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SPANNING TREES WITH MANY LEAVES IN CUBIC GRAPHS[1]

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Abstract. For a connected graph G let $L(G)$ denote the maximum number of leaves in any spanning tree of G. We give a simple construction and a complete proof of a result of Storer that if G is a connected, cubic graph on n vertices, then $L(G) \geq \lceil \frac{n}{4} + 2 \rceil$, and this is best-possible for all (even) n. The main idea is to count the number of "dead leaves" as the tree is being constructed. This method of amortized analysis is used to prove the new result that if G is also 3–connected, then $L(G) \geq \lceil \frac{n}{3} + \frac{4}{3} \rceil$, which is best-possible for many n. This bound holds more generally for any connected, cubic graph that contains no subgraph $K_4 - e$. The proof is rather elaborate since several reducible configurations need to be eliminated before proceeding with the many tricky cases in the construction.

1. Introduction and Statement of Results. We consider the problem of finding spanning trees in given graphs that contain many leaves (degree one vertices). All graphs are assumed to be simple (undirected, no loops or multiple edges). If G is a connected graph, let $L(G)$ denote the maximum number of leaves in any spanning tree of G. We are interested here in $L(G)$ for *cubic* (3–regular) graphs G.

Suppose T is a spanning tree for a connected, cubic graph G on n vertices. Necessarily, n is even. Let d_i denote the number of vertices of degree i in $T, i = 1, 2, 3$. Then the number of vertices $n = d_1 + d_2 + d_3$, while the sum of the degrees $2n - 2 = d_1 + 2d_2 + 3d_3$. It follows that $L(T) = d_1 = d_3 + 2$. Consequently, $L(G)$ is *maximized* over such graphs G when it contains T with d_2 as small as possible, that is, $d_2 = 0$. Hence, $L(G) \leq \frac{n}{2} + 1$. This bound is attained for all (even) n by taking the caterpillar in which $(n/2) - 1$ vertices form a path, and a leg (leaf vertex) is joined to each interior vertex of the path, while two legs are joined to each end of the path. This is the desired tree T, which can be embedded in a suitable graph G by adding a cycle through the leaves of T (see Fig.1).

The more interesting question then is to *minimize* $L(G)$, i.e., to obtain a lower bound on $L(G)$ over all such graphs G in terms of n. This problem was proposed and solved in 1981 by Storer.

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THEOREM 1 [4]. *If G is a connected, cubic graph on n vertices, then* $L(G) \geq \lceil \frac{n}{4} + 2 \rceil$.

This bound is best–possible for all (even) n. For example, if $n \equiv 0 \pmod 4$, then take G to be a circular "necklace" of $n/4$ "beads", where each bead is $K_4 - e$ (meaning K_4 with one edge deleted), as shown in Fig.2. If $n \equiv 2 \pmod 4$, we may take G to be a necklace in which there are $(n-6)/4$ beads that are $K_4 - e$ and one bead that is $K_{3,3} - e$. Then $L(G) = \frac{n}{4} + \frac{5}{2} = \lceil \frac{n}{4} + 2 \rceil$, as claimed.

If one permits vertices of lower degree than 3 in G, then $L(G)$ can drop dramatically, e.g., $L(P_n) = 2$, where P_n is a path on n vertices. Storer works with graphs of maximum degree 3, rather than our more restricted setting of cubic graphs. We consider the effect of introducing the stronger connectivity condition on G that it be *3-connected*, i.e., the removal of any two vertices does not disconnect it. Such graphs cannot contain $K_4 - e$, and our main result applies to this more general situation.

THEOREM 2. *If G is a connected, cubic graph on n vertices that contains no subgraph isomorphic to* $K_4 - e$, *then* $L(G) \geq \lceil \frac{n+4}{3} \rceil$.

COROLLARY. *If G is a 3–connected, cubic graph on n vertices, then* $L(G) \geq \lceil \frac{n+4}{3} \rceil$.

So the lower bound on $L(G)$ rises to over $n/3$ when $K_4 - e$ is excluded. Recall that a necklace of beads, each $K_4 - e$, was used to attain the lower bound around $n/4$ in Theorem 1.

The bound in Theorem 2 is sharp. For $n \equiv 0 \pmod 6$, say $n = 6k$, one can obtain a family of graphs G with $L(G) = 2k + 2 = \frac{n}{3} + 2 = \lceil \frac{n+4}{3} \rceil$ in the following way: Take $2k$ triangles (K_3) and add edges on these $6k$ vertices until a connected, cubic graph G is formed. To obtain a tree T in G with $L(T) = 2k + 2$, take any spanning tree T^R in the reduced graph G^R on $2k$ vertices which is obtained by contracting each of the $2k$ triangles to a point and eliminating duplicated edges. Then build T^R up to a tree T in G in the natural way, so that each vertex of degree 1 (respectively, 2, 3) in T^R gives rise to 2 (respectively, 1, 0) leaves in T.

For $n \equiv 2 \pmod 6$, say $n = 6k + 2$, the graph Q_3 of the usual 3–dimensional cube is extremal for $k = 1$. In fact, by carrying out the initial part of the proof of Theorem 2 more carefully, it can be shown that Q_3 is the *unique* extremal graph with $n \equiv 2 \pmod 6$. It follows that for G covered by the theorem, $L(G) \geq \frac{n+5}{3}$ unless $G = Q_3$.

For $n \equiv 4 \pmod 6$, say $n = 6k + 4$, we have that $L(G) \geq 2k + 3$. An extremal graph is obtained for $k = 1$ by taking two 5–cycles and pairing up their vertices, as shown in Fig.3(a). It is tempting to think that this is the only connected, cubic graph G not containing $K_4 - 3$, besides Q_3, that has $L(G) < \frac{n}{3} + 2$. (We saw above that many graphs exist with $L(G) = \frac{n}{3} + 2$.) However, Fig.3(b) shows another example with $n = 10$ and $L = 5$.

Our method of proof is to begin by finding a small tree in G with many leaves and to then grow the tree by adding several vertices in such a way that the number of leaves always grows enough to keep satisfying the theorem. A central idea is to keep track of the number of "dead leaves" as well as the number of leaves. A *dead leaf* is a leaf in the tree under construction, all of whose neighbors are already in the tree. Once a leaf is dead, it remains a leaf during the rest of the construction. To illustrate the power of this approach, we offer a new proof of Theorem 1 in the next section.

In Sec.3 we begin the proof of Theorem 2. Another fundamental idea in the proof is that certain configurations are *reducible*, which means they can be replaced by smaller configurations (which may include a special type of vertex we call a *goober*). Once the construction is completed on the reduced graph, the forest obtained can be blown up and reconnected to form the desired tree on G. Sec.4 contains the several cases that make up the proof for the reduced graph. In some instances the construction is by necessity quite elaborate.

The paper concludes in Sec.5 with some suggestions for further study including, most notably, a conjecture of N. Linial that generalizes Theorem 1.

2. Dead Leaves and Theorem 1. A natural approach to proving a result such as Theorem 1 is as follows. Starting with a small tree in G that is nice, one tries to add on some number of vertices N in such a way that the tree gains at least $N/4$ leaves. If this can always be done, then the tree eventually constructed must have at least roughly $n/4$ leaves. There is one case that is especially difficult with this approach. Consider a vertex v outside the tree that has all three of its neighbors being leaves in the tree. Then adding v causes no gain in the number of leaves. What can we do if there are many such vertices v? We do gain something by adding v, which is that v itself would have all of its edges inside the tree and that several edges involving neighbors of v are likewise accounted for. So v itself and perhaps some of its neighbors become dead leaves. At any given stage in the construction, it is obvious that the number of dead leaves, D is at most the number of leaves, L. So if we aim to show that at the end, $L \geq \frac{n}{4} + c$, for some c, or equivalently, $4L \geq n + 4c$, then it suffices to show that $aL + bD \geq n + 4c$ for some choice of $a, b \geq 0$ such that $a + b = 4$. We start off by constructing a tree on N vertices with L leaves and D dead leaves such that $\Delta(N, L, D) \geq 4c$, where $\Delta(N, L, D) = aL + bD - N$. It then suffices to show that for any constructed tree that does not yet span G, there exists some set of vertices, say N of them, that can be added in such a way as to increase the number of leaves by L and the number of dead leaves by D, where $\Delta(N, L, D) \geq 0$.

This dead leaf method is simply a type of amortized analysis in which some of the benefit of adding new leaves is postponed to the later time that the leaves die. the net gain from adding a leaf is the same. We sacrifice some advantage by using $aL, a < 4$, but not enough to case failure, while we benefit later from the term $bD, b > 0$, when we create a dead leaf. This allows us to handle a case such as a vertex v as above that is adjacent only to leaves that were already created. Suitable choices for a and b are determined by carrying out the cases in the proof and solving for a and b which make the inequalities work out, with some trial and error being necessary.

We are ready to prove Theorem 1. Besides illustrating the value of the "dead leaves" approach, it may be useful to have a complete proof written down. Storer's approach is to start with a breadth-first spanning tree and then modify it to gain leaves. His approach is natural, but the proof is merely sketched, and we were unable to work out all of the details.

Proof of Theorem 1. Let G be a connected, cubic graph on n vertices. Since n is even and $L(G)$ is integral, $L(G) \geq \lceil \frac{n}{4} + 2 \rceil$ if and only if $L(G) > \frac{n}{4} + \frac{3}{2}$. For our "dead leaves" approach, we seek a and $b, a+b = 4$, so that $aL + bD - N > 6$. It turns out that $a = 3.5$ and $b = .5$ are suitable choices, so we assume these values hereafter. Concerning the notation, vertices shall be denoted by lower case letters, and $v \sim w$ means v and w are adjacent, while $v \not\sim w$ means they are not adjacent. If a vertex v is outside a tree T but adjacent to some vertex in T, we write $\sim T$. We denote the edge between two vertices v and w by vw.

To start off, select any vertex $v \in G$, and let $w, x, y \sim v$. Begin the tree T by taking the three edges involving v. So we have $\Delta \geq \Delta(4,3,0) = 6.5 > 6$.

For subsequent stages we show that a tree T that only partially spans G can be extended by some amount such that $\Delta \geq 0$. First suppose there exists $v \in T$ such that $v \sim w, x \notin T$. Then add vw and vx to T, and $\Delta \geq \Delta(2,1,0) = 1.5$. If no such v exists, suppose these exists $v \in T$ that is not a leaf and $w \notin T$, $w \sim v$. Then add vw to T, giving $\Delta \geq \Delta(1,1,0) = 2.5$. Assume this is not possible either. So vertices $v \notin T$ with $v \sim T$ are adjacent to *no* internal vertices of T, while leaves in T that are not dead lie on only one edge outside T.

Suppose there exists $v \notin T$ such that $v \sim w, x, y \in T$. Adding vw to T creates no new leaves, but v, x, y are all dead leaves, so that $\Delta = \Delta(1,0,3) = .5$. Next suppose there exists $v \notin T$, $v \sim T$, such that v splits, i.e., $v \sim w, x \in T$. Let $t \in T$ such that $v \sim t$. Then add tv, vw, vx to T, and we have $\Delta \geq \Delta(3,1,0) = .5$. Then suppose none of the above operations is possible, so that every $v \notin T, v \sim T$, is adjacent to exactly two vertices in T. Since G is connected, there exists such a vertex v, say $v \sim s, t \in T$ and $v \sim w \notin T$. Then if also $w \sim T$, say $w \sim p, q \in T$, add edges tv, vw to T creating dead leaves at s, w, p, q, so that $\Delta = \Delta(2,0,4) = 0$. Otherwise, we have $w \not\sim T$, so w splits and $w \sim x, y \notin T$. Then add tv, vw, wx, wy to T, so that $\Delta \geq \Delta(4,1,1) = 0$, since s becomes a dead leaf. Thus in every case we can add to T so that $\Delta \geq 0$. By induction, we can eventually extend T to a spanning tree of G with at least $\lceil \frac{n}{4} + 2 \rceil$ leaves. □

The proof above is essentially a polynomial–time algorithm for constructing a tree with the desired number of leaves. This was a concern in [4], where a very different polynomial–time algorithm is provided for the more general class of connected graphs of maximum degree 3. The algorithm there begins with a breadth–first spanning tree for G and then modifies it to obtain many leaves.

3. Reducible Configurations and Theorem 2. We shall describe several reductions on G in which suitable configurations are replaced by simpler ones. Often, some vertices are replaced a new type of vertex, called a *goober*. Goobers do not count towards N, the number of vertices, but they do count towards the degrees of their neighbors. If a goober is a leaf, it does count towards L. A goober also counts as a dead leaf in D if it belongs to no edges with vertices outside the tree under construction. Goobers have degree $0, 1$, or 2. All other vertices are *ordinary* and have degree 3.

The reductions considerably simplify G. They may even disconnect G. We must carefully check that a forest with sufficiently many leaves in the reduced graph can be lifted to a spanning tree for G with sufficiently many leaves to prove Theorem 2. Then it will remain to show that for each component in the reduced graph, a spanning tree with many leaves can be constructed. This result, which we prove with many cases in Section 4, will then imply our main result, Theorem 2.

THEOREM 3. *Suppose H is a graph with $a > 0$ ordinary vertices and $b \geq 0$ goobers. Suppose H is connected and contains no $K_4 - e$ (all 4 vertices being ordinary). Suppose H is irreducible. Then $L(H) \geq \frac{a}{3} + 2$, if $b > 0$, and $L(H) \geq \frac{a}{3} + \frac{4}{3}$, if $b = 0$.*

If the original graph G is irreducible, then it has no goobers, so Theorem 2 holds from the case $a > 0, b = 0$ in Theorem 3. However, if G is reducible, one keeps reducing it until what remains has every component being irreducible and containing a goober. Those components that are not *trivial* (consist of an isolated goober) will contain at least one ordinary point, as we shall see, so that the case $a > 0, b > 0$ in Theorem 3 can be applied to each component. It may cost some leaves to reconnect separate components, but this will be offset by the gain of 2 leaves over $a/3$ in every nontrivial component, for a net gain of at least 2 over $n/3$ for reducible G.

The reductions are shown in Fig.4. In figures, goobers are always shown as open circles. Goobers usually arise by contracting triangles but other possibilities occur with Reductions (4), (5), and (6).

Some conventions must be explained in conjunction with Fig.4. Dashed lines mean that an edge may or may not be incident, and, if it is, it is retained after the reduction. In Reductions (1)–(5), two outgoing edges from vertices in the configuration may not meet to form a single edge or meet at another ordinary vertex when both originate at the left end or both at the right end of the figure. This would never happen anyway, given our hypotheses on G, with the single exception that if the edges on the right side of Reduction (5) meet at an ordinary vertex, we cannot use (5). But in this case, (6) is applicable. On the other hand, edges from opposite ends may meet, e.g., two triangles with two edges joining them (or even three) are reducible using (1). Reduction (7) is unique. We assume that no two of its three outgoing edges join or meet at another ordinary vertex. This prevents us from forming a multiple edge between ordinary vertices or a $K_4 - e$ on ordinary points, both of which are forbidden, by carrying out Reduction (7). We emphasize that outgoing edges from *opposite* sides in (1)–(6) may meet, while those from the *same* side in (1)–(7) may meet only at a goober.

Here is what some reductions do, in words. Reduction (1), for example, destroys any edges between two adjacent triangles and contracts each such triangle to a goober. Reduction (2) destroys edges between goobers. Reduction (6) gets rid of the "bead" on six vertices shown in the figure. In effect, this bead is no worse than a simple edge since we can always gain two leaves from these six vertices.

Let G^R be the graph obtained after some succession of reductions when no further reductions are possible. One can check that G^R contains no multiple edges, nor any of the forbidden graphs $K_4 - e$ (using only ordinary vertices). First suppose that G^R contains no goobers, i.e., only Reduction (7) was performed, if any were. Then G^R is connected, and Theorem 3 can be applied with $b = 0$ to produce a spanning tree for G^R with at least $\frac{a}{3} + \frac{4}{3}$ leaves, where a is the number of vertices in G^R. It remains to successively restore vertices deleted by Reduction (7), 12 vertices for each reduction, while enlarging the spanning tree each time. We shall see in Fig. 7 that the tree can be enlarged to gain 4 leaves for each reduction, which guarantees that a tree with at least $\frac{n}{3} + \frac{4}{3}$ leaves is constructed for all of G.

It remains to consider G^R that contains some goober. It can be checked easily that every component in G^R contains a goober since any reduction besides (7) gives a goober to each component if it disconnects any vertices.

Now we can apply Theorem 3 to each nontrivial component H in G^R, if any such H exist. If a denotes the number of ordinary vertices in H, then our proof will construct a spanning tree for H with at least $\frac{a}{3} + 2$ leaves. Take such a tree for every H to obtain a spanning forest for G^R.

For each isolated goober g in G^R, we seek to reattach g to some other component H in such a way that if H is nontrivial, we put back some number of original ordinary vertices $3c$, where the integer $c \geq 0$, while the tree on H is expanded to gain at least c leaves. Alternately, if H is trivial, i.e., H is another goober h, we join g and h and replace either g or h, or both, by $3c$ original ordinary vertices, while producing a tree on this component with at least $c + 2$ leaves. How this is done depends on what reduction separated g and h in the first place. By doing this, we have partially restored G^R back to G, call this intermediate graph G^S, in such a way that for every component H of G^S, H is not trivial, and if the number of ordinary vertices in H is a, then the number of leaves in the constructed tree spanning H is at least $\frac{a}{3} + 2$.

In Fig.5 we show exactly how this merging of trivial components with nontrivial ones or with each other is carried out. From Fig.4 we see that a goober was isolated when its last incident edge was destroyed and only Reduction (2) or (3) could have done this. In each row, we give the type of reduction, (2) or (3), that originally destroyed the connection we are to reinstate, followed by a picture showing edges in the forest constructed so far that involve these vertices, followed by the edges used in the forest after we reattach the isolated component. Finally, we list (c, d), where $3c$ vertices have been added and d is the number of leaves gained. One can check in each case that either c or $c+2$ leaves are gained, as necessary, corresponding to the description above. It can be seen that the vertices in the configuration are spanned by the forest and that no cycle is formed. Goobers will still have degree at most two. Cases that are identical by symmetry to ones shown are not listed.

The graph G^S typically will still contain some goobers. Our next task is to replace all goobers by the original ordinary vertices, while extending the spanning forest by an appropriate number of leaves.

First suppose a goober g was created by contracting a triangle, i.e., by Reduction (1) or (3). Depending on whether the degree of g is 1 or 2 in G^S, we have 2 or 1 leaves in the forest after restoring the triangle. Either way, we gain one leaf while adding 3 ordinary vertices. Specifically, if the triangle corresponding to g is vwx, and if there is just *one* edge in G^S to g from a vertex a that resulted by reducing the edge av, then use the edges av, vw, vx in the forest. If, instead, g is adjacent to *two* vertices, a and b, that resulted by reduction from edges av and bw, then use the edges av, vw, bw, vx in the forest after restoration.

Next consider goobers that were created by Reductions (4), (5), or (6). Such goobers occur in pairs. We shall replace each such pair of goobers by the original 6 ordinary vertices. In Fig.6, we see how to expand the forest to span the 6 new vertice, without forming cycles or connecting different components, while gaining 2 leaves in every case. In the event that the two goobers in the pair are from different components in G^S, it is not the case for (4) or (5) that the 6 ordinary vertices and 2 leaves gained are divided equally between the two components. But it is true that such additions are made if one views the forest globally over all its components rather than locally over each separate component. The design of Fig.6 is similar to Fig.5.

To restore all vertices of G, it remains to successively restore vertices deleted by applications of Reduction (7). We describe in Fig.7 the procedure to expand the spanning forest to reach all of 12 vertices restored by reversing Reduction (7) so that the number of leaves is increased by 4.

After all of the ordinary vertices have been restored, we have a spanning forest F for G in which every component has at least 2 more leaves than one-third the number of vertices (unless G is itself irreducible, so that Theorem 3 implied directly that $L(G) \geq \frac{n}{3} + \frac{4}{3}$). If the forest F is not connected, say it has k components. Then we can successively add $k-1$ edges, $e_1, \ldots, e_{k-1}$ such that for all i, edge e_i joins vertices in separate components of $F \cup \{e_1, \ldots, e_{i-1}\}$. The edge e_i can at worst destroy 2 leaves, but we have at least 2 leaves to spare for every component of F. So after adding all $k-1$ edges, there remains a spanning tree for G with at least $\frac{n}{3} + 2$ leaves.

This completes the proof of Theorem 2 from Theorem 3. We now proceed to prove Theorem 3.

4. Proof of Theorem 3. We use the dead leaf approach as in the proof of Theorem 1. This time, if the triple $S = (N, L, D)$ represents the numbers N of ordinary vertices added, L of leaves gained, and D of dead leaves gained, then we set $\Delta(S) = \Delta(N, L, D) = 2.5L + .5D - N$. Over the entire graph H we seek $\Delta \geq 4$ if $b = 0$ and $\Delta \geq 6$ if $b > 0$. At the end, $2.5L + .5D = 3L$ is integral, so Δ is integral. Thus when we prove $\Delta \geq 5.5$ if $b > 0$, it will be sufficient to prove the theorem. We first show how to start by finding a tree in H for which $\Delta = 4$ if $b = 0$ while $\Delta \geq 5.5$ if $b > 0$. Then we show that by starting with any tree in H we can always find a way to extend the tree such that $\Delta \geq 0$ for the extension. The theorem then follows.

4.1. The Initial Stage. First suppose that $b = 0$, i.e., H has no goobers. Thus, H is connected, cubic, and contains no $K_4 - e$ nor any reducible configuration. Let v a vertex in H. There exists some $w \sim v$ such that v, w belong to no triangle. Let T be the tree of all 5 edges that involved either v or w. Then we have $\Delta = \Delta(6, 4, 0) = 4$, as required.

Suppose instead that $b > 0$ in H. By hypothesis, the number of ordinary points $a > 0$. Since H is connected, there exist an ordinary vertex v and a goober g in H with $v \sim g$. If there exists another goober $h \sim v$, let T consist of all three edges incident at v, which gives $\Delta \geq \Delta(2, 3, 0) = 5.5$ as required. Otherwise, $v \sim w, x$ which are both ordinary. By Reduction (3), it must be that $w \not\sim x$. The goober g has degree at most 2, so at least one of w and x, say w, is not adjacent to g. Therefore w splits, i.e., $w \sim y, z$ which are two new vertices. Include all 5 edges that contain v or w in T. If either of y and z is a goober, we are finished, since then $\Delta \geq \Delta(4, 4, 0) = 6$.

Suppose instead that y and z are ordinary points. If any of x, y, z is adjacent to g, then g is a dead leaf in T, so $\Delta = \Delta(5, 4, 1) = 5.5$, which is good enough. Similarly, if any one of x, y, z is adjacent to the other two, it is a dead leaf and $\Delta = 5.5$. There remains the case that none of x, y, z is adjacent to g or to both of the other two. Hence one of them, e.g., y, splits into two new vertices p and q. By adding the edges yp and yq to T we obtain that $\Delta \geq \Delta(7, 5, 0) = 5.5$, which is good enough. This proves that we can always carry out the initial stage of the proof.

4.2. Simple Stages. Given any partial tree T in H, we shall describe how to carry out an additional stage that enlarges T while $\Delta \geq 0$ for the addition. We shall not need to distinguish the cases $b > 0$ and $b = 0$. In this subsection, the simplest extensions are described.

First, suppose that there exists $v \in T$ such that $v \sim w, x$, where $w, x \notin T$. Then it suffices to add vw and vx to T, gaining one leaf at the cost of adding at most two ordinary vertices. Thus, $\Delta \geq \Delta(2, 1, 0) = .5$. Whenever possible, we carry out such an extension, so assume henceforth it is impossible.

Next suppose that some internal vertex (not a leaf) v in T is adjacent to some $w \notin T$. Then if we add vw to T, it gives $\Delta \geq \Delta(1, 1, 0) = 1.5$. We may assume henceforth that internal vertices in T are adjacent only to vertices in T, while leaves in T have at most one neighbor outside T. Further, no goobers outside T are adjacent to T, or else they could be added to extend T, so that $\Delta \geq \Delta(0,0,0) = 0$.

Now consider any ordinary vertex $v \notin T$ that has all three of its neighbors, $w, x, y \in T$. Then adding vw to T creates dead leaves at v, x, and y, so that $\Delta = \Delta(1,0,3) = .5$. Therefore, we may assume ordinary vertices outside T have at most two neighbors in T.

4.3. Some Vertex v is Once Adjacent to T. In this case we suppose there exists an ordinary vertex $v \notin T$ that is adjacent to precisely one vertex in T, say $v \sim t, w, x$ where $t \in T$, and $w, x \notin T$. We shall add edges tv, vw, and vx to T. If either w or x is a goober, then we have $\Delta \geq \Delta(2,1,0) = .5$, while if either w or x is adjacent to T, a dead leaf is created in T, so this gives $\Delta \geq \Delta(3,1,1) = 0$. If either w or x splits into two new vertices, say $w \sim y, z$, then also add wy and wz to T, giving $\Delta \geq \Delta(5,2,0) = 0$.

It remains to treat the case that neither w nor x is a goober, $w, x \not\sim T$ (i.e., neither is adjacent to T), and $w \sim x$. There exists $y \sim w$, $y \notin T$. Since vwx is a triangle, $y \not\sim x$ (H contains no $K_4 - e$) and y is not a goober (Reduction (3) would apply, but H is irreducible). If y is twice adjacent to T, adding wy to T would produce 3 dead leaves, so that $\Delta = \Delta(4,1,3) = 0$. Next consider y that is adjacent only once to T, say $y \sim s \in T$ and $y \sim z \notin T$. Then we do not add vt, vx to T, but instead we add all 5 edges involving w or y, as shown in Fig.8, giving $\Delta \geq \Delta(5,2,1) = .5$. In the figure, the thin line indicates an edge in H that is not used in extending T.

Next we consider the case that $y \not\sim T$, say $y \sim z, a \notin T$. If either of z or a is a goober, then adding the 6 edges that contain either v or y gives $\Delta \geq \Delta(5,2,0) = 0$. Therefore we may assume z and a are ordinary. By Reduction (1), it must be that $z \not\sim a$. If each of z and a fails to expand, i.e., if each is adjacent either to T or to x, then adding the same 6 edges containing v or y creates at least three dead leaves. In this case, $\Delta \geq \Delta(6,2,2) = 0$. So we may next assume that at least one of a and z splits, say $z \sim b, c \notin T$. Then expand T as shown in Fig.9. If b or c is a goober, we have $\Delta \geq \Delta(7,3,0) = .5$, so assume both are ordinary. Consider the three points a, b, and c. If any one of them is adjacent to T or x, the extra dead leaf created gives $\Delta \geq \Delta(8,3,1) = 0$. Similarly, if any one of a, b, c is adjacent to the other two, this gives a dead leaf. There remains the case that there is at most one edge between vertices a, b, c, and the other edges for a, b, c go to new vertices. In particular, one of a, b, c splits into two new vertices, and the addition of the corresponding two edges gives altogether $\Delta \geq \Delta(10,4,0) = 0$. This completes the construction for v adjacent just once to T.

4.4. v is Twice Adjacent to T But Not Adjacent to a Triangle. From the subsections above, it remains to consider the case that every vertex outside of and adjacent to T is an ordinary vertex and is adjacent to T precisely twice. Unless T spans all of H, which would complete the construction, there exists such a vertex $v \sim T$, say $v \sim t, s \in T$. There exists $w \sim v$, $w \notin T$. We shall add tv, vw to T.

Assume for now that w is an ordinary point. First suppose $w \sim T$. Then w must be adjacent to T twice, say $w \sim r, q$. Then we have gained dead leaves at w, s, r, q and $\Delta = \Delta(2,0,4) = 0$. Hence, we may assume instead $w \nsim T$, so that w splits and $w \sim x, y \notin T$. We add wx, wy to T as well. If either of x or y is adjacent to T, we again get 4 dead leaves, so that $\Delta \geq \Delta(4,1,4) = .5$. Alternately, if either x or y is a goober, we have $\Delta \geq \Delta(3,1,1) = 0$. We can therefore assume x and y are ordinary points not adjacent to T. If $x \sim y$, then wxy forms a triangle of ordinary points, and there requires a difficult argument which we postpone to Sec.4.5-4.7.

We then may assume that $x \nsim y$. So x splits, and $x \sim z, a \notin T$. We add xz and xa to T as well, as shown in Fig.10. We now argue similarly to the last part of Sec.4.3, Fig.9, where now the three points are y, z, a. This takes care of the case that w is ordinary and not part of a triangle outside T.

It remains to consider the case that w is a goober. If w has degree one in H, then adding tv, vw to T creates dead leaves at s, w, so that $\Delta = \Delta(1,0,2) = 0$. Suppose instead that w is a goober of degree two in H. Then there exists $x \notin T, x \sim w$. By Reduction (2), x must be ordinary. By Reduction (3), x cannot belong to a triangle of ordinary points. Now add wx to T as well. In passing through the goober w, no contribution is made to Δ at all. Indeed, the entire argument above, where w was ordinary but not part of a triangle of ordinary points, carries over here except the role of w above is played by x here. So the tree T can always be extended in this case.

4.5. v is Adjacent to a Triangle. The proof has been reduced to the case that, along with the assumptions at the start of Sec.4.4, wxy is a triangle of ordinary points. No point is adjacent just once to T, and H contains no $K_4 - e$, so there exist $z, a \notin T$ with $z \sim x, a \sim y$. By Reduction (3), both z and a are ordinary. If both z and a are adjacent to T, then extending T by tv, vw, wx, xz, wy, ya gives $\Delta = \Delta(6,1,7) = 0$. So assume for the remainder that not both $z, a \sim T$, say $z \nsim T$. Since $z \nsim T$, z splits so that $z \sim b, c \notin T$, as shown in Fig.11. If b or c is ordinary and adjacent to T, say $b \sim T$, then the expansion shown in Fig.11 has $\Delta \geq \Delta(7,2,4) = 0$. So we may assume that that $b, c \nsim T$. Next consider that b or c, say b, is a goober. Then $b \nsim y$ due to Reduction (3). If b has degree 1 in H, or if $b \sim c$, then b is a dead leaf in Fig.11, giving $\Delta \geq \Delta(6,2,2) = 0$. If c is also a goober, then $\Delta \geq \Delta(5,2,1) = .5$. Consider instead what happens when c is ordinary and $c \nsim b, T$. If $c \sim y$, then vertices v, w, x, y, c create the configuration of Reduction (4), which is impossible since H is reducible. Therefore it must be that c splits, say $c \sim d, e$. Adding the edges cd and ce to Fig.11, we would gain a new leaf, and then $\Delta \geq \Delta(8,3,1) = 0$. Therefore, we can handle the case that b or c is a goober.

We may assume then for the rest of the proof that b and c are ordinary points not adjacent to T. By Reduction (1) $b \nsim c$ while by Reduction (4), $b, c \nsim y$. So each of b and c splits, but not necessarily disjointly. Let d, e be the other neighbors of b, and add bd, be to Fig.11. If either d or e is a goober, this gives $\Delta \geq \Delta(8,3,1) = 0$, so it remains to suppose that d and e are ordinary. If d or e is adjacent to T, then $\Delta \geq \Delta(9,3,4) = .5$, so it remains to suppose that $d, e \nsim T$.

At this time it is useful to reconsider the vertex $a \sim y$. Recall that a is ordinary. Nothing prevents a from being d or e at this stage. But suppose instead for now that $a \sim T$. Then $a \neq d, e$, so the addition of ya to T gives us $S = (10, 3, 4)$, with $\Delta(S) = -.5$, not quite enough. Refer to Fig.12. In this case, we operate on the three points c, d, e as before in Sec.4.3, that is, either one of the three is adjacent to the other two, so is a dead leaf, or else one of the three splits and creates a new leaf. Either way, we gain the necessary .5 to achieve $\Delta \geq 0$.

It remains to consider the case that $a \nsim T$, which we assume hereafter. But then all of the arguments we have been using in this section to extend out from z can be applied similarly to extend out from a. This observation will prove useful to us later on in the proof.

We now return to the expansion out from vertex z. We saw that it could be assumed that both b and c split. Arguing as with b, it may be assumed that c also splits into two ordinary vertices that are not adjacent to T. We divide the remaining possibilities into three cases. The first is that c also splits into d and e, i.e., $b, c \sim d, e$. This is shown in Fig.13(a). The second case is that c is adjacent to one of d, e, say $c \sim e$ and $c \sim f$, as shown in Fig.13(b). The third case is that $c \nsim d, e$, so that $c \sim f, g$ as shown in Fig.13(c). Of course, any of d, e, f,or g could be adjacent to y, i.e., coincide with vertex a defined earlier.

We can immediately dispense with the first of these cases, Fig.13(a). Suppose d and e have no third neighbor in common besides b and c. Then we could apply Reduction (5) to the vertices x, z, b, c, d, e, which is not possible since H is irreducible. Therefore, there exists $g \sim d, e$ with $g \neq b, c$. By Reduction (6) on z, b, c, d, e, g, the new vertex g cannot be ordinary, so it must be a goober. Then extend T by the solid lines in Fig.13(a) together with the edge dg. One then computes that $\Delta = \Delta(9, 3, 4) = .5$.

The two remaining cases are rather more involved. Sec.4.6 treats Fig.13(b) while Sec.4.7 is devoted to Fig.13(c). We shall see that these two cases are not independent, but instead 4.6 is required for 4.7.

4.6. The Case in Fig.13(b). It would appear in Fig.13(b) that we are almost through with it, since $S = (11, 4, 1)$ and $\Delta(S) = -.5$, just a hair away from working. This is deceptive. Indeed it is so tricky to fully resolve this case that it is surprising it can even be done.

Many possibilities are easy since they increase Δ by at least .5, which is all that is required. We already assume that d, e, f, g are ordinary points not adjacent to T. We are finished if any of them is adjacent to y (giving a dead leaf at y), if any one of them splits to two new vertices (giving a new leaf), or if any one of them is adjacent to some two of the other three (giving a new dead leaf). It remains to consider the situation that each of d, e, f, g is adjacent to one of the others. By relabelling, it can be assumed that the matching on d, e, f, g, is one of the two shown in Fig.14. We treat these two major cases separately, beginning with the configuration in Fig.14(a).

4.6.1. The Case in Fig.14(a). By arguments presented in Sec.4.5, it follows that a is adjacent to two ordinary points outside T besides y. Suppose a is adjacent to two points among d, e, f, g. Because $K_4 - e$ is not allowed in H, we cannot have either $a \sim d, e$ (both or $a \sim f, g$ (both). So we may assume, say, $a \sim e$ and $a \sim g$. Then extending T in the obvious way with leaves at d, e, f, g, a gives $\Delta = \Delta(12,4,4) = 0$.

Next consider the case that a is adjacent to just one of d, e, f, g, say $a \sim g$, so that we also have $a \sim h$, where h is a new ordinary vertex. Suppose that h is adjacent to d, e, or f. If $h \sim f$, then z, c, f, g, a, h could have been reduced by Reduction (4), so this is not possible. Since $K_4 - e$ is forbidden, it cannot be that $h \sim d, e$ (both). Thus, h is adjacent to exactly one of d, e, say $h \sim e$, and also $h \sim i$, a new vertex which must be ordinary (by what we assume when extending outward from y). In Fig.15 it is shown how to deal with this situation to achieve $\Delta = \Delta(14,5,4) = .5$.

Still assuming that $a \sim g, a \sim h$, we must next discuss the case that $h \nsim d, e, f, g$, say h splits with $h \sim i, j$, two new vertices not shown in Fig.14 (a). Extending T out from y instead of x we are finished by arguments in the last section unless i and j are ordinary and not adjacent to T, i.e., Fig.13(b) applies working out from y. Then by the discussion at the start of this section, we are finished unless there is a matching on the vertices c, f, i, j. If so, since $c \sim f$, it must be that $i \sim j$ and $i, j \nsim c, f$. This configuration is shown in Fig.16. One can compute that the best we can do now is $S = (15,5,2)$ and $\Delta(S) = -1.5$, even worse than Fig.13(b).

In Fig.16 we have that f is not adjacent to any other vertices besides c and g, so there is a new vertex $k \sim f$. By Reduction (3), k cannot be a goober. Suppose the other two neighbors of k are outside T but are among vertices shown in Fig.16, i.e., among d, e, i, j. Since $K_4 - e$ is forbidden, we may assume $k \sim e, i$. In Fig.17 the current situation is displayed. The graph to the right of w, including w itself, is isomorphic to configuration (7) in Fig.4. So the edge leaving d and the edge leaving j either join or meet at an ordinary vertex since otherwise we could have used Reduction (7). Suppose first these edges join, i.e., $d \sim j$. An extension is shown in Fig.18(a) for this case which has $\Delta = \Delta(16,5,7) = 0$. On the other hand, if there exists an ordinary vertex $l \sim d, j$, let m be the third neighbor of l. An extension of T all the way out to l is shown in Fig.18(b) in which $\Delta \geq \Delta(18,6,7) = .5$.

Next we may consider the case that vertex k is adjacent to just one of d, e, i, j. By relabelling we may assume $k \sim i$ and that there exists a new vertex $l \sim k$. In this case an appropriate extension of T exists, shown in Fig.19. It has $\Delta \geq \Delta(17,6,4) = 0$.

Next suppose that vertex k is not adjacent to any of d, e, i, j. It could be that $k \sim T$. In this case, there seems to be no appropriate extension of T that uses all labelled vertices, but there is one that uses fewer vertices. It is shown in Fig.20, and it has $\Delta = \Delta(12,4,4) = 0$.

The remaining possibility for k is that it be ordinary and not adjacent to T or to any labelled vertex besides f. Then it splits, so there exist new vertices $l, m \sim k$. The obvious tree for Fig.16 extended by the edges involving k has $S = (18, 6, 2)$, which is not very good yet. If l and m are each adjacent twice to d, e, i, j, we pick up 6 more dead leaves, so $\Delta = \Delta(18, 6, 8) = 1$, which is suitable. If this is not the case, then at least one of l and m fails to be adjacent to a vertex in at least one of the pairs d, e and i, j. We may assume $m \not\sim d, e$. Consider the tree shown in Fig.21, in which several labelled vertices have been omitted. If either l or m is a goober, the tree has $\Delta \geq \Delta(13, 5, 1) = 0$, while if either l or $m \sim T$, it gives $\Delta \geq \Delta(14, 5, 4) = .5$. We already know that $m \not\sim y, g, d, e$. Further, $m \not\sim l$ by Reduction (1). So it remains here to consider the case that m splits in Fig.21 into two vertices, call them n and o. If we add mn and mo to the tree shown, it gives $\Delta \geq \Delta(16, 6, 1) = -.5$. Then we may argue as usual on the three leaves l, n, o to produce an extension of T with $\Delta \geq 0$.

Finally consider the case that $a \not\sim d, e, f, g$, so that a splits into new ordinary vertices h and i. If h and i are each adjacent to d, e, f, or g, then the tree in Fig.14(a) together with edges ah and ai gives $\Delta \geq \Delta(14, 5, 3) = 0$. If h or i is a goober, we have $\Delta \geq \Delta(13, 5, 1) = 0$. On the other hand, suppose h and i are ordinary and at least one of them, say h, is not adjacent to any of d, e, f, g. Then h splits into new vertices j and k. Adding ah, ai, hj, hk to the tree in Fig.14(a) gives $\Delta \geq \Delta(16, 6, 1) = -.5$. Then we can argue on the three leaves i, j, k as usual to extend T with $\Delta \geq 0$.

4.6.2. The Case in Fig.14(b). As with the beginning of Sec.4.6.1, we may assume in Fig.14(b) that a is adjacent to two ordinary points outside T. If a is adjacent to any two of d, e, f, g, then the obvious extension of T for Fig.14(b) with leaves at d, e, f, g, a has $\Delta = \Delta(12, 4, 4) = 0$.

We next consider the case that a is adjacent to just one of d, e, f, g. By symmetry, we may assume $a \sim g$. Then there exists a new vertex adjacent to a, call it h, which be ordinary. We treat several cases depending on how h relates to d, e, f. First suppose $h \sim e, f$. Extending outward from y rather than x produces Fig. 22. The tree in Fig.22 has $\Delta = \Delta(11, 4, 2) = 0$. Suppose next that we have $h \sim d, f$. Then Fig.23 shows a suitable extension with $\Delta = \Delta(13, 4, 6) = 0$. The case that $h \sim d, e$ is similar to the last one. So now suppose h is adjacent to just one of d, e, f, so that also $h \sim i$, which is an ordinary point by the argument in Sec.4.5 concerning points near a. Treating each case separately, first suppose $h \sim d$. Fig.24(a) presents an extension with $\Delta \geq \Delta(14, 5, 4) = .5$. The case $h \sim e$ is treated by Fig.24(b), in which $\Delta \geq \Delta(14, 5, 5) = 1$. The case $h \sim f$ is similar to $h \sim e$. Finally, if h is adjacent to none of d, e, f, then it splits, say $h \sim i, j$, and there is an extension with $\Delta \geq \Delta(13, 5, 2) = .5$ shown in Fig.24(c). This completes the cases in which a is adjacent to any of d, e, f, g.

Next suppose $a \not\sim d, e, f, g$, so that a splits into new ordinary vertices h and i in Fig.13(b). If any two of d, e, f, g are adjacent either to h or i, they become dead leaves in the obvious tree extension, which yields $\Delta \geq \Delta(14, 5, 3) = 0$. Otherwise, at least one of

h, i, say h, is not adjacent to any of d, e, f, g, which means it splits into vertices j and k. If any of i, j, k is a goober, or adjacent to T, or adjacent to any of d, e, f, g, then we have $\Delta \geq 0$. Otherwise, can pick up at least .5 by treating the triple i, j, k as in Sec.4.3, Fig.9. This completes the treatment of Fig.13(b).

4.7. The Case in Fig.13(c). Referring to Fig.13(c), Sec.4.5, we claim that $e \not\sim d, f$. For suppose, say, that $e \sim d$. There is a third neighbor of d, call it g, which may or may not be new. It cannot be that g is goober, due to Reduction (3). But g cannot be ordinary either, or else Reduction (4) could be applied to g, d, b, e, z, c. So our claim holds.

If $e \sim y$, then we can extend T as shown in Fig.25 to achieve $\Delta = \Delta(9,3,3) = 0$. It should be pointed out that vertex f is not involved. We may assume for the remainder that $e \not\sim y$.

If $f \sim y$, it is also possible to extend T. First, consider the case that we also have $f \sim d$. Then d splits to a third vertex, call it g, which cannot be any of the others in Fig.13(b). So T can be carefully extended as shown in Fig.26 to attain $\Delta \geq \Delta(11,4,4) = 1$. Second, consider the case that $f \sim y$ and $f \not\sim d$. Then f must split to a third vertex, again call it g, which does not appear in Fig.13(b). Since $f \sim y, f$ is the vertex called a, which we learned about in Sec.4.5. It follows from this information about $f = a$ that g is ordinary and $g \not\sim T$. Further, all neighbors of g are ordinary points. Suppose that $g \not\sim e$. Then g splits into vertices, all them h and i, which do not appear elsewhere in Fig.13(c) except possibly at d, which we disregard for now. So growing outward from y instead of x gives us Fig.27. There may be other edges on these vertices in Fig.27, which has been redrawn so that its isomorphism to Fig.13(b) is evident. Therefore, we can continue outward from y (instead of x in Fig.13(b)) and extend T as described in Sec.4.6. (For this reason, we needed to treat Fig.13(b) before Fig.13(c).)

We are still assuming $f \sim y$ and $f \not\sim d$. Above we dealt with the case that $g \not\sim e$. Now assume instead that $g \sim e$. If it also happens that $g \sim d$, then only d among the the labelled vertices outside T does not have all of its neighbors described. So there exists another vertex $h \sim d$. In Fig.28, we present an extension for this case with $\Delta \geq \Delta(12,4,5) = .5$ We last assumed $g \sim d$. Instead consider the case $g \not\sim d$, so there exists a new vertex $h \not\sim g$. The extension shown in Fig.29 has $\Delta \geq \Delta(12,4,4) = 0$.

We have treated now all cases with $e \sim y$ or $f \sim y$, and the cases $d \sim y$ are the same as $f \sim y$ up to relabelling (refer to Fig.13(c)). Assume for the remainder that $d, e, f, \not\sim y$. Suppose next that $d \sim f$. There must be a third neighbor of d, call it g. Then the extension shown in Fig.30 gives $\Delta \geq \Delta(11,4,2) = 0$.

It remains to proceed from Fig.13(c), where no two of y, d, e, f are adjacent. It must be that d splits, say $d \sim g, h$. Ignoring f for the moment, the tree with leaves at y, c, e, g, h has $S = (11,4,1)$ and $\Delta(S) = -.5$, just short of working. Indeed, if either g or h is a goober, is adjacent to T, is adjacent to y or e, or splits, then we pick up the extra .5 we need in $\Delta(S)$. The remaining case is that g and h are ordinary and $g \sim h$. Similarly, working out from f and disregarding d, the remaining case has that f splits into two adjacent ordinary points. Since H contains no $K_4 - e$ it cannot be that $f \sim g, h$. Therefore, f splits into new points, say $f \sim i, j$ where $i \sim j$. As with g, h, we may assume that $i, j \not\sim y$. Therefore, a new vertex, call it a, splits out from y, as shown in Fig.31.

Unless H expands from a in a way that is isomorphic to the way it expands from z, we are finished by one of the earlier cases in this proof. Therefore, we may suppose the expansions from z and a are isomorphic, although not necessarily disjoint. In particular, the two neighbors of a besides y belong to a 4–cycle along with a.

We claim that $a \not\sim g, h, i, j$. For if this were not so, we could assume by symmetry that $a \sim j$. A 4–cycle through a, j could only be a, j, f, i (which creates $K_4 - e$ in H) or a, j, i, k for some new point $k \sim a$ (which creates Reduction (4) on c, f, i, j, k, a). Either way, it is not allowed.

We can also exclude $a \sim e$ because no 4–cycle could pass through a, e. Therefore, a splits into new vertices k, l. Then a new vertex m must be adjacent to k, l to create the 4–cycle through a, so that $m \sim k, l$. Next recall that for k, l (working out from a) to be like b, c (working out from z), it must be that k and l are each adjacent to some triangle. Consider the triangle next to k. If it contained any points from the z–side, they would have to be among d, g, h or f, i, j. By symmetry we could suppose $k \sim j$. Then consider the amazing tree extension shown in Fig.32 in which $\Delta \geq \Delta(18, 7, 3) = 1$. There remains the case the $k \not\sim g, h, i, j$. Then k splits to a new triangle, say nop where $k \sim n$. Then use Fig.32 except replace triangle fij by nop, using edges kn, no, np in the tree. In Fig.32, the dead leaves at c, f become live leaves at c, o, but we still achieve $\Delta \geq \Delta(18, 7, 1) = 0$, which is just good enough to complete the entire proof of Theorem 3. □

5. Directions for Further Study. It clearly would be most enlightening if one could find a proof of Theorem 2 which is not so lengthy, elaborate, and delicate as the one presented here. We are not particularly optimistic that this is possible. However, there may be a simpler proof, more like the one we provided for Theorem 1, of the weaker result that there exists some constant c such that for all G described by Theorem 2, $L(G) \geq \frac{n}{3} + c$. Perhaps restricting it to 3–connected, cubic G will make it easier.

As we noted earlier, we can show that the cubic graph, Q_3, is the only G in Theorem 2 such that $L(G) = \frac{n}{3} + \frac{4}{3}$. In view of Theorem 3, any graph G in Theorem 2 that has a reduction using any of the reductions (1)–(6) in Fig.4 satisfies $L(G) \geq \frac{n}{3} + 2$. It remains to consider G that are irreducible or can be fully reduced using Reduction (7). One can check the initial cases as in Sec.4.1, only being more careful, and show that unless $G = Q_3$, then $\Delta \geq 4.5$ at the start, which forces in the end $L(G) \geq \frac{n}{3} + \frac{5}{3}$. No graph can be reduced using Reduction (7) to Q_3, as it contains no triangles, so that only Q_3 has $L(G) = \frac{n}{3} + \frac{4}{3}$.

The next question then is what other graphs besides Q_3 have $L(G) < \frac{n}{3} + 2$, i.e., $L(G) = \frac{n}{3} + \frac{5}{3}$? It suffices to consider irreducible graphs or graphs that are irreducible after using only Reduction (7). Previously in Fig.3 we saw two examples of such graphs.

The examples of large graphs G with $L(G)$ about $n/3$ that we constructed in Sec.1 contained many triangles. It could be that triangle–free graphs contain significantly more leaves, that is, there may exist $c > \frac{1}{3}$ such that for some constant d, $L(G) \geq cn + d$ for all triangle–free, connected, cubic graphs G. This is a subclass of the graphs in Theorem 2. One could further restrict attention to connected, cubic graphs that contain no triangle nor C_4. In this case we conjecture that $L(G)$ is at least $\frac{2}{5}\, n + d$ for some constant d.

The most interesting and largest open problem is a conjecture of N. Linial [3] that generalizes Theorem 1: Suppose G is a connected graph that is regular of degree $r \geq 2$. Then there exists some constant d, depending only on r, such that

$$L(G) \geq \frac{r-2}{r+1}\, n + d\ .$$

If true, an extremal graph would be the necklace where each bead is $K_{d+1} - e$. This conjecture has been verified for $r \leq 4$ by Kleitman and West[2], who prove the bound holds for the larger class of all connected graphs of minimum degree r.

For graphs that are not regular, Linial suspects an even stronger bound holds. If a graph G has degree sequence $(d_1 \geq d_2 \geq \cdots \geq d_n \geq 2)$ and is connected, then Linial conjectures that

$$L(G) \geq \sum_i \frac{d_i - 2}{d_i + 1}\ .$$

Such a lower bound on $L(G)$ in terms of the degree sequence would be analogous to a known bound on the independence number of G due to Wei [5,cf.1].

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Fig. 1. A Cubic Graph G with $L(G)=(n/2)+1$

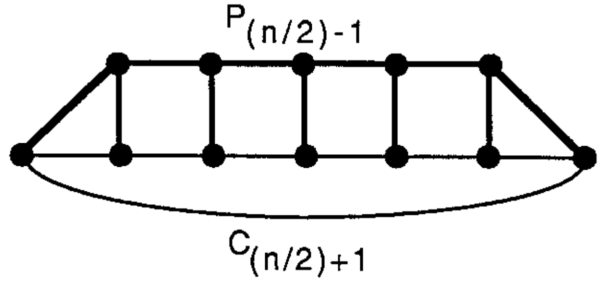


Fig. 2. Extremal Graph for Theorem 1

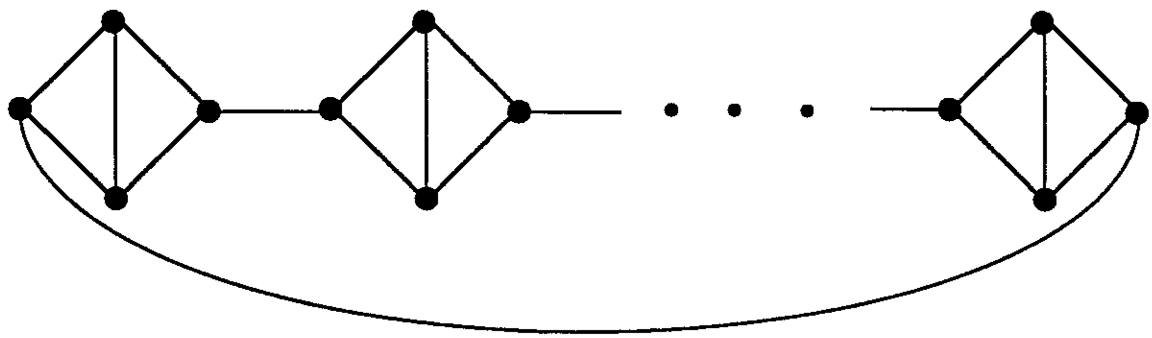


Fig.3. Graphs with n=10 and L=5

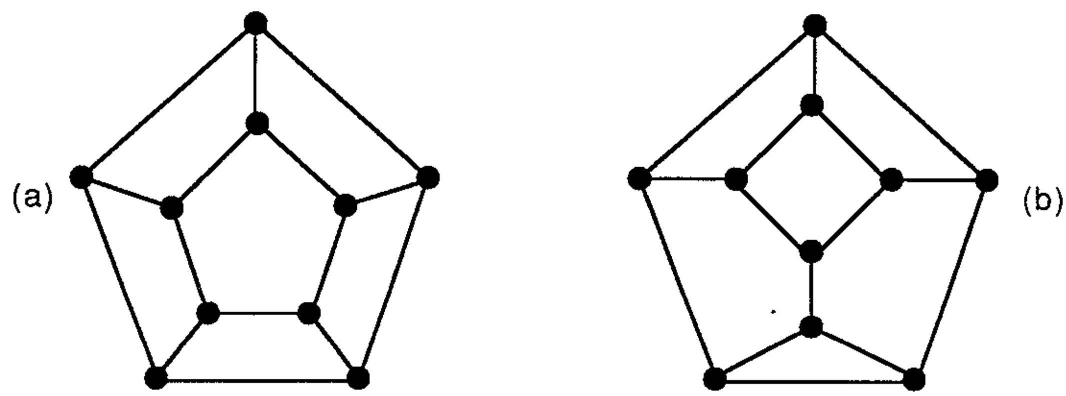


Fig. 4. List of Reductions

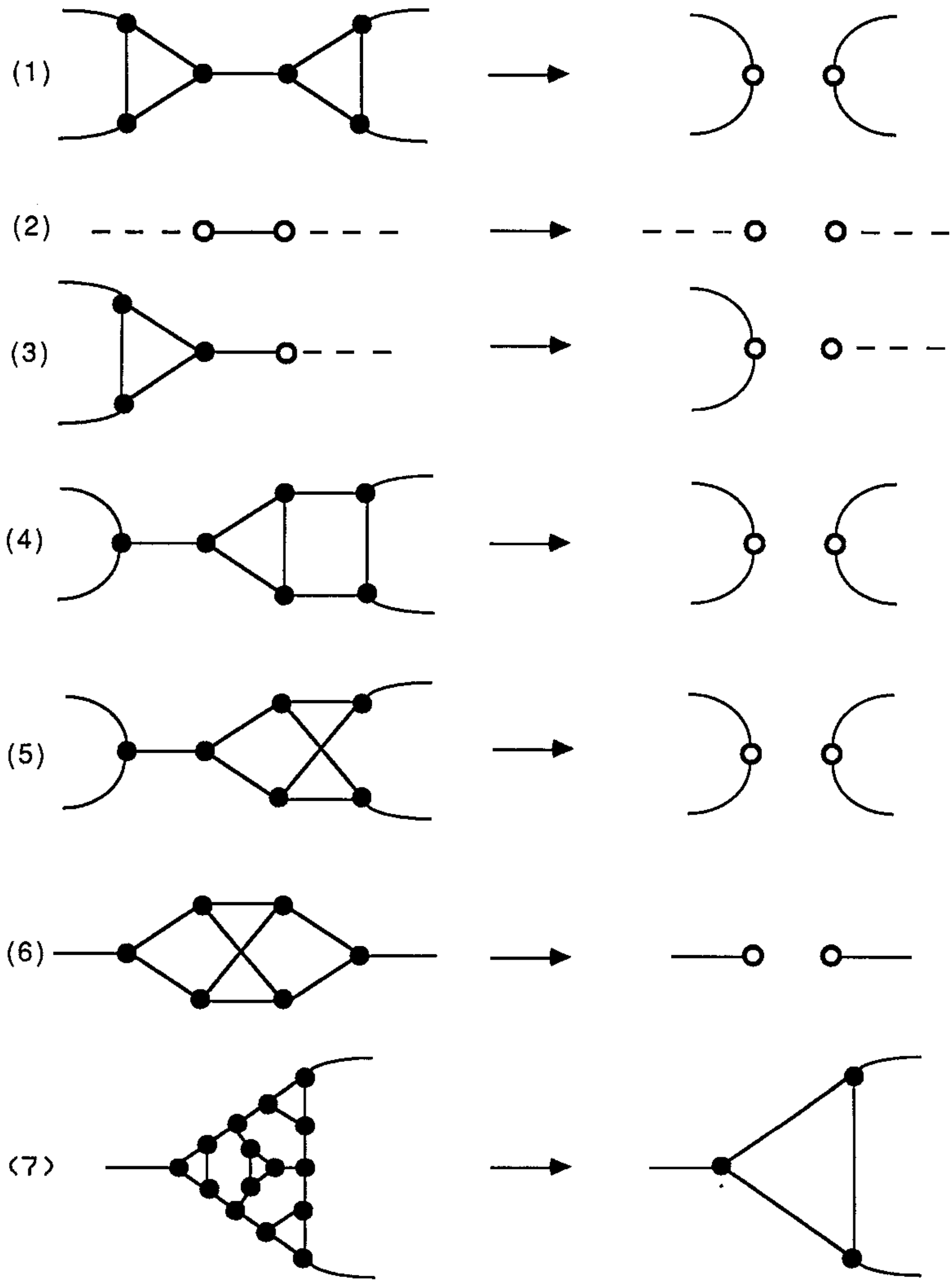


Fig. 5. Merging Trivial Components

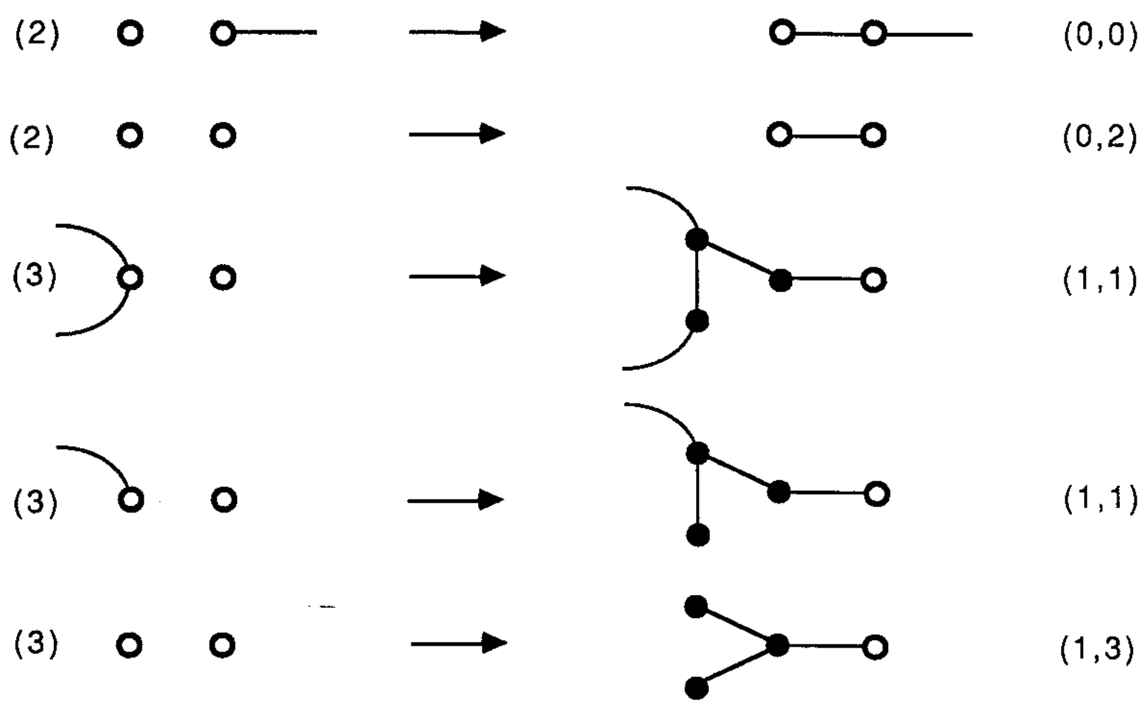


Fig. 6. Restoring Vertices from Goobers

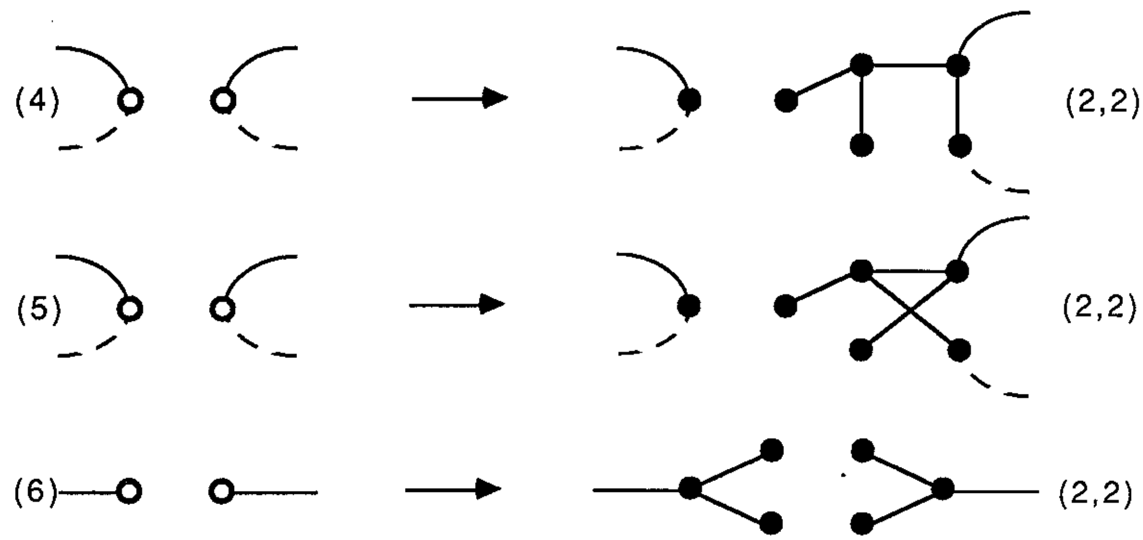


Fig. 7. Restoration of Reduction (7)

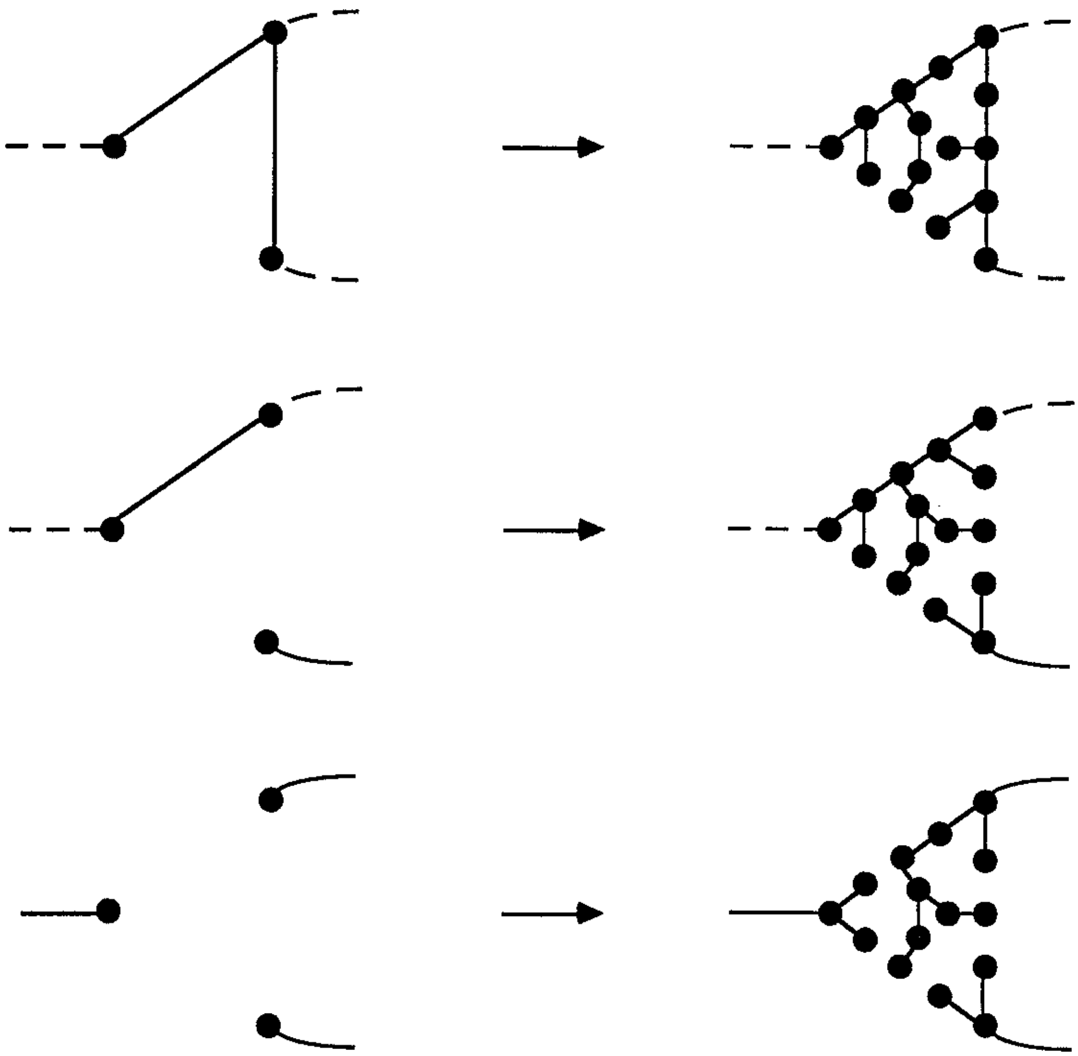


Fig. 8. The Case y~s∈T, y~z∉T

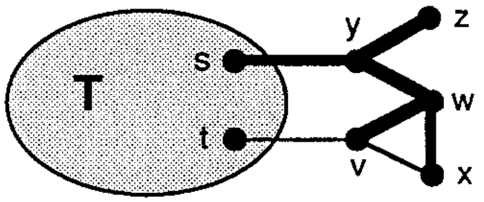


Fig. 9. z~b,c∉T

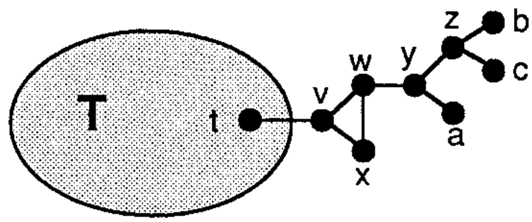


Fig. 10. x~z,a

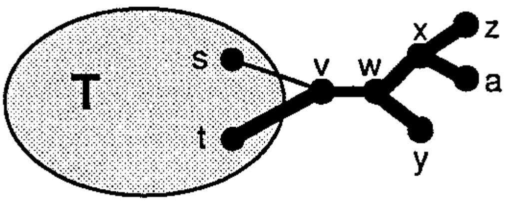


Fig. 11. z Splits

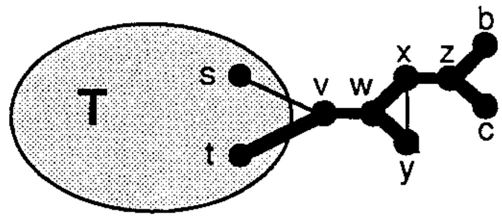


Fig. 12. a ~ T

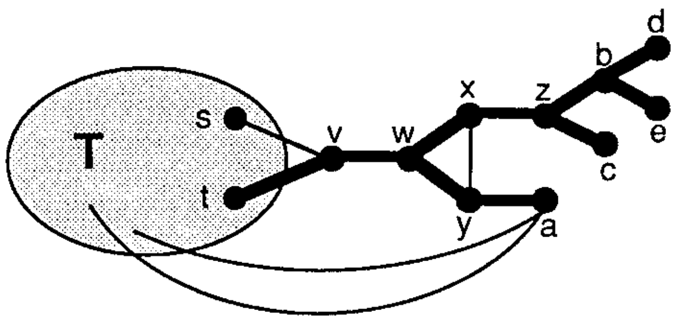


Fig. 13. The Remaining Cases

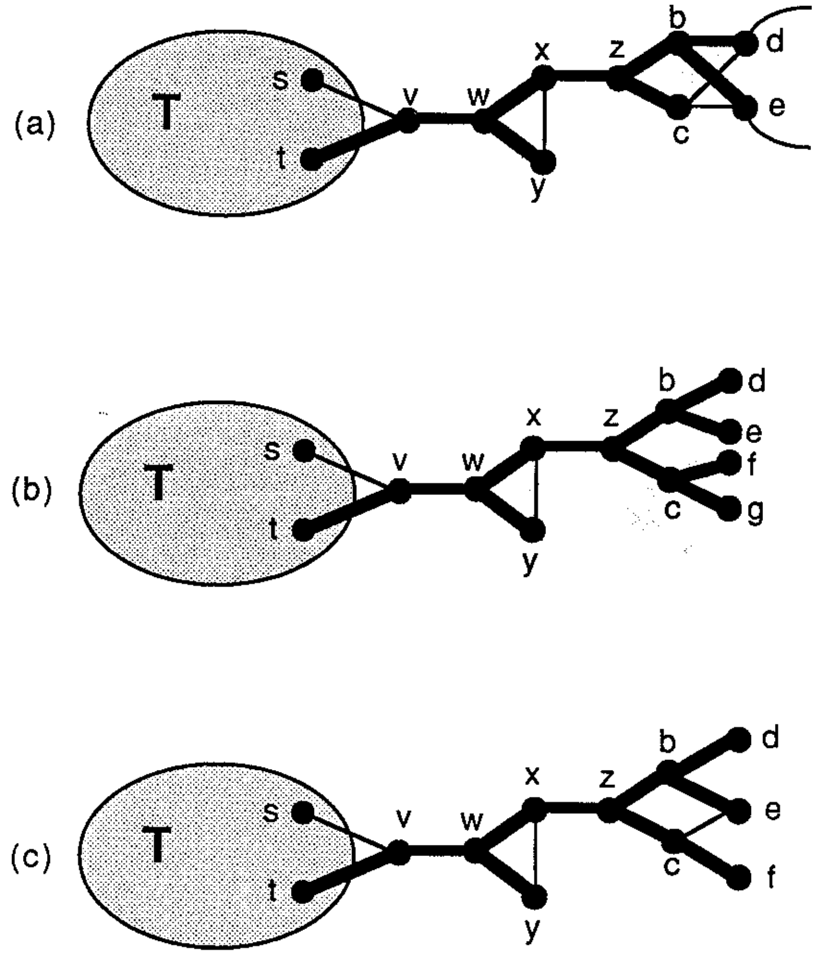


Fig. 14. There is a Matching on d,e,f,g

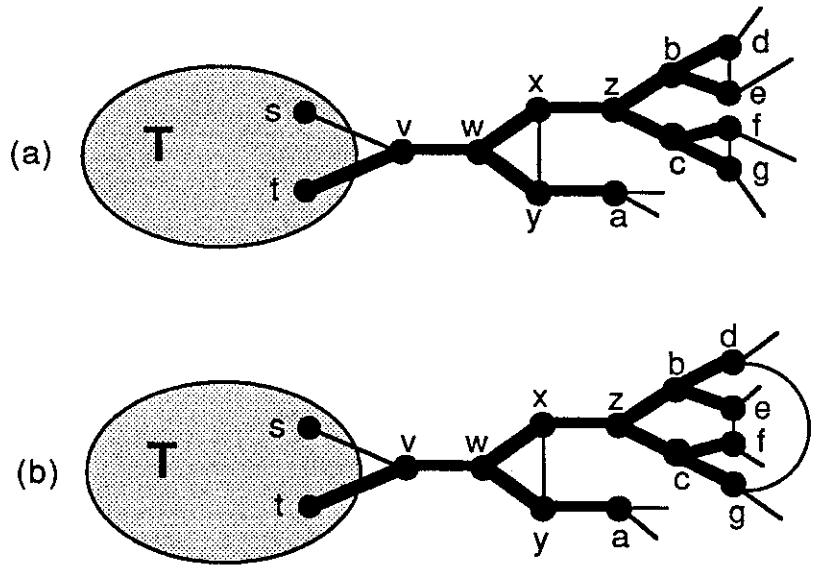


Fig. 15. a ~ g, a ~ h, h ~ e

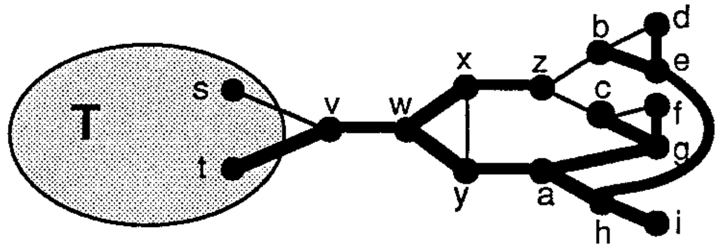


Fig. 16. h Splits

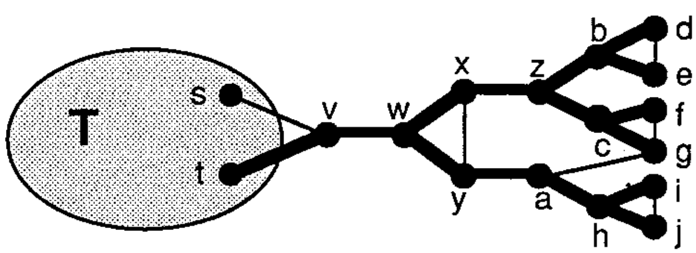


Fig. 17. f ~ k, k ~ e,i

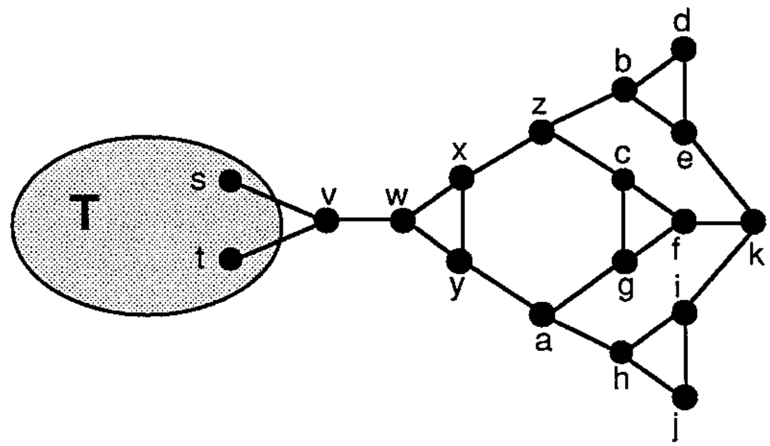


Fig. 18. Extensions for Fig. 17

(a)

(b)

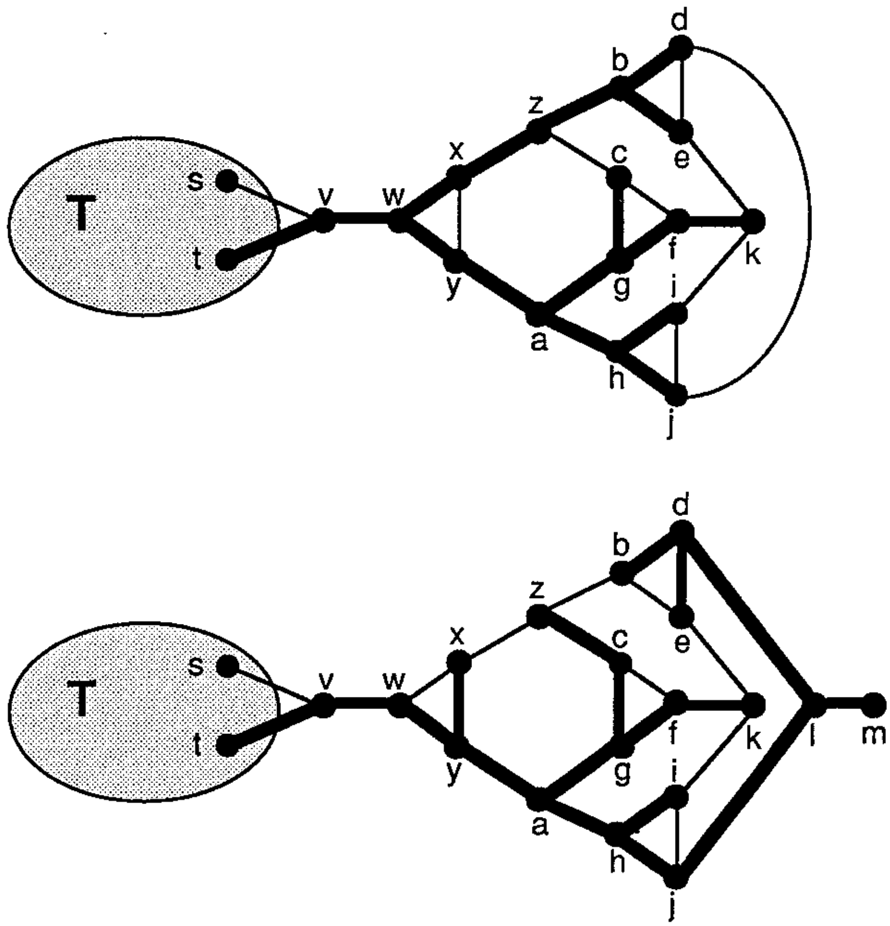


Fig. 19. k is Once Adjacent to d,e,i,j

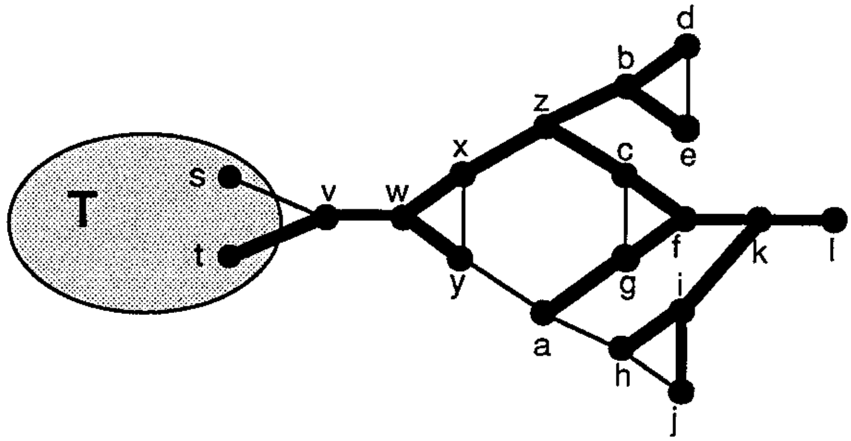


Fig. 20. k ~ T

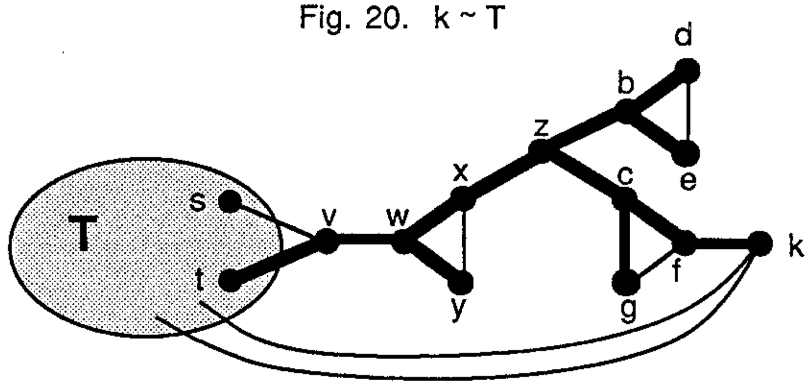


Fig. 21. m ≁ d,e

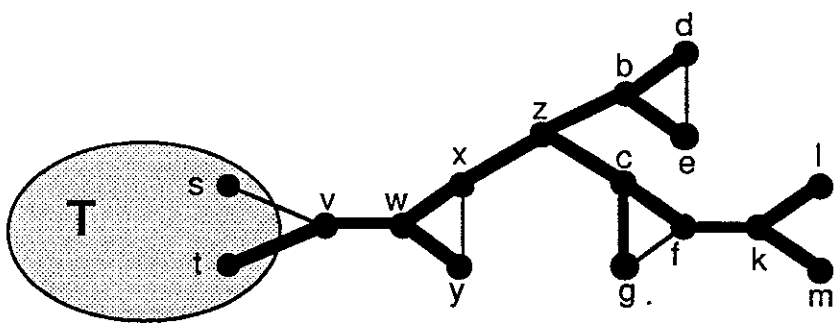


Fig. 22. a ~ g, h ~ e,f

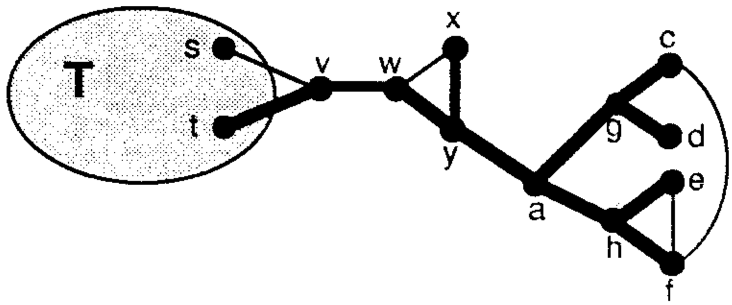


Fig. 23. a ~ g, h ~ d,f

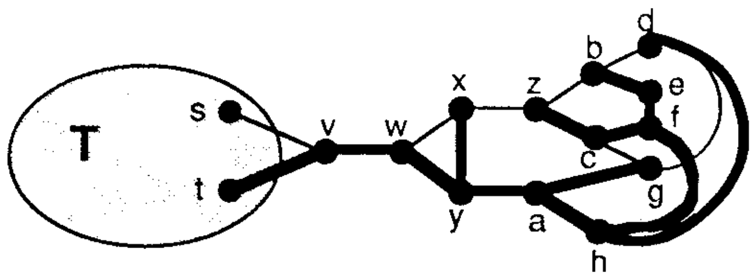
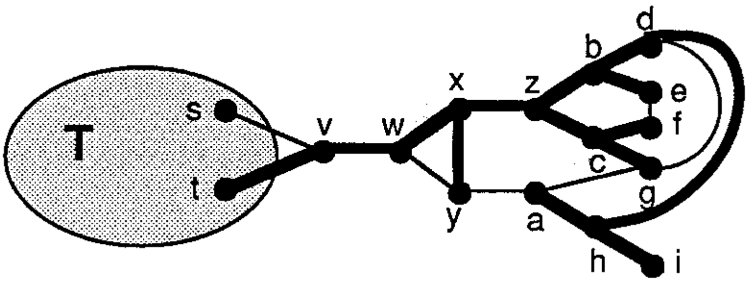
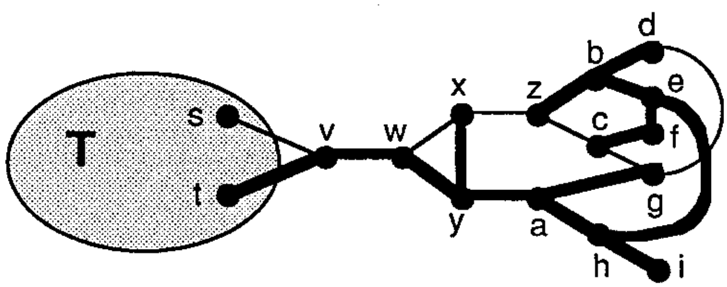


Fig. 24. a ~ g, h ~ at most one of d,e,f

(a) h ~ d



(b) h ~ e



(c) $h \not\sim d,e,f$

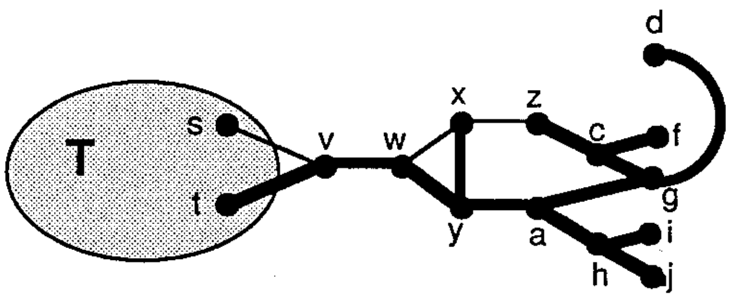


Fig. 25. e ~ y

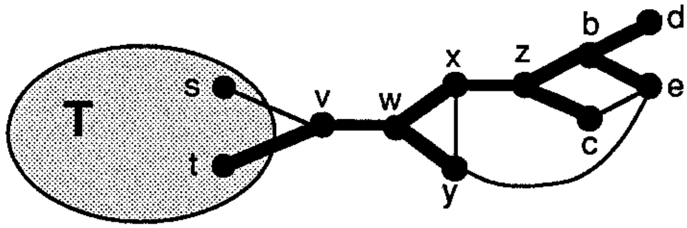


Fig. 26. f ~ y,d

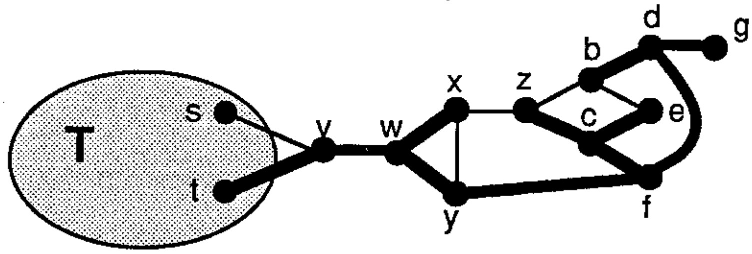


Fig. 27. f ~ y, f ≁ d, g ≁ e

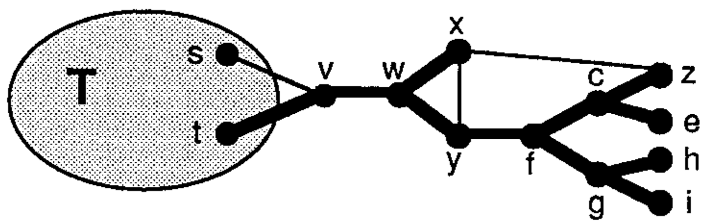


Fig. 28. f ~y, f ≁d, g ~ e,d

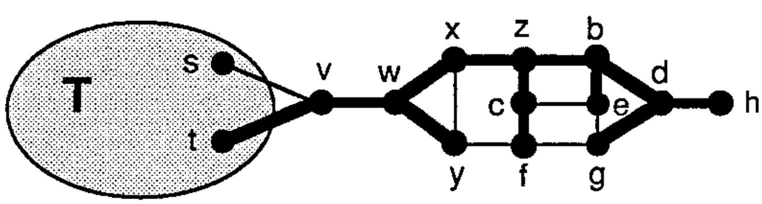


Fig. 29. f ~ y, f ≁ d, g ~ e, g ≁ d

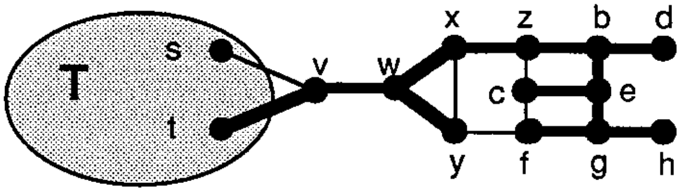


Fig. 30. d ~ f

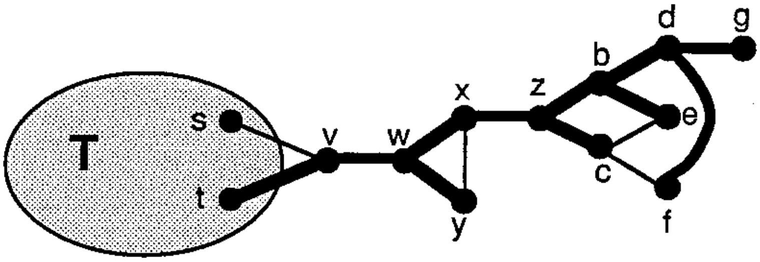


Fig. 31. No Two of y,d,e,f are Adjacent

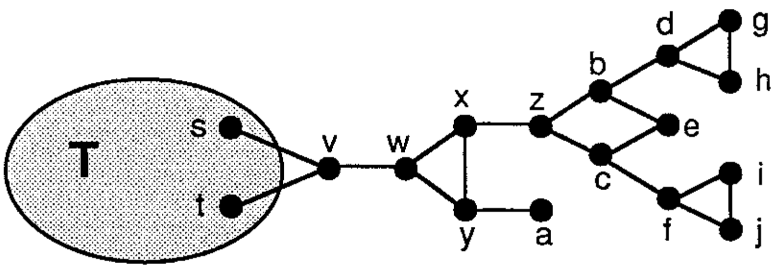


Fig. 32. k ~ j