Sparse reconstruction by convex relaxation: Fourier and Gaussian measurements

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Abstract— **This paper proves best known guarantees for exact reconstruction of a sparse signal f from few non-adaptive universal linear measurements. We consider Fourier measurements (random sample of frequencies of f) and random Gaussian measurements. The method for reconstruction that has recently gained momentum in the Sparse Approximation Theory is to relax this highly non-convex problem to a convex problem, and then solve it as a linear program. What are best guarantees for the reconstruction problem to be equivalent to its convex relaxation is an open question. Recent work shows that the number of measurements $k(r, n)$ needed to exactly reconstruct any r-sparse signal f of length n from its linear measurements with convex relaxation is usually $O(r \operatorname{polylog}(n))$. However, known guarantees involve huge constants, in spite of very good performance of the algorithms in practice. In attempt to reconcile theory with practice, we prove the first guarantees for universal measurements (i.e. which work for all sparse functions) with reasonable constants. For Gaussian measurements, $k(r, n) \lesssim 11.7\, r\big[1.5 + \log(n/r)\big]$, which is optimal up to constants. For Fourier measurements, we prove the best known bound $k(r, n) = O(r \log(n) \cdot \log^2(r) \log(r \log n))$, which is optimal within the $\log \log n$ and $\log^3 r$ factors. Our arguments are based on the technique of Geometric Functional Analysis and Probability in Banach spaces.**

I. INTRODUCTION

During the last two years, the Sparse Approximation Theory benefited from a rapid development of methods based on the Linear Programming. The idea was to relax a sparse recovery problem to a convex optimization problem. The convex problem can be further be rendered as a linear program, and analyzed with all available methods of Linear Programming.

Convex relaxation of sparse recovery problems can be traced back in its rudimentary form to mid-seventies; references to its early history can be found in [26]. With the development of fast methods of Linear Programming in the eighties, the idea of convex relaxation became truly promising. It was put forward most enthusiastically and successfully by Donoho and his collaborators since the late eighties, starting from the seminal paper [15] (see Theorem 8, attributed there to Logan, and Theorem 9). There is extensive work being carried out, both in theory and in practice, based on the convex relaxation [8], [14], [16], [17], [13], [19], [24], [25], [26], [11], [9], [10], [12], [2], [1], [4], [5], [23], [3], [6], [20].

To have theoretical guarantees for the convex relaxation method, one needs to show that *the sparse approximation problem is equivalent to its convex relaxation*. Proving this presents a mathematical challenge. Known theoretical guarantees work only for random measurements (e.g. random Gaussian and Fourier measurements). Even when there is a theoretical guarantee, it involves intractable or very large constants, far worse than in the observed practical performances.

In this paper, we substantially improve best known theoretical guarantees for random Gaussian and Fourier (and nonharmonic Fourier) measurements. For the first time, we are able to prove guarantees with reasonable constants (although only for Gaussian measurements). Our proofs are based on methods of Geometric Functional Analysis, Such methods were recently successfully used for related problems [23], [20]. As a result, our proofs are reasonably short (and hopefully, transparent).

In Section II, we state the sparse reconstruction problem and describe the convex relaxation method. A guarantee of its correctness is a very general *restricted isometry condition* on the measurement ensemble, due to Candes and Tao ([5], see [3]). Under this condition, the reconstruction problem with respect to these measurements is equivalent to its convex relaxation. In Sections III and IV, we improve best known guarantees for the sparse reconstruction from random Fourier (and nonharmonic Fourier) measurements and Gaussian measurements (Theorem 3.1 and 4.1 respectively).

II. THE SPARSE RECONSTRUCTION PROBLEM AND ITS CONVEX RELAXATION

We want to reconstruct an unknown signal $f \in \mathbb{C}^n$ from linear measurements $\Phi f \in \mathbb{C}^k$, where Φ is some known $k \times n$ matrix, called the *measurement matrix*. In the interesting case $k < n$, the problem is underdetermined, and we are interested in the sparsest solution. We can state this as the optimization problem

$$\text{minimize } \|f^*\|_0 \text{ subject to } \Phi f^* = \Phi f, \qquad (1)$$

where $\|f\|_0 = |\operatorname{supp} f|$ is the number of nonzero coefficients of f. This problem is highly non-convex. So we will consider its *convex relaxation*:

$$\text{minimize } \|f^*\|_1 \text{ subject to } \Phi f^* = \Phi f, \qquad (2)$$

where $\|f\|_p$ denotes the ℓ_p norm throughout this paper, $(\sum_{i=1}^n |f_i|^p)^{1/p}$. Problem (2) can be classically reformulated as the *linear program*

$$\text{minimize } \sum_{i=1}^n t_i \text{ subject to } -t \le f^* \le t, \ \Phi f^* = \Phi f,$$

which can be efficiently solved using general or special methods of Linear Programming. Then the main question is:

> *Under what conditions on Φ are problems* (1) *and* (2) *equivalent?*

In this paper, we will be interested in the *exact reconstruction*, i.e. we expect that the solutions to (1) and (2) are equal to each other and to f. Results for approximate reconstruction can be derived as consequences, see [4].

For exact reconstruction to be possible at all, one has to assume that the signal f is r-sparse, that is $\mathrm{supp}(f) \le r$, and that the number of measurements $k = k(r,n)$ has to be at least twice the sparsity r. Our goal will be to find sufficient conditions (guarantees) for the exact reconstruction. The number of measurements $k(r,n)$ should be kept as small as possible. Intuitively, the number of measurements should be of the order of r, which is the 'true' dimension of f, rather than the nominal dimension n.

Various results that appeared over the last two years demonstrate that many natural measurement matrices Φ yield exact reconstruction, with the number of measurements $k(r,n) = O(r \cdot \mathrm{polylog}(n))$, see [2], [4], [5], [23]. In Sections III and IV, we improve best known estimates on k for Fourier (and, more generally, nonharmonic Fourier) and Gaussian matrices respectively.

A general sufficient condition for exact reconstruction is the *restricted isometry condition* on Φ, due to Candes and Tao ([5], see [3]). It roughly says that the matrix Φ acts as an almost isometry on all $O(r)$-sparse vectors. Precisely, we define the restricted isometry constant δ_r to be the smallest positive number such that the inequality

$$C(1 - \delta_r)\|x\|_2^2 \le \|\Phi_T x\|_2^2 \le C(1 + \delta_r)\|x\|_2^2 \quad (3)$$

holds for some number $C > 0$ and for all x and all subsets $T \subset \{1, \dots, n\}$ of size $|T| \le r$, where Φ_T denotes the $k \times |T|$ matrix that consists of the columns of Φ indexed by T. The following theorem is due to Candes and Tao ([5], see [3]).

Theorem 2.1 (Restricted Isometry Condition): Let Φ be a measurement matrix whose restricted isometry constant satisfies

$$\delta_{3r} + 3\delta_{4r} \le 2. \quad (4)$$

Let f be an r-sparse signal. Then the solution to the linear program (2) *is unique and is equal to f.*

This theorem says that under the restricted isometry condition (4) on the measurement matrix Φ, the reconstruction problem (1) is equivalent to its convex relaxation (2) for all r-sparse functions f.

A problem with the use of Theorem 2.1 is that the restricted isometry condition (4) is usually difficult to check. Indeed, the number of sets T involved in this condition is exponential in r. As a result, no explicit construction of a measurement matrix is presently known that obeys the restricted isometry condition (4). All known constructions of measurement matrices are randomized.

III. RECONSTRUCTION FROM FOURIER MEASUREMENTS

Our goal will be to reconstruct an r-sparse signal $f \in \mathbb{C}^n$ from its discrete Fourier transform evaluated at $k = k(r,n)$ points. These points will be chosen at random and uniformly in $\{0, \dots, n-1\}$, forming a set Ω.

The Discrete Fourier transform $\hat{f} = \Psi f$ is defined by the DFT matrix Ψ with entries

$$\Psi_{\omega,t} = \frac{1}{\sqrt{n}} \exp(-i 2\pi \omega t / n), \quad \omega, t \in \{0, \dots, n-1\}.$$

So, our measurement matrix Φ is the submatrix of Ψ consisting of random rows (with indices in Ω). To be able to apply Theorem 2.1, it is enough to check that the restricted isometry condition (4) holds for the random matrix Φ with high probability. The problem is – what is the smallest number of rows $k(r,n)$ of Φ for which this holds? With that number, Theorem 2.1 immediately implies the following reconstruction theorem for Fourier measurements:

Theorem 3.1 (Reconstruction from Fourier measurements): A random set $\Omega \in \{0, \dots, n-1\}$ of size $k(r,n)$ satisfies the following with high probability. Let f be an r-sparse signal in $\mathbb{C}^n$. Then f can be exactly reconstructed from the values of its Fourier transform on Ω as a solution to the linear program

$$\text{minimize } \|f^*\|_1 \text{ subject to } \hat{f}^*(\omega) = \hat{f}(\omega), \quad \omega \in \Omega.$$

The central remaining problem, what is the smallest value of $k(r,n)$, is still open. The best known estimate is due to Candes and Tao [4]:

$$k(r,n) = O(r \log^6 n). \quad (5)$$

The conjectured optimal estimate would be $O(r \log n)$, which is known to hold for nonuniveral measuremets, i.e. for *one* sparse signal f and for a random set Ω [2].

In this paper, we improve on the best known bound (5):

Theorem 3.2 (Sample size): Theorem 3.1 holds with

$$k(r,n) = O(r \log(n) \cdot \log^2(r) \log(r \log n)).$$

The dependence on n is thus optimal within the $\log \log n$ factor and the dependence on r is optimal within the $\log^3 r$ factor. So, our estimate is especially good for small r, but our estimate always yields $k(r,n) = O(r \log^4 n)$.

Remark 3.3: Our results hold for transforms more general than the discrete Fourier transform. One can replace the DFT matrix Ψ by any orthogonal matrix with entries of magnitude $O(1/\sqrt{n})$. Theorems 3.1 and 3.2 hold for any such matrix.

In the remainder of this section, we prove Theorem 3.2. Let Ω be a random subset of $\{0, \dots, n\}$ of size k. Recall that the

measurement matrix Φ that consists of the rows of Ψ whose indices are in Ω). In view of Theorem 3, it suffices to prove that the restricted isometry constant δ_r of Φ satisfies

$$\mathbb{E}\delta_r \leq \varepsilon \tag{6}$$

whenever

$$k \geq C\Big(\frac{r\log n}{\varepsilon^2}\Big)\log\Big(\frac{r\log n}{\varepsilon^2}\Big)\log^2 r, \tag{7}$$

where $\varepsilon > 0$ is arbitrary, and C is some absolute constant.

Let $y_1, \ldots, y_k$ denote the rows of the matrix Ψ. Dualizing (3) we see that (6) is equivalent to the following inequality:

$$\mathbb{E}\sup_{|T|\leq r}\Big\|id_{\mathbb{C}^T} - C'\sum_{i\in\Omega} y_i^T \otimes y_i^T\Big\| \leq \varepsilon$$

with $C' = 1/\sqrt{C}$. Here and thereafter, for vectors $x, y \in \mathbb{C}^n$ the tensor $x \otimes y$ is the rank-one linear operator given by $(x \otimes y)(z) = \langle x, y \rangle z$, where $\langle \cdot \rangle$ is the canonical inner product on $\mathbb{C}^n$. The notation x^T stands for the restriction of a vector x on its coordinates in the set T. The operator $id_{\mathbb{C}^T}$ in (8) is the identity on $\mathbb{C}^T$, and the norm is the operator norm for operators on ℓ_2^T.

The orthogonality of Ψ can be expressed as $id_{\mathbb{C}^n} = \sum_{i=0}^{n-1} y_i \otimes y_i$. We shall re-normalize the vectors y_i, letting $x_i = \sqrt{n}\ y_{i-1}$. Now we have $\|x_i\|_\infty = O(1)$ for all i. The proof has now reduced to the following probabilistic statement, which we interpret as a law of large numbers for random operators.

Theorem 3.4 (Uniform Operator Law of Large Numbers): Let $x_1, \ldots, x_n$ be vectors in $\mathbb{C}^n$ with uniformly bounded entries: $\|x_i\|_\infty \leq K$ for all i. Assume that $id_{\mathbb{C}^n} = \frac{1}{n}\sum_{i=1}^n x_i \otimes x_i$. Let Ω be a random subset of $\{1, \ldots, n\}$ of size k. Then

$$\mathbb{E}\sup_{|T|\leq r}\Big\|id_{\mathbb{C}^T} - \frac{1}{k}\sum_{i\in\Omega} x_i^T \otimes x_i^T\Big\| \leq \varepsilon \tag{8}$$

provided k satisfies (7) (with constant C that may depend on K).

Theorem 3.4 is proved by the techniques developed in Probability in Banach spaces. The general roadmap is similar ton [21], [22]. We first observe that

$$\mathbb{E}\frac{1}{k}\sum_{i\in\Omega} x_i^T \otimes x_i^T = \frac{1}{n}\sum_{i=1}^n x_i^T \otimes x_i^T = id_{\mathbb{C}^n},$$

so the random operator whose norm we estimate in (8) has mean zero. Then the standard symmetrization (see [27] Lemma 6.3) implies that the left-hand side of (8) does not exceed

$$2\mathbb{E}\sup_{|T|\leq r}\Big\|\frac{1}{k}\sum_{i\in\Omega}\varepsilon_i\ x_i^T \otimes x_i^T\Big\|$$

where (ε_i) are independent symmetric $\{-1, 1\}$-valued random variables; also (jointly) independent of Ω. Then the conclusion of Theorem 3.4 will be easily deduced from the following lemma.

Lemma 3.5: Let $x_1, \ldots, x_k$, $k \leq n$, be vectors in $\mathbb{C}^n$ with uniformly bounded entries, $\|x_i\|_\infty \leq K$ for all i. Then

$$\mathbb{E}\sup_{|T|\leq r}\Big\|\sum_{i=1}^k \varepsilon_i\ x_i^T \otimes x_i^T\Big\| \leq k_1 \sup_{|T|\leq r}\Big\|\sum_{i=1}^k x_i^T \otimes x_i^T\Big\|^{\frac{1}{2}} \tag{9}$$

where $k_1 \leq C_1(K)\sqrt{r}\log(r)\sqrt{\log n}\sqrt{\log k}$.

Let us show how Lemma 3.5 implies Theorem 3.4. We first condition on a choice of Ω and apply Lemma 3.5 for x_i, $i \in \Omega$. Then we take the expectation with respect to Ω. We then use the a consequence of Hölder inequality, $\mathbb{E}(|X|^{\frac{1}{2}}) \leq (\mathbb{E}|X|)^{\frac{1}{2}}$ and the triangle inequality. Let us denote the left hand side of (8) by E. We obtain:

$$E \leq \frac{2k_1}{\sqrt{k}}\mathbb{E}\sup_{|T|\leq n}\Big\|\frac{1}{k}\sum_{i\in\Omega} x_i^T \otimes x_i^T\Big\|^{\frac{1}{2}} \leq \frac{2k_1}{\sqrt{k}}(E+1)^{\frac{1}{2}}.$$

It follows that $E \leq C_2\frac{2k_1}{\sqrt{k}}$, provided that $\frac{2k_1}{\sqrt{k}} = O(1)$. Theorem 3.4 now follows from our choice of $k = k(r, n)$.

Hence it is only left to prove Lemma 3.5. Throughout the proof, B_p^n and B_p^T denote the unit ball of the norm $\|\cdot\|_p$ on $\mathbb{C}^n$. To this end, we first replace Bernoulli r.v.'s ε_i by standard independent normal random variables g_i, using a comparison principle (inequality (4.8) in [27]). Then our problem becomes to bound the Gaussian process, indexed by the union of the unit Euclidean balls B_2^T in $\mathbb{C}^T$ for all subsets I of $\{1, \ldots, n\}$ of size at most r. We apply Dudley's inequality (Theorem 11.17 in [27]), which is a general upper bound on Gaussian processes. Let us denote the left hand side of (8) by E_1. We obtain:

$$E_1 \leq C_3\mathbb{E}\sup_{|T|\leq r}\Big\|\sum_{i=1}^k g_i\ x_i^T \otimes x_i^T\Big\|$$

$$= C_3\mathbb{E}\sup_{\substack{|T|\leq r\\x\in B_2^T}}\Big|\sum_{i=1}^k g_i\langle x_i, x\rangle^2\Big|$$

$$\leq C_4\int_0^\infty \log^{1/2} N\big(\cup_{|T|\leq r} B_2^T, \delta, u\big)\ du,$$

where $N(Z, \delta, u)$ denotes the minimal number of balls of radius u in metric δ centered in points of Z, needed to cover the set Z. The metric δ in Dudley's inequality is defined by the Gaussian process, and in our case it is

$$\delta(x, y) = \Big[\sum_{i=1}^M \big(\langle x_i, x\rangle^2 - \langle x_i, y\rangle^2\big)^2\Big]^{\frac{1}{2}}$$

$$\leq \Big[\sum_{i=1}^k \big(\langle x_i, x\rangle + \langle x_i, y\rangle\big)^2\Big]^{\frac{1}{2}}\max_{i\leq k}|\langle x_i, x-y\rangle|$$

$$\leq 2\max_{\substack{|T|\leq r\\z\in B_2^T}}\Big[\sum_{i=1}^k\langle x_i, z\rangle^2\Big]^{\frac{1}{2}}\max_{i\leq k}|\langle x_i, x-y\rangle|$$

$$= 2R\max_{i\leq k}|\langle x_i, x-y\rangle|,$$

where

$$R := \sup_{|T| \leq r} \Big\| \sum_{i=1}^{k} x_i^T \otimes x_i^T \Big\|^{\frac{1}{2}}.$$

Hence

$$E_1 \leq C_5 R \sqrt{r} \int_0^\infty \log^{1/2} N\big(\tfrac{1}{\sqrt{r}} D_2^{r,n}, \|\cdot\|_X, u\big)\, du. \quad (10)$$

Here

$$D_p^{r,n} = \bigcup_{|T| \leq r} B_p^T, \quad \|x\|_X = \max_{i \leq k} |\langle x_i, x\rangle|.$$

We will use containments

$$\frac{1}{\sqrt{r}} D_2^{r,n} \subseteq D_1^{r,n} \subseteq K B_X, \quad D_1^{r,n} \subseteq B_1^n, \quad (11)$$

where B_X denotes the unit ball of the norm $\|\cdot\|_X$. The second containment follows from the uniform boundedness of (x_i). We can thus replace $\frac{1}{\sqrt{r}} D_2^{r,n}$ in (10) by $D_1^{r,n}$. Comparing (10) to the right hand side of (9) we see that, in order to complete the proof of Lemma 3.5, it suffices to show that

$$\int_0^K \log^{1/2} N\big(D_1^{r,n}, \|\cdot\|_X, u\big)\, du \leq C_6 \log(r) \sqrt{\log n} \sqrt{\log k}, \tag{12}$$

with $C_6 = C_6(K)$. To this end, we will estimate the covering numbers in this integral in two different ways. For big u, we will just use the second containment in (11), which allows us to replace $D_1^{r,n}$ by B_1^n.

Lemma 3.6: Let $x_1, \dots, x_k$, $k \leq n$, be vectors as in Lemma 3.5. Then for all $u > 0$ we have

$$N(B_1^n, \|\cdot\|_X, u) \leq (2n)^m,$$

where $m = C_7 K^2 \log(k)/u^2$.

Proof: We use the empirical method of Maurey. Fix a vector $y \in B_1^n$. Define a random vector $Z \in \mathbb{R}^n$ that takes values $(0, \dots, 0, \text{sign}(y(i)), 0, \dots, 0)$ with probability $|y(i)|$ each, $i = 1, \dots, n$ (all entries of that vector are zero except i-th). Here $\text{sign}(z) = z/|z|$, whenever $z \neq 0$, and 0 otherwise. Note that $\mathbb{E}Z = y$. Let $Z_1, \dots, Z_m$ be independent copies of Z. Using symmetrization as before, we see that

$$E_3 := \mathbb{E}\Big\| y - \frac{1}{m} \sum_{j=1}^{m} Z_j \Big\|_X \leq \frac{2}{m}\, \mathbb{E}\Big\| \sum_{j=1}^{m} \varepsilon_j Z_j \Big\|_X.$$

Now we condition on a choice of (Z_j) and take the expectation with respect to random signs (ε_j). Using comparison to Gaussian variables as before, we obtain

$$E_4 := \mathbb{E}\Big\| \sum_{j=1}^{m} \varepsilon_j Z_j \Big\|_X \leq C_7 \mathbb{E}\Big\| \sum_{j=1}^{m} g_j Z_j \Big\|_X$$

$$= C_7 \mathbb{E} \max_{i \leq k} \Big| \sum_{j=1}^{m} g_j \langle Z_j, x_i\rangle \Big|.$$

For each i, $\gamma_i := \sum_{j=1}^{m} g_j \langle Z_j, x_i\rangle$ is a Gaussian random variable with zero mean and with variance

$$\sigma_i = \Big(\sum_{j=1}^{m} |\langle Z_j, x_i\rangle|^2 \Big)^{1/2} \leq K\sqrt{m},$$

since $|\langle Z_j, x_i\rangle| \leq \|x_i\|_\infty \leq K$. Using a simple bound on the maximum of Gaussian random variables (see (3.13) in [27]), we obtain

$$E_4 \leq C_7 \mathbb{E} \max_{i \leq k} |\gamma_i| \leq C_8 \sqrt{\log k} \max_{i \leq k} \sigma_i \leq C_8 \sqrt{\log k} K \sqrt{m}.$$

Taking the expectation with respect to (Z_j) we obtain

$$E_3 \leq \frac{2}{m} \mathbb{E}(E_4) \leq \frac{2 C_8 K \sqrt{\log k}}{\sqrt{m}}.$$

With the choice of m made in the statement of the lemma, we conclude that $E_3 \leq u$. We have shown that for every $y \in B_1^n$, there exists a $z \in \mathbb{C}^n$ of the form $z = \frac{1}{m} \sum_{j=1}^{m} Z_j$ such that $\|y - z\|_X \leq u$. Each Z_j takes $2n$ values, so z takes $(2n)^m$ values. Hence B_1^n can be covered by at $(2n)^m$ balls of norm $\|\cdot\|_X$ of radius u. A standard argument shows that we can assume that these balls are centered in points of B_1^n. This completes the proof of Lemma 3.6. ∎

For small u, we will use a simple volumetric estimate. The diameter of B_1^r considered as a set in $\mathbb{C}^n$ is at most K with respect to the norm $\|\cdot\|_X$ (this was stated as the last containment in (11)). It follows that $N(B_1^r, \|\cdot\|, u) \leq (1 + 2K/u)^r$ for all $r > 0$, see (5.7) in [Pi]. The set $D_1^{r,n}$ consists of $d(r,n) = \sum_{j=1}^{r} \binom{n}{i}$ balls of form B_1^T, thus

$$N\big(D_1^{r,n}, \|\cdot\|_X, u\big) \leq d(n,r)(1 + 2K/u)^r. \tag{13}$$

Now we combine the estimate of the covering number $N(u) = \log^{1/2} N\big(D_1^{r,n}, \|\cdot\|_X, u\big)$ of Lemma 3.6, and the volumetric estimate (13), to bound the integral in (12). Using Stirling's approximation, we see that $d(r,n) \leq (C_9 n/r)^r$. Thus

$$N(u) \leq C_{10} \sqrt{r} \big[\sqrt{\log(n/r)} + \sqrt{\log(1 + 2/u)} \big] =: N_1(u),$$
$$N(u) \leq \frac{C_{10}}{u} \sqrt{\log k} \sqrt{\log n} =: N_2(u),$$

where $C_{10} = C_{10}(K)$. Then we bound the integral in (12) as

$$\int_0^K N(u)\, du \leq \int_0^A N_1(u)\, du + \int_A^K N_2(u)\, du$$
$$\leq C_{11} A \sqrt{r} \big[\sqrt{\log(n/r)} + \log(1 + 2/A) \big]$$
$$+ C_{11} \log(1/A) \sqrt{\log k} \sqrt{\log n},$$

where $C_{11} = C_{11}(K)$. Choosing $A = 1/\sqrt{r}$, we conclude that the integral in (12) is at most $\sqrt{\log(n/r)} + \log r + \log(r)\sqrt{\log k}\sqrt{\log n}$. This proves (12), which completes the proof of Lemma 3.5 and thus of Theorems 3.4 and 3.2. ∎

IV. RECONSTRUCTION FROM GAUSSIAN MEASUREMENTS

Our goal will be to reconstruct an r-sparse signal $f \in \mathbb{R}^n$ from $k = k(r,n)$ Gaussian measurements. These are given by $\Phi f \in \mathbb{R}^k$, where Φ is a $k \times n$ random matrix ('Gaussian matrix' in the sequel), whose entries are independent $N(0,1)$ random variables. The reconstruction will be achieved by solving the linear program (2).

The problem again is to find the smallest number of measurements $k(r,n)$ for which, with high probability, we have an exact reconstruciton of every r-sparse signal f from its measurements Φf? It has recently been shown in [5], [23], [3] that

$$k(r,n) = O(r \log(n/r)), \tag{14}$$

and was extended in [20] to sub-gaussian measurements. This is asymptotically optimal. However, the constant factor implicit in (14) has not been known; previous proofs of (14) yield unreasonably weak constants (of order $2,000$ and higher). In fact, *there has not been known any theoretical guarantees with reasonable constants for Linear Programming based reconstructions*. So, there is presently a gap between theoretical guarantees and good practical performance of reconstruction (2) (see e.g. [3]). Here we shall prove a first practically reasonable guarantee of the form (14):

$$k(r,n) \leq c_1 r \big[c_2 + \log(n/r)\big](1 + o(1)), \tag{15}$$
$$c_1 = 6 + 4\sqrt{2} \approx 11.66, \quad c_2 = 1.5.$$

Theorem 4.1 (Reconstruction from Gaussian measurements): A $k \times n$ Gaussian matrix Φ with $k > k(r,n)$ satisfies the following with probability

$$1 - 3.5 \, \exp\Big(-\big(\sqrt{k} - \sqrt{k(r,n)}\big)^2 / 18 \Big).$$

Let f be an r-sparse signal in $\mathbb{R}^n$. Then f can be exactly reconstructed from the measurements Φf as a unique solution to the linear program (2).

Our proof of Theorem 4.1 is direct, we will not use the Restricted Isometry Theorem 2.1. The first part of this argument follows a general method of [20]. One interprets the exact reconstruction as the fact that the (random) kernel of Φ misses the cone generated by the (shifted) ball of ℓ_1. Then one embeds the cone in a universal set D, which is easier to handle, and proves that the random subspace does not intersect D. However, to obtain good constants as in (15), we will need to (a) improve the constant of embedding into D from [20], and (b) use Gordon's Escape Through the Mesh Theorem [18], which is tight in terms of constants. In Gordon's theorem, one measures the size of a set S in $\mathbb{R}^n$ by its *Gaussian width*

$$w(D) = \mathbb{E} \sup_{x \in S} \langle g, x \rangle,$$

where g is a random vector in $\mathbb{R}^n$ whose components are independent $N(0,1)$ random variables (Gaussian vector). The following is Gordon's theorem [18].

Theorem 4.2 (Escape Through the Mesh (Gordon)): Let S be a subset of the unit Euclidean sphere S^{n-1} in $\mathbb{R}^n$. Let Y be a random $(n-k)$-dimensional subspace of $\mathbb{R}^n$, distributed uniformly in the Grassmanian with respect to the Haar measure. Assume that $w(S) > \sqrt{k}$. Then $Y \cap S = \emptyset$ with probability at least

$$1 - 3.5 \, \exp\Big(-\big(k/\sqrt{k+1} - w(S)\big)^2 / 18 \Big).$$

We will now prove Theorem 4.1. First note that the function f is the unique solution of (2) if and only if 0 is the unique solution of the problem

$$\text{minimize } \|f - g^*\|_1 \text{ subject to } \Phi g^* \in \mathrm{Ker}(\Phi) =: Y. \tag{16}$$

Y is a $(n-k)$-dimensional subspace of $\mathbb{R}^n$. Due to the rotation invariance of the Gaussian random vectors, Y is distributed uniformly in the Grassmanian $G_{n-k,n}$ of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$, with respect to the Haar measure.

Now, 0 is the unique solution to (16) if and only if 0 is the unique metric projection of f onto the subspace Y in the norm $\|\cdot\|_1$. This in turn is equivalent to the fact that 0 is the unique contact point between the subspace Y and the ball of the norm $\|\cdot\|_1$ centered at f:

$$(f + \|f\|_1 B_1^n) \cap Y = \{0\}. \tag{17}$$

(Recall that B_p^n is the unit ball of the norm $\|\cdot\|_p$.) Let $\mathcal{C}_f$ be the cone in $\mathbb{R}^n$ generated by the set $f + \|f\|_1 B_1^n$ (the cone of a set $A \in \mathbb{R}^n$ is defined as $\{ta \mid a \in A, \ t \in \mathbb{R}^+\}$). Then the statement that (17) holds for all r-sparse functions f is clearly equivalent to

$$\mathcal{C}_f \cap Y = \{0\} \text{ for all } r\text{-sparse functions } f. \tag{18}$$

We can represent the cone $\mathcal{C}_f$ as follows. Let

$$T^+ = \{i \mid f(i) > 0\}, \ T^- = \{j \mid f(i) < 0\}, \ T = T^+ \cup T^-.$$

Then

$$\mathcal{C}_f = \Big\{ t \in \mathbb{R}^n \mid \sum_{i \in T^-} t(i) - \sum_{i \in T^+} t(i) + \sum_{i \in T^c} |t(i)| \leq 0 \Big\}.$$

We will now bound the cone $\mathcal{C}_f$ by a universal set, which does not depend on f.

Lemma 4.3: Consider the spherical part of the cone, $K_f = \mathcal{C}_f \cap S^{n-1}$. Then $K_f \subset (\sqrt{2}+1)D$, where

$$D = \mathrm{conv}\{x \in S^{n-1} \mid |\mathrm{supp}(x)| \leq r\}.$$

Proof: Fix a point $x \in \mathcal{C} \cap S^{n-1}$. We have

$$\sum_{i \in T} |x(i)| \leq \sqrt{|I|} \leq \sqrt{r}, \quad \sum_{i \in T^c} |x(i)| \leq \sum_{i \in T} |x(i)| \leq \sqrt{r}.$$

The norm $\|\cdot\|_D$ on $\mathbb{R}^n$ whose unit ball is D can be computed as

$$\|x\|_D = \sum_{l=1}^{L} \Big(\sum_{i \in I_l} (x(i)^*)^2 \Big)^{1/2},$$

where $L = \lceil n/r \rceil$, $I_l = \{r(l-1)+1, \ldots, rl\}$, for $l < L$, $I_L = \{r(L-1)+1, \ldots, n\}$, and $(x(i)^*)$ is a non-decreasing rearrangement of the sequence $(|x(i)|)$.

Set $F = F(x) = \{i \mid |x(i)| \geq 1/\sqrt{r}\}$. Since $x \in S^{n-1}$, we have $|F| \leq r$. Hence, for any $x \in K$ there exists a set $E = E(x) \subset \{1, \ldots, m\}$, which consists of $2r$ elements and such that $E \supseteq F \cup I$. Therefore, x can be represented as $x = x' + x''$ so that $\mathrm{supp}(x') \subseteq E$, $\|x\|_2 \leq 1$, $\mathrm{supp}(x'') \subseteq E^c$, $\|x''\|_\infty \leq 1/\sqrt{r}$. Set

$$V_E = B_2^E \times \left(\sqrt{r} B_1^{E^c} \cap \frac{1}{\sqrt{r}} B_\infty^{E^c} \right).$$

Then the above argument shows that $K_f \subset \bigcup_{|E|=2r} V_E =: W$.

The maximum of $\|x\|_D$ over $x \in W$ is attained at the extreme points of the sets V_E, which have the form $x = x' + x''$, where $x' \in S^E$, and x'' has coordinates 0 and $\pm 1/\sqrt{r}$ with r non-zero coordinates. Notice that since $|\mathrm{supp}(x')| \leq 2r$, $\|x'\|_D \leq \sqrt{2}\|x'\|_2$. Thus, for any extreme point x of V_E,

$$\|x\|_D \leq \|x'\|_D + \|x''\|_D \leq \sqrt{2}\|x'\|_2 + \|x''\|_2 \leq \sqrt{2} + 1.$$

The second inequality follows from $\mathrm{supp}(x') \leq 2r$ and $\mathrm{supp}(x'') = r$. This completes the proof of the lemma. ∎

To use Gordon's escape through the mesh theorem, we have to estimate the Gaussian width of D.

Lemma 4.4:

$$w(D) \leq \sqrt{2r \log(e^{3/2} n/r)}(1 + o(1)).$$

Proof: By definition,

$$w(D) = \sup_{|J|=r} \left(\sum_{i \in J} |g(i)|^2 \right)^{1/2}.$$

Let $p > 1$ be a number to be chosen later. By Hölder's inequality, we have

$$w(D) \leq \mathbb{E}\left(\sum_{|J|=r} \left(\sum_{i \in J} |g(i)|^2 \right)^{p/2} \right)^{1/p}$$

$$\leq \binom{n}{r}^{1/p} \left(\mathbb{E}\left(\sum_{i=1}^{r} |g(i)|^2 \right)^{p/2} \right)^{1/p}$$

$$\leq \left(\frac{en}{r} \right)^{r/p} \left(2^{p/2} \cdot \frac{\Gamma(p/2 + r/2)}{\Gamma(r/2)} \right)^{1/p}.$$

By the Stirling's formula,

$$2^{p/2} \cdot \frac{\Gamma(p/2 + r/2)}{\Gamma(r/2)} = \left(1 + \frac{p}{r} \right)^{\frac{r+1}{2}} \left(\frac{p+r}{e} \right)^{p/2} (1 + o(1)).$$

Therefore, $w(D) \leq \left(\frac{en}{r} \right)^{r/p} \left(\frac{p+r}{e} \right)^{1/2} (1 + o(1))$. Now set $p = 2r \log(\frac{en}{r})$. Then

$$w(D) \leq (p+r)^{1/2}(1 + o(1)) = \sqrt{2r \log \frac{e^{3/2} n}{r}}(1 + o(1)).$$

∎

To deduce (18) we define $S = \bigcup_f K_f$, where the union is over all r-sparse functions f. Then (18) is equivalent to

$$S \cap Y = \emptyset. \tag{19}$$

Lemma 4.3 implies that $S \subseteq (\sqrt{2}+1)D$. Then by Lemma 4.4,

$$w(S) \leq (\sqrt{2} + 1)w(D) = (1 - o(1))\sqrt{k(r, n)}.$$

Then (19) follows Gordon's Theorem 4.2. This completes the proof of Theorem 4.1. ∎

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