Space-Times Admitting Isolated Horizons

Jerzy Lewandowski
Instytut Fizyki Teoretycznej, Uniwersytet Warszawski,
ul. Hoża 69, 00-681 Warszawa, Poland, lewand@fuw.edu.pl

Abstract

We characterize a general solution to the vacuum Einstein equations which admits isolated horizons. We show it is a non-linear superposition – in precise sense – of the Schwarzschild metric with a certain free data set propagating tangentially to the horizon. This proves Ashtekar's conjecture about the structure of spacetime near the isolated horizon. The same superposition method applied to the Kerr metric gives another class of vacuum solutions admitting isolated horizons. More generally, a vacuum spacetime admitting any null, non expanding, shear free surface is characterized. The results are applied to show that, generically, the non-rotating isolated horizon does not admit a Killing vector field and a spacetime is not spherically symmetric near a symmetric horizon.

The quantum geometry considerations applied to black hole entropy [1] led Ashtekar et al to a new approach to black hole mechanics. The idea is to consider a null surface which locally has the properties of the Schwarzschild horizon, but is not necessarily infinitely extendible, so the spacetime metric in a neighborhood is not necessarily that of Schwarzschild. Such a surface was called a non rotating isolated horizon (NRIH). The number of degrees of freedom describing a spacetime admitting a NRIH is much larger than that describing a static black hole (see below). In a series of works the laws of the black hole thermodynamics and mechanics were extended to this case [2, 3].

In this letter we completely characterize a general solution to the Einstein vacuum equations which admits an isolated horizon and, in particular, a NRIH. For that purpose, we use Friedrich's characteristic Cauchy problem defined on null surfaces [4] (The idea of constructing solutions to the Einstein's equations starting with data defined on a null surface was first formulated by Newman [5]). The null Cauchy problem formulation gives rise to our superposition method: Given a local solution to the Einstein vacuum equations and the data it defines on a null surface, a new solution can be constructed from the null surface data and certain new data freely defined on a transversal null surface. We show that a general solution which admits a NRIH is given by the superposition of the data defined by the Schwarzschild metric on the horizon and the data defined freely on a transversal null surface. This result is then applied to prove that a generic NRIH does not admit a Killing vector field. Even though there are vector fields defined on the horizon which Lie annihilate the metric tensor [7], none of them, generically, can be extended to a neighborhood. The statement concerns the null vector fields as well as the space like vectors generating symmetries of the internal geometry induced on the 2 dimensional cross sections of the NRIH.

We also characterize a general solution to the Einstein vacuum equations which admits a null non expanding surface. An interesting subclass of spacetimes is obtained by superposing, in our sense, the data defined by the Kerr metric on its horizon with the data freely defined on a transversal null surface. By analogy to the non rotating case, the resulting null surface equipped with the data corresponding to those of the Kerr metric may be thought of as a rotating isolated horizon.

Another way to extend our results is to admit matter fields in spacetime. In particular the Maxwell field fits the null surfaces formulations of the Cauchy problem very well. We use Newman-Penrose spin connection and curvature coefficients in the notation of [8]. All our considerations and results will be *local* in the following sense: Given two null 3-surfaces N_0 and N_1 intersecting in the their future, by *locally* we mean 'in the past part of a suitable neighborhood of a $N_0 \cap N_1$ bounded by the incoming parts of the surfaces'.

Isolated horizons: definitions. Consider a null 3-submanifold N_0 of a 4-dimensional spacetime M diffeomorphic to

$$S_2 \times [v_0, v_1], \tag{1}$$

where 2-spheres S_2 can be identified with space-like cross sections and the intervals $[v_0, v_1]$ lie along the null generators of N_0. We say that N_0 is an isolated horizon if the intrinsic, degenerate metric tensor induced in N_0 is annihilated by the Lie derivative with respect to any vector field

$$l = -o^A o^{A'} \tag{2}$$

tangent to the null generators of N_0. In other words, l is non expanding and shear free,

$$\rho = \sigma = 0. \tag{3}$$

An isolated horizon N_0 equipped with a foliation by space-like 2-cross sections is called non rotating isolated horizon (NRIH) whenever a transversal, future oriented null vector field

$$n \ = \ -\iota^A \iota^{A'}, \tag{4}$$

defined on N_0 by the gradient of a function v labeling the leaves of the foliation[1] satisfies the following conditions on N_0:

i) n is shear free, and its expansion is a negative function of v,

$$\lambda = 0, \ \ \mu = f(v) < 0; \tag{5}$$

ii) moreover, it is assumed that the Newman-Penrose spin-coefficient π vanishes

$$\pi = 0; \tag{6}$$

[1] That is, for every vector X tangent to N_0, we have $X^a n_a = X^a v_{,a}$.

iii) the Ricci tensor component $R_{\mu\nu}m^\mu\bar{m}^\nu$ is a function, say K, of the function v only;

$$R_{\mu\nu}m^\mu\bar{m}^\nu = K(v), \tag{7}$$

where m is a null, complex valued vector field tangent to the slices $v = const$ normalized by $m^\mu\bar{m}_\mu = 1$;

iv) The vector field $k^\mu = G^{\mu\nu}l_\nu$, where $G_{\mu\nu}$ is the Einstein tensor, is causal, $k^\mu k_\mu \leq 0$.

The vanishing of the shear and of the expansion of l is rescaling invariant. We normalize l such that

$$l^\mu n_\mu = -1. \tag{8}$$

A NRIH will be denoted by $(N_0, [(l,n)])$ where the bracket indicates, that the vector fields (l,n) are defined up to the foliation preserving transformations $v \mapsto v'(v)$.

Space-times admitting NRIH.
Suppose now, that $(N_0, [(l,n)])$, is a NRIH and the Einstein vacuum equations hold in the past of a neighborhood of N_0. To characterize (locally) a general solution we need to introduce another null surface, N_1 say. Let N_1 be a surface generated by finite segments of the incoming null geodesics which intersect N_0 at $v = v_1$ (see (1): we are assuming that the cartesian product corresponds to the foliation and the variable v) and are parallel to the vector field n at the intersection points. Thus the intersection,

$$N_0 \cap N_1 =: S \tag{9}$$

is the cross section $v = v_1$ of N_0. Locally (see above for the definition of 'locally'), the metric tensor is uniquely characterized (up to diffeomorphisms) by Friedrich's reduced data:

$$\text{on } S \ : \ m, \mathrm{Re}\rho,\ \mathrm{Re}\mu,\ \sigma,\ \lambda, \pi, \tag{10}$$

$$\text{on } N_0 \ : \ \Psi_0, \tag{11}$$

$$\text{on } N_1 \ : \ \Psi_4, \tag{12}$$

where m is a complex valued vector field tangent to S. The resulting solution is given by a null frame which satisfies the following gauge conditions,

$$\nu = \gamma = \tau = \pi - \alpha - \bar{\beta} = \mu - \bar{\mu} = 0, \tag{13}$$

locally in the spacetime, and

$$\epsilon = 0 \text{ on } \quad N_0. \tag{14}$$

Conversely, given submanifolds $N_0 \cup N_1$ of a time oriented 4-manifold M, the triple (M, N_0, N_1) being diffeomorphic (by the time orientation preserving diffeomorphism) to the one above, every freely chosen data (10, 11, 12) corresponds to a unique solution to the vacuum Einstein equations.

Let $(N_0, [(l, n)])$ be a NRIH . To calculate Friedrich's data, we need to satisfy the gauge conditions (14), (13). Since N_0 is non-diverging and shear-free, we can choose on N_0 a normalized complex vector field m tangent to the foliation, such that m is Lie constant along the null generators of N_0. This implies the vanishing of $\epsilon - \bar{\epsilon}$. From the generalized '0th law' [3] we know that, if we parameterize the foliation of N_0 by a function v' such that

$$\mu' = \text{const} \quad \text{on} \quad N_0, \tag{15}$$

then owing to the vacuum Einstein's equations

$$\epsilon' + \bar{\epsilon'} = \text{const} \quad \text{on} \quad N_0. \tag{16}$$

The geometric meaning of this law is that another function

$$v \ := \ \exp(2\epsilon v') \tag{17}$$

defines an affine parameter along the null generators of N_0. Therefore, if we use the pair (l, n) corresponding to the function v, then

$$l^\mu l_{\nu;\mu} \ = \ 0, \text{ hence } \ \epsilon = 0, \quad \text{on} \quad N_0. \tag{18}$$

(Incidently, in this normalization, $l^\mu n_{\nu;\mu} \ = \ 0$ due to $\pi = 0$). The vanishing of $\pi - \alpha - \bar{\beta}$, $\mu - \bar{\mu}$ is automatically ensured on N_0 by the pullback of n on N_0 being dv. Finally, the gauge conditions (13) can be satisfied locally in M by appropriate rotations of a null frame along the incoming geodesics, not affecting the data already fixed on N_0.

Now, we can consider the reduced data of the horizon. It follows directly from the definition, that

$$\text{on } S: \ \sigma = \lambda = \text{Re}(\rho) = \pi = 0, \ \mu = \text{const} < 0 \ \text{ and } \ \text{on } N_0: \ \Psi_0 \ = \ 0. \tag{19}$$

We also know [3] that the property (iii) in the definition of NRIH implies that the 2-metric tensor induced on S is spherically symmetric. The above conditions are necessary for the reduced data to define a NRIH.

Conversely, suppose that reduced data (10), (11) satisfy the conditions (19) and that they define a spherically symmetric 2-metric on the slice S. Then, it follows from the Einstein vacuum equations that, locally, N_0 is a NRIH. To summarize, locally, N_0 is a NRIH if and only if the vacuum spacetime is given by the reduced data (10), (11), (12) such that (19) holds and the vector field m defines on the slice S a homogeneous 2-metric tensor. The degrees of freedom are: i) the radius r_0 of the 2-metric of S, and ii) a complex valued function Ψ_4 freely defined on N_1. The constant $\mu_{|S}$ can be rescaled to be any fixed $\mu_0 < 0$.

Non existence of Killing vector fields for NRIH. Let us apply now our very result to the issue of the existence of Killing vectors. The usual way one addresses that problem is writing the Killing equation and trying to solve it. Another way is to look for invariant objects and see if those have a common symmetry[2]. (Perhaps the first way is a little better to prove the existence whereas the second way may be more useful to disprove it.) We will apply the second one. As it was indicated in [2], a null surface admits *at most one* structure of NRIH. Moreover, let as fix a number $\mu_0 < 0$ and use the rescaling freedom to fix the null vector fields (l, n) representing the NRIH structure $[(l, n)]$, such that

$$\mu = \mu_0, \quad \text{on} \quad N_0. \tag{20}$$

There is exactly one pair (l, n) on N_0 which satisfies the NRIH properties and the normalization of μ. Every isometry of spacetime preserving N_0, preserves the value of μ. Therefore, it necessarily preserves the vector fields l and n. Hence, the potential local isometry preserves also the function

$$|\Psi_4|^2 \;=\; |C_{\nu\mu\alpha\beta} n^\nu m^\mu n^\alpha m^\beta|^2, \tag{21}$$

where C is the Weyl tensor.

Let us use the above isometry invariant to see whether N_0 admits a tangential null Killing vector field. On N_0, the (would be) Killing vector is of

[2]Scalar invariants can be defined on M or even on the bundle of null directions, see Nurowski *et al* [6]

the form

$$\xi \;=\; b_0 l \tag{22}$$

where b_0 is a function. The following should be true

$$0 \;=\; b_0 l^\mu (|\Psi_4|^2)_{;\mu} \;=\; -8b_0\epsilon|\Psi_4|^2 \;=\; -4b_0(\text{surface gravity})|\Psi_4|^2, \tag{23}$$

the second equality being the consequence of the Einstein equations and the Bianchi identities. Since the surface gravity is not zero, this contradicts the existence of a null Killing vector field on the horizon unless

$$\Psi_4 \;=\; 0. \tag{24}$$

The general formula for a possible Killing vector field tangential to N_0 is

$$\xi \;=\; b_0 l + K \tag{25}$$

where K is tangent to the leaves of the foliation and together with b_0 is subject to the following restrictions. Since the isometry generated by ξ has to preserve the foliation of N_0, and the flow generated by l already does, the function b_0 is constant on each leaf of the foliation. Since the symmetry has to preserve the vector field l, b_0 is constant on N_0 and K commutes with l. Finally, because the symmetry should preserve the internal degenerate metric tensor on N_0, K on each leaf is a Killing vector field. On the other hand, the equation

$$\xi(|\Psi_4|^2) \;=\; 0, \tag{26}$$

implies

$$b_0 \;=\; \frac{1}{8\epsilon} K(\ln|\Psi_4|^2). \tag{27}$$

For a generic Ψ_4 defined on the cross section S of N_0, the right hand side of (27) is not constant on S for any Killing vector field of S.[3] So, generically, there is no Killing vector field in a past neighborhood of a NRIH which is tangent to N_0. (Sufficient conditions for the existence of a Killing symmetry of an isolated horizon will be derived in a forthcoming paper.)

---

[3]If it were constant, on the other hand, then necessarily $b_0 = 0$ provided the orbits of K in the 2-sphere are closed.

General isolated horizons.

A solution admitting the general isolated horizon can also be characterized using the reduced data. One can easily check that, whenever

$$\sigma \ = \ \rho \ = 0, \ \ \text{on} \ \ S, \ \ \text{and} \ \ \Psi_0 \ = \ 0, \text{on} \ \ N_0, \tag{28}$$

in the reduced data set (10), (11, (12), then the corresponding solution satisfies $\sigma = \rho = \Psi_0 = 0$ on N_0, hence N_0 is an isolated horizon. Of course the above data is also necessarily an isolated horizon data.

Therefore:

N_0 is an isolated horizon in Einstein's vacuum space-time, if and only if it is locally given by the reduced data (10, 11, 12) and the conditions (28), the remaining data $\mathrm{Re}\mu$, λ, π *on S and Ψ_4 on N_1 being arbitrary.*

The superposition method. There is one feature of the characteristic Cauchy problem of [4] we would like to emphasize more strongly here because of its relevance to the generalization of BH mechanics. Given a reduced data (10), (11), (12) one can evolve it, in particular, along the surface N_0. The data determines at each point of N_0 a vacuum solution: a null 4-frame, the spin connection and the Weyl tensor. Remarkably, the evolution of every field along the null generators of N_0 is independent of Ψ_4 except the evolution of Ψ_4 itself. We tend to think of this construction as a non-linear superposition of a vacuum solution given near N_0 with the contribution coming from data $\Delta\Psi_4$ given on N_1 and evolved tangentially to N_0. If we know a spacetime whose Newman-Penrose coefficients on N_0 we particularly like, but Ψ_4 is not relevant for us, by varying Ψ_4 on a transversal null surface N_1 we obtain a large family of solutions each of which has the desired properties on N_0. For example, let us take the Schwarzschild metric as the preferred solution, N_0 being a part of its horizon. The family of solutions obtained by the superposition with Ψ_4 coming in tangentially to N_0, is *exactly the set of general vacuum solutions admitting a NRIH which we have derived in this paper.* For every member of this family, on N_0, the 4-metric tensor, and all, except Ψ_4, Newman-Penrose coefficients are the same, as those of Schwarzschild. Ashtekar and collaborators wrote the laws of BH mechanics of Schwarzshild horizon purely in terms of the spin and curvature coefficients on N_0, not involving Ψ_4. That is why the laws hold automatically for a general NRIH [3].

Kerr like isolated horizons. The superposition can be well applied to the Kerr metric. Consider reduced data given by the following recipe:

a) take a reduced data for Kerr, such that N_0 is an isolated horizon, and N_1 is an arbitrary transversal null surface;

b) Keep Ψ_0 on N_0, and $\mathrm{Re}\rho, \mathrm{Re}\mu, \sigma, \lambda, \pi$ on the intersection S, but use an arbitrary function for Ψ_4 on N_1.

The resulting solution will be described by the same 4-metric tensor and the Newman-Penrose coefficients on N_0, except Ψ_4, as the original Kerr metric.

Following this example and the definition of NRIH, we propose to define a *Kerr like isolated horizon* to be a null surface N_0 equipped with an induced (degenerate) intrinsic metric tensor and the Newman-Penrose spin connection coefficients of the Kerr solution. This will determine the Weyl tensor spin coefficients except Ψ_4. Therefore, if we formulate the laws of rotating BH exclusively in terms of this data on the horizon, the same laws will hold for every metric tensor admitting the Kerr like isolated horizon. Since an analogous Schwarzschild like horizon would be exactly a NRIH, the above definition is a natural step toward defining a rotating case.

NRIH in the Einstein-Maxwell case. In a non-vacuum case, the conditions imposed on a NRIH imply restrictions on the stress energy tensor of the matter. They are [2]

$$\Phi_{00} \ = \ \Phi_{01} \ = \ \Phi_{02} \ = \ 0 \ = \ \delta(\Phi_{11} + \frac{1}{8}R) \tag{29}$$

the last equation being the condition *iii*) in the definition of NRIH. Those conditions are met by an electro magnetic field such that

$$\Phi_0 = 0, \tag{30}$$

and $|\Phi_1|^2$ is constant on the leaves of the foliation of N_0.

If we assume the Einstein-Maxwell equations to hold on $N_0 \cup N_1$, by looking at the Newman-Penrose version of the Maxwell equations, it is easy to complete the vacuum free data with suitable data for the electro-magnetic field. Indeed, for Φ_0 given on N_0, Φ_1 defined on S and Φ_2 defined on N_1, the Einstein-Maxwell equations determine the metric tensor, connection, curvature and electro magnetic field on the null surfaces N_0 and N_1, as well as their rates of change in the transversal directions. Then, N_0 is a NRIH if and only if the Einstein-Maxwell data is given by the reduced data (10-12) of the vacuum NRIH case, and

$$\Phi_0 \ = \ 0, \text{ on } \ N_0, \ |\Phi_1| \ = \ \text{const}, \ \ \text{on} S, \tag{31}$$

Φ_2 being arbitrary on N_1.

The electro magnetic field affects only the evolution of the gravitational data in the direction transversal to $N_0 \cup N_1$. In this case the existence/uniqueness statements can be found in Friedrich's contribution in [9].

Acknowledgments. I am most grateful to Abhay Ashtekar for the introduction to his idea of the isolated horizons and for numerous discussions, and to Helmut Friedrich for drawing my attention to his characteristic Cauchy problem which was crucial for this paper. I have also benefited from conversations with Alan Rendall, Ted Newman, Istvan Racz, Pawel Nurowski, Bob Wald, Christopher Beetle and Stephen Fairhurst. Finally, I would like to thank MPI for Gravitational Physics in Potsdam-Golm and the organizers of the workshop Strong Gravitational Fields held in Santa Barbara for their hospitality. This research was supported in part by Albert Einstein MPI, University of Santa Barbara, and the Polish Committee for Scientific Research under grant no. 2 P03B 060 17.

References

[1] Ashtekar A, Baez J, Corichi A and Krassnov K 1998 *Phys. Rev. Let.* **80**, N0. 5, 904-907

[2] Ashtekar A, Corichi A and Krassnov K Isolated Horisons: The Classical Phase Space Preprint gr-qc/9905089

[3] Ashtekar A, Beetle C and Fairhurst S Mechanics of Isolated Horizons Preprint gr-qc/9907068

[4] Friedrich H 1981 *Proc. R. Soc. Lond.* **A 375** 169 and Rendall A D 1900 *Proc. R. Soc. Lond.* **A 427** 221

[5] Newman E T 1971 *J. Math. Phys.* **12** 1929

[6] Nurowski P, Hughston L, and Robinson D 1999 *Class.Quantum Grav.* **16** 255

[7] Rácz I On Further Generalization of the Rigidity Theorem for Spacetimes with a Stationary Event Horizon or a Compact Cauchy Horizon Preprint gr-qc/9901029

[8] Kramer D, Stephani H, MacCallum M, Herlt E 1980 *Exact solutions of Einstein's field equations*, (VEB Deutscher Verlag der Wissenschaften, Berlin) and Erratum!

[9] Drerulle N and Piran T (eds.) 1983 *Gravitational Radiation* North-Holland Amsterdam