cobordism invariant of M. This is illustrated by Example 6.1, for which $c_2(\mathbb{C}P^2) = 3$ and $q_-(\mathbb{C}P^2) = 0$. It follows by additivity that $c_2(M) = 6$ for any omnioriented toric manifold representing $2[\mathbb{C}P^2]$; and therefore that the quotient polytope has 6 or more vertices. In particular, as observed by Feldman, M cannot be constructed over an oriented copy of $\varDelta(2) \# \varDelta(2)$, which is a square!

An independent additive generator of $\varOmega_4$ is represented by $(\mathbb{C}P^1)^2$, which has second Chern number 4, and may certainly be realised over the square.

Our third example shows a related 4–dimensional situation in which the connected sum of the quotient polytopes *does* support a suitable orientation.

Example 6.4. Let $\overline{\mathbb{C}P}^2$ denote the underlying toric manifold of (6.1), in which the dicharacteristic is unaltered but the orientation of $\Delta(2)$ is reversed. Therefore every vertex has sign -1, and we may construct $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ as an omnioriented toric manifold over $\Delta(2) \# \Delta(2)$. Figure 2 illustrates the procedure diagramatically, in terms of dicharacteristics and orientations.

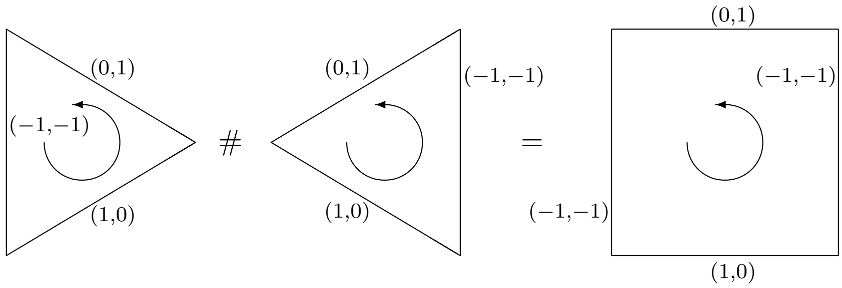

FIGURE 2. The omnioriented connected sum $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

Of course $[\overline{\mathbb{C}P}^2] = -[\mathbb{C}P^2]$. So $[\mathbb{C}P^2] + [\overline{\mathbb{C}P}^2] = 0$ in Ω_4^U, and the resulting manifold bounds by Proposition 5.6.

One other observation on 2–dimensional box sums is also worth making. Given k'- and k''-gons P' and P'' in $\mathbb{R}^2$, it follows from (5.1) that

$$q(P' \square P'') = q(P') + q(P'') \quad \text{and} \quad m(P' \square P'') = m(P') + m(P'').$$

Thus $q(P' \square P'') = m(P' \square P'') = k' + k''$. So $P' \square P''$ is a $(k' + k'')$-gon, and is combinatorially equivalent to the Minkowski sum $P' + P''$ whenever P' and P'' are in general position.

A situation similar to that of Example 6.2 arises in higher dimensions, when we consider the problem of representing complex cobordism classes by smooth projective toric varieties. For any such V, the top Chern number coincides with the Euler characteristic, and is therefore equal to the number of vertices of the quotient polytope P; so $q_-(V) = 0$, by (6.3). Moreover, the Todd genus satisfies $Td(V) = 1$.

Remarks 6.5. Suppose that smooth projective toric varieties V_1 and V_2 are of dimension $\geqslant 4$, and have quotient polytopes P_1 and P_2 respectively. Then $c_n(V_1) = q(P_1)$ and $c_n(V_2) = q(P_2)$, yet $q(P_1 \# P_2) = q(P_1) + q(P_2) - 2$, from (5.1). Since c_n is additive, no omnioriented toric manifold over $P_1 \# P_2$ can possibly represent $[V_1] + [V_2]$. This objection vanishes for $P_1 \square P_2$, because it enjoys an additional $2^n - 2$ vertices.

The fact that no smooth projective toric varienty can represent $[V_1] + [V_2]$ follows immediately from the Todd genus.

Example 6.6. For any non-negative integers r and s such that $r + s > 0$, the cobordism class $r[\mathbb{C}P^2] + s[\mathbb{C}P^1]^2$ is represented by an omnioriented toric manifold $M(r,s)$. Its quotient polytope is the iterated box sum

$$P(r,s) \;=\; \big(\square^r \varDelta(2)\big)\,\square\,\big(\square^s I^2\big),$$

which satisfies $q_-(P(r,s)=0$. Applying the Todd genus once more, we deduce that $M(r,s)$ cannot be cobordant to any smooth toric variety, so long as $(r,s)\neq(1,0)$ or $(0,1)$.

Higher dimensional examples of this phenomena are given by ... These examples suggest that we might study omnioriented toric manifolds for which $q_-(M)=0$, as a natural generalisation of smooth projective toric varieties. Indeed, by considering q_- in more detail, we may ask further and deeper questions about the representability of cobordism classes.

Remarks 6.7. Suppose that an omnioriented toric manifold L has quotient polytope Q, and that its associated stably complex structure bounds. Since $c_n(L)=0$, it follows from (6.3) that $q(Q)$ is even, and that half the vertices have sign $+1$, and half -1. We may then construct a generalised version of our connected sum, by forming the omnioriented toric manifold $M_1 \# L \# M_2$ over $P_1 \# Q \# P_2$. Interesting possibilities for L include the bounded flag manifolds B_n of [4], in which case Q is also I^n, and products such as $S^2\times\mathbb{C}P^{n-1}$, in which case Q is $I\times\varDelta(n-1)$.

We may further generalise the procedure by connecting three or more pM_j over appropriate vertices of Q. Mention q_- here ...

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