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# Spacetime $b$-Boundaries

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**Abstract.** It is shown that Schmidt's $b$-boundary for a spacetime can be analyzed using a submanifold of the tangent bundle, rather than the principal bundle or the bundle of orthogonal frames.

## 1. Introduction

Schmidt [1] has shown that every spacetime can be assigned a boundary, called the $b$-boundary. Roughly speaking, the boundary points are ideal endpoints for those inextendible curves which do not escape to infinity. Though useful in general arguments, such as those in Hawking and Ellis [2], the $b$-boundary is hard to construct in specific examples. The purpose of this paper is to point out that the construction can be carried out using only a submanifold of the tangent bundle. Section 2 states the results, Section 3 supplies the proofs, and Section 4 gives 2 examples.

In discussing differential geometry, the notation and terminology of Bishop and Goldberg [3] will usually be used. Hu [4] will be taken as the standard topology reference. Throughout the paper $(M, g, D)$ will denote a *spacetime*: a real, 4-dimensional, connected, Hausdorff, oriented, time-oriented, $C^\infty$ Lorentzian manifold $(M, g)$ together with the Levi-Civita connection $D$ of $g$. $TM$ denotes the tangent bundle, with projection $\pi: TM \to M$. The main idea is the following. Suppose $\alpha: E \to M$ is an inextendible $C^\infty$ curve. $\alpha$ may be lightlike and need not be geodesic so, in general, neither arc length nor an affine parameter supplies an adequate criterion for when $\alpha$ fails to escape to infinity. But suppose we had a unit timelike vector field $P: M \to TM$ available. Then we could use arc length with respect to the positive definite metric $g + 2g(P, \cdot) \otimes g(P, \cdot)$. The game is to introduce $P$ and then amputate it back out.

## 2. The Unit Future

The *unit future* $UM$ of $M$ is the following $C^\infty$ submanifold of the tangent bundle: $UM = \{(x, P) \in TM \mid g(P, P) = -1, P \text{ is future-pointing}\}$. Thus $U$, defined by $U = \pi|_{UM}$, is a $C^\infty$ onto map $U: UM \to M$. As in

Bishop and Goldberg [3] we can regard the identity map of $UM$ onto itself as a $C^\infty$ vector field $P: UM \to TM$ over the map $U$. For example, suppose $x \in M$ and $y, z \in U^{-1}\{x\}$. Then $Py$, $Pz \in M_{Uy} = M_x$ and $g(Py, Pz) \leqq -1$, where equality holds iff $y = z$. As pointed out in Bishop and Goldberg [3], $U^*D$ is a $C^\infty$ connection over ("on") the map $U$. For example, suppose $y \in UM$ and $Y \in (UM)_y$. Then $Y$ is *vertical* iff $U_* Y = 0$, *horizontal* iff $U^* D_Y P = 0$, and zero iff it is both horizontal and vertical.

**Proposition 2.1.** *There is a unique Riemannian metric $G$ on $UM$ such that for all $(y, Y) \in TUM$, $G(Y, Y) = g(U_* Y, U_* Y) + 2[g(U_* Y, Py)]^2 + g(U^* D_Y P, U^* D_Y P)$.*

*Proof.* Since $g$ is Lorentzian, $g(Uy) + 2g(Py, .) \otimes g(Py, .)$ is a positive definite quadratic form on $M_{Uy}$. Thus if $Y$ is horizontal, $G(Y, Y) \geqq 0$, with equality holding iff $Y = 0$. Moreover, for any $Y$, $g(U^* D_Y P, Py) = \frac{1}{2} U^* D_Y[(g \circ U)(P, P)] = \frac{1}{2} Y[-1] = 0$. Thus $U^* D_Y P \in (Py)^\perp \subset M_{Uy}$ for any $Y$. But $g$ restricted to $(Py)^\perp$ is positive definite. Thus if $Y$ is vertical $G(Y, Y) \geqq 0$, with equality holding iff $Y = 0$. Thus $G$ is positive definite. The rest is straightforward. □

Let $d: UM^2 \to [0, \infty)$ be the topological metric determined by $G$, as in Helgason [5; Section 1.9]. Let $(\boldsymbol{UM}, \boldsymbol{d})$ be the complete metric space in which $(UM, d)$ is dense. Denote the positive integers by $Z^+$. Define a relation $R \subset \boldsymbol{UM}^2$ as follows. $wRy$ iff there are Cauchy sequences $w': Z^+ \to UM$ and $y': Z^+ \to UM$ such that: (A) $w'$ converges to $w$ and $y'$ converges to $y$; (B) the projections coincide, i.e. $U \circ w' = U \circ y'$; and (C) there is a uniform lower bound $A \in (-\infty, -1]$ such that, for all $n \in Z^+$, $g(Pw'n, Py'n) \geqq A$. We now show that $R$ is an equivalence relation and that the decomposition space $\boldsymbol{UM}/R$ is homeomorphic to the union of $M$ with the $b$-boundary of $M$.

## 3. Proofs

To give the proofs and relate $\boldsymbol{UM}/R$ to the space defined by Schmidt we first review a standard definition of the $b$-boundary. Let $OM$ be the bundle, above $M$, of those (Lorentzian-) orthonormal frames whose orientation and time-orientation is that determined by $(M, g)$. Let $\theta: OM \to M$ be the projection. Let $P_i$ $(i = 1, \ldots, 4)$ be the four standard vector fields over $\theta$. Thus for each $\delta \in (1, 2, 3)$, $(g \circ \theta)(P_\delta, P_\delta) = 1 = -(g \circ \theta)(P_4, P_4)$, with the other dot products zero. Let $V: OM \to UM$ be the projection onto the unit future. Then $\theta = U \circ V$ and $P_4 = P \circ V$. Define a $C^\infty$, (0, 2) tensor field $H$ on $OM$ by $H(Q, Q) = \sum_{\delta=1}^{3} \{g(\theta^* D_Q P_\delta, \theta^* D_Q P_\delta) + [g(\theta^* D_Q P_\delta, P_4 q)]^2\}$ for all $(q, Q) \in TOM$. By an argument similar to that of Section 2, $G_0 = V^* G + H$ is a Riemannian metric on

$OM$. Let $d_0$ be the topological metric determined by $G_0$, $(\mathbf{OM}, \mathbf{d}_0)$ be the complete metric space in which $(OM, d_0)$ is dense.

One can extend $\theta$ to $\mathbf{OM}$ by using the structure group $L$ of $OM$. The elements of $L$ are real $4 \times 4$ matrices and $L$ is isomorphic to that component of the Lorentz group which contains the identity; here and throughout $L$ is assigned its standard topology. The action of $l \in L$ on $OM$ will be denoted by $R_l : OM \to OM$. Thus if $q, r \in OM$ then $\theta q = \theta r$ iff there is an $l \in L$ such that $R_l q = r$. Each $R_l$ has a uniformly continuous extension $\mathbf{R}_l : \mathbf{OM} \to \mathbf{OM}$. For $q, r \in \mathbf{OM}$, $\mathbf{R}_l q = r$ iff there are Cauchy sequences $q' : Z^+ \to OM$ converging to $q$ and $r' : Z^+ \to OM$ converging to $r$ with $R_l \circ q'$ converging to $r$. There is an equivalence relation $\sim$ on $\mathbf{OM}$, defined as follows: $q \sim r$ iff there is an $l \in L$ such that $\mathbf{R}_l q = r$. The decomposition space $\mathbf{M} = \mathbf{OM}/\sim$ is called the *spacetime with b-boundary* $\mathbf{M}$. The *b-boundary* of $(M, g, D)$ is the topological space $\mathbf{M} - \boldsymbol{\theta}(OM)$ $= \mathbf{M} - \theta(OM) = \mathbf{M} - M$, where $\boldsymbol{\theta}$ is the projection.

We can see the relation of these definitions to the discussion of Section 2 by filling in the two missing maps, $\mathbf{V}$ and $\mathbf{U}$, in the following diagram.

$$
\begin{array}{ccccc}
 & & \theta & & \\
OM & \xrightarrow{V} & UM & \xrightarrow{U} & M \\
\cap & & \cap & & \cap \\
\mathbf{OM} & \xrightarrow{\mathbf{V}} & \mathbf{UM} & \xrightarrow{\mathbf{U}} & \mathbf{M} \\
 & & \boldsymbol{\theta} & &
\end{array}
$$

**Proposition 3.1.** *For all* $q, r \in OM$ *and* $y \in UM$:

(A) $d_0(q, r) \geqq d(Vq, Vr)$;

(B) *there is an* $s \in V^{-1}\{y\}$ *such that* $d_0(s, q) = d(y, Vq)$.

*Proof*. The tensor field $H$ defined above is positive semi-definite. Since $G_0 = H + V^* G$, assertion (A) follows. To prove (B) we shall construct an "optimum lift" into $OM$ of each curve into $UM$. The following notation will be convenient. Let $\beta$ be a $C^\infty$ curve into $OM$. Abbreviate $(\theta \circ \beta)^* D_{(d/dt)}(P_4 \circ \beta)$, where $t$ is the curve parameter, by $\dot{P}_4$, etc. Now let $\alpha : [0, a] \to UM$ be a $C^\infty$ curve from $Vq$ to $y$. Then there is a unique $C^\infty$ curve $\beta : [0, a] \to OM$ such that: (i) $V \circ \beta = \alpha$; (ii) $\beta 0 = q$; and (iii) $\beta$ obeys the Fermi-Walker transport law in the sense that for all $\delta \in (1, 2, 3)$ $\dot{P}_\delta = [(g \circ \theta \circ \beta)(\dot{P}_4, P_\delta \circ \beta)](P_4 \circ \beta)$. From the form of $H$, the length of $\beta$ is the same as the length of $\alpha$. Moreover $V^{-1}\{y\}$ is compact. (B) above now follows by considering a sequence of curves $\alpha_1, \alpha_2, \ldots$ into $UM$ whose lengths approach $d(y, Vq)$, with each $\alpha_i$ going from $Vq$ to $y$. □

**Theorem 3.2.** *There is a unique, uniformly continuous, uniformly open, onto extension* $\mathbf{V} : \mathbf{OM} \to \mathbf{UM}$ *of* $V : OM \to UM$.

*Proof*. $V$ is uniformly continuous by 3.1.A. Therefore, as shown in Kelley [6; Chapter 6], $V$ has a unique uniformly continuous extension $\boldsymbol{V}: \boldsymbol{OM} \to \boldsymbol{UM}$. If we can show that 3.1.B extends, the uniform openess of $\boldsymbol{V}$ will follow; compare Kelley [6; Chapter 6]. Suppose that $q \in OM$ and $y \in \boldsymbol{V}(\boldsymbol{OM}) \subset \boldsymbol{UM}$. Let $r' : Z^+ \to OM$ be a Cauchy sequence such that $V \circ r'$ converges to $y$. For each $n \in Z^+$ we can, by 3.1.B, choose $s'(n) \in V^{-1}\{Vr'n\}$ such that $d_0(s'(n), q) = d(Vr'n, Vq)$. This determines a sequence $s' : Z^+ \to OM$; it also determines a sequence $l' : Z^+ \to L$ by the rule $R_{l'n}s'n = r'n$ for all such $n$. Now $P_4(s'n) = P(Vs'n) = P(Vr'n) = P_4(r'n)$. Thus the image of $l'$ is contained in a compact subset of $L$ and there is at least one cluster point, say $l \in L$. Then $R_l^{-1}r'$ is a Cauchy sequence; let $s \in \boldsymbol{OM}$ be its limit. Then $s \in \boldsymbol{V}^{-1}\{y\}$ and $\boldsymbol{d}_0(s, q) = \boldsymbol{d}(\boldsymbol{V}s, \boldsymbol{V}q) = \boldsymbol{d}(y, \boldsymbol{V}q)$. Thus 3.1.B extends to this case. 3.1.B also extends to the more general case $y \in \boldsymbol{V}(\boldsymbol{OM})$, $q \in \boldsymbol{OM}$; the proof is so similar to that just given it is omitted. Thus $\boldsymbol{V}$ is uniformly open. Kelley [6; Chapter 6] shows that the range of a continuous, uniformly open map of a complete metric space into a Hausdorff uniform space is complete. It follows that $\boldsymbol{V}$ is onto. □

Since $\boldsymbol{V}$ is open and continuous it is an identification. Moreover, note that any Cauchy sequence $y' : Z^+ \to UM$ can be lifted to a Cauchy sequence $r' : Z^+ \to OM$, with $V \circ r' = y'$. For let $s' : Z^+ \to OM$ be a Cauchy sequence such that $V \circ s'$ converges to the limit $y \in UM$ of $y'$. For each $n \in Z^+$, choose $r'n$ such that $d_0(r'n, s'n) = d(y'n, Vs'n)$ and $r'n \in V^{-1}\{y'n\}$. Then $r'$ is Cauchy. Having extended $V$ we can now extend $U$. Suppose $y \in \boldsymbol{UM}$ and $q, r \in \boldsymbol{V}^{-1}\{y\}$.

**Proposition 3.3.** $\boldsymbol{\theta} q = \boldsymbol{\theta} r$.

*Proof*. Suppose $y'$ is a Cauchy sequence which converges to $y$. Lift $y'$ to a Cauchy sequence $q'$ which converges to $q$, using the method just discussed; also lift $y'$ to a Cauchy sequence $r'$ which converges to $r$. Define a sequence $l' : Z^+ \to L$ by $R_{l'n}q'n = r'n$. As in the theorem, there is a cluster point $l \in L$. $\boldsymbol{R}_l q = r$ so $\boldsymbol{\theta} q = \boldsymbol{\theta} r$. □

Thus we can define $\boldsymbol{U} : \boldsymbol{UM} \to \boldsymbol{M}$ by $\boldsymbol{U}y = \boldsymbol{\theta}(\boldsymbol{V}^{-1}\{y\})$ for all $y \in \boldsymbol{UM}$. Since $\boldsymbol{\theta}$ and $\boldsymbol{V}$ are identifications, $\boldsymbol{U}$ is an identification. The last step is to describe $\boldsymbol{U}$ wholly in terms of structures defined on $UM$. Suppose $w, y \in \boldsymbol{UM}$; let $R$ be as in 2.

**Proposition 3.4.** $\boldsymbol{U}w = \boldsymbol{U}y$ *iff* $wRy$.

*Proof*. Suppose $\boldsymbol{U}w = \boldsymbol{U}y$. Thus if $q'$ is a Cauchy sequence which converges to $q \in \boldsymbol{V}^{-1}\{w\}$ and $r'$ is a Cauchy sequence which converges to $r \in \boldsymbol{V}^{-1}\{y\}$ then there is an $l \in L$ such that $R_l \circ q'$ converges to $r$. Form a Cauchy sequence which converges to $w$ by alternating terms from $V \circ q'$ and $V \circ R_l^{-1} \circ r'$ and a Cauchy sequence which converges to $y$ by alternating terms from $V \circ R_l \circ q'$ and $V \circ r'$. The projections of these two Cauchy sequences into $M$ coincide and the existence of a uniform lower bound is implied by the fact that $l$ is fixed. Thus $wRy$. Conversely, suppose $wRy$. Suppose $w'$ converges to $w$ and $y'$ converges to $y$, with the projections of $w'$ and $y'$ identical. Lift $w'$ to a Cauchy sequence $q'$ into $OM$, $y'$ to a Cauchy sequence $r'$ into $OM$. Define $l' : Z^+ \to L$ by $R_{l'n} q' n = r' n$. The existence of a uniform lower bound on $g(Pw'n, Pr'n)$ implies that $l'$ has at least one cluster point $l \in L$. Then $R_l \circ q'$ converges to the same point as $r'$ so $\boldsymbol{U}w = \boldsymbol{U}y$. □

As corollaries we have that $R$ is an equivalence relation and that $\boldsymbol{UM}/R = \boldsymbol{M}$, as claimed in Section 2.

## 4. Examples and Comments

The first example shows that the condition of a uniform lower bound in the definition of $R$, Section 2, cannot be dropped. Let $\alpha : (-\infty, 0) \to M$ be a lightlike geodesic with the following property. There is an $x \in M$ such that, for all $n \in Z^+, \alpha(-1/2^n) = x$ and $\alpha_*(-1/2^n) = 2^n \alpha_*(-1) \in M_x$. Thus the image of $\alpha$ is $\alpha[-1, -\frac{1}{2}) \subset M$ and $\alpha$ winds around an infinite number of times as the affine parameter, say $t$, approaches zero from below. Hawking and Ellis [2] show this rather peculiar behavoir can in fact occur. Now let $X_0 \in M_x$ be unit, timelike, and future-pointing. The constant sequence $y' : Z^+ \to UM$ given by $y'n = (x, X_0)$ has $y = (x, X_0)$ as limit and $\boldsymbol{U}y = \boldsymbol{U}y = x$. Next define a vector field over $\alpha$, $X : (-\infty, 0) \to TM$, as follows: $X$ is parallel, i.e. $\alpha^* D_{d/dt}(X \circ \alpha) = 0$; and $X(-1) = X_0$. Then the sequence $w'$ defined by $w'n = (x, X(-1/2^n))$ is also Cauchy. For the only contribution to the arc length of the curve $\beta = (\alpha, X \circ \alpha) :(-\infty, 0) \to UM$ comes from the term $2[(g \circ \alpha)(\alpha_*, X)]^2$, which is constant; let $w$ be the limit of $w'$. $w'$ has the same projection into $M$ as $y'$, but $\boldsymbol{U}w \neq \boldsymbol{U}y$, as discussed in Hawking and Ellis [2]. The catch is that $g(Pw'n, Py'n) = g(X(-1/2^n), X_0)$ is not bounded from below.

The second example shows that, at least in one artificially constructed case, working with $UM$ rather than $OM$ gives a major simplification. Let $N$ be $R^3$ with the origin $(0,0,0)$ deleted. Let $h$ be a $C^0$ Riemannian metric on $R^3$ which is $C^\infty$ on $N$. Let $M = N \times (-\infty, \infty)$, with projections $S : M \to N$ and $T : M \to (-\infty, \infty)$. Define $g$ on $M$ by $g = S^* h - dT \otimes dT$. Supply $(M, g)$ with the natural orientation, natural time-orientation, and the Levi-Civita connection $D$. Then $(M, g, D)$ is a spacetime. The claim is that $\boldsymbol{M}$ is homeomorphic to $R^4$; roughly speaking, the $b$-boundary consists simply of the "missing points" $(0,0,0) \times (-\infty, \infty)$. Only an outline of the rather tedious proof will be given.

Let $\alpha : [0, a] \to UM$ be a $C^\infty$ curve. Then $(G \circ \alpha)(\alpha_*, \alpha_*) \geqq (g \circ U \circ \alpha)(\dot{P}, \dot{P})$, with $\dot{P}$ essentially as in 3.1. Define the vector field $X : M \to TM$

by $T_* X = d/ds$, $S_* X = 0$. $X$ is unit, timelike, future pointing, and parallel ("covariant constant"). Let $f$: $[0, a] \to [0, \infty)$ be the function defined by $\cosh f = -(g \circ U \circ \alpha)(P \circ \alpha, X \circ U \circ \alpha)$. Using the inequality mentioned above and the fact that $X$ is parallel one finds that the arc length of $\alpha$ is at least $|f0 - fa|$. Now let $w' : Z^+ \to UM$ be a Cauchy sequence with limit $w$. The above estimate shows there is a uniform lower bound on $g(Pw'n, XUw'n)$. Next one can work with the "horizontal part" $g(U_* Y, U_* Y) + 2[g(U_* Y, Py)]^2$ of $G$, rather than the "vertical part" $g(U^* D_Y P, U^* D_Y P)$ used above. One finds that the sequence $y'$, defined by $y'n = (Uw'n, XUw'n) \in UM$ is also Cauchy and that its limit $y$ obeys $\boldsymbol{U}y = \boldsymbol{U}w$. Thus one can confine attention to sequences $y'$ with the property $P \circ y' = X \circ U \circ y'$. The rest is straightforward and gives the result already mentioned.

If one tries to work directly with $OM$ in this second example a terrible mess results. Unfortunately, in more realistic cases even using $UM$ still leads to quite difficult computations. Whether one can develop effective techniques for computing the $b$-boundaries of the various physically interesting spacetimes remains to be seen. If not, the physical relevance of $b$-boundary techniques may remain rather obscure.

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