FULL LENGTH PAPER

Sparse noncommutative polynomial optimization

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Abstract

This article focuses on optimization of polynomials in noncommuting variables, while taking into account sparsity in the input data. A converging hierarchy of semidefinite relaxations for eigenvalue and trace optimization is provided. This hierarchy is a noncommutative analogue of results due to Lasserre (SIAM J Optim 17(3):822–843, 2006) and Waki et al. (SIAM J Optim 17(1):218–242, 2006). The Gelfand–Naimark–Segal construction is applied to extract optimizers if flatness and irreducibility conditions are satisfied. Among the main techniques used are amalgamation results from operator algebra. The theoretical results are utilized to compute lower bounds on minimal eigenvalue of noncommutative polynomials from the literature.

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1 Introduction

The goal of this article is to handle a specific class of sparse polynomial optimization problems with noncommuting variables (e.g., polynomials in matrices). Applications of interest include control theory and linear systems in engineering [54], quantum theory and quantum information science [47]. For example, in the latter context, noncommutative polynomial optimization provides upper bounds on the maximum violation level of Bell inequalities [51]. Further motivations relate to the generalized Lax conjecture [33], with computational proof attempts relying on noncommutative sums of squares (in Clifford algebras) [48]. The problem of verifying noncommutative polynomial inequalities has also occurred in a conjecture formulated by Bessis, Moussa and Villani (BMV) in 1975 [7], and restated by Lieb and Seiringer [35]. The initial conjecture boils down to verifying that the (univariate) polynomial $t \mapsto \operatorname{tr}(A + tB)^m$ has only nonnegative coefficients, for all positive semidefinite matrices A and B, and all $m \in \mathbb{N}$. Even though the BMV conjecture has been established by Stahl [55] for all m, one can rely on computational proofs for a fixed value of m. Schweighofer and the first author derived a computational proof [27] of the conjecture for $m \leq 13$. Recently, noncommutative polynomial optimization has been used in [16] to study optimization problems related to bipartite quantum correlations, and in [17] to derive hierarchies of lower bounds for several matrix factorization ranks, including nonnegative rank, positive semidefinite rank as well as their symmetric analogues.

In the commutative setting, *polynomial optimization* focuses on minimizing or maximizing a polynomial over a *semialgebraic* set, that is, a set defined by a finite conjunction/disjunction of polynomial inequalities. In general, computing the exact solution of a polynomial optimization problem is an NP-hard problem [32]. In practice, one can at least compute an approximation of the solution by considering a relaxation of the problem. In the seminal 2001 paper [29], Lasserre introduced a nowadays famous hierarchy of relaxations, called the *moment-sums of squares hierarchy* allowing us to obtain a converging sequence of lower bounds for the minimum of a polynomial over a compact semialgebraic set. Each lower bound is computed by solving a semidefinite program (SDP). In SDP, one optimizes a linear function under a linear matrix inequality constraint. SDP itself appears in a wide range of applications (combinatorial optimization [34], control theory [4], matrix completion [31], etc.) and can be solved efficiently (up to a few thousand optimization variables) by freely available software, e.g. SeDuMi [56], SDPT3 [58], SDPA [70] or Mosek [42]. For optimization problems involving n-variate polynomials of degree less than d, the size of the matrices involved at step $k \geq d$ of Lasserre's hierarchy of SDP relaxations is proportional to $\binom{n+k}{n}$. Therefore, the size of the SDP problems arising from the hierarchy grows rapidly.

For unconstrained problems involving a large number of variables n, a remedy consists of reducing the size of the SDP matrices by discarding the monomials which never appear in the support of the SOS decompositions. This technique, based on a result by Reznick [52], consists of computing the *Newton polytope* of the input polynomial (that is, the convex hull of the support of this polynomial) and selecting only monomials with support lying in half of this polytope. For constrained optimization, another workaround is based on exploiting a potential sparsity/symmetry pattern arising in the input polynomials. In [30] (see also [64] and the related `SparsePOP` solver [63]), the

author derives a sparse version of Putinar's representation [50] for polynomials positive on compact semialgebraic sets. See also [18] for a simpler proof. This variant can be used for cases where the objective function can be written as a sum of polynomials, each of them involving a small number of variables. Sparse polynomial optimization techniques enable us to successfully handle various concrete applications. The frameworks [37,39], coming with the Real2Float software library, rely on these techniques to produce a hierarchy of upper bounds converging to the absolute roundoff error of a numerical program involving polynomial operations. In energy networks, it is now possible to compute the solution of large-scale power flow problems with up to thousand variables [25]. In [59], the authors derive the sparse analogue of [22] to obtain a hierarchy of upper bounds for the volume of large-scale semialgebraic sets. Recently, sparse polynomial optimization has been used to bound the Lipschitz constants of ReLU networks [14] and to handle sparse positive definite forms [41]. In the same spirit, the symmetry pattern of the polynomial optimization problem can be exploited [53]. More recent progress focused on the use of alternative hierarchies, including the so-called *bounded degree SOS hierarchy* (BSOS) [36]. Here, one represents a positive polynomial as the sum of two terms: an SOS polynomial of a priori fixed degree, and a term lying in the set of *Krivine-Stengle* representations [26], that is, a combination of positive linear cross-products of the polynomials involved in the set of constraints. The BSOS hierarchy can handle bigger instances than the standard moment-SOS hierarchy. In addition, sparsity can be exploited in the same way as for the sparse SOS hierarchy, which allows us to tackle even larger problems [65].

In the noncommutative context, a given noncommutative polynomial in n variables and degree d is positive semidefinite if and only if it decomposes as a *sum of hermitian squares* (SOHS) [19,38]. In practice, an SOHS decomposition can be computed by solving an SDP of size $O(n^d)$, which is even larger than the size of the matrices involved in the commutative case. SOHS decompositions are also used for constrained optimization, either to minimize eigenvalues or traces of noncommutative polynomial objective functions, under noncommutative polynomial (in)equality constraints. The optimal value of such constrained problems can be approximated, as closely as desired, while relying on the noncommutative analogue of Lasserre's hierarchy [3,13,49]. The NCSOStools [5,12] library can compute such approximations for optimization problems involving polynomials in noncommuting variables. By comparison with the commutative case, the size $O(n^k)$ of the SDP matrices at a given step k of the noncommutative hierarchy becomes intractable even faster, typically for $k, n \simeq 6$ on a standard laptop.

A remedy for unconstrained problems is to rely on the adequate noncommutative analogue of the standard Newton polytope method, which is called the *Newton chip method* (see, e.g. [5, §2.3]) and can be further improved with the *augmented Newton chip method* (see, e.g., [5, §2.4]), by removing certain terms which can never appear in an SOHS decomposition of a given input. As in the commutative case, the Newton polytope method cannot be applied for constrained problems. When one cannot go from step k to step $k + 1$ in the hierarchy because of the computational burden, one can always consider matrices indexed by all terms of degree k plus a fixed percentage of terms of degree $k + 1$. This is used for instance to compute tighter upper bounds for maximum violation levels of Bell inequalities [51]. Another trick, implemented in the

`Ncpol2sdpa` library [62], consists of exploiting simple equality constraints, such as "$X^2 = Y$", to derive substitution rules for variables involved in the SDP relaxations. Similar substitutions are performed in the commutative case by `Gloptipoly` 3 [21].

Apart from such heuristic procedures, there is, to the best of our knowledge, no general method to exploit additional structure, such as sparsity, of (un)constrained noncommutative polynomial optimization problems.

Contributions We state and prove in Sect. 3 a sparse variant of the noncommutative version of Putinar's Positivstellensatz, under the same sparsity pattern assumptions as the ones used in the commutative case [30,64]; these conditions are known as the *running intersection property* (RIP) in graph theory [15,45]. Our proof relies on amalgamation results for operator algebras. Then, we present in Sect. 4 the sparse Gelfand–Naimark–Segal (GNS) construction yielding representations for linear functionals positive w.r.t. sparsity. This allows us to extract minimizers, providing that flatness and irreducibility conditions hold. We rely on this sparse representation to design algorithms performing eigenvalue optimization (Sect. 5) and trace optimization of noncommutative sparse polynomials (Sect. 6), both in the unconstrained and constrained case. Along the way we exhibit an example showing that the Helton-McCullough [19,38] Sum of Squares theorem (every positive nc polynomial is a sum of hermitian squares) fails in the sparse setting, see Lemma 5.2. Finally, we provide in Sect. 7 experimental comparisons between the bounds given by the dense relaxations and the ones produced by our algorithms, currently implemented in the `NCSOStools` software library.

We also point out to the interested reader that the second author has also recently developed in [66] a noncommutative analog of the procedures exploiting monomial term sparsity [67–69]. In particular, correlative and term sparsity can be combined to address eigenvalue and trace optimization problems with up to thousands of variables.

2 Notation and definitions

This section gives the basic definitions and introduces notation used in the rest of the article.

2.1 Noncommutative polynomials

Let us denote by $\mathbb{M}_n(\mathbb{R})$ (resp. $\mathbb{S}_n$) the space of all real (resp. symmetric) matrices of order n, and by $\mathbb{S}_n^k$ the set of k-tuples $\underline{A} = (A_1, \ldots, A_k)$ of symmetric matrices A_i of order n. The normalized trace of an $n \times n$ matrix $A = (a_{i,j})_{i,j}$ is given by $\operatorname{tr} A = \frac{1}{n} \sum_{i=1}^{n} a_{i,i}$. Given $A \in \mathbb{S}_n$, we write $A \succeq 0$ (resp. $A \succ 0$) when A is positive semidefinite (resp. positive definite), i.e., has only nonnegative (resp. positive) eigenvalues. Let $\mathbf{I}_n$ stands for the identity matrix of order n. For a fixed $n \in \mathbb{N}$, we consider a finite alphabet $X_1, \ldots, X_n$ of symmetric letters and generate all possible words of finite length in these letters. The empty word is denoted by 1. The resulting set of words is $\langle \underline{X} \rangle$, with $\underline{X} = (X_1, \ldots, X_n)$. We denote by $\mathbb{R}\langle \underline{X} \rangle$ the set of real polynomials in noncommutative variables, abbreviated as *nc polynomials*. A *monomial*

is an element of the form $a_w w$, with $a_w \in \mathbb{R}\setminus\{0\}$ and $w \in \langle \underline{X} \rangle$. The *degree* of an nc polynomial $f \in \mathbb{R}\langle \underline{X} \rangle$ is the length of the longest word involved in f. For $d \in \mathbb{N}$, $\langle \underline{X} \rangle_d$ is the set of all words of degree at most d. Let us denote by $\mathbf{W}_d$ the vector of all words of $\langle \underline{X} \rangle_d$ w.r.t. to the lexicographic order. Note that the dimension of $\mathbb{R}\langle \underline{X} \rangle_d$ is the length of $\mathbf{W}_d$, which is $\sigma(n, d) := \sum_{i=0}^{d} n^i = \frac{n^{d+1}-1}{n-1}$. The set $\mathbb{R}\langle \underline{X} \rangle$ is equipped with the involution $\star$ that fixes $\mathbb{R} \cup \{X_1, \ldots, X_n\}$ point-wise and reverses words, so that $\mathbb{R}\langle \underline{X} \rangle$ is the $\star$-algebra freely generated by n symmetric letters $X_1, \ldots, X_n$. The set of all *symmetric elements* is defined as $\operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle := \{f \in \mathbb{R}\langle \underline{X} \rangle : f = f^\star\}$.

Sums of hermitian squares An nc polynomial of the form $g^\star g$ is called a *hermitian square*. A given $f \in \mathbb{R}\langle \underline{X} \rangle$ is a *sum of hermitian squares* (SOHS) if there exist nc polynomials $h_1, \ldots, h_r \in \mathbb{R}\langle \underline{X} \rangle$, with $r \in \mathbb{N}$, such that $f = h_1^\star h_1 + \cdots + h_r^\star h_r$. Let $\Sigma\langle \underline{X} \rangle$ stands for the set of SOHS. We denote by $\Sigma\langle \underline{X} \rangle_d \subseteq \Sigma\langle \underline{X} \rangle$ the set of SOHS polynomials of degree at most $2d$. We now recall how to check whether a given $f \in \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle$ is an SOHS. The existing procedure, known as the *Gram matrix method*, relies on the following proposition (see, e.g., [19, §2.2]):

Proposition 2.1 *Assume that $f \in \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle_{2d}$. Then $f \in \Sigma\langle \underline{X} \rangle$ if and only if there exists $G_f \succeq 0$ satisfying*

$$f = \mathbf{W}_d^\star \, G_f \, \mathbf{W}_d. \tag{2.1}$$

Conversely, given such $G_f \succeq 0$ with rank r, one can construct $g_1, \ldots, g_r \in \mathbb{R}\langle \underline{X} \rangle_d$ such that $f = \sum_{i=1}^{r} g_i^\star g_i$.

Any symmetric matrix G_f (not necessarily positive semidefinite) satisfying (2.1) is called a *Gram matrix* of f.

Semialgebraic sets and quadratic modules Given a positive integer m and $S = \{s_1, \ldots, s_m\} \subseteq \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle$, the *semialgebraic* set D_S associated to S is defined as follows:

$$D_S := \bigcup_{k\in\mathbb{N}} \{\underline{A} = (A_1, \ldots, A_n) \in \mathbb{S}_k^n : s_j(\underline{A}) \succeq 0, \quad j = 1 \ldots, m\}. \tag{2.2}$$

When considering only tuples of $N \times N$ symmetric matrices, we use the notation $D_S^N := D_S \cap \mathbb{S}_N^n$. The *operator semialgebraic set* D_S^∞ is the set of all bounded self-adjoint operators $\underline{A}$ on a Hilbert space $\mathcal{H}$ endowed with a scalar product $\langle \cdot \mid \cdot \rangle$, making $g(\underline{A})$ a positive semidefinite operator for all $g \in S$, i.e., $\langle g(\underline{A})v \mid v \rangle \geq 0$, for all $v \in \mathcal{H}$. We say that a noncommutative polynomial f is positive (denoted by $f \succ 0$) on D_S^∞ if for all $\underline{A} \in D_S^\infty$ the operator $f(\underline{A})$ is positive definite, i.e., $\langle f(\underline{A})v \mid v \rangle > 0$, for all nonzero $v \in \mathcal{H}$. The *quadratic module* $\mathcal{M}(S)$, generated by S, is defined by

$$\mathcal{M}(S) := \left\{ \sum_{i=1}^{K} a_i^\star s_i' a_i : K \in \mathbb{N}, a_i \in \mathbb{R}\langle \underline{X} \rangle, s_i' \in S \cup \{1\} \right\}. \tag{2.3}$$

Given $d \in \mathbb{N}$, the truncated *quadratic module* $\mathcal{M}(S)_d$ of order d, generated by S, is

$$\mathcal{M}(S)_d := \left\{ \sum_{i=1}^{K} a_i^\star s_i' a_i : K \in \mathbb{N}, a_i \in \mathbb{R}\langle \underline{X} \rangle, s_i' \in S \cup \{1\}, \deg(a_i^\star s_i' a_i) \leq 2d \right\}.$$
(2.4)

Let **1** stands for the unit polynomial. A quadratic module $\mathcal{M}$ is called *archimedean* if for each $a \in \mathbb{R}\langle \underline{X} \rangle$, there exists $N \in \mathbb{R}^{\geq 0}$ such that $N \cdot \mathbf{1} - a^\star a \in \mathcal{M}$. One can show that this is equivalent to the existence of an $N \in \mathbb{R}^{\geq 0}$ such that $N \cdot \mathbf{1} - \sum_{i=1}^{n} X_i^2 \in \mathcal{M}$.

The noncommutative analog of Putinar's Positivstellensatz [50] describing non-commutative polynomials positive on D_S^∞ with archimedean $\mathcal{M}(S)$ is due to Helton and McCullough:

Theorem 2.2 ([23, Theorem 1.2]) *Let* $S \cup \{f\} \subseteq \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle$ *and assume that* $\mathcal{M}(S)$ *is archimedean. If* $f(\underline{A}) \succ 0$ *for all* $\underline{A} \in D_S^\infty$, *then* $f \in \mathcal{M}(S)$.

2.2 Sparsity patterns

Let $I_0 := \{1, \ldots, n\}$. For $p \in \mathbb{N}$ consider $I_1, \ldots, I_p \subseteq I_0$ satisfying $\bigcup_{k=1}^{p} I_k = I_0$. Let n_k be the size of I_k, for each $k = 1, \ldots, p$.

We denote by $\langle \underline{X}(I_k) \rangle$ (resp. $\mathbb{R}\langle \underline{X}(I_k) \rangle$) the set of words (resp. nc polynomials) in the n_k variables $\underline{X}(I_k) = \{X_i : i \in I_k\}$. The dimension of $\mathbb{R}\langle \underline{X}(I_k) \rangle_d$ is $\sigma(n_k, d) = \frac{n_k^{d+1} - 1}{n_k - 1}$. Note that $\mathbb{R}\langle \underline{X}(I_0) \rangle = \mathbb{R}\langle \underline{X} \rangle$. We also define $\operatorname{Sym} \mathbb{R}\langle \underline{X}(I_k) \rangle := \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle \cap \mathbb{R}\langle \underline{X}(I_k) \rangle$, let $\Sigma \langle \underline{X}(I_k) \rangle$ stands for the set of SOHS in $\mathbb{R}\langle \underline{X}(I_k) \rangle$ and we denote by $\Sigma \langle \underline{X}(I_k) \rangle_d$ the restriction of $\Sigma \langle \underline{X}(I_k) \rangle$ to nc polynomials of degree at most $2d$. In the sequel, we will rely on two specific assumptions. The first one is as follows:

Assumption 2.3 *(Boundedness)* Let D_S be as in (2.2). There is $N \in \mathbb{R}^{>0}$ such that $\sum_{i=1}^{n} X_i^2 \preceq N \cdot \mathbf{1}$, for all $\underline{X} = (X_1, \ldots, X_n) \in D_S^\infty$.

Then, Assumption 2.3 implies that $\sum_{j \in I_k} X_j^2 \preceq N \cdot \mathbf{1}$, for all $k = 1, \ldots, p$. Thus we define

$$s_{m+k} := N \cdot \mathbf{1} - \sum_{j \in I_k} X_j^2, \quad k = 1, \ldots, p,$$
(2.5)

and set $m' = m + p$ in order to describe the same set D_S again as:

$$D_S := \bigcup_{k \in \mathbb{N}} \{\underline{A} \in \mathbb{S}_k^n : s_j(\underline{A}) \succeq 0, \quad j = 1, \ldots, m'\},$$
(2.6)

as well as the operator semialgebraic set D_S^∞.

The second assumption is as follows:

Assumption 2.4 *(RIP)* Let D_S be as in (2.6) and let $f \in \mathbb{R}\langle \underline{X} \rangle$. The index set $J := \{1, \ldots, m'\}$ is partitioned into p disjoint sets $J_1, \ldots, J_p$ and the two collections $\{I_1, \ldots, I_p\}$ and $\{J_1, \ldots, J_p\}$ satisfy:

(i) For all $j \in J_k$, $g_j \in \operatorname{Sym} \mathbb{R}\langle \underline{X}(I_k) \rangle$.

(ii) The objective function can be decomposed as $f = f_1 + \cdots + f_p$, with $f_k \in \mathbb{R}\langle \underline{X}(I_k) \rangle$, for all $k = 1, \ldots, p$.

(iii) The running intersection property (RIP) holds, i.e., for all $k = 1, \ldots, p - 1$, one has

$$I_{k+1} \cap \bigcup_{j \leq k} I_j \subseteq I_\ell \quad \text{for some } \ell \leq k. \tag{2.7}$$

Even though we assume that $I_1, \ldots, I_p$ are explicitly given, one can compute such subsets using the procedure in [64]. Roughly speaking, this procedure consists of two steps. The first step provides the correlation sparsity pattern (csp) graph of the variables involved in the input polynomial data. The second step computes the maximal cliques of a chordal extension of this csp graph. Even if the computation of all maximal cliques of a graph is an NP hard problem in general, it turns out that this procedure is efficient in practice, due to the properties of chordal graphs (see, e.g., [9] for more details on the properties of chordal graphs).

2.3 Hankel and localizing matrices

To $g \in \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle$ and a linear functional $L : \mathbb{R}\langle \underline{X} \rangle_{2d} \to \mathbb{R}$, one associates the following two matrices:

(1) the *noncommutative Hankel matrix* $\mathbf{M}_d(L)$ is the matrix indexed by words $u, v \in \langle \underline{X} \rangle_d$, with $(\mathbf{M}_d(L))_{u,v} = L(u^\star v)$;

(2) the *localizing matrix* $\mathbf{M}_{d - \lceil \deg g / 2 \rceil}(gL)$ is the matrix indexed by words $u, v \in \langle \underline{X} \rangle_{d - \lceil \deg g / 2 \rceil}$, with $(\mathbf{M}_{d - \lceil \deg g / 2 \rceil}(gL))_{u,v} = L(u^\star g v)$.

The functional L is called *unital* if $L(1) = 1$ and is called *symmetric* if $L(f^\star) = L(f)$, for all f belonging to the domain of L. We also recall the following useful facts together with their proofs for the sake of completeness.

Lemma 2.5 ([5, Lemma 1.44]) *Let $g \in \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle$ and let $L : \mathbb{R}\langle \underline{X} \rangle_{2d} \to \mathbb{R}$ be a symmetric linear functional. Then, one has:*

(1) $L(h^\star h) \geq 0$ for all $h \in \mathbb{R}\langle \underline{X} \rangle_d$, if and only if, $\mathbf{M}_d(L) \succeq 0$;

(2) $L(h^\star g h) \geq 0$ for all $h \in \mathbb{R}\langle \underline{X} \rangle_{d - \lceil \deg g / 2 \rceil}$, if and only if, $\mathbf{M}_{d - \lceil \deg g / 2 \rceil}(gL) \succeq 0$.

Proof For $h = \sum_w h_w w \in \mathbb{R}\langle \underline{X} \rangle_d$, let us denote by $\mathbf{h} \in \mathbb{R}^{\sigma(n,d)}$ the vector consisting of all coefficients h_w of h. The first statement now follows from

$$L(h^\star h) = \sum_{u,v} h_u h_v L(u^\star v) = \sum_{u,v} h_u h_v (\mathbf{M}_d(L))_{u,v} = \mathbf{h}^T \mathbf{M}_d(L) \mathbf{h}.$$

The second statement follows after checking that $L(h^\star g h) = \mathbf{h}^T \mathbf{M}_{d - \lceil \deg g / 2 \rceil}(gL) \mathbf{h}$. $\square$

Definition 2.6 Suppose $L : \mathbb{R}\langle \underline{X} \rangle_{2d+2\delta} \to \mathbb{R}$ is a linear functional with restriction $\tilde{L} : \mathbb{R}\langle \underline{X} \rangle_{2d} \to \mathbb{R}$. We associate to L and $\tilde{L}$ the Hankel matrices $\mathbf{M}_{d+\delta}(L)$ and $\mathbf{M}_d(\tilde{L})$ respectively, and get the block form

$$\mathbf{M}_{d+\delta}(L) = \begin{bmatrix} \mathbf{M}_d(\tilde{L}) & B \\ B^T & C \end{bmatrix}.$$

We say that L is δ-*flat* or that L is a *flat extension* of $\tilde{L}$, if $\mathbf{M}_{d+\delta}(L)$ is flat over $\mathbf{M}_d(\tilde{L})$, i.e., if rank $\mathbf{M}_{d+\delta}(L) = $ rank $\mathbf{M}_d(\tilde{L})$.

For a subset $I \subseteq \{1, \dots, p\}$, let us define $\mathbf{M}_d(L, I)$ to be the Hankel submatrix obtained from $\mathbf{M}_d(L)$ after retaining only those rows and columns indexed by $w \in \langle \underline{X}(I) \rangle_d$. When $I \subseteq I_k$ and $g \in \mathbb{R}\langle \underline{X}(I_k) \rangle$, for $k \in \{1, \dots, p\}$, we define the localizing submatrix $\mathbf{M}_{d-\lceil \deg g/2 \rceil}(gL, I)$ in a similar fashion. In particular, $\mathbf{M}_d(L, I_k)$ and $\mathbf{M}_{d-\lceil \deg g/2 \rceil}(gL, I_k)$ can be seen as Hankel and localizing matrices with rows and columns indexed by a basis of $\mathbb{R}\langle \underline{X}(I_k) \rangle_d$ and $\mathbb{R}\langle \underline{X}(I_k) \rangle_{d-\lceil \deg g/2 \rceil}$, respectively.

3 Sparse representations of noncommutative positive polynomials

In this section, we prove our main theoretical result, which is a sparse version of the Helton-McCullough archimedean Positivstellensatz (Theorem 2.2). For this, we rely on amalgamation theory for $C^\star$-algebras, see e.g. [6,61].

Given a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$. A $C^\star$-algebra is a complex Banach algebra $\mathcal{A}$ (thus also a Banach space), endowed with a norm $\| \cdot \|$, and with an involution $\star$ satisfying $\|xx^\star\| = \|x\|^2$ for all $x \in \mathcal{A}$. Equivalently, it is a norm closed subalgebra with involution of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Given a $C^\star$-algebra $\mathcal{A}$, a *state* φ is defined to be a positive linear functional of unit norm on $\mathcal{A}$, and we write often $(\mathcal{A}, \varphi)$ when $\mathcal{A}$ comes together with the state φ. Given two $C^\star$-algebras $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_2)$, a homomorphism $\iota : \mathcal{A}_1 \to \mathcal{A}_2$ is called *state-preserving* if $\varphi_2 \circ \iota = \varphi_1$. Given a $C^\star$-algebra $\mathcal{A}$, a *unitary representation* of $\mathcal{A}$ in $\mathcal{H}$ is a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ which is *strongly continuous*, i.e., the mapping $\mathcal{A} \to \mathcal{H}, g \mapsto \pi(g)\xi$ is continuous for every $\xi \in \mathcal{H}$.

**Theorem 3.1** ([6] or [61, Section 5]) *Let* $(\mathcal{A}, \varphi_0)$ *and* $\{(\mathcal{B}_k, \varphi_k) : k \in I\}$ *be* $C^\star$-*algebras with states, and let* ι_k *be a state-preserving embedding of* $\mathcal{A}$ *into* $\mathcal{B}_k$, *for each* $k \in I$. *Then there exists a* $C^\star$-*algebra* $\mathcal{D}$ *amalgamating the* $(\mathcal{B}_k, \varphi_k)$ *over* $(\mathcal{A}, \varphi_0)$. *That is, there is a state* φ *on* $\mathcal{D}$, *and state-preserving homomorphisms* $j_k : \mathcal{B}_k \to \mathcal{D}$, *such that* $j_k \circ \iota_k = j_i \circ \iota_i$, *for all* $k, i \in I$, *and such that* $\bigcup_{k \in I} j_k(\mathcal{B}_k)$ *generates* D.

Theorem 3.1 is illustrated in Fig. 1 in the case $I = \{1, 2\}$.

We also recall the construction by Gelfand–Naimark–Segal (GNS) establishing a correspondence between $\star$-representations of a $C^\star$-algebra and positive linear functionals on it. In our context, the next result [5, Theorem 1.27] restricts to linear functionals on $\mathbb{R}\langle \underline{X} \rangle$ which are positive on an archimedean quadratic module.

Theorem 3.2 *Let* $S \subseteq \mathrm{Sym}\, \mathbb{R}\langle \underline{X} \rangle$ *be given such that its quadratic module* $\mathcal{M}(S)$ *is archimedean. Let* $L : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}$ *be a nontrivial linear functional with* $L(\mathcal{M}(S)) \subseteq$

Fig. 1 Illustration of Theorem 3.1 in the case $I = \{1, 2\}$
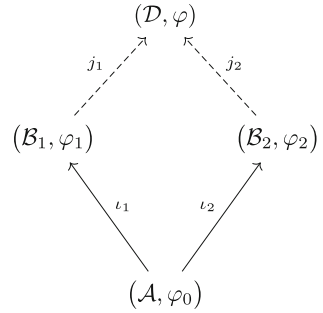
$\mathbb{R}^{\geq 0}$. *Then there exists a tuple* $\underline{A} = (A_1, \ldots, A_n) \in D_S^\infty$ *and a vector* $\mathbf{v}$ *such that* $L(f) = \langle f(\underline{A})\mathbf{v}, \mathbf{v} \rangle$, *for all* $f \in \mathbb{R}\langle \underline{X} \rangle$.

For $k = 1, \ldots, p$, let us define

$$\mathcal{M}(S)^k := \left\{ \sum_{i=1}^{K} a_i^\star s_i a_i \, : \, K \in \mathbb{N}, \, a_i \in \mathbb{R}\langle \underline{X}(I_k) \rangle, \, s_i \in (S \cap \mathrm{Sym}\, \mathbb{R}\langle \underline{X}(I_k) \rangle) \cup \{1\} \right\},$$

and

$$\mathcal{M}(S)^{\mathrm{sparse}} := \mathcal{M}(S)^1 + \cdots + \mathcal{M}(S)^p. \tag{3.1}$$

Next, we state the main foundational result of this paper.

Theorem 3.3 *Let* $S \cup \{f\} \subseteq \mathrm{Sym}\, \mathbb{R}\langle \underline{X} \rangle$ *and let* D_S *be as in* (2.6) *with the additional quadratic constraints* (2.5). *Suppose Assumption* 2.4 *holds. If* $f(\underline{A}) \succ 0$ *for all* $\underline{A} \in D_S^\infty$, *then* $f \in \mathcal{M}(S)^{\mathrm{sparse}}$.

Proof The proof is by contradiction: suppose that $f(\underline{A}) \succ 0$ for all $\underline{A} \in D_S^\infty$, and that $f \notin \mathcal{M}(S)^{\mathrm{sparse}}$. By the Hahn-Banach separation theorem, also known as the Eidelheit-Kakutani Theorem in this context (see [2, Corollary III.1.7] or [24, §0.2.4]), there exists a linear functional $L : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}$ with $L(f) \leq 0$ and $L(\mathcal{M}(S)^{\mathrm{sparse}}) \subseteq \mathbb{R}^{\geq 0}$. Since $\mathbf{1}$ belongs to the algebraic interior of $\mathcal{M}(S)^{\mathrm{sparse}}$ by archimedeanity and L is nonzero, one has $L(\mathbf{1}) > 0$.

Here, we cannot directly apply Theorem 3.2 since $\mathcal{M}(S)^{\mathrm{sparse}}$ is not the quadratic module of $\mathbb{R}\langle \underline{X} \rangle$ generated by the polynomials involved in S. Nevertheless, we will prove that there exists a tuple $\underline{A} = (A_1, \ldots, A_n) \in D_S^\infty$ and a nonzero vector $\mathbf{w}$ such that $L(f) = \langle f(\underline{A})\mathbf{w}, \mathbf{w} \rangle$. Since $f \succ 0$ implies that $\langle f(\underline{A})\mathbf{w}, \mathbf{w} \rangle > 0$, this will contradict the fact that $L(f) \leq 0$.

For $k = 1, \ldots, p$, let us denote by $L^k : \mathbb{R}\langle \underline{X}(I_k) \rangle \to \mathbb{R}$ the restriction of L to $\mathbb{R}\langle \underline{X}(I_k) \rangle$. Observe that $L^k(\mathcal{M}(S)^k) \subseteq \mathbb{R}^{\geq 0}$. Each linear functional L^k induces a sesquilinear form

$$(g, h) \mapsto \langle g, h \rangle_k := L^k(g^\star h)$$

on $\mathbb{R}\langle\underline{X}(I_k)\rangle$, which is positive semidefinite since L^k is positive on sums of hermitian squares, allowing us to apply the Cauchy-Schwarz inequality. Let $\mathcal{N}^k :=$ $\{h \in \mathbb{R}\langle\underline{X}(I_k)\rangle : \langle h, h\rangle_k = 0\}$ be the nullvectors corresponding to L^k. By using again the Cauchy-Schwarz inequality, one can show that $\mathcal{N}^k$ is a vector subspace of $\mathbb{R}\langle\underline{X}(I_k)\rangle$, and the sesquilinear form L^k induces an inner product on the quotient space $\mathbb{R}\langle\underline{X}(I_k)\rangle/\mathcal{N}^k$. Let us denote by $\mathcal{H}(I_k)$ the Hilbert space completion of $\mathbb{R}\langle\underline{X}(I_k)\rangle/\mathcal{N}^k$ and denote by $\langle\cdot, \cdot\rangle_k$ its inner product. Since $L(\mathbf{1}) > 0$, one has $\mathbf{1} \notin \mathcal{N}^k$, and $\mathcal{H}(I_k)$ is nontrivial and separable. By using the fact that L^k is nonnegative on the archimedean quadratic module $\mathcal{M}(S)^k$, there exists $N \in \mathbb{N}$ such that $L^k(g^\star(N - X_i^2)g) \geq 0$, for all $g \in \mathbb{R}\langle\underline{X}(I_k)\rangle$ and $i \in I_k$. Therefore, one has

$$0 \leq \langle X_i g, X_i g\rangle_k = L^k(g^\star X_i^2 g) \leq N L^k(g^\star g), \tag{3.2}$$

implying that $\mathcal{N}^k$ is a left ideal. Therefore, the left multiplication operator $\hat{X}_i^k : g \mapsto X_i g$ is well-defined on $\mathbb{R}\langle\underline{X}(I_k)\rangle/\mathcal{N}^k$, for all $i \in I_k$. By (3.2), this operator is also bounded and can be extended uniquely to a bounded operator on $\mathcal{H}(I_k)$. We fix an orthonormal basis of $\mathcal{H}(I_k)$ and denote by $\hat{A}_i^k$ the corresponding representative of the left multiplication by X_i in $\mathcal{B}(\mathcal{H}(I_k))$ with respect to this basis. Let us denote $\hat{A}^k := (\hat{A}_i^k)_{i\in I_k}$. Then, one has for all $g \in \mathbb{R}\langle\underline{X}(I_k)\rangle$

$$L^k(g) = \langle g(\hat{A}^k)\mathbf{v}^k, \mathbf{v}^k\rangle_k, \tag{3.3}$$

where $\mathbf{v}^k \in \mathcal{H}(I_k)$ is the image of the identity polynomial. We denote by φ_k the state induced by $\mathbf{v}^k$ on $\mathcal{B}(\mathcal{H}(I_k))$, that is, $\varphi_k(B) = \langle B\mathbf{v}^k, \mathbf{v}^k\rangle$ for $B \in \mathcal{B}(\mathcal{H}(I_k))$. In particular, $\varphi_k(g(\hat{A}^k)) = L^k(g)$, for all $g \in \mathbb{R}\langle\underline{X}(I_k)\rangle$. We refer to Figure 2 for an illustration for the case $p = 2$. Note also that given a polynomial $u \in \mathbb{R}\langle\underline{X}(I_k)\rangle$ with associated vector $\mathbf{u} \in \mathcal{H}(I_k)$, there exists a polynomial $g \in \mathbb{R}\langle\underline{X}(I_k)\rangle$ (by construction), such that $\mathbf{u} = g(\hat{A}^k)\mathbf{v}^k$.

Now, the proof proceeds by induction on p. With $p = 1$, this corresponds to the dense representation result stated in Theorem 2.2.

Case $p = 2$

First, note that the running intersection property (2.7) always holds in this case. Let us define the sesquilinear form $(g, h) \mapsto \langle g, h\rangle_{12} := L^1(g^\star h)$ on $\mathbb{R}\langle\underline{X}(I_1 \cap I_2)\rangle$. As above, we obtain $\mathcal{N}^{12} := \{h \in \mathbb{R}\langle\underline{X}(I_1 \cap I_2)\rangle : \langle h, h\rangle_{12} = 0\}$ and the Hilbert space completion $\mathcal{H}(I_1 \cap I_2)$ of $\mathbb{R}\langle\underline{X}(I_1 \cap I_2)\rangle/\mathcal{N}^{12}$. We denote by L^{12} the restriction of L^1 (or, equivalently, L^2) to $\mathbb{R}\langle\underline{X}(I_1 \cap I_2)\rangle$, and by φ_{12} the induced state on $\mathcal{B}(\mathcal{H}(I_1 \cap I_2))$. Let us denote by $\hat{A}_i^{12}$ the corresponding representative of the left multiplication by X_i in $\mathcal{B}(\mathcal{H}(I_1 \cap I_2))$ with respect to this basis, for $i \in I_1 \cap I_2$. For $k \in \{1, 2\}$, let us denote by $i_k : \mathbb{R}\langle\underline{X}(I_1 \cap I_2)\rangle \to \mathbb{R}\langle\underline{X}(I_k)\rangle$ the canonical embedding. Next we apply Theorem 3.1 with $I = \{1, 2\}$, $\mathcal{A} = \mathcal{B}(\mathcal{H}(I_1 \cap I_2))$ endowed with φ_{12}, $\mathcal{B}_k = \mathcal{B}(\mathcal{H}(I_k))$ endowed with φ_k, and $\iota_k : \mathcal{B}(\mathcal{H}(I_1 \cap I_2)) \to \mathcal{B}(\mathcal{H}(I_k))$ being the canonical embedding, satisfying $\iota_k(\hat{A}_i^{12}) = \hat{A}_i^k$ for all $i \in I_1 \cap I_2$ (observe that $\mathcal{B}(\mathcal{H}(I_1 \cap I_2))$ contains the algebra generated by $\hat{A}_i^{12}$ as a dense subset). If $I_1 \cap I_2 = \emptyset$, then we amalgamate them

Fig. 2 Amalgamation for the case $p = 2$
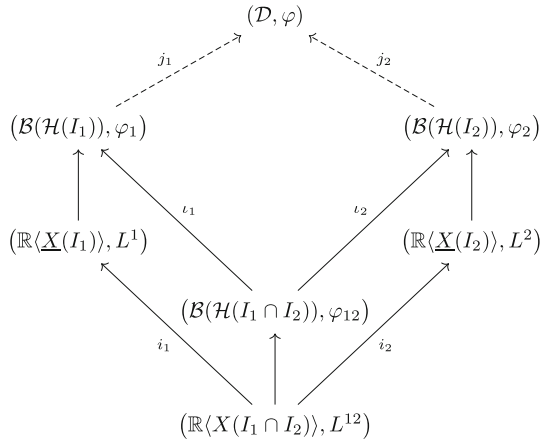
over $\mathbb{R}$, and otherwise over $(\mathcal{B}(\mathcal{H}(I_1 \cap I_2)), \varphi_{12})$. Note that ι_k is state-preserving by construction. As displayed in Fig. 2, we obtain an amalgamation $\mathcal{D}$ with state φ and homomorphisms $j_k : \mathcal{B}(\mathcal{H}(I_k)) \to \mathcal{D}$ such that $j_1 \circ \iota_1 = j_2 \circ \iota_2$.

Next perform the GNS construction with $(\mathcal{D}, \varphi)$. There is a Hilbert space $\mathcal{K}$, representation $\pi : \mathcal{D} \to \mathcal{B}(\mathcal{K})$ and vector $\xi \in \mathcal{K}$ so that $\varphi(a) = \langle \pi(a)\xi, \xi \rangle$. Then, let us define $\underline{A} := (A_1, \ldots, A_n)$, with

$$A_i := \begin{cases} \pi(j_1(\hat{A}_i^1)) & \text{if } i \in I_1, \\ \pi(j_2(\hat{A}_i^2)) & \text{if } i \in I_2. \end{cases}$$

By the amalgamation property, this is well-defined since $j_1(\hat{A}_i^1) = j_1 \circ \iota_1(\hat{A}_i^{12}) = j_2 \circ \iota_2(\hat{A}_i^{12}) = j_2(\hat{A}_i^2)$ if $i \in I_1 \cap I_2$.

For all $g \in \mathbb{R}\langle \underline{X} \rangle$, we now set $\tilde{L}(g) := \langle g(\underline{A})\xi, \xi \rangle$. We claim that $\tilde{L}$ extends L^k. Indeed, for $g \in \mathbb{R}\langle \underline{X}(I_k) \rangle$ we have

$$\tilde{L}(g) = \langle g(\underline{A})\xi, \xi \rangle = \langle g(\pi(j_k(\hat{A}^k)))\xi, \xi \rangle = \langle \pi(g(j_k(\hat{A}^k)))\xi, \xi \rangle$$
$$= \varphi(g(j_k(\hat{A}^k))) = \varphi(j_k(g(\hat{A}^k))) = \varphi_k(g(\hat{A}^k)) = L^k(g).$$

The above equalities come from the fact that nc polynomials commute with homomorphisms (here π, ι_k), since they are linear combination of products of letters and homomorphisms are addition, multiplication as well as unit (multiplicative identity) preserving.

Therefore,

$$\langle f(\underline{A})\xi, \xi \rangle = \tilde{L}(f) = \tilde{L}(f_1) + \tilde{L}(f_2) = L^1(f_1) + L^2(f_2) = L(f) \leq 0.$$

It only remains to prove that $\underline{A} \in D_S^\infty$, i.e., that $s(\underline{A}) \succeq 0$, for all $s \in S$. By assumption, $s \in \text{Sym } \mathbb{R}\langle \underline{X}(I_k) \rangle$ for some $k \in \{1, 2\}$, so

$$s(\underline{A}) = s((\pi \circ j_k)(\hat{A}^k)) = (\pi \circ j_k)(s(\hat{A}^k)). \tag{3.4}$$

Since $\mathbb{R}\langle \underline{X}(I_k)\rangle/\mathcal{N}^k$ is dense in $\mathcal{H}(I_k)$, one can approximate as closely as desired $\mathbf{u} \in \mathcal{H}(I_k)$ by elements of $\mathbb{R}\langle \underline{X}(I_k)\rangle/\mathcal{N}^k$. We prove that $\langle s(\hat{A}^k)\mathbf{u}, \mathbf{u}\rangle_k \geq 0$, where $\mathbf{u}$ stands for a vector representative of $u \in \mathbb{R}\langle \underline{X}(I_k)\rangle/\mathcal{N}^k$. Given such a vector $\mathbf{u}$, there exists a polynomial $g \in \mathbb{R}\langle \underline{X}(I_k)\rangle$, such that $\mathbf{u} = g(\hat{A}^k)\mathbf{v}^k$. Next, the following holds:

$$\langle s(\hat{A}^k)\mathbf{u}, \mathbf{u}\rangle_k = \langle s(\hat{A}^k)g(\hat{A}^k)\mathbf{v}^k, g(\hat{A}^k)\mathbf{v}^k\rangle_k = \langle (sg)(\hat{A}^k)\mathbf{v}^k, g(\hat{A}^k)\mathbf{v}^k\rangle_k = L^k(g^\star sg),$$

where the last equality comes from (3.3). Since $g^\star sg \in \mathcal{M}(S)^k$ and $L^k(\mathcal{M}(S)^k) \subseteq \mathbb{R}^{\geq 0}$, one has $\langle s(\hat{A}^k)\mathbf{u}, \mathbf{u}\rangle_k \geq 0$, which implies that $c := s(\hat{A}^k) \succeq 0$. Since c is a nonnegative element of the $C^\star$-algebra $\mathcal{B}(\mathcal{H}(I_k))$, there exists $b \in \mathcal{B}(\mathcal{H}(I_k))$ such that $c = b^\star b$. Eventually by (3.4), one has $s(\underline{A}) = (\pi \circ j_k)(s(\hat{A}^k)) = \pi(j_k(c)) = \pi(j_k(b^\star b)) = \pi(j_k(b))^\star \pi(j_k(b)) \succeq 0$, yielding $\underline{A} \in D_S^\infty$, the desired result.

General case

Now assume $p > 2$. For each $m \leq p$ we will construct a Hilbert space $\mathcal{H}(\cup_{j\leq m} I_j)$ with state $\tilde{\varphi}_m$ acting on $\mathcal{B}(\mathcal{H}(\cup_{j\leq m} I_j))$, a tuple $\underline{A}^m \in D_S^\infty$ as well as a unit vector ξ^m such that the linear functional $\tilde{L}^m : \mathbb{R}\langle \underline{X}\rangle \to \mathbb{R}$, defined by

$$\tilde{L}^m(g) := \langle g(\underline{A}^m)\xi^m, \xi^m\rangle, \tag{3.5}$$

extends L_j for each $j \leq m$, implying that $\tilde{L}^m(g) = L(g)$ for all $g \in \sum_{j\leq m} \mathbb{R}\langle \underline{X}(I_j)\rangle$.

The basis for the induction, $p \leq 2$, has been established above.

Let $p > m \geq 2$ and assume by induction that we have $\mathcal{H}(\cup_{j\leq m} I_j)$, $\tilde{\varphi}_m$, $\underline{A}^m \in D_S^\infty$ and $\tilde{L}^m$ as above. By the running intersection property (2.7), there is $k \leq m$ with $\left(\cup_{j\leq m} I_j\right) \cap I_{m+1} \subseteq I_k$. Recall that L^{m+1} is defined as the restriction of L to $\mathbb{R}\langle \underline{X}(I_{m+1})\rangle$. Let L^0 be the restriction of L (or, equivalently, of $\tilde{L}^m$) to $\mathbb{R}\langle \underline{X}(I_k)\rangle$. As before, Theorem 3.2 produces Hilbert spaces $\mathcal{H}(I_{m+1})$, $\mathcal{H}_0 := \mathcal{H}((\cup_{j\leq m} I_j)\cap I_{m+1})$, operators $\hat{A}^{m+1}$, $\hat{A}^0$ and states φ_{m+1}, φ_0 acting on $\mathcal{B}(\mathcal{H}(I_{m+1}))$, $\mathcal{B}(\mathcal{H}_0)$.

The operator $\hat{A}^{m+1}$ and state φ_{m+1} satisfy $\varphi_{m+1}(g(\hat{A}^{m+1})) = L^{m+1}(g)$, for all $g \in \mathbb{R}\langle \underline{X}(I_{m+1})\rangle$. In addition, the canonical embeddings $\iota : \mathcal{B}(\mathcal{H}_0) \to \mathcal{B}(\mathcal{H}(\cup_{j\leq m} I_j))$, $\iota_{m+1} : \mathcal{B}(\mathcal{H}_0) \to \mathcal{B}(\mathcal{H}(I_{m+1}))$, and the operator $\hat{A}^0$ satisfy $\iota(\hat{A}_i^0) = A_i^m$ and $\iota_{m+1}(\hat{A}_i^0) = \hat{A}_i^{m+1}$, for all $i \in (\cup_{j\leq m} I_j) \cap I_{m+1}$.

The remaining part of the proof is very similar to the case $p = 2$. We amalgamate $\mathcal{B}(\mathcal{H}(\cup_{j\leq m} I_j))$ and $\mathcal{B}(\mathcal{H}(I_{m+1}))$; if $\left(\cup_{j\leq m} I_j\right)\cap I_{m+1} = \emptyset$, then we amalgamate them over $\mathbb{R}$, and otherwise over $(\mathcal{B}(\mathcal{H}_0), \varphi_0)$. Doing so, we obtain an amalgamation $\mathcal{D}_{m+1}$ and two homomorphisms $j : \mathcal{B}(\mathcal{H}(\cup_{j\leq m} I_j)) \to \mathcal{D}_{m+1}$ and $j_{m+1} : \mathcal{B}(\mathcal{H}(I_{m+1})) \to \mathcal{D}_{m+1}$. Applying the GNS construction to the amalgamated $C^\star$-algebra then yields a Hilbert space $\mathcal{K}_{m+1}$, a representation $\pi_{m+1} : \mathcal{D}_{m+1} \to \mathcal{B}(\mathcal{K}_{m+1})$, a unit vector $\xi^{m+1} \in \mathcal{K}_{m+1}$ and we can define $\underline{A}^{m+1}$ with

$$A_i^{m+1} := \begin{cases} \pi_{m+1}(j(A_i^m)) & \text{if } i \in \cup_{j \leq m} I_j, \\ \pi_{m+1}(j_{m+1}(\hat{A}_i^{m+1})) & \text{if } i \in I_{m+1}, \end{cases}$$

as well as $\tilde{L}^{m+1}(g) := \langle g(\underline{A}^{m+1}) \xi^{m+1}, \xi^{m+1} \rangle$.

As in the case $p = 2$, by the amalgamation property, $\underline{A}^{m+1}$ is well-defined since $j(A_i^m) = j \circ \iota(\hat{A}_i^0) = j_{m+1} \circ \iota_{m+1}(\hat{A}_i^0) = j_{m+1}(\hat{A}_i^{m+1})$ if $i \in (\cup_{j \leq m} I_j) \cap I_{m+1}$. One proves as before that $\underline{A}^{m+1} \in D_S^\infty$. In addition, $\tilde{L}^{m+1}(g) = \tilde{L}^m(g) = L(g)$ for all $g \in \sum_{j \leq m} \mathbb{R}\langle \underline{X}(I_j) \rangle$, where the first equality comes from the definition of $\underline{A}^{m+1}$ and the second one comes from the induction hypothesis. One has $\tilde{L}^{m+1} = L^{m+1} = L(g)$ for all $g \in \mathbb{R}\langle \underline{X}(I_{m+1}) \rangle$, which implies that $L(g) = \langle g(\underline{A}^{m+1}) \xi^{m+1}, \xi^{m+1} \rangle$ for all $g \in \sum_{j \leq m+1} \mathbb{R}\langle \underline{X}(I_j) \rangle$.

For $m = p$, we obtain $\underline{A}^p \in D_S^\infty$ and a unit vector ξ^p such that $\langle f(\underline{A}^p) \xi^p, \xi^p \rangle = L(f) \leq 0$, yielding the desired conclusion. $\qquad\square$

The reader will notice that the RIP property is used subtly in the proof of Theorem 3.3. Next, we provide an example demonstrating that sparsity without a RIP-type condition is not sufficient to deduce sparsity in SOHS decompositions.

Example 3.4 Consider the case of three variables $\underline{X} = (X_1, X_2, X_3)$ and the polynomial

$$f = (X_1 + X_2 + X_3)^2$$
$$= X_1^2 + X_2^2 + X_3^2 + X_1X_2 + X_2X_1 + X_1X_3 + X_3X_1 + X_2X_3 + X_3X_2 \in \Sigma\langle \underline{X} \rangle.$$

Then $f = f_1 + f_2 + f_3$, with

$$f_1 = \frac{1}{2}X_1^2 + \frac{1}{2}X_2^2 + X_1X_2 + X_2X_1 \in \mathbb{R}\langle X_1, X_2 \rangle,$$
$$f_2 = \frac{1}{2}X_2^2 + \frac{1}{2}X_3^2 + X_2X_3 + X_3X_2 \in \mathbb{R}\langle X_2, X_3 \rangle,$$
$$f_3 = \frac{1}{2}X_1^2 + \frac{1}{2}X_3^2 + X_1X_3 + X_3X_1 \in \mathbb{R}\langle X_1, X_3 \rangle.$$

However, the sets $I_1 = \{1, 2\}$, $I_2 = \{2, 3\}$ and $I_3 = \{1, 3\}$ do not satisfy the RIP condition (2.7) and $f \notin \Sigma\langle \underline{X} \rangle^{\text{sparse}} := \Sigma\langle X_1, X_2 \rangle + \Sigma\langle X_2, X_3 \rangle + \Sigma\langle X_1, X_3 \rangle$ since it has a unique Gram matrix by homogeneity.

Now consider $S = \{1 - X_1^2, 1 - X_2^2, 1 - X_3^2\}$. Then D_S is as in (2.6), $\mathcal{M}(S)^{\text{sparse}}$ is as in (3.1) and $f|_{D_S^\infty} \succeq 0$. However, we claim that $f - \lambda \in \mathcal{M}(S)^{\text{sparse}}$ iff $\lambda \leq -3$. Clearly,

$$f + 3 = (X_1 + X_2)^2 + (X_1 + X_3)^2 + (X_2 + X_3)^2 + (1 - X_1^2)$$
$$+ (1 - X_2^2) + (1 - X_3^2) \in \mathcal{M}(S)^{\text{sparse}}.$$

So one has $-3 \leq \sup\{\lambda : f - \lambda \in \mathcal{M}(S)^{\text{sparse}}\}$, and the dual of this latter problem is given by

$$
\begin{aligned}
\inf_{L_k} \quad & \sum_{k=1}^{3} L_k(f_k) \\
\text{s.t.} \quad & L_k(1) = 1, \quad k = 1, \ldots, 3, \\
& L_k(h^\star h) \succeq 0 \quad \forall h \in \mathbb{R}\langle \underline{X}(I_k)\rangle, \quad k = 1, \ldots, 3, \\
& L_k(h^\star(1 - X_k^2)h) \succeq 0 \quad \forall h \in \mathbb{R}\langle \underline{X}(I_k)\rangle, \quad k = 1, \ldots, 3, \\
& L_j|_{\mathbb{R}\langle \underline{X}(I_j \cap I_k)\rangle} = L_k|_{\mathbb{R}\langle \underline{X}(I_j \cap I_k)\rangle}, \quad j, k = 1, \ldots, 3.
\end{aligned}
\tag{3.6}
$$

Hence, by weak duality, it suffices to show that there exist linear functionals $L_k : \mathbb{R}\langle \underline{X}(I_k)\rangle \to \mathbb{R}$ satisfying the constraints of problem (3.6) and such that $\sum_k L_k(f_k) = -3$. Define

$$
A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = -A
$$

and let

$$
L_k(g) = \operatorname{tr} g(A, B) \quad \text{for } g \in \mathbb{R}\langle \underline{X}(I_k)\rangle.
$$

Since $L_k(f_k) = -1$, the three first constraints of problem (3.6) are easily verified and $\sum_k L_k(f_k) = -3$. For the last one, given, say $h \in \mathbb{R}\langle \underline{X}(I_1)\rangle \cap \mathbb{R}\langle \underline{X}(I_2)\rangle = \mathbb{R}\langle X_2\rangle$, we have

$$
\begin{aligned}
L_1(h) &= \operatorname{tr} h(B), \\
L_2(h) &= \operatorname{tr} h(A),
\end{aligned}
$$

since L_1 (resp. L_2) is defined on $\mathbb{R}\langle X_1, X_2\rangle$ (resp. $\mathbb{R}\langle X_2, X_3\rangle$) and h depends only on the second (resp. first) variable X_2 corresponding to B (resp. A).

But matrices A and B are orthogonally equivalent as $U A U^T = B$ for

$$
U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
$$

whence $h(B) = h(U A U^T) = U h(A) U^T$ and $h(A)$ have the same trace.

4 Sparse GNS construction and optimizer extraction

The aim of this section is to provide a general algorithm to extract solutions of sparse noncommutative optimization problems. We will apply this algorithm below to eigenvalue optimization (Sect. 5) and trace optimization (Sect. 6). For this purpose, we first present sparse noncommutative versions of theorems by Curto and Fialkow. In

the commutative case, Curto and Fialkow provided sufficient conditions for linear functionals on the set of degree $2d$ polynomials to be represented by integration with respect to a nonnegative measure. The main sufficient condition to guarantee such a representation is flatness (see Definition 2.6) of the corresponding Hankel matrix. This notion was exploited in a noncommutative setting for the first time by McCullough [38] in his proof of the Helton-McCullough Sums of Squares theorem, cf. [38, Lemma 2.2].

In the dense case [49] (see also [1, Chapter 21] and [5, Theorem 1.69]) provides a first noncommutative variant for the eigenvalue problem. See [3] for a similar construction for the trace problem. As this will be needed in the sequel, we recall this theorem and a sketch of its proof, which relies on a finite-dimensional GNS construction.

Theorem 4.1 *Let $S \subseteq \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle$ and set $\delta := \max\{\lceil \deg(g)/2 \rceil : g \in S \cup \{1\}\}$. For $d \in \mathbb{N}$, let $L : \mathbb{R}\langle \underline{X} \rangle_{2d+2\delta} \to \mathbb{R}$ be a unital linear functional satisfying $L(\mathcal{M}(S)_{d+\delta}) \subseteq \mathbb{R}^{\geq 0}$. If L is δ-flat, then there exist $\hat{A} \in D_S^r$ for some $r \leq \sigma(n, d)$ and a unit vector $\mathbf{v}$ such that*

$$L(g) = \langle g(\hat{A})\mathbf{v}, \mathbf{v} \rangle, \tag{4.1}$$

for all $g \in \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle_{2d}$.

Proof Let $r := \operatorname{rank} \mathbf{M}_{d+\delta}(L)$. Since $\mathbf{M}_{d+\delta}(L)$ is a positive semidefinite matrix, we obtain the Gram matrix decomposition $\mathbf{M}_{d+\delta}(L) = [\langle \mathbf{u}, \mathbf{w} \rangle]_{u,w}$ with vectors $\mathbf{u}, \mathbf{w} \in \mathbb{R}^r$, where the labels are words of degree at most $d + \delta$. Then, we define the following finite-dimensional Hilbert space

$$\mathcal{H} := \operatorname{span}\{\mathbf{w} \mid \deg w \leq d + \delta\} = \operatorname{span}\{\mathbf{w} \mid \deg w \leq d\},$$

where the equality comes from the flatness assumption. Afterwards, one can directly consider the operators $\hat{A}_i$ representing the left multiplication by X_i on $\mathcal{H}$, i.e., $\hat{A}_i \mathbf{w} = \mathbf{X}_i \mathbf{w}$. Thanks to the flatness assumption, the operators $\hat{A}_i$ are well-defined and one can show that they are symmetric. Let $\hat{A} := (\hat{A}_1, \ldots, \hat{A}_n)$. As in the GNS construction of Theorem 3.3, one has $L(g) = \langle g(\hat{A})\mathbf{v}, \mathbf{v} \rangle$, with $\mathbf{v}$ being the vector representing $\mathbf{1}$ in $\mathcal{H}$. Given $s \in S$, let us prove that $\langle s(\hat{A})\mathbf{w}, \mathbf{w} \rangle \geq 0$, for all $\mathbf{w} \in \mathcal{H}$. By construction, any vector $\mathbf{w} \in \mathcal{H}$ can be written as $g(\hat{A})\mathbf{v}$, for some polynomial $g \in \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle_{2d}$. Thus, one has $\langle s(\hat{A})\mathbf{w}, \mathbf{w} \rangle = \langle s(\hat{A})g(\hat{A})\mathbf{v}, g(\hat{A})\mathbf{v} \rangle = L(g^\star sg) \geq 0$ since $g^\star sg \in \mathcal{M}(S)_{d+\delta}$. Thus, one has $\hat{A} \in D_S^r$, the desired result. □

We now give the sparse version of Theorem 4.1.

Theorem 4.2 *Let $S \subseteq \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle_{2d}$, and assume D_S is as in (2.6) with the additional quadratic constraints (2.5). Suppose Assumption 2.4(i) holds. Set $\delta := \max\{\lceil \deg(g)/2 \rceil : g \in S \cup \{1\}\}$. Let $L : \mathbb{R}\langle \underline{X} \rangle_{2d+2\delta} \to \mathbb{R}$ be a unital linear functional satisfying $L(\mathcal{M}(S)_d^{\operatorname{sparse}}) \subseteq \mathbb{R}^{\geq 0}$. Assume that the following holds:*

(H1) $\mathbf{M}_{d+\delta}(L, I_k)$ *and* $\mathbf{M}_{d+\delta}(L, I_k \cap I_j)$ *are δ-flat, for all $j, k \in \{1, \ldots, p\}$.*

Then, there exist finite-dimensional Hilbert spaces $\mathcal{H}(I_k)$ with dimension r_k, for all $k \in \{1, \ldots, p\}$, Hilbert spaces $\mathcal{H}(I_j \cap I_k) \subseteq \mathcal{H}(I_j), \mathcal{H}(I_k)$ for all pairs (j, k) with $I_j \cap I_k \neq 0$, and operators $\hat{A}^k, \hat{A}^{jk}$, acting on them, respectively. Further, there are unit vectors $\mathbf{v}^j \in \mathcal{H}(I_j)$ and $\mathbf{v}^{jk} \in \mathcal{H}(I_j \cap I_k)$ such that

$$
\begin{aligned}
L(f) &= \langle f(\hat{A}^j)\mathbf{v}^j, \mathbf{v}^j \rangle \quad \text{for all } f \in \mathbb{R}\langle \underline{X}(I_j)\rangle_{2d}, \\
L(g) &= \langle g(\hat{A}^{jk})\mathbf{v}^{jk}, \mathbf{v}^{jk} \rangle \quad \text{for all } g \in \mathbb{R}\langle \underline{X}(I_j \cap I_k)\rangle_{2d}.
\end{aligned}
\tag{4.2}
$$

Assuming that for all pairs (j, k) with $I_j \cap I_k \neq \emptyset$, one has

(H2) the matrices $(\hat{A}_i^{jk})_{i \in I_j \cap I_k}$ have no common complex invariant subspaces,

then there exist $\underline{A} \in D_S^r$, with $r := r_1 \cdots r_p$, and a unit vector $\mathbf{v}$ such that

$$
L(f) = \langle f(\underline{A})\mathbf{v}, \mathbf{v} \rangle,
\tag{4.3}
$$

for all $f \in \sum_j \mathbb{R}\langle \underline{X}(I_j)\rangle_{2d}$.

In the proof of Theorem 4.2 we will make use of the following simple linear algebra observation.

Lemma 4.3 *Let $Z \in M_n(\mathbb{R})$. If $\operatorname{tr}(ZA) = 0$ for all $A \in M_n(\mathbb{R})$, then $Z = 0$.*

Proof We have $\operatorname{tr}(ZZ^T) = 0$ whence $ZZ^T = 0$ and thus $Z = 0$. □

Proof of Theorem 4.2 Start by applying Theorem 4.1 to $L|_{\mathbb{R}\langle \underline{X}(I_j)\rangle}$ and $L|_{\mathbb{R}\langle \underline{X}(I_j \cap I_k)\rangle}$ to obtain the desired (real) Hilbert spaces $\mathcal{H}(I_j), \mathcal{H}(I_j \cap I_k)$, unit vectors $\mathbf{v}^j, \mathbf{v}^{jk}$ and operators $\hat{A}^j, \hat{A}^{jk}$ satisfying (4.2). Note that we may assume $\mathcal{H}(I_j \cap I_k) \subseteq \mathcal{H}(I_j), \mathcal{H}(I_k)$ as the map $f(\hat{A}^{jk})\mathbf{v}^{jk} \mapsto f(\hat{A}^j)\mathbf{v}^j$ is an isometry by construction. Then

$$
\hat{A}^{jk} = \hat{A}^j|_{\mathcal{H}(I_j \cap I_k)} = \hat{A}^k|_{\mathcal{H}(I_j \cap I_k)}.
\tag{4.4}
$$

Let us denote by $\mathcal{A}(I_j)$ and $\mathcal{A}(I_j \cap I_k)$ the algebras generated by $\hat{A}^j$, and $\hat{A}^{jk}$, respectively. By (4.4), the map $\hat{A}_\ell^{jk} \mapsto \hat{A}_\ell^j$ is a $\star$-homomorphism $\mathcal{A}(I_j) \to \mathcal{A}(I_j \cap I_k)$. With $r_k = \dim \mathcal{H}(I_k)$ and $r_{jk} = \dim \mathcal{H}(I_j \cap I_k)$, one has $\mathcal{A}(I_k) \subseteq \mathbb{M}_{r_k}(\mathbb{R})$ and $\mathcal{A}(I_j \cap I_k) \subseteq \mathbb{M}_{r_{jk}}(\mathbb{R})$.

We next want to find a finite-dimensional $C^\star$-algebra $\mathcal{A}$, i.e., a subalgebra of $\mathbb{M}_m(\mathbb{R})$ for some $m \in \mathbb{N}$, making the diagram in Fig. 3 commute.

In the sequel, the proof proceeds by induction on p and we focus specifically on the case $p = 2$, as the general case then follows by a simple inductive argument. By the amalgamation property of $C^\star$-algebras stated in Theorem 3.1, we can always find such an *infinite-dimensional* $\mathcal{A}$. However, as shown in Example 4.4, there may not be a suitable finite-dimensional $\mathcal{A}$.

To ensure this, we assume that (H2) holds, namely that the matrices $(\hat{A}_i^{12})_{i \in I_1 \cap I_2}$ have no common complex invariant subspaces, which implies by Burnside's theorem (see, e.g., [10, Corollary 5.23]) that $\mathcal{A}(I_1 \cap I_2) = \mathbb{M}_{r_{12}}(\mathbb{R})$. Then, for all $A \in \mathcal{A}(I_1 \cap I_2) = \mathbb{M}_{r_{12}}(\mathbb{R})$, $\iota_k(A)$ is just a direct sum of copies of A, up to orthogonal equivalence

Fig. 3 Amalgamation of finite-dimensional $C^\star$-algebras
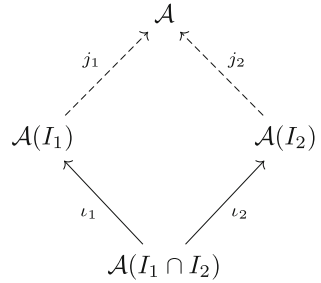
(by the Skolem-Noether theorem [8, Section III.3]), i.e., there are orthogonal matrices U_k such that

$$\iota_k(A) = U_k^T (\mathbf{I}_{r_k/r_{12}} \otimes A) U_k,$$

for all $k \in \{1, 2\}$. By replacing $\hat{A}^k$ with their conjugates $U_k^T \hat{A}^k U_k$, and $\mathbf{v}^k$ by $U_k^T \mathbf{v}^k$, we may without loss of generality assume $\iota_k(A) = \mathbf{I}_{r_k/r_{12}} \otimes A$.

The linear functional L induces linear functionals $\check{L}^k$, $\check{L}^{12}$ on $\mathcal{A}(I_k)$, $\mathcal{A}(I_1 \cap I_2)$ given by $B \mapsto \operatorname{tr}(B\mathbf{v}^k(\mathbf{v}^k)^T)$ and $C \mapsto \operatorname{tr}(C\mathbf{v}^{12}(\mathbf{v}^{12})^T)$, respectively. Write $\mathbf{v}^k = \sum_{j=1}^{r_k/r_{12}} e_j^k \otimes u_j^k$ for the standard basis vectors $e_j^k \in \mathbb{R}^{r_k/r_{12}}$ and some vectors $u_j^k \in \mathbb{R}^{r_{12}}$. Then for $C \in \mathcal{A}(I_1 \cap I_2) = M_{r_{12}}(\mathbb{R})$ we have

$$
\begin{aligned}
\check{L}^{12}(C) &= \operatorname{tr}(C\mathbf{v}^{12}(\mathbf{v}^{12})^T) \\
&= \check{L}^k(\mathbf{I}_{r_k/r_{12}} \otimes C) = \operatorname{tr}((\mathbf{I} \otimes C)\mathbf{v}^k(\mathbf{v}^k)^T) \\
&= \operatorname{tr}\left((\mathbf{I} \otimes C)(\sum_{j=1}^{r_k/r_{12}} e_j^k \otimes u_j^k)(\sum_{j=1}^{r_k/r_{12}} e_j^k \otimes u_j^k)^T\right) \\
&= \operatorname{tr}\left((\mathbf{I} \otimes C)\sum_{i,j=1}^{r_k/r_{12}} (e_j^k(e_i^k)^T) \otimes (u_j^k(u_i^k)^T)\right) \\
&= \sum_{i,j=1}^{r_k/r_{12}} \operatorname{tr}\left((e_j^k(e_i^k)^T) \otimes C u_j^k(u_i^k)^T\right) \\
&= \sum_{i,j=1}^{r_k/r_{12}} \operatorname{tr}\left(e_j^k(e_i^k)^T\right) \operatorname{tr}\left(C u_j^k(u_i^k)^T\right) \\
&= \operatorname{tr}\left(C \sum_{j=1}^{r_k/r_{12}} u_j^k(u_j^k)^T\right).
\end{aligned}
$$

From the equality $\operatorname{tr}(C\mathbf{v}^{12}(\mathbf{v}^{12})^T) = \operatorname{tr}\left(C \sum_{j=1}^{r_k/r_{12}} u_j^k(u_j^k)^T\right)$ for all $C \in M_{r_{12}}(\mathbb{R})$ we deduce using Lemma 4.3 that $\mathbf{v}^{12}(\mathbf{v}^{12})^T = \sum_{j=1}^{r_k/r_{12}} u_j^k(u_j^k)^T$. Since the left-hand

side outer product is rank one, each of the u_j^k must be a scalar multiple of $\mathbf{v}^{12}$, say $u_j^k = \lambda_{jk} \mathbf{v}^{12}$. Thus $\mathbf{v}^k = \sum_{j=1}^{r_k/r_{12}} \lambda_{jk} e_j^k \otimes \mathbf{v}^{12}$ and $\sum_j \lambda_{jk}^2 = 1$ since $\|\mathbf{v}^k\| = 1$.

Now set $\mathcal{A} := \mathbb{M}_{r_1 r_2}(\mathbb{R})$ and define $j_1(A) := \mathbf{I}_{r_2} \otimes A$, for all $A \in \mathcal{A}(I_1)$, and

$$j_2(B) := (U^T \otimes \mathbf{I}_{r_{12}})(\mathbf{I}_{r_1} \otimes B)(U \otimes \mathbf{I}_{r_{12}}), \tag{4.5}$$

for all $B \in \mathcal{A}(I_2)$. Here U is an $r_1 r_2/r_{12}$ orthogonal matrix to be determined later. This amalgamates the diagram in Fig. 3 (independently of the choice of U).

Each extension of the linear functional $\check{L}^k$ to a linear functional on $\mathcal{A}$ is of the form

$$C \mapsto \operatorname{tr}\left(C \sum_{\ell=1}^{r_{3-k}} \mu_{\ell k} e_\ell^{3-k} \otimes \mathbf{v}^k\right) = \operatorname{tr}\left(C \underbrace{\sum_\ell \sum_{j=1}^{r_k/r_{12}} \mu_{\ell k} e_\ell^{3-k} \otimes \lambda_{jk} e_j^k \otimes \mathbf{v}^{12}}_{\mathbf{w}^k}\right), \tag{4.6}$$

where $\sum_\ell \mu_{\ell k}^2 = 1$. Since the vectors $\mathbf{w}^k$ are norm one, there is a unitary U with $U \mathbf{w}_1 = \mathbf{w}_2$. Using this U in the definition (4.5), the extension (4.6) of $\check{L}^1$ to a linear functional $\check{L} : \mathcal{A} \to \mathbb{R}$ also extends $\check{L}^2$ (via j_2).

Now define the operators $\underline{A} := (A_1, \ldots, A_n)$, with

$$A_i := \begin{cases} j_1(\hat{A}_i^1) & \text{if } i \in I_1, \\ j_2(\hat{A}_i^2) & \text{if } i \in I_2. \end{cases}$$

Then $L(f) = \langle f(\underline{A})(\mathbf{w}^1 \otimes \mathbf{v}^{12}), \mathbf{w}^1 \otimes \mathbf{v}^{12}\rangle$ for all $f \in \mathbb{R}\langle \underline{X}(I_1)\rangle_{2d} + \mathbb{R}\langle \underline{X}(I_2)\rangle_{2d}$.

To conclude the proof note that each A_i is symmetric and that $\underline{A} \in D_S^{r_1 r_2}$. For the latter we use the fact that each constraint g is either in $\mathbb{R}\langle \underline{X}(I_1)\rangle$ or $\mathbb{R}\langle \underline{X}(I_2)\rangle$, and that $\star$-subalgebras of matrix algebras admit square roots of positive semidefinite operators. $\qquad\square$

Example 4.4 *(Non-amalgamation in the category of finite-dimensional algebras)* For given I_1, I_2, suppose $\mathcal{A}(I_1 \cap I_2)$ is generated by the 2×2 diagonal matrix

$$A^{12} = \begin{pmatrix} 1 & \\ & 2 \end{pmatrix},$$

and assume $\mathcal{A}(I_1) = \mathcal{A}(I_2) = \mathbb{M}_3(\mathbb{R})$. (Observe that $\mathcal{A}(I_1 \cap I_2)$ is the algebra of all diagonal matrices.) For each $k \in \{1, 2\}$, let us define $\iota_k(A) := A \oplus k$, for all $A \in \mathcal{A}(I_1 \cap I_2)$. We claim that there is no finite-dimensional $C^\star$-algebra $\mathcal{A}$ amalgamating the above Fig. 3. Indeed, by the Skolem-Noether theorem, every homomorphism $\mathbb{M}_n(\mathbb{R}) \to \mathbb{M}_m(\mathbb{R})$ is of the form $x \mapsto P^{-1}(x \otimes \mathbf{I}_{m/n}) P$ for some invertible P; in particular, n divides m. If a desired $\mathcal{A}$ existed, then the matrices $(A^{12} \oplus 1) \otimes \mathbf{I}_k$ and $(A^{12} \oplus 2) \otimes \mathbf{I}_k$ would be similar. But they are not as is easily seen from eigenvalue multiplicities.

Remark 4.5 Theorem 4.2 can be seen as a noncommutative variant of the result by Lasserre stated in [30, Theorem 3.7], related to the minimizers extraction in the context

of sparse polynomial optimization. In the sparse commutative case, Lasserre assumes flatness of each moment matrix indexed in the canonical basis of $\mathbb{R}[\underline{X}(I_k)]_d$, for each $k \in \{1, \ldots, p\}$, which is similar to our flatness condition (H1). The difference is that this technical flatness condition on each I_k adapts to the degree of the constraints polynomials on variables in I_k, resulting in an adapted parameter δ_k instead of global δ. We could assume the same in Theorem 4.2 but for the sake of simplicity, we assume that these parameters are all equal. In addition, Lasserre assumes that each moment matrix indexed in the canonical basis of $\mathbb{R}[\underline{X}(I_j \cap I_k)]_d$ is rank one, for all pairs (j, k) with $I_j \cap I_k \neq \emptyset$, which is the commutative analog of our irreducibility condition (H2).

4.1 Implementing the sparse GNS construction

As in the dense case, we can summarize the sparse GNS construction procedure described in the proof of Theorem 4.2 into an algorithm, called `SparseGNS`, stated below in Algorithm 4.6, for the case $p = 2$ (the general case is similar).

This algorithm describes how to compute the tuple $\underline{A} = (A_1, \ldots, A_n)$ of amalgamated matrices acting on $\mathcal{H} = \mathbb{R}^{r_1 r_2} \cong \mathcal{H}(I_1) \otimes \mathbb{R}^{r_2} \cong \mathbb{R}^{r_1} \otimes \mathcal{H}(I_2)$, and a vector $\mathbf{v}$ satisfying (4.3). To check the irreducibility (H2) condition, in Line 3 of the algorithm, one relies on the Burnside theorem from matrix theory (see, e.g., [10, Corollary 5.23]): the algebra generated by $e \times e$ (real) symmetric matrices is irreducible if and only if it is isomorphic to $\mathbb{M}_e(\mathbb{R})$. So one only needs to check the dimension of the algebra.

Algorithm 4.6 `SparseGNS`

Require: $\mathbf{M}_d(L)$, Hankel matrix of L.
1: *Apply the GNS construction to obtain $\mathcal{H}(I_1)$, $\mathcal{H}(I_2)$ and $\mathcal{H}(I_1 \cap I_2)$ of respective dimensions r_1, r_2 and r_{12}, associated to $\mathbf{M}_d(L, I_1)$, $\mathbf{M}_d(L, I_2)$ and $\mathbf{M}_d(L, I_1 \cap I_2)$, as well as $\hat{A}^1$, $\hat{A}^2$ and $\hat{A}^{12}$ acting on $\mathcal{H}(I_1)$, $\mathcal{H}(I_2)$ and $\mathcal{H}(I_1 \cap I_2)$, respectively. ▷ the dense GNS algorithm is implemented in e.g.* `NCSOStools` *[12]*
2: *Find the corresponding unit vectors $\mathbf{v}^1 \in \mathcal{H}(I_1)$, $\mathbf{v}^2 \in \mathcal{H}(I_2)$ and $\mathbf{v}^{12} \in \mathcal{H}(I_1 \cap I_2)$ so that (4.2) holds.*
3: **if** *The flatness (H1) and irreducibility (H2) conditions from Theorem 4.2 do not hold* **then**
4:     *Stop*
5: **end if**
6: **for** $k \in \{1, 2\}$, $i \in I_1 \cap I_2$ **do**
7:     *Compute $(\chi_{i,\ell}^k)_\ell$ such that the block diagonalization $\hat{A}_i^k = \text{diag}(\chi_{i,\ell}^k)_\ell$ holds. ▷ e.g., by [40, Algorithm 4.1]*
8:     *Compute invertible matrices $(P_\ell)_{\ell>1}$ such that $P_\ell^{-1} \chi_{i,\ell}^k P_\ell = \chi_{i,1}^k$*
9:     *Normalize each P_ℓ to make it orthogonal. Use them to change the basis in the blocks $(\chi_{i,\ell}^k)_{\ell>1}$*   ▷ *Thus, one has $\hat{A}_i^k = \mathbf{I} \otimes \chi_i^k$*
10:     *Compute an orthogonal P such that $P^{-1} \chi_i^k P = \hat{A}_i^{12}$ ▷ Hence, without loss of generality, $\hat{A}_i^{12} = \chi_i^k$*
11:     *Decompose $\mathbf{v}^k = \sum_j \lambda_{jk} e_j^k \otimes \mathbf{v}^{12}$*
12: **end for**
13: *Find an orthogonal matrix U sending $e_1^1 \otimes \sum_j \lambda_{j1} e_j^1 \mapsto e_1^2 \otimes \sum_j \lambda_{j2} e_j^2$*

14: **for** $i \in \{1, \ldots, n\}$ **do**

15: $\quad A_i := \begin{cases} \mathbf{I}_{r_2} \otimes \hat{A}_i^1 & \text{if } i \in I_1, \\ (U^T \otimes \mathbf{I}_{r_{12}})(\mathbf{I}_{r_1} \otimes \hat{A}_i^2)(U \otimes \mathbf{I}_{r_{12}}) & \text{if } i \in I_2 \end{cases}$

16: **end for**

17: *Compute* $\mathbf{v} = e_1^1 \otimes \mathbf{v}^1$

Output: *$\underline{A} = (A_1, \ldots, A_n)$ and $\mathbf{v}$.*

Corollary 4.6 *The procedure* `SparseGNS` *described in Algorithm 4.6 is sound and returns the tuple $\underline{A}$ and the vector $\mathbf{v}$ from Theorem 4.2.*

Proof Correctness of the algorithm has been essentially established in the proof of Theorem 4.2. Both computation in Line 8 and Line 10 can be performed since the only homomorphisms out of full matrix algebras are ampliations composed with a conjugation (by the Skolem-Noether theorem). One can perform an orthogonal change of basis in Line 9, and $\hat{A}_i^k = \mathbf{I} \otimes \chi_i^k$, for all $k \in \{1, 2\}$ and $i \in I_1 \cap I_2$. Indeed, let us assume that a matrix P is invertible, and the map $\phi : A \mapsto P^{-1}AP$ from $\mathbb{M}_n(\mathbb{R})$ to $\mathbb{M}_n(\mathbb{R})$ preserves transposes. Then, the following equalities

$$\phi(A^T) = P^{-1}A^T P = (P^{-1}AP)^T = P^T A^T P^{-T}$$

imply that PP^T commutes with all $n \times n$ matrices. Therefore, PP^T is a scalar matrix, and P is a scalar multiple of an orthogonal matrix, the desired result. Eventually, each component of the tuple A, given in Line 15, is well defined by construction and gives rise to the desired amalgamation. Line 17 constructs the vector $\mathbf{v}$ needed for (4.2) to hold. $\qquad\square$

5 Eigenvalue optimization of noncommutative sparse polynomials

The aim of this section is to provide SDP relaxations allowing one to under-approximate the smallest eigenvalue that a given nc polynomial can attain on a tuple of symmetric matrices from a given semialgebraic set. The unconstrained case is handled in Sect. 5.1, where we show how to compute a lower bound on the smallest eigenvalue via solving an SDP. The constrained case is handled in Sect. 5.2, where we derive a hierarchy of lower bounds converging to the minimal eigenvalue, assuming that the quadratic module is archimedean and that RIP holds (Assumption 2.4).

We first recall the celebrated Helton-McCullough Sums of Squares theorem [19, 38] stating the equivalence between sums of hermitian squares (SOHS) and positive semidefinite nc polynomials.

Theorem 5.1 *Given $f \in \mathbb{R}\langle \underline{X} \rangle$, we have $f(\underline{A}) \succeq 0$, for all $\underline{A} \in \mathbb{S}^n$, if and only if $f \in \Sigma\langle \underline{X} \rangle$.*

In contrast with the constrained case where we obtain the analog of Putinar's Positivstellensatz in Theorem 3.3, there is no sparse analog of Theorem 5.1, as shown in the following example.

Lemma 5.2 *There exist polynomials which are sparse sums of hermitian squares but are not sums of sparse hermitian squares.*

Proof Let $v = \begin{bmatrix} X_1 & X_1 X_2 & X_2 & X_3 & X_3 X_2 \end{bmatrix}$,

$$G = \begin{bmatrix} 1 & -1 & -1 & 0 & \alpha \\ -1 & 2 & 0 & -\alpha & 0 \\ -1 & 0 & 3 & -1 & 9 \\ 0 & -\alpha & -1 & 6 & -27 \\ \alpha & 0 & 9 & -27 & 142 \end{bmatrix}, \quad \alpha \in \mathbb{R}, \tag{5.1}$$

and consider

$$\begin{aligned} f &= vGv^\star \\ &= X_1^2 - X_1 X_2 - X_2 X_1 + 3X_2^2 - 2X_1 X_2 X_1 + 2X_1 X_2^2 X_1 \\ &\quad - X_2 X_3 - X_3 X_2 + 6X_3^2 + 9X_2^2 X_3 + 9X_3 X_2^2 - 54X_3 X_2 X_3 + 142X_3 X_2^2 X_3. \end{aligned} \tag{5.2}$$

The polynomial f is clearly sparse w.r.t. $I_1 = \{x_1, x_2\}$ and $I_2 = \{x_2, x_3\}$. Note that the matrix G is positive semidefinite if and only if $0.270615 \lesssim \alpha \lesssim 1.1075$, whence f is a sparse polynomial that is an SOHS.

We claim that $f \notin \Sigma\langle \underline{X}(I_1)\rangle + \Sigma\langle \underline{X}(I_2)\rangle$, i.e., f is not a sum of sparse hermitian squares. By the Newton chip method [5, Section 2.3] only monomials in v can appear in a sum of squares decomposition of f. Further, every Gram matrix of f (with border vector v) is of the form (5.1). However, the matrix G with $\alpha = 0$ is not positive semidefinite, hence $f \notin \Sigma\langle \underline{X}(I_1)\rangle + \Sigma\langle \underline{X}(I_2)\rangle$. □

5.1 Unconstrained eigenvalue optimization with sparsity

Let $\mathbf{I}$ stands for the identity matrix. Given $f \in \mathrm{Sym}\,\mathbb{R}\langle \underline{X}\rangle$ of degree $2d$, the smallest eigenvalue of f is obtained by solving the following optimization problem

$$\lambda_{\min}(f) := \inf\{\langle f(\underline{A})\mathbf{v}, \mathbf{v}\rangle : \underline{A} \in \mathbb{S}^n, \|\mathbf{v}\| = 1\}. \tag{5.3}$$

The optimal value $\lambda_{\min}(f)$ of Problem (5.3) is the greatest lower bound on the eigenvalues of $f(\underline{A})$ over all n-tuples $\underline{A}$ of real symmetric matrices. Problem (5.3) can be rewritten as follows:

$$\begin{aligned} \lambda_{\min}(f) = \sup_{\lambda} \quad & \lambda \\ \text{s.t.} \quad & f(\underline{A}) - \lambda\mathbf{I} \succeq 0, \quad \forall \underline{A} \in \mathbb{S}^n, \end{aligned} \tag{5.4}$$

which is in turn equivalent to

$$\begin{aligned} \lambda_{\min,d}(f) = \sup_{\lambda} \quad & \lambda \\ \text{s.t.} \quad & f(\underline{X}) - \lambda \in \Sigma\langle \underline{X}\rangle_d, \end{aligned} \tag{5.5}$$

as a consequence of Theorem 5.1.

The dual of SDP (5.5) is

$$
\begin{aligned}
L_{\text{sohs},d}(f) = \inf_{L} \quad & \langle \mathbf{M}_d(L), G_f \rangle \\
\text{s.t.} \quad & L(1) = 1, \quad \mathbf{M}_d(L) \succeq 0, \\
& L : \mathbb{R}\langle \underline{X} \rangle_{2d} \to \mathbb{R} \text{ linear,}
\end{aligned}
\tag{5.6}
$$

where G_f is a Gram matrix for f (see Proposition 2.1).

One can compute $\lambda_{\min}(f)$ by solving a single SDP, either SDP (5.6) or SDP (5.5), since there is no duality gap between these two programs (see, e.g., [5, Theorem 4.1]), that is, one has $L_{\text{sohs},d}(f) = \lambda_{\min,d}(f) = \lambda_{\min}(f)$.

Now, we address eigenvalue optimization for a given sparse nc polynomial $f = f_1 + \cdots + f_p$ of degree $2d$, with $f_k \in \text{Sym}\,\mathbb{R}\langle \underline{X}(I_k) \rangle_{2d}$, for all $k = 1, \ldots, p$. For all $k = 1, \ldots, p$, let G_{f_k} be a Gram matrix associated to f_k. The sparse variant of SDP (5.6) is

$$
\begin{aligned}
L_{\text{sohs},d}^{\text{sparse}}(f) = \inf_{L} \quad & \sum_{k=1}^{p} \langle \mathbf{M}_d(L, I_k), G_{f_k} \rangle \\
\text{s.t.} \quad & L(1) = 1, \quad \mathbf{M}_d(L, I_k) \succeq 0, \quad k = 1, \ldots, p, \\
& L : \mathbb{R}\langle \underline{X}(I_1) \rangle_{2d} + \cdots + \mathbb{R}\langle \underline{X}(I_p) \rangle_{2d} \to \mathbb{R} \text{ linear,}
\end{aligned}
\tag{5.7}
$$

whose dual is the sparse variant of SDP (5.5):

$$
\begin{aligned}
\lambda_{\min,d}^{\text{sparse}}(f) = \sup_{\lambda} \quad & \lambda \\
\text{s.t.} \quad & f - \lambda \in \Sigma\langle \underline{X}(I_1) \rangle_{2d} + \cdots + \Sigma\langle \underline{X}(I_p) \rangle_{2d},
\end{aligned}
\tag{5.8}
$$

To prove that there is no duality gap between SDP (5.7) and SDP (5.8), we need a sparse variant of [43, Proposition 3.4], which says that $\Sigma\langle \underline{X} \rangle_d$ is closed in $\mathbb{R}\langle \underline{X} \rangle_{2d}$:

Proposition 5.3 *The set* $\Sigma\langle \underline{X} \rangle_d^{\text{sparse}}$ *is a closed convex subset of* $\mathbb{R}\langle \underline{X}(I_1) \rangle_{2d} + \cdots + \mathbb{R}\langle \underline{X}(I_p) \rangle_{2d}$.

Proof For each $k \in \{1, \ldots, p\}$, we endow each $\mathbb{R}\langle \underline{X}(I_k) \rangle_{2d}$ with a norm $\| \cdot \|_k$. For each $f \in \mathbb{R}\langle \underline{X}(I_1) \rangle_{2d} + \cdots + \mathbb{R}\langle \underline{X}(I_p) \rangle_{2d}$, we set

$$
\|f\| := \inf \left\{ \|f_1\|_1 + \cdots + \|f_p\|_2 : f = f_1 + \cdots + f_p, \, f_k \in \mathbb{R}\langle \underline{X}(I_k) \rangle_{2d} \right\}
$$

Let us consider an element $h = h_1 + \cdots + h_p \in \Sigma\langle \underline{X} \rangle_d^{\text{sparse}}$, with $h_k \in \Sigma\langle \underline{X}(I_k) \rangle_d$. For each $k \in \{1, \ldots, p\}$, h_k can be written as a sum of at most $\sigma(n_k, d)$ hermitian squares of degree at most $2d$ by Proposition 2.1. Define the mapping

$$
\phi_k : (\mathbb{R}\langle \underline{X}(I_k) \rangle_d)^{\sigma(n_k, d)} \to \mathbb{R}\langle \underline{X}(I_k) \rangle_{2d}
$$

$$(h_{kj})_{j=1}^{\sigma(n_k,d)} \mapsto \sum_{j=1}^{\sigma(n_k,d)} h_{kj}^\star h_{kj},$$

and let us denote $\mathbf{h}_k = (h_{kj})_{j=1}^{\sigma(n_k,d)}$. Then, the image of the map ϕ, defined by

$$\phi : \prod_{k=1}^{p} (\mathbb{R}\langle \underline{X}(I_k)\rangle_d)^{\sigma(n_k,d)} \to \mathbb{R}\langle \underline{X}(I_1)\rangle_{2d} + \cdots + \mathbb{R}\langle \underline{X}(I_p)\rangle_{2d}$$

$$(\mathbf{h}_1, \ldots, \mathbf{h}_p) \mapsto \phi_1(\mathbf{h}_1) + \cdots + \phi_p(\mathbf{h}_p),$$

is equal to $\Sigma\langle \underline{X}\rangle_d^{\text{sparse}}$.

Let us define the subset $\mathcal{V} \subset \prod_{k=1}^{p}(\mathbb{R}\langle \underline{X}(I_k)\rangle_d)^{\sigma(n_k,d)}$ by

$$\mathcal{V} := \left\{ \mathbf{h} = \mathbf{h}_1 + \cdots + \mathbf{h}_p : \sum_{k=1}^{p} \left\| \sum_{j=1}^{\sigma(n_k,d)} h_{kj}^\star h_{kj} \right\|_k = 1 \right\}.$$

Since $\mathcal{V}$ is compact, then $\phi(\mathcal{V})$ is also compact. Note that $0 \notin \mathcal{V}$, implying that $0 \notin \phi(\mathcal{V})$. Next, let us consider a sequence $(f^\ell)_{\ell \geq 1}$ of elements of $\Sigma\langle \underline{X}\rangle_d^{\text{sparse}}$, converging to $f \in \mathbb{R}\langle \underline{X}(I_1)\rangle_{2d} + \cdots + \mathbb{R}\langle \underline{X}(I_p)\rangle_{2d}$. One can write $f^\ell = \lambda^\ell v^\ell$ for $\lambda^\ell \in \mathbb{R}^{\geq 0}$ and $v^\ell \in \phi(\mathcal{V})$. By compactness of $\phi(\mathcal{V})$, there exists a subsequence of $(v^\ell)_\ell$, (also denoted $(v^\ell)_\ell$), which converges to $v \in \phi(\mathcal{V}) \subset \Sigma\langle \underline{X}\rangle_d^{\text{sparse}}$. By definition of $\|\cdot\|$ and $\mathcal{V}$, one has $\|v^\ell\| \leq 1$, for all $\ell \geq 1$. Since $0 \notin \phi(\mathcal{V})$ and $\phi(\mathcal{V})$ is compact, there exists an $\varepsilon > 0$ such that $\|v^\ell\| > \varepsilon$, for all $\ell \geq 1$. Therefore, $\lambda^\ell = \frac{\|f^\ell\|}{\|v^\ell\|}$ converges to $\frac{\|f\|}{\|v\|}$, as ℓ goes to infinity. From this, we deduce that f^ℓ converges to $f = \frac{\|f\|}{\|v\|}v \in \Sigma\langle \underline{X}\rangle_d^{\text{sparse}}$, yielding the desired result. $\qquad\square$

From Proposition 5.3, we obtain the following theorem which does not require Assumption 2.4.

Theorem 5.4 *Let* $f \in \text{Sym}\,\mathbb{R}\langle \underline{X}\rangle$ *of degree* $2d$, *with* $f = f_1 + \cdots + f_p$, $f_k \in \text{Sym}\,\mathbb{R}\langle \underline{X}(I_k)\rangle_{2d}$, *for all* $k = 1, \ldots, p$. *Then, one has* $\lambda_{\min,d}^{\text{sparse}}(f) = L_{\text{sohs},d}^{\text{sparse}}(f)$, *i.e., there is no duality gap between SDP* (5.7) *and SDP* (5.8).

Proof The strong duality is obtained exactly as for the dense case [5, Theorem 4.1], and relies on the closedness of $\Sigma\langle \underline{X}\rangle_d^{\text{sparse}}$, stated in Proposition 5.3. $\qquad\square$

Remark 5.5 By contrast with the dense case, it is not enough to compute the solution of SDP (5.7) to obtain the optimal value $\lambda_{\min}(f)$ of the unconstrained optimization problem (5.3). However, one can still compute a certified lower bound $\lambda_{\min}^{\text{sparse},d}(f)$ by solving a single SDP, either in the primal form (5.7) or in the dual form (5.8). Note that the related computational cost is potentially much less expensive. Indeed, SDP (5.8) involves $\sum_{k=1}^{p} \sigma(n_k, 2d)$ equality constraints and $\sum_{k=1}^{p} \sigma(n_k, d) + 1$ variables. This is in contrast with the dense version (5.5), which involves $\sigma(n, 2d)$ equality constraints and $1 + \sigma(n, d)$ variables.

5.2 Constrained eigenvalue optimization with sparsity

Here, we focus on providing lower bounds for the constrained eigenvalue optimization of nc polynomials. Given $f \in \mathrm{Sym}\,\mathbb{R}\langle \underline{X} \rangle$ and $S := \{g_1, \ldots, g_m\} \subset \mathrm{Sym}\,\mathbb{R}\langle \underline{X} \rangle$ as in (2.2), let us define $\lambda_{\min}(f, S)$ as follows:

$$\lambda_{\min}(f, S) := \inf\{\langle f(\underline{A})\mathbf{v}, \mathbf{v} \rangle : \underline{A} \in D_S^\infty, \|\mathbf{v}\| = 1\}, \tag{5.9}$$

which is, as for the unconstrained case, equivalent to

$$\begin{aligned} \lambda_{\min}(f, S) = \sup_\lambda \quad &\lambda \\ \text{s.t.} \quad &f(\underline{A}) - \lambda \mathbf{I} \succeq 0, \quad \forall \underline{A} \in D_S^\infty. \end{aligned} \tag{5.10}$$

Let $d_j := \lceil \deg g_j / 2 \rceil$, for each $j = 1, \ldots, m$ and $d := \max\{\lceil \deg f / 2 \rceil, d_1, \ldots, d_m\}$. As shown in [13,49] (see also [23]), one can approximate $\lambda_{\min}(f, S)$ from below via the following hierarchy of SDP programs, indexed by $s \geq d$:

$$\begin{aligned} \lambda_s(f, S) := \sup_\lambda \quad &\lambda \\ \text{s.t.} \quad &f - \lambda \in \mathcal{M}(S)_s. \end{aligned} \tag{5.11}$$

The dual of SDP (5.11) is

$$\begin{aligned} L_s(f, S) := \inf_L \quad &\langle \mathbf{M}_s(L), G_f \rangle \\ \text{s.t.} \quad &L(1) = 1, \\ &\mathbf{M}_s(L) \succeq 0, \quad \mathbf{M}_{s-d_j}(g_j L) \succeq 0, \quad j = 1, \ldots, m, \\ &L : \mathbb{R}\langle \underline{X} \rangle_{2d} \to \mathbb{R} \text{ linear}, \end{aligned} \tag{5.12}$$

Under additional assumptions, this hierarchy of primal-dual SDP (5.11)–(5.12) converges to the value of the constrained eigenvalue problem.

Corollary 5.6 *Assume that D_S is as in (2.6) with the additional quadratic constraints (2.5) and that the quadratic module M_S is archimedean. Then the following holds for each $f \in \mathrm{Sym}\,\mathbb{R}\langle \underline{X} \rangle$:*

$$\lim_{s \to \infty} L_s(f, S) = \lim_{s \to \infty} \lambda_s(f, S) = \lambda_{\min}(f, S). \tag{5.13}$$

The main ingredient of the proof (see, e.g., [5, Corollary 4.11]) is the nc analog of Putinar's Positivstellensatz, stated in Theorem 2.2.

Let $S \cup \{f\} \subseteq \mathrm{Sym}\,\mathbb{R}\langle \underline{X} \rangle$ and let D_S be as in (2.6) with the additional quadratic constraints (2.5). Let $\mathcal{M}(S)^{\mathrm{sparse}}$ be as in (3.1) and let us define $\mathcal{M}(S)_s^{\mathrm{sparse}}$ in the same way as the truncated quadratic module $\mathcal{M}(S)_s$ in (2.4). Now, let us state the sparse variant of the primal-dual hierarchy (5.11)–(5.12) of lower bounds for $\lambda_{\min}(f, S)$.

For all $s \geq d$, the sparse variant of SDP (5.12) is

$$L_s^{\text{sparse}}(f, S) := \inf_L \ \sum_{k=1}^p \langle \mathbf{M}_s(L, I_k), G_{f_k} \rangle$$

$$\text{s.t.} \quad L(1) = 1,$$
$$\mathbf{M}_s(L, I_k) \succeq 0, \quad k = 1, \ldots, p,$$
$$\mathbf{M}_{s-d_j}(g_j L, I_k) \succeq 0, \quad j = 1, \ldots, m, \quad k = 1, \ldots, p,$$
$$L : \mathbb{R}\langle \underline{X}(I_1) \rangle_{2d} + \cdots + \mathbb{R}\langle \underline{X}(I_p) \rangle_{2d} \to \mathbb{R} \ \text{linear},$$

$$(5.14)$$

whose dual is the sparse variant of SDP (5.11):

$$\lambda_s^{\text{sparse}}(f, S) := \sup_\lambda \ \lambda$$

$$(5.15)$$

$$\text{s.t.} \quad f - \lambda \in \mathcal{M}(S)_s.$$

Recall that an ε-neighborhood of 0 is the set $\mathcal{N}_\varepsilon$ defined for a given $\varepsilon > 0$ by:

$$\mathcal{N}_\varepsilon := \bigcup_{k \in \mathbb{N}} \left\{ \underline{A} := (A_1, \ldots, A_n) \in \mathbb{S}_k^n : \varepsilon^2 - \sum_{i=1}^n A_i^2 \succeq 0 \right\}.$$

Lemma 5.7 *If $h \in \mathbb{R}\langle \underline{X} \rangle$ vanishes on an ε-neighborhood of 0, then $h = 0$.*

Proof See [5, Lemma 1.35]. □

Proposition 5.8 *Let $S \cup \{f\} \subseteq \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle$, assume that D_S contains an ε-neighborhood of 0 and that D_S is as in (2.6) with the additional quadratic constraints (2.5). Then SDP (5.14) admits strictly feasible solutions.*

Proof This proof being almost the same as the one of [5, Proposition 4.9] is presented for the sake of completeness. By Lemma 2.5, it is enough to build a linear map $L : \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle_{2s} \to \mathbb{R}$ such that for all $k = 1, \ldots, p$ one has:

- $L(h^\star h) > 0$, for all nonzero $h \in \mathbb{R}\langle \underline{X}(I_k) \rangle_s$;
- for all $j \in J_k$, one has $L(h^\star g_j h) > 0$, for all nonzero $h \in \mathbb{R}\langle \underline{X}(I_k) \rangle_{s - \lceil \deg g_j / 2 \rceil}$.

Let us pick $N > s$ and let $\mathcal{U}$ stands for the set of all $N \times N$ matrices from D_S with rational entries:

$$\mathcal{U} := \{ \underline{A}^{(r)} := (A_1^{(r)}, \ldots, A_n^{(r)}) : r \in \mathbb{N}, \underline{A}^{(r)} \in D_S^N \}$$

Note that this set $\mathcal{U}$ contains a dense subset of $\mathcal{N}_\varepsilon$. Let us associate to $\underline{A} \in \mathcal{U}$ the linear map $L_{\underline{A}} : \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle_{2d} \to \mathbb{R}$ defined by $L_{\underline{A}}(h) := \operatorname{tr}(h(\underline{A}))$. From this, we define L as follows:

$$L := \sum_{r=1}^\infty 2^{-r} \frac{L_{\underline{A}^{(r)}}}{\|L_{\underline{A}^{(r)}}\|}.$$

Now let us fix $k \in \{1, \ldots, p\}$. Obviously, one has $L(h^\star h) \geq 0$, for all nonzero $h \in \mathbb{R}\langle \underline{X}(I_k)\rangle_d$. Let us suppose that $L(h^\star h) = 0$ for some $h \in \mathbb{R}\langle \underline{X}(I_k)\rangle_d$. Then, one has $L_{\underline{A}^{(r)}}(h^\star h) = 0 = \operatorname{tr}(h^\star(\underline{A}^{(r)})h(\underline{A}^{(r)}))$, for all $r \in \mathbb{N}$. This implies that for all $r \in \mathbb{N}$, one has $h^\star(\underline{A}^{(r)})h(\underline{A}^{(r)}) = 0$, which in turn yields $h(\underline{A}^{(r)}) = 0$. Since $\mathcal{U}$ contains a dense subset of $\mathcal{N}_\varepsilon$, this implies that h vanishes on a ε-neighborhood of 0. As a consequence of Lemma 5.7, one has $h = 0$.

In a similar way, we prove that if $L(h^\star g_j h) = 0$ for some $h \in \mathbb{R}\langle \underline{X}(I_k)\rangle_{s-\lceil \deg g_j/2 \rceil}$, then one necessarily has $h = 0$. $\qquad\square$

Corollary 5.9 *Let $S \cup \{f\} \subseteq \operatorname{Sym}\mathbb{R}\langle \underline{X}\rangle$, assume that D_S is as in (2.6) with the additional quadratic constraints (2.5). Let Assumption 2.4 hold. Then, one has*

$$\lim_{s\to\infty} L_s^{\mathrm{sparse}}(f, S) = \lim_{s\to\infty} \lambda_s^{\mathrm{sparse}}(f, S) = \lambda_{\min}(f, S). \tag{5.16}$$

Proof The proof is similar to the one in the dense case. Let us take $\lambda := \lambda_{\min}(f, S) - \varepsilon$, where $\varepsilon > 0$. Then, one has $f - \lambda \succ 0$ on D_S^∞, so Theorem 3.3 implies that $f - \lambda \in \mathcal{M}(S)^{\mathrm{sparse}}$. Hence, there exists s such that $f - \lambda \in \mathcal{M}(S)_s^{\mathrm{sparse}}$, yielding a feasible solution for SDP (5.15), so $\lambda_{\min}(f, S) - \varepsilon \leq \lambda_s^{\mathrm{sparse}}(f, S)$. By weak duality between SDP (5.14) and SDP (5.15), one has $\lambda_s^{\mathrm{sparse}}(f, S) \leq L_s^{\mathrm{sparse}}(f, S)$. Therefore, one obtains $\lambda_{\min}(f, S) - \varepsilon \leq \lambda_s^{\mathrm{sparse}}(f, S) \leq L_s^{\mathrm{sparse}}(f, S) \leq \lambda_{\min}(f, S)$, yielding the desired result, after taking limits as $\varepsilon \to 0$. $\qquad\square$

As for the unconstrained case, there is no sparse variant of the "perfect" Positivstellensatz stated in [5, §4.4] or [20], for constrained eigenvalue optimization over convex nc semialgebraic sets, such as those associated either to the sparse nc ball $\mathbb{B}^{\mathrm{sparse}} := \{1 - \sum_{i\in I_1} X_i^2, \ldots, 1 - \sum_{i\in I_p} X_i^2\}$ or the nc polydisc $\mathbb{D} := \{1 - X_1^2, \ldots, 1 - X_n^2\}$. Namely, for an nc polynomial f of degree $2d+1$, computing only SDP (5.7) with optimal value $\lambda_{\min,d+1}^{\mathrm{sparse}}(f, S)$ when $S = \mathbb{B}^{\mathrm{sparse}}$ or $S = \mathbb{D}^{\mathrm{sparse}}$ does not suffice to obtain the value of $\lambda_{\min}(f, S)$. This is explained in Example 5.10 below, which implies that there is no sparse variant of [5, Corollary 4.18] when $S = \mathbb{B}^{\mathrm{sparse}}$.

Example 5.10 Let us consider a randomly generated cubic polynomial $f = f_1 + f_2$ with

$$\begin{aligned}
f_1 =& 4 - X_1 + 3X_2 - 3X_3 - 3X_1^2 - 7X_1X_2 + 6X_1X_3 - X_2X_1 - 5X_3X_1 + 5X_3X_2 \\
& - 5X_1^3 - 3X_1^2X_3 + 4X_1X_2X_1 - 6X_1X_2X_3 + 7X_1X_3X_1 + 2X_1X_3X_2 - X_1X_3^2 \\
& - X_2X_1^2 + 3X_2X_1X_2 - X_2X_1X_3 - 2X_2^3 - 5X_2^2X_3 - 4X_2X_3^2 - 5X_3X_1^2 \\
& + 7X_3X_1X_2 + 6X_3X_2X_1 - 4X_3X_2X_2 - X_3^2X_1 - 2X_3^2X_2 + 7X_3^3,
\end{aligned}$$

$$\begin{aligned}
f_2 =& -1 + 6X_2 + 5X_3 + 3X_4 - 5X_2^2 + 2X_2X_3 + 4X_2X_4 - 4X_3X_2 + X_3^2 - X_3X_4 \\
& + X_4X_2 - X_4X_3 + 2X_4^2 - 7X_2^3 + 4X_2X_3^2 + 5X_2X_3X_4 - 7X_2X_4X_3 - 7X_2X_4^2 \\
& + X_3X_2^2 + 6X_3X_2X_3 - 6X_3X_2X_4 - 3X_3^2X_2 - 7X_3^2X_4 + 6X_3X_4X_2 \\
& - 3X_3X_4X_3 - 7X_3X_4^2 + 3X_4X_2^2 - 7X_4X_2X_3 - X_4X_2X_4 - 5X_4X_3^2 \\
& + 7X_4X_3X_4 + 6X_4^2X_2 - 4X_4^3,
\end{aligned}$$

and the nc polyball $S = \mathbb{B}^{\text{sparse}} = \{1 - X_1^2 - X_2^2 - X_3^2, 1 - X_2^2 - X_3^2 - X_4^2\}$ corresponding to $I_1 = \{1, 2, 3\}$ and $I_2 = \{2, 3, 4\}$. Then, one has $\lambda_2^{\text{sparse}}(f, S) \simeq -27.536 < \lambda_3^{\text{sparse}}(f, S) \simeq -27.467 \simeq \lambda_{\min,2}(f, S) = \lambda_{\min}(f, S)$.

5.3 Extracting optimizers

Here, we explain how to extract a pair of optimizers $(\underline{A}, \mathbf{v})$ for the eigenvalue optimization problems when the flatness and irreducibility conditions of Theorem 4.2 hold. We apply the `SparseGNS` procedure described in Algorithm 4.6 on the optimal solution of SDP (5.7) in the unconstrained case or SDP (5.14) in the constrained case. In the unconstrained case, we have the following sparse variant of [5, Proposition 4.4].

Proposition 5.11 *Given f as in Theorem 5.4, let us assume that SDP (5.7) yields an optimal solution $\mathbf{M}_{d+1}(L)$ associated to $L_{\text{sohs},d+1}^{\text{sparse}}(f)$. If the linear functional L underlying $\mathbf{M}_{d+1}(L)$ satisfies the flatness (H1) and irreducibility (H2) conditions stated in Theorem 4.2, then one has*

$$\lambda_{\min}(f) = L_{\text{sohs},d+1}^{\text{sparse}}(f) = \sum_{k=1}^{p} \langle \mathbf{M}_{d+1}(L, I_k), G_{f_k} \rangle.$$

Proof The first equality comes from Theorem 5.4. Let us assume that each moment matrix satisfies the assumptions of Theorem 4.2. Then, we obtain a tuple $\underline{A}$ of symmetric matrices and a unit vector $\mathbf{v}$ such that $L(f) = \langle f(\underline{A})\mathbf{v} \mid \mathbf{v} \rangle$. Since one has $L(f) = \sum_{k=1}^{p} \langle \mathbf{M}_{d+1}(L, I_k), G_{f_k} \rangle = \lambda_{\min}(f)$, the desired result holds. $\square$

We can extract optimizers for the unconstrained minimal eigenvalue problem (5.3) thanks to the following algorithm.

Algorithm 5.12 `SparseEigGNS`

Require: $f \in \text{Sym} \, \mathbb{R}\langle \underline{X} \rangle_{2d}$ satisfying Assumption 2.4.
1: *Compute $L_{\text{sohs},d+1}^{\text{sparse}}(f)$ by solving SDP (5.7)*
2: **if** *SDP (5.7) is unbounded or its optimum is not attained* **then**
3:     *Stop*
4: **end if**
5: *Let $\mathbf{M}_{d+1}(L)$ be an optimizer of SDP (5.7). Compute $\underline{A}$, $\mathbf{v} := $ `SparseGNS` $(\mathbf{M}_{d+1}(L))$.*
Output: $\underline{A}$ *and* $\mathbf{v}$.

In the constrained case, the next result is the sparse variant of [5, Theorem 4.12] and is a direct corollary of Theorem 4.2.

Corollary 5.12 *Let $S \cup \{f\} \subseteq \text{Sym} \, \mathbb{R}\langle \underline{X} \rangle$, assume that D_S is as in (2.6) with the additional quadratic constraints (2.5). Suppose Assumptions 2.4(i)–(ii) hold. Let $\mathbf{M}_s(L)$ be an optimal solution of SDP (5.14) with value $L_s(f, S)$, for $s \geq d + \delta$, such that L satisfies the assumptions of Theorem 4.2. Then, there exist $r \in \mathbb{N}$, $\underline{A} \in D_S^r$ and a unit vector $\mathbf{v}$ such that*

$$\lambda_{\min}(f, S) = \langle f(\underline{A})\mathbf{v}, \mathbf{v} \rangle = L_s(f, S).$$

Remark 5.13 As in the dense case [5, Algorithm 4.2], one can provide a randomized algorithm to look for flat optimal solutions for the constrained eigenvalue problem (5.9). The underlying reason which motivates this randomized approach is work by Nie, who derives in [46] a hierarchy of SDP programs, with a random objective function, that converges to a flat solution (under mild assumptions).

Example 5.14 Consider the sparse polynomial $f = f_1 + f_2$ from Example 5.10. The Hankel matrix $\mathbf{M}_3(L)$ obtained when computing $\lambda_3^{\mathrm{sparse}}$ by solving (5.14) for $s = 3$ satisfies the flatness (H1) and irreducibility (H2) conditions of Theorem 4.2. We can thus apply the SparseGNS algorithm yielding

$$
A_1 = \begin{bmatrix}
0.0059 & 0.0481 & 0.1638 & 0.4570 \\
0.0481 & -0.2583 & 0.5629 & -0.2624 \\
0.1638 & 0.5629 & 0.3265 & -0.3734 \\
0.4570 & -0.2624 & -0.3734 & -0.2337
\end{bmatrix}
$$

$$
A_2 = \begin{bmatrix}
-0.3502 & 0.0080 & 0.1411 & 0.0865 \\
0.0080 & -0.4053 & 0.2404 & -0.1649 \\
0.1411 & 0.2404 & -0.0959 & 0.3652 \\
0.0865 & -0.1649 & 0.3652 & 0.4117
\end{bmatrix}
$$

$$
A_3 = \begin{bmatrix}
-0.7669 & -0.0074 & -0.1313 & -0.0805 \\
-0.0074 & -0.4715 & -0.2238 & 0.1535 \\
-0.1313 & -0.2238 & 0.0848 & -0.3400 \\
-0.0805 & 0.1535 & -0.3400 & -0.2126
\end{bmatrix}
$$

$$
A_4 = \begin{bmatrix}
0.3302 & -0.1839 & 0.1811 & -0.0404 \\
-0.1839 & -0.1069 & 0.5114 & -0.0570 \\
0.1811 & 0.5114 & 0.1311 & -0.3664 \\
-0.0404 & -0.0570 & -0.3664 & 0.4440
\end{bmatrix}
$$

where

$$
f(\underline{A}) = \begin{bmatrix}
-10.3144 & 3.9233 & -5.0836 & -7.7828 \\
3.9233 & 1.8363 & 4.5078 & -7.5905 \\
-5.0836 & 4.5078 & -19.5827 & 13.9157 \\
-7.7828 & -7.5905 & 13.9157 & 8.3381
\end{bmatrix}
$$

has minimal eigenvalue -27.4665 with unit eigenvector

$$
\mathbf{v} = \begin{bmatrix} 0.1546 & -0.2507 & 0.8840 & -0.3631 \end{bmatrix}^T .
$$

In this case all the ranks involved were equal to four. So A_2 and A_3 were computed already from $\mathbf{M}_3(L, I_1 \cap I_2)$, after an appropriate basis change A_1 (and the same A_2, A_3) was obtained from $\mathbf{M}_3(L, I_1)$, and finally A_4 was computed from $\mathbf{M}_3(L, I_2)$.

6 Trace optimization of noncommutative sparse polynomials

The aim of this section is to provide SDP relaxations allowing one to under-approximate the smallest trace of an nc polynomial on a semialgebraic set. In Sect. 6.1, we provide a sparse tracial representation for tracial linear functionals. In Sect. 6.2, we address the unconstrained trace minimization problem. As in Sect. 5.1, we compute a lower bound on the smallest trace via SDP. The constrained case is handled in Sect. 6.3, where we derive a hierarchy of lower bounds converging to the minimal trace, assuming that the quadratic module is archimedean and that RIP holds (Assumption 2.4). Most proofs are similar to the ones of eigenvalue problems addressed in Sect. 5, so our treatment here is more concise.

We start this section by introducing useful notations about commutators and trace zero polynomials. Given $g, h \in \mathbb{R}\langle \underline{X} \rangle$, the nc polynomial $[g, h] := gh - hg$ is called a *commutator*. Two nc polynomials $g, h \in \mathbb{R}\langle \underline{X} \rangle$ are called *cyclically equivalent* ($g \overset{\text{cyc}}{\sim} h$) if $g - h$ is a sum of commutators. Given $S \subseteq \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle$ with corresponding quadratic module M_S and truncated variant $\mathcal{M}(S)_d$, one defines $\Theta_{S,d} := \{g \in \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle_{2d} : g \overset{\text{cyc}}{\sim} h \text{ for some } h \in \mathcal{M}(S)_d\}$ and $\Theta_S := \bigcup_{d \in \mathbb{N}} \Theta_{S,d}$. In this case, Θ_S stands for the *cyclic quadratic module* generated by S and $\Theta_{S,d}$ stands for the *truncated cyclic quadratic module* generated by S.

For $S \subseteq \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle$ and D_S as in (2.6) with the additional quadratic constraints (2.5), let us define $\Theta_{S,d}^k := \{g \in \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle_{2d} : g \overset{\text{cyc}}{\sim} h \text{ for some } h \in M_{S,d}^k\}$, $\Theta_S^k := \bigcup_{d \in \mathbb{N}} \Theta_{S,d}^k$, for all $k = 1, \dots, p$ and the sum

$$\Theta_{S,d}^{\text{sparse}} := \Theta_{S,d}^1 + \cdots + \Theta_{S,d}^p, \tag{6.1}$$

as well as $\Theta_S^{\text{sparse}} := \bigcup_{d \in \mathbb{N}} \Theta_{S,d}^{\text{sparse}}$. If S is empty, we drop the S in the above notations. An nc polynomial $g \in \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle$ is called a *trace zero* nc polynomial if $\operatorname{tr}(g(\underline{A})) = 0$, for all $\underline{A} \in \mathbb{S}^n$. This is equivalent to $g \overset{\text{cyc}}{\sim} 0$ (see e.g. [27, Proposition 2.3]). For a given nc polynomial g, the cyclic degree of g, denoted by $\operatorname{cdeg} g$, is the smallest degree of a polynomial cyclically equivalent to g.

6.1 Sparse tracial representations

The next theorem allows one to obtain a sparse tracial representation of a tracial linear functional, under the same flatness and irreducibility conditions stated in Theorem 4.2. This is a sparse variant of [5, Theorem 1.71].

Theorem 6.1 *Let $S \subseteq \operatorname{Sym} \mathbb{R}\langle \underline{X} \rangle_{2d}$, and assume that the semialgebraic set D_S is as in (2.6) with the additional quadratic constraints (2.5). Let Assumption 2.4(i) hold. Set $\delta := \max\{\lceil \deg(g)/2 \rceil : g \in S \cup \{1\}\}$. Let $L : \mathbb{R}\langle \underline{X} \rangle_{2d+2\delta} \to \mathbb{R}$ be a unital tracial linear functional satisfying $L(\Theta_{S,d}^{\text{sparse}}) \subseteq \mathbb{R}^{\geq 0}$. Assume that the flatness (H1) and irreducibility (H2) conditions of Theorem 4.2 hold. Then there are finitely many n-tuples $\underline{A}^{(j)}$ of symmetric matrices in D_S^r for some $r \in \mathbb{N}$, and positive scalars λ_j with $\sum_j \lambda_j = 1$, such that for all $f \in \mathbb{R}\langle \underline{X}(I_1) \rangle_{2d} + \cdots + \mathbb{R}\langle \underline{X}(I_p) \rangle_{2d}$, one has:*

$$L(f) = \sum_j \lambda_j \operatorname{tr} f(\underline{A}^{(j)}). \tag{6.2}$$

Proof As in Theorem 4.2, we perform the finite-dimensional GNS construction to obtain a tuple $\underline{A} \in D_S^r$, for some $r \in \mathbb{N}$, and unit vector $\mathbf{v}$ such that (6.2) holds. To obtain the tracial representation, the proof is essentially the same as the one of [5, Theorem 1.71] and relies on the Wedderburn theorem, see e.g. [28, Chapter 1] for more details. $\square$

6.2 Unconstrained trace optimization with sparsity

Given $f \in \operatorname{Sym} \mathbb{R}\langle\underline{X}\rangle$, the *trace-minimum* of f is obtained by solving the following optimization problem

$$\operatorname{tr}_{\min}(f) := \inf\{\operatorname{tr} f(\underline{A}) : \underline{A} \in \mathbb{S}^n\}, \tag{6.3}$$

which is equivalent to

$$\operatorname{tr}_{\min}(f) = \sup\{a : \operatorname{tr}(f - a)(\underline{A}) \geq 0, \forall \underline{A} \in \mathbb{S}^n\}, \tag{6.4}$$

If the cyclic degree of f is odd, then $\operatorname{tr}_{\min}(f) = -\infty$, thus let us assume that $2d = \operatorname{cdeg} f$. To approximate $\operatorname{tr}_{\min}(f)$ from below, one considers the following relaxation:

$$\operatorname{tr}_{\Theta}(f) = \sup\{a : f - a \in \Theta_d\}, \tag{6.5}$$

whose dual is

$$\begin{aligned}
L_{\Theta}(f) := \inf_L \quad & \langle \mathbf{M}_d(L), G_f \rangle \\
\text{s.t.} \quad & (\mathbf{M}_d(L))_{u,v} = (\mathbf{M}_d(L))_{w,z}, \quad \text{for all } u^\star v \stackrel{\text{cyc}}{\sim} w^\star z, \\
& L(1) = 1, \quad \mathbf{M}_d(L) \succeq 0, \\
& L : \mathbb{R}\langle\underline{X}\rangle_{2d} \to \mathbb{R} \text{ linear},
\end{aligned} \tag{6.6}$$

One has $\operatorname{tr}_{\Theta}(f) = L_{\Theta}(f) \leq \operatorname{tr}_{\min}(f)$, where the inequality comes from [5, Lemma 5.2] and the equality results from the strong duality between SDP (6.6) and SDP (6.5), see e.g. [5, Theorem 5.3] for a proof. In addition, if the optimizer $\mathbf{M}_d(L)^{\text{opt}}$ of SDP (6.6) satisfies the flatness condition, i.e., the linear functional underlying $\mathbf{M}_d(L)^{\text{opt}}$ is 1-flat (see Definition 2.6), then the above relaxations are exact and one has $\operatorname{tr}_{\Theta}(f) = L_{\Theta}(f) = \operatorname{tr}_{\min}(f)$. This exactness result is stated in [5, Theorem 5.4].

For a given nc polynomial $f = f_1 + \cdots + f_p$, with $f_k \in \operatorname{Sym} \mathbb{R}\langle\underline{X}(I_k)\rangle_{2d}$, for all $k = 1, \ldots, p$, we consider the following sparse variant of SDP (6.6):

$$L_\Theta^{\text{sparse}}(f) = \inf_L \ \sum_{k=1}^{p} \langle \mathbf{M}_d(L, I_k), G_{f_k} \rangle$$

$$\text{s.t.} \quad (\mathbf{M}_d(L, I_k))_{u,v} = (\mathbf{M}_d(L, I_k))_{w,z}, \quad \text{for all } u^\star v \overset{\text{cyc}}{\sim} w^\star z, \quad (6.7)$$

$$L(1) = 1, \quad \mathbf{M}_d(L, I_k) \succeq 0, \quad k = 1, \ldots, p,$$

$$L : \mathbb{R}\langle \underline{X}(I_1) \rangle_{2d} + \cdots + \mathbb{R}\langle \underline{X}(I_p) \rangle_{2d} \to \mathbb{R} \ \text{linear},$$

whose dual is the sparse variant of SDP (6.5):

$$\text{tr}_\Theta^{\text{sparse}}(f) = \sup_\lambda \ \lambda$$

$$\text{s.t.} \quad f - \lambda \in \Theta_d^{\text{sparse}}. \tag{6.8}$$

Now, we are ready to state the sparse variant of [5, Theorem 5.3].

Theorem 6.2 *Let* $f \in \text{Sym} \mathbb{R}\langle \underline{X} \rangle$ *of degree* $2d$, *with* $f = f_1 + \cdots + f_p$, $f_k \in \text{Sym} \mathbb{R}\langle \underline{X}(I_k) \rangle_{2d}$, *for all* $k = 1, \ldots, p$. *There is no duality gap between SDP* (6.7) *and SDP* (6.8), *namely* $\text{tr}_\Theta^{\text{sparse}}(f) = L_\Theta^{\text{sparse}}(f)$.

Proof The proof of strong duality is essentially the same as the one of Theorem 5.4. It relies on the closedness of the convex cone Θ_d^{sparse} which comes from the closedness of Σ_d^{sparse}, proved in Proposition 5.3. ∎

As for unconstrained eigenvalue optimization, one can retrieve the solution of the initial trace minimization problem under the same assumptions as Theorem 4.2. This is stated in the next proposition, which is the sparse variant of [5, Theorem 5.4].

Proposition 6.3 *Let* f *be as in Theorem* 6.2, *and assume that SDP* (6.7) *admits an optimal solution* $\mathbf{M}_d(L)$. *If the linear functional* L *underlying* $\mathbf{M}_d(L)$ *satisfies the flatness (H1) and irreducibility (H2) conditions stated in Theorem* 4.2, *then*

$$\text{tr}_\Theta^{\text{sparse}}(f) = L_\Theta^{\text{sparse}}(f) = \text{tr}_{\min}(f).$$

Proof The first equality comes from Theorem 6.2. By Theorem 6.1, there exist finitely many n-tuples of symmetric matrices $\underline{A}^{(j)}$ and positive scalars λ_j with $\sum_j \lambda_j = 1$ such that $L(f) = \sum_j \lambda_j \, \text{tr} \, f(\underline{A}^{(j)})$. Since $L(f) = \sum_{k=1}^{p} \langle \mathbf{M}_d(L, I_k), G_{f_k} \rangle = L_\Theta^{\text{sparse}}(f)$ and $\text{tr}_{\min}(f) = \sum_j (\lambda_j \, \text{tr}_{\min}(f)) \leq \sum_j \lambda_j \, \text{tr} \, f(\underline{A}^{(j)}) = L(f)$, one has $\text{tr}_{\min}(f) \leq L_\Theta^{\text{sparse}}(f)$. The desired result then follows from weak duality between SDP (6.7) and SDP (6.8). ∎

In practice, Proposition 6.3 allows one to derive an algorithm similar to the `SparseEigGNS` procedure (described in Algorithm 5.12) to find flat optimal solutions for the unconstrained trace problem.

6.3 Constrained trace optimization with sparsity

In this subsection, we provide the sparse tracial version of Lasserre's hierarchy to minimize the trace of a noncommutative polynomial on a semialgebraic set. Given $f \in \text{Sym}\,\mathbb{R}\langle \underline{X} \rangle$ and $S := \{g_1, \ldots, g_m\} \subset \text{Sym}\,\mathbb{R}\langle \underline{X} \rangle$ as in (2.2), let us define $\text{tr}_{\min}(f, S)$ as follows:

$$\text{tr}_{\min}(f, S) := \inf\{\text{tr}\, f(\underline{A}) : \underline{A} \in D_S\}. \tag{6.9}$$

Since an infinite-dimensional Hilbert space does not admit a trace, we obtain lower bounds on the minimal trace by considering a particular subset of D_S^∞. This subset is obtained by restricting from the algebra of all bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ to finite von Neumann algebras [57] of type I and type II. We introduce $\text{tr}_{\min}(f, S)^{\text{II}_1}$ as the trace-minimum of f on $D_S^{\text{II}_1}$. This latter set is defined as follows (see [5, Definition 1.59]):

Definition 6.4 Let $\mathcal{F}$ be a type-II_1-von Neumann algebra [57, Chapter 5]. Let us define $\mathcal{D}_S^\mathcal{F}$ as the set of all tuples $\underline{A} = (A_1, \ldots, A_n) \in \mathcal{F}^n$ making $s(\underline{A})$ a positive semidefinite operator for every $s \in S$. The von Neumann semialgebraic set $D_S^{\text{II}_1}$ generated by S is defined as

$$D_S^{\text{II}_1} := \bigcup_\mathcal{F} \mathcal{D}_S^\mathcal{F},$$

where the union is over all type-II_1-von Neumann algebras with separable predual.

By [5, Proposition 1.62], if $f \in \Theta_S$, then $\text{tr}\, f(\underline{A}) \geq 0$, for all $A \in D_S$ and $A \in D_S^{\text{II}_1}$. Since D_S can be modeled by $D_S^{\text{II}_1}$, one has $\text{tr}_{\min}(f, S)^{\text{II}_1} \leq \text{tr}_{\min}(f, S)$. With d being defined as in Sect. 5.2, one can approximate $\text{tr}_{\min}(f, S)^{\text{II}_1}$ from below via the following hierarchy of SDP programs, indexed by $s \geq d$:

$$\text{tr}_{\Theta,s}(f, S) = \sup\{a : f - a \in \Theta_{S,d}\}, \tag{6.10}$$

whose dual is

$$
\begin{aligned}
L_{\Theta,s}(f, S) := \inf_L \quad & \langle \mathbf{M}_s(L), G_f \rangle \\
\text{s.t.} \quad & (\mathbf{M}_s(L))_{u,v} = (\mathbf{M}_s(L))_{w,z}, \quad \text{for all } u^\star v \overset{\text{cyc}}{\sim} w^\star z, \\
& L(1) = 1, \\
& \mathbf{M}_s(L) \succeq 0, \quad \mathbf{M}_{s-d_j}(g_j L) \succeq 0, \quad j = 1, \ldots, m, \\
& L : \mathbb{R}\langle \underline{X} \rangle_{2d} \to \mathbb{R} \text{ linear.}
\end{aligned} \tag{6.11}
$$

If the quadratic module M_S is archimedean, the resulting hierarchy of SDP programs provides a sequence of lower bounds $\text{tr}_{\Theta,s}(f, S)$ monotonically converging to $\text{tr}_{\min}(f, S)^{\text{II}_1}$, see e.g. [5, Corollary 3.5].

Next, we present a sparse variant hierarchy of SDP programs providing a sequence of lower bounds $\mathrm{tr}^{\mathrm{sparse}}_{\Theta,s}(f, S)$ monotonically converging to $\mathrm{tr}_{\min}(f, S)^{\mathrm{II}_1}$. Let $S \cup \{f\} \subseteq \mathrm{Sym}\, \mathbb{R}\langle \underline{X} \rangle$ and let D_S be as in (2.6) with the additional quadratic constraints (2.5). Let us define the sparse variant of SDP (6.11), indexed by $s \geq d$:

$$
\begin{aligned}
L^{\mathrm{sparse}}_{\Theta,s}(f, S) = \inf_{L} \quad & \sum_{k=1}^{p} \langle \mathbf{M}_s(L, I_k), G_{f_k} \rangle \\
\text{s.t.} \quad & (\mathbf{M}_s(L, I_k))_{u,v} = (\mathbf{M}_s(L, I_k))_{w,z}, \quad \text{for all } u^\star v \overset{\mathrm{cyc}}{\sim} w^\star z, \\
& L(1) = 1, \\
& \mathbf{M}_s(L, I_k) \succeq 0, \quad k = 1, \ldots, p, \\
& \mathbf{M}_{s-d_j}(g_j L, I_k) \succeq 0, \quad j = 1, \ldots, m, \quad k = 1, \ldots, p, \\
& L : \mathbb{R}\langle \underline{X}(I_1) \rangle_{2d} + \cdots + \mathbb{R}\langle \underline{X}(I_p) \rangle_{2d} \to \mathbb{R} \text{ linear.}
\end{aligned}
\tag{6.12}
$$

whose dual is the sparse variant of SDP (6.10):

$$
\mathrm{tr}^{\mathrm{sparse}}_{\Theta,s}(f, S) = \sup\{a : f - a \in \Theta^{\mathrm{sparse}}_{S,d}\},
\tag{6.13}
$$

With the same conditions as the ones assumed in Proposition 5.8 for constrained eigenvalue optimization, SDP (6.12) admits strictly feasible solutions, so there is no duality gap between SDP (6.12) and SDP (6.13). The proof is the same since the constructed linear functional in Proposition 5.8 is tracial. In order to prove convergence of the hierarchy of bounds given by the SDP (6.12)–(6.13), we need the following proposition, which is the sparse variant of [5, Proposition 1.63].

Proposition 6.5 *Let $S \cup \{f\} \subseteq \mathrm{Sym}\, \mathbb{R}\langle \underline{X} \rangle$ and let D_S be as in (2.6) with the additional quadratic constraints (2.5). Let Assumption 2.4 hold. Then the following are equivalent:*

(i) $\mathrm{tr}\, f(\underline{A}) \geq 0$ for all $\underline{A} \in D_S^{\mathrm{II}_1}$;

(ii) for all $\varepsilon > 0$, there exists $g \in \mathcal{M}(S)^{\mathrm{sparse}}$ with $f + \varepsilon \overset{\mathrm{cyc}}{\sim} g$.

Proof The implication (ii) $\implies$ (i) is trivial. For the converse implication, let us fix $\varepsilon > 0$ such that the conclusion of (ii) does not hold. By the Hahn-Banach separation theorem, there exists a linear functional $L : \mathrm{Sym}\, \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}$ with $L(f + \varepsilon) \leq 0$ and $L(\mathcal{M}(S)^{\mathrm{sparse}}) \subseteq \mathbb{R}^{\geq 0}$. As in Theorem 3.3, the GNS construction leads to operator algebras $\mathcal{A}_k, \mathcal{A}_{jk}$ for $j, k = 1, \ldots, p$ and $j \neq k$, with $\mathcal{A}_{jk} \subseteq \mathcal{A}_j, \mathcal{A}_k$. However, in this case the GNS construction yields tracial states on these, whence they are all finite von Neumann algebras. Now amalgamate in the category of von Neumann algebras (cf. [60]) to obtain a finite von Neumann algebra $\mathcal{A}$ with trace τ so that $\tau(f) \leq -\varepsilon < 0$. $\qquad\square$

Proposition 6.5 implies the following convergence property.

Corollary 6.6 *Let $S \cup \{f\} \subseteq \mathrm{Sym}\, \mathbb{R}\langle \underline{X} \rangle$ and let D_S be as in (2.6) with the additional quadratic constraints (2.5). Let Assumption 2.4 hold. Then*

$$
\lim_{s \to \infty} \mathrm{tr}^{\mathrm{sparse}}_{\Theta,s}(f, S) = \lim_{s \to \infty} L^{\mathrm{sparse}}_{\Theta,s}(f, S) = \mathrm{tr}_{\min}(f, S)^{\mathrm{II}_1}.
$$

Proof By weak duality, one has $\mathrm{tr}_{\Theta,s}^{\mathrm{sparse}}(f, S) \le L_{\Theta,s}^{\mathrm{sparse}}(f, S) \le \mathrm{tr}_{\min}(f, S)^{\mathrm{II}_1}$. In addition, Proposition 6.5 implies that for each each $m \in \mathbb{N}$, there exists $s(m) \in \mathbb{N}$ such that $f - \mathrm{tr}_{\min}(f, S)^{\mathrm{II}_1} + \frac{1}{m} \in \Theta_{S,s(m)}^{\mathrm{sparse}}$. This implies that

$$\mathrm{tr}_{\min}(f, S)^{\mathrm{II}_1} - \frac{1}{m} \le \mathrm{tr}_{\Theta,s(m)}^{\mathrm{sparse}}(f, S),$$

yielding the desired conclusion. $\qquad\square$

To extract solutions of constrained trace minimization problems, we rely on the following variant of Theorem 6.1. It is, in turn, the tracial analog of Theorem 4.2.

Proposition 6.7 *Let* $S \subseteq \mathrm{Sym}\,\mathbb{R}\langle \underline{X} \rangle_{2d}$, *and assume that the semialgebraic set* D_S *is as in* (2.6) *with the additional quadratic constraints* (2.5). *Let Assumption* 2.4(i) *hold. Set* $\delta := \max\{\lceil \deg(g)/2 \rceil : g \in S \cup 1\}$. *Let* $\mathbf{M}_s(L)$ *be an optimal solution of SDP* (6.7) *with value* $L_{\Theta,s}^{\mathrm{sparse}}(f, S)$, *for* $s \ge d + \delta$, *such that* L *satisfies the flatness (H1) and irreducibility (H2) conditions of Theorem* 4.2. *Then there are finitely many* n-*tuples* $\underline{A}^{(j)}$ *of symmetric matrices in* D_S^r *for some* $r \in \mathbb{N}$, *and positive scalars* λ_j *with* $\sum_j \lambda_j = 1$ *such that*

$$L(f) = \sum_j \lambda_j \,\mathrm{tr}\, f(\underline{A}^{(j)}).$$

In particular, one has $\mathrm{tr}_{\min}(f, S) = \mathrm{tr}_{\min}(f, S)^{\mathrm{II}_1} = L_{\Theta,s}^{\mathrm{sparse}}(f, S)$.

As in the dense case [5, Algorithm 5.1], one can rely on Proposition 6.7 to provide a randomized algorithm to look for flat optimal solutions for the constrained trace problem (6.9).

7 Numerical experiments

The aim of this section is to provide experimental comparison between the bounds given by the dense relaxations (using `NCeigMin` under `NCSOStools`) and the ones produced by our sparse variants. For the sake of conciseness, we focus on minimal eigenvalue computation.

In Sect. 7.1 we focus on the unconstrained case. For a given nc polynomial f of degree $2d$, we compare the smallest eigenvalue $\lambda_{\min}(f) = \lambda_{\min,d}(f) = L_{\mathrm{sohs},d}(f)$ computed via SDP (5.6) (or equivalently SDP (5.5)) with $\lambda_{\min,d}^{\mathrm{sparse}}(f) = L_{\mathrm{sohs},d}^{\mathrm{sparse}}(f)$, computed via SDP (5.7) (or equivalently SDP (5.8)).

In Sect. 7.2 we focus on the constrained case. We compare the values of $\lambda_s(f, S) = L_s(f, S)$, obtained in the dense setting via SDP (5.12) (or equivalently SDP (5.11)), with the values of $\lambda_s^{\mathrm{sparse}}(f) = L_s^{\mathrm{sparse}}(f)$, obtained in the sparse setting via SDP (5.14) (or equivalently SDP (5.15)), for various sets of constraints S and increasing values of the relaxation order s.

The resulting algorithm, denoted by `NCeigMinSparse`, is currently implemented in `NCSOStools` [12]. This software library is available within Matlab and interfaced

Table 1 NCeigMin versus NCeigMinSparse for unconstrained minimal eigenvalues of the chained singular and generalized Rosenbrock functions

f	n	NCeigMin				NCeigMinSparse			
		m_{sdp}	n_{sdp}	$\lambda_{\min,2}(f)$	Time (s)	m_{sdp}	n_{sdp}	$\lambda_{\min,2}^{\text{sparse}}(f)$	Time (s)
f_{cs}	4	78	169	0	0.42	78	169	0	0.37
	8	398	841	0	1.33	165	1323	0	3.69
	12	974	2025	0	4.35	298	2205	0	6.28
	16	1806	3721	0	14.29	413	3087	0	9.18
	20	2894	5929	0	52.47	537	3969	0	12.78
	24	4238	8649	0	152.17	661	4851	0	17.65
f_{gR}	10	200	400	0	0.56	95	441	0	1.39
	12	288	576	0	0.81	117	539	0	1.78
	14	392	784	0	1.12	139	637	0	2.20
	16	512	1024	0	1.46	161	735	0	2.67
	18	648	1296	0	2.15	183	833	0	3.26
	20	800	1600	0	2.92	205	931	0	4.10

with the SDP solver Mosek 8.1 [42], which turned out to yield better performance than SeDuMi 1.3 [56]. All numerical results were obtained using a cluster available at the Faculty of mechanical engineering, University of Ljubljana, which has 30 TFlops computing performance. For our computations we used only one computing node which consisted of 2 Intel Xeon X5670 2,93GHz processors, each with 6 computing cores; 48 GB DDR3 memory; 500 GB hard drive. We ran Matlab in a plain (sequential) mode, without imposing any paralelization.

7.1 Unconstrained optimization

In Table 1, we report results obtained for minimizing the eigenvalue of the nc variants of the following functions:

- The chained singular function [11]:

$$f_{\text{cs}} := \sum_{i \in J}((X_i + 10X_{i+1})^2 + 5(X_{i+2} - X_{i+3})^2 + (X_{i+1} - 2X_{i+2})^4$$
$$+10(X_i - 10X_{i+3})^4),$$

where $J = \{1, 3, 4, \ldots, n-3\}$ and n is a multiple of 4. In this case, one can choose $I_k = \{k, k+1, k+2, k+3\}$ for all $k = 1, \ldots, n-3$ so that the associated sparsity pattern satisfies (2.7).
- The generalized Rosenbrock function [44]:

$$f_{\text{gR}} := 1 + \sum_{i=1}^{n-1} \left(100(X_{i+1} - X_i^2)^2 + (1 - X_{i+1})^2\right).$$

In this case, one can choose $I_k = \{k, k+1\}$ for all $k = 1, \ldots, n-1$ so that the associated sparsity pattern satisfies (2.7).

We compute bounds on the minimal eigenvalues of $f = f_{cs}$ for each $n \in \{4, \ldots, 24\}$ being a multiple of 4, and f_{gR} for even values of $n \in \{2, \ldots, 20\}$. For both functions, the minimal eigenvalue is 0. We indicate in Table 1 the data related to the semidefinite programs solved by Mosek. For each value of n, m_{sdp} stands for the total number of constraints and n_{sdp} stands for the total number of variables either of the SDP program (5.6) solved to compute $\lambda_{\min}(f)$ or the SDP program (5.7) solved to compute $\lambda_{\min,2}^{sparse}(f)$. As emphasized in the columns corresponding to m_{sdp}, the size of the SDP programs can be significantly reduced after exploiting sparsity, which is consistent with Remark 5.5. While the procedure NCeigMin does not take sparsity into account, it relies on the Newton chip method [5, §2.3] to reduce the number of variables involved in the Hankel matrix from SDP (5.6). This explains why n_{sdp} is smaller for some values of n (e.g. $n = 8$ for f_{cs}) when running NCeigMin. However, the sparse procedure NCeigMinSparse turns out to be very often more efficient to compute the minimal eigenvalue. So far, our NCeigMinSparse procedure is limited by the computational abilities of current SDP solvers (such as Mosek) to handle matrices with more constraints and variables than the ones obtained e.g. for the chained singular function at $n = 24$ (see the related values of m_{sdp} and n_{sdp} in the corresponding column). It turns out that exploiting the sparsity pattern yields SDP programs with significantly fewer variables than the ones obtained after running the Newton chip method.

In the column reporting timings, we indicate the time needed to prepare *and* solve the SDP relaxation. For values of $n, d \gtrsim 8$, our current implementation in (interpreted) Matlab happens to be rather inefficient to construct the SDP problem itself, mainly because we rely on a naive nc polynomial arithmetic. To overcome this computational burden, we plan to interface NCSOStools with a C library implementing a more sophisticated monomial arithmetic. We also emphasize that for these unconstrained problems, each function is a sum of sparse hermitian squares, thus the sparse procedure NCeigMinSparse always retrieves the same optimal value as the dense procedure NCeigMin. However, the bound computed via the sparse procedure can be a strict lower bound of the minimal eigenvalue, as shown in Lemma 5.2.

7.2 Constrained optimization

In Table 2, we report results obtained for minimizing the eigenvalue of the nc chained singular function on the semialgebraic set $S_{cs} := \{1 - X_1^2, \ldots, 1 - X_n^2, X_1 - 1/3, \ldots, X_n - 1/3\}$ for $n \in \{4, 8, 12, 16, 20, 24\}$. Since f has degree 4, it follows from [5, Corollary 4.18] that it is enough to solve SDP (5.7) with optimal value $\lambda_2(f, S_{cs})$ to compute the minimal eigenvalue $\lambda_{\min}(f, S_{cs})$. For the experiments described in Table 2, we cannot rely on the Newton chip method as in the unconstrained case. Thus the dense procedure NCeigMin suffers from a severe computational burden for $n > 10$; the symbol "−" in a column entry indicates that the calculation did not finish in a couple of hours. As already observed before for the unconstrained case, the sparse procedure NCeigMinSparse performs much bet-

Table 2 `NCeigMin` versus `NCeigMinSparse` for minimal eigenvalue of the chained singular function on the nc polydisc S_{cs}

n	`NCeigMin`				`NCeigMinSparse`			
	m_{sdp}	n_{sdp}	$\lambda_2(f_{cs}, S_{cs})$	Time (s)	m_{sdp}	n_{sdp}	$\lambda_2^{sparse}(f_{cs}, S_{cs})$	Time (s)
4	161	641	315.21	3.25	161	641	315.21	2.95
8	1009	6625	965.48	146.99	525	1923	965.48	4.66
12	3121	28705	1615.7	7891.6	889	3205	1615.7	7.43
16	–				1253	4487	2266.05	13.20
20	–				1617	5769	2916.32	18.50
24	–				1981	7051	3566.56	26.38

ter than `NCeigMin`. Surprisingly, `NCeigMinSparse` yields the same bounds as `NCeigMin` at the minimal relaxation order $s = 2$, for all values of $n \leq 10$.

As shown in Example 5.10, there is no guarantee to obtain the above mentioned convergence behavior in a systematic way. We consider randomly generated cubic n-variate polynomials f_{rand} satisfying Assumption 2.4 with $I_k = \{k, k+1, k+2\}$, for all $k = 1, \ldots, n-2$. The corresponding nc polyball is given by $\mathbb{B}^{sparse} := \{1 - X_1^2 - X_2^2 - X_3^2, \ldots, 1 - X_{n-2}^2 - X_{n-1}^2 - X_n^2\}$. In Table 3, we report results obtained for minimizing the eigenvalue of f_{rand} on $\mathbb{B}^{sparse}$, for each value of $n \in \{4, \ldots, 10\}$. Here again, the sparse procedure `NCeigMinSparse` yields better performance than `NCeigMin`. Moreover, the sparse bound obtained for each $n \leq 10$ at minimal relaxation order $s = 2$ already gives an accurate approximation of the optimal bound provided by the dense procedure. We emphasize that the value of the third order relaxation obtained with the sparse procedure is almost equal to the optimal bound. In addition, the dense procedure cannot handle to solve the minimal order relaxation for $n > 10$, while we can always obtain a lower bound of the eigenvalue with `NCeigMinSparse`.

8 Conclusion and perspectives

We have presented a sparse variant of Putinar's Positivstellensatz for positive noncommutative polynomials, yielding a converging hierarchy of semidefinite relaxations for eigenvalue and trace optimization. We also designed a general algorithm to extract solutions of such sparse problems, thanks to a sparse variant of the Gelfand–Naimark–Segal construction and amalgamation properties of operator algebras. Experimental results obtained with `NCSOStools` prove that one can obtain accurate lower bounds via these semidefinite relaxations in an efficient way.

An obvious direction of further research is to investigate whether and how one can benefit from sparsity exploitation in other application fields, for instance to compute certified approximations of quantum graph parameters or maximum violation bounds of Bell inequalities in quantum information theory.

We have proved that there is no sparse analog of the Helton-McCullough Sums of Squares theorem. Thus, another interesting track of research is to look for alter-

Table 3 NCeigMin versus NCeigMinSparse for minimal eigenvalue of random cubic polynomials on the nc polyball $S = \mathbb{B}^{\text{sparse}}$

n	NCeigMin				NCeigMinSparse				
	m_{sdp}	n_{sdp}	$\lambda_2(f_{\text{rand}}, S)$	Time (s)	s	m_{sdp}	n_{sdp}	$\lambda_s^{\text{sparse}}(f_{\text{rand}}, S)$	Time (s)
4	71	491	-53.64	3.31	2	79	370	-53.72	1.18
					3	729	3538	-53.64	12.64
6	239	2045	-142.52	26.79	2	179	740	-142.62	2.33
					3	1535	7076	-142.52	29.52
8	559	5815	-165.89	171.30	2	279	1110	-166.32	3.73
					3	2341	10614	-165.91	62.70
10	1079	13289	-199.62	857.95	2	379	1480	-200.51	5.43
					3	3147	14152	-199.66	139.22
11	1429	18985	-180.39	2111.26	2	429	1665	-180.93	6.58
					3	3550	15921	-180.40	209.73
12	–	–	–	–	2	479	1850	-385.89	7.82
					3	3953	17690	-384.87	289.12
16	–	–	–	–	2	679	2590	-344.31	15.46
					3	5565	24766	-342.15	975.43
20	–	–	–	–	2	879	3330	-504.36	31.41
					3	7177	31842	-503.02	2587.61

native representations of sparse positive polynomials, e.g., representations involving noncommutative rational functions.

Apart from sparsity, we also intend to pursue research efforts to take into account other properties of structured noncommutative polynomials, such as symmetry.

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