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Spacetimes with Killing Tensors*

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Abstract. For Einstein-Maxwell fields for which the Weyl spinor is of type $\{2, 2\}$, and the electromagnetic field spinor is of type $\{1, 1\}$ with its principal null directions coaligned with those of the Weyl spinor, the integrability conditions for the existence of a certain valence two Killing tensor are shown to reduce to a simple criterion involving the ratio of the amplitude of the Weyl spinor to the amplitude of a certain test solution of the spin two zero restmass field equations. The charged Kerr solution provides an example of a spacetime for which the criterion is satisfied; the charged C-metric provides an example for which it is not.

I. Introduction

The geodesic equation is said to admit a *quadratic first integral* if there exists on spacetime a symmetric tensor field K_{ab} which satisfies the *Killing equation*[1]

$$\nabla_{(a}K_{bc)} = 0\,.$$

Let t^a be tangent to an affinely parametrized geodesic,

$$t^a \nabla_a t^b = 0\,;$$

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[1] The Battelle conventions of Penrose [1] are used here for the denotation and manipulation of tensor and spinor indices.

then the quadratic first integral obtained from K_{ab} is

$$K_{ab}t^a t^b.$$

As a consequence of the geodesic equation and the Killing equation, this quantity is conserved along the geodesic orbit:

$$t^c \nabla_c(K_{ab}t^a t^b) = 0.$$

The Killing equation splits naturally into two parts. When it is transvected with the metric g^{ab}, the resulting *trace equation* can be put in the form

$$\nabla^a P_{ab} + \tfrac{3}{4}\nabla_b K = 0.$$

Here the definition $K := K_a{}^a$ is used; P_{ab} is the tracefree part of K_{ab},

$$K_{ab} = P_{ab} + \tfrac{1}{4}K g_{ab}.$$

The remaining *tracefree part* of the Killing equation is

$$\nabla_{(c}P_{ab)} - \tfrac{1}{3}g_{(ab}\nabla^d P_{c)d} = 0.$$

A tensor P_{ab} for which this equation holds is referred to as a *conformal Killing tensor.*

A conformal Killing tensor is of interest inasmuch as it gives rise to a quadratic first integral for *null* geodesic orbits. To see the manner in which this is the case note that if k^a is tangent to an affinely parametrized null geodesic,

$$k^a \nabla_a k^b = 0, \qquad k_a k^a = 0,$$

then $P_{ab}k^a k^b$ is constant along the geodesic since

$$\begin{aligned} k^c \nabla_c(P_{ab}k^a k^b) &= k^a k^b k^c \nabla_{(c}P_{ab)} \\ &= \tfrac{1}{3}(k_a k^a)\, k^c \nabla^b P_{bc}, \end{aligned}$$

which vanishes on account of the fact that k^a is null.

These notes concern themselves with the class of Einstein-Maxwell fields for which the Weyl tensor is of type {2, 2}, and the electromagnetic field tensor is of type {1, 1} with its principal null directions coaligned with those of the Weyl tensor. The purpose of the upcoming discussion is to display a concise criterion for the determination of whether an Einstein-Maxwell field of this kind admits a certain Killing tensor of valence two. The argument proceeds first with a demonstration of the fact that each of these fields admits a *conformal* Killing tensor. Then the circumstances are examined under which the full Killing tensor may subsequently be constructed. The charged Kerr solution is presented as an affirmative example; the charged C-metric as a non-affirmative example.

II. The Einstein-Maxwell Equations

Consider some consequences of the Bianchi identity

$$\nabla^A_{A'}\Psi_{ABCD} = \nabla^{B'}_{(B}\Phi_{CD)A'B'}$$

taken together with the Einstein-Maxwell equations

$$\Phi_{ab} = \varphi_{AB}\bar{\varphi}_{A'B'}\,; \qquad \nabla^A_{A'}\varphi_{AB} = 0\,.$$

It is helpful to introduce a normalized spinor dyad

$$\{o_A, \iota_A : o_A \iota^A = 1\}\,,$$

with each member aligned along one of the special null rays in accordance with the conditions mentioned at the end of Section I. Writing for the electromagnetic field spinor

$$\varphi_{AB} = \varphi\, o_{(A}\iota_{B)}\,,$$

and for the Weyl spinor

$$\Psi_{ABCD} = \Psi\, o_{(A}\iota_B o_C \iota_{D)}\,,$$

one defines, thereby, the complex field amplitudes φ and Ψ. As a consequence of a generalized Goldberg-Sachs theorem [2] the null directions defined by o^A and ι^A are geodesic and shearfree,

$$o^A o^B \nabla_{A'A} o_B = 0\,; \qquad \iota^A \iota^B \nabla_{A'A} \iota_B = 0\,.$$

Quite useful in what follows is the introduction of two auxiliary test fields. The first of these[2] is a solution of the *vacuum spin 2 zero restmass field equation,*

$$\nabla^A_{A'} \Upsilon_{ABCD} = 0\,.$$

Defining, up to a disposable constant factor,

$$\Upsilon := \varphi^{3/2}\,,$$

the solution is

$$\Upsilon_{ABCD} = \Upsilon\, o_{(A}\iota_B o_C \iota_{D)}\,.$$

The second auxiliary field, of central significance to the discussion, is a solution of the *twistor equation*

$$\nabla_{A'(A}X_{BC)} = 0\,.$$

In this instance the solution [3, 4] is

$$X_{AB} = \Upsilon^{-1}\varphi_{AB}\,.$$

[2] The authors wish to express their gratitude to Ivor Robinson for discussions concerning the construction of this field.

III. The Bianchi Identity

Here it is observed that *the vector $\Phi_{ab}\nabla^b\Upsilon^{-1}$ is the gradient of a complex scalar.*

Define a scalar α by the ratio

$$\alpha := \Psi/\Upsilon .$$

The Bianchi identity, as a consequence of the field equations for φ_{AB} and Υ_{ABCD}, is seen to take the form

$$\Upsilon_{ABCD}\nabla^A_{A'}\alpha = \bar{\varphi}_{A'B'}\nabla^{B'}_{(B}\varphi_{CD)} .$$

Now use the fact that φ_{AB} is proportional to the solution X_{AB} of the twistor equation in order to write

$$\Upsilon_{ABCD}\nabla^A_{A'}\alpha = \bar{\varphi}_{A'B'}X_{(CD}\nabla^{B'}_{B)}\Upsilon .$$

Transvecting each side with Υ^{BCDE} gives the desired relation

$$-\tfrac{1}{4}\nabla_a\alpha = \Phi_{ab}\nabla^b\Upsilon^{-1} .$$

During the course of Section V special use will be made of a real concomitant of this last relation:

$$-\tfrac{1}{4}[\bar{\Upsilon}^{-1}\nabla_a\alpha + \Upsilon^{-1}\nabla_a\bar{\alpha}] = \Phi_{ab}\nabla^b(\Upsilon\bar{\Upsilon})^{-1} .$$

IV. The Conformal Killing Tensor

At this point one has at his disposal all that is necessary to construct a tracefree conformal Killing tensor P_{ab}. This end is achieved with the remark [3] that

$$P_{ab} = X_{AB}\bar{X}_{A'B'} .$$

That P_{ab} is a conformal Killing tensor can be seen to follow from the identity

$$\nabla_{(a}P_{bc)} - \tfrac{1}{3}g_{(ab}\nabla^d P_{c)d} = \nabla_{(A'(A}P_{BC)B'C')}$$

taken in conjunction with the twistor equation for X_{AB}:

$$\nabla^{(A'}_{(A}P^{B'C')}_{BC)} = \nabla^{(A'}_{(A}X_{BC)}\bar{X}^{B'C')} = 0 .$$

V. The Killing Equation

The conformal Killing equation having been solved, what remains is the *trace equation* of Section I, the condition that the vector $\nabla^a P_{ab}$ should be a gradient. Seeing that

$$P_{ab} = (\Upsilon\bar{\Upsilon})^{-1}\Phi_{ab} ,$$

the contracted Bianchi identity

$$\nabla^a \Phi_{ab} = 0$$

allows the reexpression of the trace equation with the statement that there should exist a scalar K such that

$$\Phi_{ab} \nabla^b (\Upsilon \bar{\Upsilon})^{-1} = -\tfrac{3}{4} \nabla_a K .$$

Use the concluding remark of Section III to render this statement into the simple form

$$\bar{\Upsilon}^{-1} \nabla_a \alpha + \Upsilon^{-1} \nabla_a \bar{\alpha} = 3 \nabla_a K .$$

The integrability condition for this last equation is that

$$\alpha = \beta(\bar{\Upsilon}) + \int \Upsilon^{-2} \gamma(\Upsilon, \bar{\Upsilon}) \, d\Upsilon$$

where β is a disposable function of $\bar{\Upsilon}$, and γ is a *real* function of Υ and $\bar{\Upsilon}$. This is the condition for the existence of the valence two Killing tensor.

VI. Concluding Remarks

The problem has now been reduced to a tractable form. All that is called for is a systematic analysis of these Einstein-Maxwell fields to determine precisely for which members of the class the integrability criterion of Section V is satisfied. As it turns out, *all members of this class of spacetimes*[3] *admit the Killing tensor with the notable exceptions of the C-metric and the twisting generalizations thereof due to Kinnersley* [5].

This point can be elucidated by means of two examples:

i) *The Charged Kerr Solution.* The line element in this case is [6]

$$\begin{aligned} ds^2 = {} & \varrho^2 d\theta^2 - 2a \sin^2\theta \, dr \, d\varphi + 2 dr \, du \\ & + \varrho^{-2} [(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta] \sin^2\theta \, d\varphi^2 \\ & - 2a\varrho^{-2} (2mr - e^2) \sin^2\theta \, d\varphi \, du \\ & - [1 - \varrho^{-2}(2mr - e^2)] \, du^2 \end{aligned}$$

with

$$\varrho^2 := r^2 + a^2 \cos^2\theta$$

$$\Delta := r^2 - 2mr + a^2 + e^2 .$$

The pertinent field amplitudes are

$$\Psi = m(r + ia\cos\theta)^{-3} - e^2 (r + ia\cos\theta)^{-3} (r - ia\cos\theta)^{-1} .$$

$$\Upsilon = (r + ia\cos\theta)^{-3} .$$

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[3] The result stated here may be easily seen to be valid as well for the associated vacuum metrics. The techniques introduced in these notes apply with equal force to the vacuum case, provided the interpretation of the special fields be adjusted: the field Υ_{ABCD} becomes the Weyl spinor up to an arbitrary constant factor; the field φ_{AB} becomes a test Maxwell field; and, the fields Ψ_{ABCD} and Φ_{ab} may be regarded together as a test solution of the Einstein-Maxwell Bianchi identity.

Upon construction of the ratio

$$\alpha = m - e^2 \bar{\Upsilon}^{1/3},$$

it is manifest that the integrability condition is satisfied.

ii) *The Charged C-Metric.* Here the line element may be put in the form [7]:

$$ds^2 = H\,du^2 + 2\,du\,dr + 2Ar^2\,du\,dx - r^2(G^{-1}dx^2 + G\,dz^2),$$

where

$$H := -A^2 r^2 G(x - A^{-1}r^{-1}),$$

$$G(x) := 1 - x^2 - 2mAx^3 - e^2 A^2 x^4.$$

The field amplitudes

$$\Psi = (m + 2e^2 Ax)\,r^{-3} - e^2 r^{-4},$$

$$\Upsilon = r^{-3}$$

have the ratio

$$\alpha = (m + 2e^2 Ax) - e^2 \bar{\Upsilon}^{1/3}.$$

It is clear that the integrability condition is not satisfied, and thus, that the charged C-metric does not admit the full Killing tensor.

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