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SPATIAL BEHAVIOUR IN A MINDLIN-TYPE THERMOELASTIC PLATE

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Abstract. Spatial behaviour is studied for the transient solutions in the bending of a Mindlin-type thermoelastic plate. Some appropriate time-weighted line-integral measures are associated with the transient solutions and the spatial estimates are established for these measures describing spatial behaviour results of the Saint-Venant and Phragmén-Lindelöf type. A complete description of the spatial behaviour is obtained by combining the spatial estimates with time-independent and time-dependent decay and growth rates. For a thermoelastic plate whose middle surface is like a semi-infinite strip, it is shown, by means of the maximum principle, that at infinity a sharper spatial decay holds and it is dominated by the thermal characteristics only. Uniqueness results are also established.

1. Introduction. A linear thermoelastic thin plate model has been developed by Schiavone and Tait [1]. For such a model the bending of a Mindlin-type thermoelastic plate is examined in [2] when the source terms are harmonic in time and sufficient time has elapsed for the system to have reached a steady-state. A uniqueness result is established for exterior boundary value problems subject to certain regularity assumptions and some appropriate radiation conditions and a condition on the angular frequency of oscillation.

In this paper we study the spatial behaviour of the transient solutions of the thermoelastic plate model introduced in [1]. We associate with the transient solutions some appropriate time-weighted line-integral measures and then we establish the spatial estimates describing their spatial behaviour. In fact, we establish spatial estimates of two sorts: one is characterized by a time-dependent decay rate which is suitable for appropriate short values of time and the other has a time-independent decay rate which is indicated for appropriate large values of time. By combining the two sorts of spatial estimates we get a complete description of the spatial behaviour of the transient solutions. In particular, for a bounded plate we establish some spatial decay estimates of Saint-Venant

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type, while for an unbounded plate we establish alternatives of Phragmén-Lindelöf type. For literature reviews on these topics, we refer to the surveys by Horgan [3,4].

For the case of a thermoelastic plate whose middle surface is like a semi-infinite strip we use the maximum principle in order to improve the above spatial decay estimates. Thus, we prove that at large spatial distances of the end of the strip a sharper spatial decay holds and it is controlled by the thermal coefficients only.

As a direct consequence of the above results we obtain some uniqueness results.

2. Basic formulation. Throughout this paper Greek and Latin subscripts take the values $1, 2$, and $1, 2, 3$, respectively, summation is carried out over repeated indices, and $x = (x_1, x_2)$ and $x = (x_1, x_2, x_3)$ are generic points referred to orthogonal Cartesian coordinates in R^2 and R^3, respectively. A superposed dot denotes $\dfrac{\partial}{\partial t}$ and the suffix "k" denotes $\dfrac{\partial}{\partial x_k}$.

Let $\overline{S} \times \left[-\dfrac{h_0}{2}, \dfrac{h_0}{2}\right]$ be the region occupied by a homogeneous thin thermoelastic plate, where S is a domain in R^2 bounded by a simple C^2-curve ∂S and $0 < h_0 = \text{const} << \operatorname{diam} S$ is the thickness. We assume that, in addition to mechanical loads, the plate is subject to an unknown temperature distribution $\tau(x_1, x_2, x_3, t)$ measured from a reference state of uniform temperature distribution τ_0, at which temperature the plate is free from thermal stresses and strains. It is further assumed that the plate is elastically and thermally isotropic.

The equations of motion for the bending of a Mindlin-type thermoelastic plate are [1]:

$$h^2(\lambda + \mu)u_{\rho,\rho\alpha} + \mu(h^2 u_{\alpha,\rho\rho} - u_\alpha - u_{3,\alpha}) - u_{4,\alpha} = \rho h^2 \ddot{u}_\alpha - H_\alpha, \tag{1}$$

$$\mu(u_{3,\rho\rho} + u_{\alpha,\alpha}) = \rho \ddot{u}_3 - F_3, \tag{2}$$

$$u_{4,\rho\rho} - \frac{1}{K}\dot{u}_4 - \eta\alpha h^2(3\lambda + 2\mu)\dot{u}_{\rho,\rho} = N. \tag{3}$$

Here we have used the following notation:

$$\begin{gathered} u = u(x_\alpha, t) = (u_1, u_2, 0), \quad u_3 = u_3(x_\alpha, t), \\ u_4 = u_4(x_\alpha, t) = \tfrac{3\lambda+2\mu}{h_0} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} x_3 \varepsilon_\tau dx_3, \\ H = H(x_\alpha, t) = (H_1, H_2, 0), \quad F_3 = F_3(x_\alpha, t), \quad N = N(x_\alpha, t), \quad h^2 = \tfrac{h_0^2}{12}, \end{gathered} \tag{4}$$

where λ and μ are the Lamé constants, ε_τ denotes the thermal strain, α is the coefficient of thermal expansion,

$$\eta = \frac{(3\lambda + 2\mu)\,\alpha\tau_0}{\lambda_0}, \quad K = \frac{\lambda_0}{\rho c}, \tag{5}$$

ρ and c are, respectively, the constant mass density and specific heat of the plate, and $\lambda_0 > 0$ is the (constant) coefficient of thermal conductivity. It should be noted that H and F_3 characterize resultant body forces and couples, as well as forces and couples on the plate's faces, and N is a known quantity that represents heat generation within the plate and measurements of temperatures of the faces and in the surrounding medium. In accordance with the plate assumptions [1], u_i characterize displacement and u_4 the resultant "thermal moment" on the plate's middle surface.

Furthermore, we consider the boundary-initial value problem $(\mathcal{P})$ defined by the relations (1) to (3) and the initial conditions

$$\begin{gathered} u_\alpha(x,0) = u_\alpha^0(x), \quad \dot{u}_\alpha(x,0) = \dot{u}_\alpha^0(x), \\ u_3(x,0) = u_3^0(x), \quad \dot{u}_3(x,0) = \dot{u}_3^0(x), \quad u_4(x,0) = u_4^0(x), \quad x \in \bar{S}, \end{gathered} \tag{6}$$

and the following boundary conditions

$$u_\alpha(x,t) = \tilde{u}_\alpha(x,t), \quad u_3(x,t) = \tilde{u}_3(x,t), \quad u_4(x,t) = \tilde{u}_4(x,t) \quad \text{on} \quad \partial S \times [0,\infty). \tag{7}$$

In the above relations u_α^0, $\dot{u}_\alpha^0$, u_3^0, $\dot{u}_3^0$, u_4^0, $\tilde{u}_\alpha$, $\tilde{u}_3$, and $\tilde{u}_4$ are prescribed continuous functions.

By a transient solution of the boundary-initial value problem $(\mathcal{P})$, corresponding to the given data $\mathcal{D} = \left\{H_\alpha, F_3, N; u_\alpha^0, \dot{u}_\alpha^0, u_3^0, \dot{u}_3^0, u_4^0; \tilde{u}_\alpha, \tilde{u}_3, \tilde{u}_4\right\}$, we mean the ordered array $\{u_\alpha, u_3, u_4\}$ satisfying the basic equations (1) to (3), the initial conditions (6) and the boundary conditions (7).

Throughout this paper we assume that

$$\rho > 0, \ c > 0, \ \alpha > 0, \ \mu > 0, \ 3\lambda + 2\mu > 0, \ \tau_0 > 0. \tag{8}$$

This implies that the quadratic form

$$W(\xi) = \frac{1}{2}\lambda\xi_{rr}\xi_{ss} + \mu\xi_{ij}\xi_{ij}, \quad \xi = (\xi_{ij}), \quad \xi_{ij} = \xi_{ji}, \tag{9}$$

is positive definite. It follows then that there exist the minimum elastic modulus $\mu_m > 0$ and the maximum elastic modulus $\mu_M > 0$ so that we have [5]

$$\mu_m\xi_{ij}\xi_{ij} \le 2W(\xi) \le \mu_M\xi_{ij}\xi_{ij}, \quad \xi = (\xi_{ij}), \quad \xi_{ij} = \xi_{ji}. \tag{10}$$

Further, if we set

$$t_{ij}(\xi) = \lambda\xi_{rr}\delta_{ij} + 2\mu\xi_{ij}, \quad \xi = (\xi_{ij}), \quad \xi_{ij} = \xi_{ji}, \tag{11}$$

then we have [5]

$$t_{ij}(\xi)t_{ij}(\xi) \le 2\mu_M W(\xi), \quad \xi = (\xi_{ij}), \quad \xi_{ij} = \xi_{ji}. \tag{12}$$

Let us consider the state of bending, that is [1] $v_1 = x_3u_1$, $v_2 = x_3u_2$, $v_3 = u_3$. We set

$$\xi_{ij} = e_{ij}(V) = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad V = (v_1, v_2, v_3), \tag{13}$$

into the relation (11) so that we get

$$t_{\alpha\beta}(\xi) = x_3\tau_{\alpha\beta}, \quad t_{\alpha 3}(\xi) = \tau_{\alpha 3}, \quad t_{33}(\xi) = x_3\tau_{33}, \tag{14}$$

$$\tau_{\alpha\beta} = \lambda u_{\rho,\rho}\delta_{\alpha\beta} + \mu(u_{\alpha,\beta} + u_{\beta,\alpha}), \ \tau_{\alpha 3} = \mu(u_\alpha + u_{3,\alpha}), \ \tau_{33} = \lambda u_{\rho,\rho}\,. \tag{15}$$

We further substitute the relation (14) into (12) and then we integrate the result with respect to x_3 over $\left[-\frac{h_0}{2}, \frac{h_0}{2}\right]$. Thus, we deduce that

$$h^2\left(\tau_{\alpha\beta}\tau_{\alpha\beta} + \tau_{33}^2\right) + 2\tau_{\alpha 3}\tau_{\alpha 3} \le 2\mu_M W^*(U), \tag{16}$$

where

$$W^*(U) = \frac{1}{h_0}\int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} W(e)dx_3 \tag{17}$$

$$= h^2\left\{\frac{1}{2}\lambda u_{\rho,\rho}u_{\alpha,\alpha} + \mu\left[u_{1,1}^2 + u_{2,2}^2 + \frac{1}{2}(u_{1,2}+u_{2,1})^2\right]\right\}$$

$$+\frac{1}{2}\mu(u_\alpha + u_{3,\alpha})(u_\alpha + u_{3,\alpha})\ , \qquad U = (u_1, u_2, u_3)\,.$$

Finally, we note that the relations (15) and (17) give

$$\dot{W}^*(U) = h^2\tau_{\alpha\beta}\dot{u}_{\alpha,\beta} + \tau_{\alpha 3}(\dot{u}_\alpha + \dot{u}_{3,\alpha})\ . \tag{18}$$

3. Spatial estimates with time-independent decay and growth rates. Let us consider a given time $T > 0$ and let $\{u_\alpha, u_3, u_4\}$ be a transient solution of the boundary-initial value problem $(\mathcal{P})$ corresponding to the given data

$$\mathcal{D} = \left\{H_\alpha, F_3, N; u_\alpha^0, \dot{u}_\alpha^0, u_3^0, \dot{u}_3^0, u_4^0; \tilde{u}_\alpha, \tilde{u}_3, \tilde{u}_4\right\} \text{ on } [0,T]\,.$$

We denote by $\hat{\mathcal{D}}_T$ the set of all $x \in \overline{S}$ such that

i) if $x \in S$, then

$$u_\alpha^0(x) \neq 0 \ \text{ or } \ \dot{u}_\alpha^0(x) \neq 0 \ \text{ or } \ u_3^0(x) \neq 0 \ \text{ or } \ \dot{u}_3^0(x) \neq 0 \ \text{ or } \ u_4^0(x) \neq 0\ , \tag{19}$$

or

$$H_\alpha(x,t) \neq 0 \ \text{ or } \ F_3(x,t) \neq 0 \ \text{ or } \ N(x,t) \neq 0 \ \text{ for some } t \in [0,T]\,; \tag{20}$$

or

ii) if $x \in \partial S$, then

$$\tilde{u}_\alpha(x,t) \neq 0 \ \text{ or } \ \tilde{u}_3(x,t) \neq 0 \ \text{ or } \ \tilde{u}_4(x,t) \neq 0 \ \text{ for some } t \in [0,T]\,. \tag{21}$$

Roughly speaking, $\hat{\mathcal{D}}_T$ represents the support of the initial and boundary data and the supply terms on the time interval $[0,T]$. Throughout this section we shall assume that $\hat{\mathcal{D}}_T$ is a bounded set.

We consider next a nonempty set $\hat{\mathcal{D}}_T^*$ so that $\hat{\mathcal{D}}_T \subset \hat{\mathcal{D}}_T^* \subset \overline{S}$ such that:

(i) if $\hat{\mathcal{D}}_T \cap S \neq \emptyset$ then we choose $\hat{\mathcal{D}}_T^*$ to be the smallest bounded regular region in $\overline{S}$ that includes $\hat{\mathcal{D}}_T$; in particular, we set $\hat{\mathcal{D}}_T^* = \hat{\mathcal{D}}_T$ if $\hat{\mathcal{D}}_T$ also happens to be a regular region;

(ii) if $\emptyset \neq \hat{\mathcal{D}}_T \subset \partial S$, then we choose $\hat{\mathcal{D}}_T^*$ to be the smallest regular subcurve of ∂S that includes $\hat{\mathcal{D}}_T$; in particular, we set $\hat{\mathcal{D}}_T^* = \hat{\mathcal{D}}_T$ if $\hat{\mathcal{D}}_T$ is a regular subcurve of ∂S;

(iii) if $\hat{\mathcal{D}}_T = \emptyset$, then we choose $\hat{\mathcal{D}}_T^*$ to be an arbitrary nonempty regular subcurve of ∂S.

On this basis we introduce the set $\mathcal{D}_r$, $r > 0$, by

$$\mathcal{D}_r = \left\{x \in \overline{S} : \hat{\mathcal{D}}_T^* \cap \overline{\Sigma(x,r)} \neq \emptyset\right\}, \tag{22}$$

where $\Sigma(x,r)$ is the open disk with radius r and center at x. Furthermore, we shall use the notation S_r for the part of S contained in $S\backslash\mathcal{D}_r$, and we set $S(r_1,r_2) = S_{r_2}\backslash S_{r_1}$, $r_1 > r_2$. Moreover, we shall denote by L_r the subcurve of ∂S_r contained inside of S and whose outward unit normal vector is forwarded to the exterior of D_r .

We associate with the solution $\{u_\alpha, u_3, u_4\}$ the following time-weighted line-integral function [6]:

$$I(r,t) = -\int_0^t \int_{L_r} e^{-\sigma s}\left[h^2\tau_{\alpha\beta}(s)\dot{u}_\beta(s) + \tau_{\alpha 3}(s)\dot{u}_3(s) + Au_4(s)u_{4,\alpha}(s) - u_4(s)\dot{u}_\alpha(s)\right]n_\alpha d\sigma ds \ , \ r \geq 0 \ , \ t \in [0,T], \tag{23}$$

where σ is a prescribed strictly positive parameter, $\tau_{\alpha\beta}$ and $\tau_{\alpha 3}$ are defined by the relation (15) and

$$A = \frac{\lambda_0}{\tau_0 \alpha^2 h^2 (3\lambda + 2\mu)^2}. \tag{24}$$

On the basis of the relations (21) to (23) and the divergence theorem, we have

$$\begin{aligned} &I(r_1,t) - I(r_2,t) \\ &= -\int_0^t \int_{\partial S(r_1,r_2)} e^{-\sigma s}\left[h^2\tau_{\alpha\beta}(s)\dot{u}_\beta(s)+\tau_{\alpha 3}(s)\dot{u}_3(s)+Au_4(s)u_{4,\alpha}(s)-u_4(s)\dot{u}_\alpha(s)\right]n_\alpha d\sigma ds \\ &= -\int_0^t \int_{S(r_1,r_2)} e^{-\sigma s}\left[h^2\tau_{\alpha\beta,\alpha}(s)\dot{u}_\beta(s) + h^2\tau_{\alpha\beta}(s)\dot{u}_{\beta,\alpha}(s) + \tau_{\alpha 3,\alpha}(s)\dot{u}_3(s) + \tau_{\alpha 3}(s)\dot{u}_{3,\alpha}(s)\right. \\ &\quad \left. + Au_{4,\alpha}(s)u_{4,\alpha}(s) + Au_4(s)u_{4,\alpha\alpha}(s) - u_{4,\alpha}(s)\dot{u}_\alpha(s) - u_4(s)\dot{u}_{\alpha,\alpha}(s)\right] da ds, \\ &\quad t \in [0,T], \ r_1 > r_2. \end{aligned} \tag{25}$$

Then by using the relations (15), (18), (20) to (23) and the basic equations (1) to (3), from the relation (25) we deduce

$$I(r_1,t) - I(r_2,t) = -\int_0^t \int_{S(r_1,r_2)} e^{-\sigma s}\left\{\frac{1}{2}\frac{\partial}{\partial s}\left[\rho h^2\dot{u}_\alpha(s)\dot{u}_\alpha(s) + \rho\dot{u}_3^2(s) + \frac{A}{K}u_4^2(s) + 2W^*(U(s))\right] + Au_{4,\rho}(s)u_{4,\rho}(s)\right\} da ds \ , \ t \in [0,T], \ r_1 > r_2, \tag{26}$$

so that, by an integration by parts, we get

$$\begin{aligned} I(r_1,t) - I(r_2,t) = &-\int_{S(r_1,r_2)} e^{-\sigma t}\frac{1}{2}\left[\rho h^2\dot{u}_\alpha(t)\dot{u}_\alpha(t)+\rho\dot{u}_3^2(t)+\frac{A}{K}u_4^2(t)+2W^*(U(t))\right] da \\ &-\int_0^t \int_{S(r_1,r_2)} e^{-\sigma s}\{\frac{\sigma}{2}[\rho h^2\dot{u}_\alpha(s)\dot{u}_\alpha(s) + \rho\dot{u}_3^2(s) + \frac{A}{K}u_4^2(s) + 2W^*(U(s))] \\ &+ Au_{4,\rho}(s)u_{4,\rho}(s)\} da ds \ , \ t \in [0,T], \ r_1 > r_2. \end{aligned} \tag{27}$$

Thus, from the relation (27), we obtain

$$\begin{aligned} \frac{\partial I}{\partial r}(r,t) = &-\int_{L_r} e^{-\sigma t}\frac{1}{2}\left[\rho h^2\dot{u}_\alpha(t)\dot{u}_\alpha(t) + \rho\dot{u}_3^2(t) + \frac{A}{K}u_4^2(t) + 2W^*(U(t))\right] d\sigma \\ &-\int_0^t \int_{L_r} e^{-\sigma s}\{\frac{\sigma}{2}[\rho h^2\dot{u}_\alpha(s)\dot{u}_\alpha(s) + \rho\dot{u}_3^2(s) + \frac{A}{K}u_4^2(s) + 2W^*(U(s))] \\ &+ Au_{4,\rho}(s)u_{4,\rho}(s)\} d\sigma ds \ , \ t \in [0,T], \ r \geq 0. \end{aligned} \tag{28}$$

We proceed now to obtain an appropriate estimate for $|I(r,t)|$. In this aim we write the relation (23) in the form

$$I(r,t) = -\int_0^t \int_{L_r} e^{-\sigma s} \left[S_{\alpha i}(s)\dot{\psi}_i(s) + Au_4(s)u_{4,\alpha}(s) \right] n_\alpha d\sigma ds, \tag{29}$$

where

$$S_{\alpha\beta} = h\tau_{\alpha\beta} - h^{-1}u_4\delta_{\alpha\beta} \quad , \quad S_{\alpha 3} = \tau_{\alpha 3} \quad , \tag{30}$$

and

$$\psi_\alpha = hu_\alpha \quad , \quad \psi_3 = u_3 \ . \tag{31}$$

Then we use the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality in (29) in order to obtain

$$|I(r,t)| = \left| \int_0^t \int_{L_r} e^{-\sigma s} \left\{ \left[\frac{1}{\sqrt{\rho}} S_{\alpha i}(s) \right] \left[\sqrt{\rho}\dot{\psi}_i(s) n_\alpha \right] + \left[\sqrt{A}u_4(s)n_\alpha \right] \left[\sqrt{A}u_{4,\alpha}(s) \right] \right\} d\sigma ds \right| \tag{32}$$

$$\leq \frac{1}{2} \int_0^t \int_{L_r} e^{-\sigma s} \left[\frac{1}{\varepsilon_1 \rho} S_{\alpha i}(s)S_{\alpha i}(s) + \varepsilon_1 \rho \dot{\psi}_i(s)\dot{\psi}_i(s) + \frac{1}{\varepsilon_2} Au_4^2(s) + \varepsilon_2 Au_{4,\alpha}(s)u_{4,\alpha}(s) \right] d\sigma ds,$$
$$t \in [0,T], \ r \geq 0, \ \forall \varepsilon_1, \ \varepsilon_2 > 0.$$

With a view toward establishing an estimate for $S_{\alpha i}S_{\alpha i}$ we use the relations (16) and (30) and the Cauchy-Schwarz inequality in order to obtain

$$S_{\alpha i}S_{\alpha i} = S_{\alpha\beta}\left(h\tau_{\alpha\beta} - h^{-1}u_4\delta_{\alpha\beta}\right) + S_{\alpha 3}\tau_{\alpha 3} = \left[S_{\alpha\beta}\left(h\tau_{\alpha\beta}\right) + S_{\alpha 3}\tau_{\alpha 3}\right] \tag{33}$$
$$-S_{\alpha\beta}\left(h^{-1}u_4\delta_{\alpha\beta}\right) \leq \left(S_{\alpha i}S_{\alpha i}\right)^{\frac{1}{2}} \left(h^2\tau_{\rho\beta}\tau_{\rho\beta} + \tau_{\rho 3}\tau_{\rho 3}\right)^{\frac{1}{2}} + \left(S_{\alpha i}S_{\alpha i}\right)^{\frac{1}{2}} \left(h^{-2}u_4^2\delta_{\rho\beta}\delta_{\rho\beta}\right)^{\frac{1}{2}}$$
$$\leq \left(S_{\alpha i}S_{\alpha i}\right)^{\frac{1}{2}} \left[\left(2\mu_M W^*(U)\right)^{\frac{1}{2}} + \sqrt{2}h^{-1}|u_4| \right],$$

so that we deduce that

$$\left(S_{\alpha i}S_{\alpha i}\right)^{\frac{1}{2}} \leq \left[2\mu_M W^*(U)\right]^{\frac{1}{2}} + \sqrt{2}h^{-1}|u_4| \,. \tag{34}$$

By using the arithmetic-geometric mean inequality in (34), we get

$$S_{\alpha i}S_{\alpha i} \leq 2(1+\varepsilon)\mu_M W^*(U) + 2(1+\frac{1}{\varepsilon})h^{-2}u_4^2 \quad , \quad \forall\varepsilon > 0. \tag{35}$$

By combining the relations (31) and (35) with the relation (32), we deduce that

$$|I(r,t)| \leq \int_0^t \int_{L_r} e^{-\sigma s} \{ \frac{\varepsilon_1}{\sigma} [\frac{\sigma}{2} \left(\rho h^2 \dot{u}_\alpha(s)\dot{u}_\alpha(s) + \rho\dot{u}_3^2(s)\right)] \tag{36}$$
$$+ [\frac{K}{\sigma\varepsilon_2} + \frac{2K}{\sigma A\varepsilon_1\rho h^2}(1+\frac{1}{\varepsilon})] \frac{\sigma}{2} \frac{A}{K} u_4^2(s) + \frac{\mu_M(1+\varepsilon)}{\sigma\varepsilon_1\rho} [\sigma W^*(U(s))]$$
$$+ \frac{\varepsilon_2}{2} Au_{4,\alpha}(s)u_{4,\alpha}(s) \} d\sigma ds \ , \qquad \forall\varepsilon, \ \varepsilon_1, \ \varepsilon_2 > 0, \ t \in [0,T], \ r \geq 0.$$

We now equate the coefficients of the various energetic terms in the last integral in (36), that is we set

$$\frac{\varepsilon_1}{\sigma} = \frac{K}{\sigma\varepsilon_2} + \frac{2K}{\sigma A\varepsilon_1\rho h^2}(1+\frac{1}{\varepsilon}) = \frac{\mu_M(1+\varepsilon)}{\sigma\varepsilon_1\rho} = \frac{\varepsilon_2}{2}. \tag{37}$$

Therefore, we choose

$$\varepsilon_1 = k \ , \quad \varepsilon_2 = \frac{2k}{\sigma} \ , \quad k = \sqrt{\frac{\mu_M(1+\varepsilon)}{\rho}} \ , \tag{38}$$

where ε is the positive root of the algebraic equation

$$\varepsilon^2 + \varepsilon\left(1 - \frac{K\sigma\rho}{2\mu_M} - \frac{2K}{Ah^2\mu_M}\right) - \frac{2K}{Ah^2\mu_M} = 0. \tag{39}$$

With the above choices, the inequality (36) becomes

$$\begin{aligned}|I(r,t)| \leq \frac{k}{\sigma}\int_0^t \int_{L_r} e^{-\sigma s}\{\frac{\sigma}{2}[\rho h^2 \dot{u}_\alpha(s)\dot{u}_\alpha(s) + \rho\dot{u}_3^2(s) + \frac{A}{K}u_4^2(s) + 2W^*(U(s))] \\ + Au_{4,\alpha}(s)u_{4,\alpha}(s)\}\, d\sigma ds \ , \ t \in [0,T], \ r \geq 0.\end{aligned} \tag{40}$$

By taking into account the relations (8), (10), (17), (28), and (40), we deduce the following first-order partial differential inequality

$$\frac{\sigma}{k}|I(r,t)| + \frac{\partial I}{\partial r}(r,t) \leq 0, \quad r \geq 0, \ \ 0 \leq t \leq T. \tag{41}$$

At this time we note that our further analysis requires a separate discussion for bounded and unbounded thermoelastic plates.

Let us first consider the case of a bounded thermoelastic plate. It follows then that r ranges over $[0,\Delta]$ where Δ is the diameter of the region $S_{(r=0)}$. In view of the relations (21) to (23), we deduce that

$$I(\Delta,t) = 0 \ , \qquad t \in [0,T]. \tag{42}$$

On the other hand, by means of the relations (8), (16), and (24), from (28) it follows that

$$\frac{\partial I}{\partial r}(r,t) \leq 0, \quad r \in [0,\Delta], \ \ t \in [0,T]. \tag{43}$$

Then the relations (42) and (43) prove that

$$I(r,t) \geq 0, \quad r \in [0,\Delta], \ \ t \in [0,T], \tag{44}$$

and moreover, if we set $r_1 = \Delta, \ r_2 = r \in [0,\Delta]$ in (27), we get

$$I(r,t) = \mathcal{E}(r,t), \tag{45}$$

where

$$\begin{aligned}\mathcal{E}(r,t) = \int_{S_r} e^{-\sigma t}\frac{1}{2}\left[\rho h^2\dot{u}_\alpha(t)\dot{u}_\alpha(t) + \rho\dot{u}_3^2(t) + \frac{A}{K}u_4^2(t) + 2W^*(U(t))\right] da \\ + \int_0^t \int_{S_r} e^{-\sigma s}\{\frac{\sigma}{2}[\rho h^2\dot{u}_\alpha(s)\dot{u}_\alpha(s) + \rho\dot{u}_3^2(s) + \frac{A}{K}u_4^2(s) + 2W^*(U(s))] \\ + Au_{4,\rho}(s)u_{4,\rho}(s)\}\, dads \ , \ t \in [0,T], \ r \in [0,\Delta].\end{aligned} \tag{46}$$

Thus, the relation (41) can be written in the form

$$\frac{\partial}{\partial r}\left[\exp\left(\frac{\sigma}{k}r\right)\mathcal{E}(r,t)\right] \leq 0, \quad r \in [0,\Delta], \ \ t \in [0,T], \tag{47}$$

so that by an integration with respect to r we deduce the following estimate of Saint-Venant type

$$\mathcal{E}(r,t) \le \mathcal{E}(0,t)\exp\left(-\frac{\sigma}{k}r\right), \quad r\in[0,\Delta], \quad t\in[0,T]. \tag{48}$$

Let us now consider the case of an unbounded thermoelastic plate. Then r ranges over $[0,\infty)$. Let t be fixed in $[0,T]$. Since $I(r,t)$ is a non-increasing function with respect to r, it follows that we can have only two possibilities:

(a) $I(r,t)\ge 0$ for all $r\ge 0$

or

(b) there exists $r_t \ge 0$ so that $I(r_t,t)<0$.

For part (a) we can proceed as in obtaining the estimate (48) to get

$$I(r,t) \le I(0,t)\exp\left(-\frac{\sigma}{k}r\right), \quad r\in[0,\infty), \quad t\in[0,T]. \tag{49}$$

For part (b), by means of the relation (43), we deduce that

$$I(r,t)\le I(r_t,t)<0 \quad \text{for all } r\ge r_t,$$

and, therefore, the differential inequality (41) implies

$$\frac{\partial}{\partial r}\left[\exp\left(-\frac{\sigma}{k}r\right)I(r,t)\right]\le 0, \quad r\ge r_t>0, \quad t\in[0,T], \tag{50}$$

so that by an integration with respect to r we get

$$-I(r,t)\ge -I(r_t,t)\exp\left[\frac{\sigma}{k}(r-r_t)\right]>0, \quad r\ge r_t>0, \quad t\in[0,T]. \tag{51}$$

Thus, the relations (49) and (51) describe an alternative of Phragmén-Lindelöf type for the spatial behaviour of the transient solutions in a Mindlin-type thermoelastic unbounded plate. In view of the relations (49) and (51) and by making r_1 tend to infinity into the relation (27), this means that for a transient solution either $\mathcal{E}(r,t)$ is bounded and then $I(r,t)=\mathcal{E}(r,t)$ decays faster than an exponential decaying function or $\mathcal{E}(r,t)$ is unbounded and then $I(r,t)$ grows faster than an exponential growing function.

The above spatial decay estimates are characterized by the fact that the decay rate is independent of time. We proceed in the next section to improve the above spatial description for short values of time by establishing spatial estimates with time-dependent decay and growth rates.

4. Spatial estimates with time-dependent decay and growth rates. We introduce the measure

$$\mathcal{I}(r,t)=\int_0^t I(r,s)ds, \quad r\ge 0, \quad t\in[0,T], \tag{52}$$

where $I(r,t)$ is defined by the relation (23).

On the basis of the inequality

$$\int_0^t\int_0^s f^2(z)dzds\le t\int_0^t f^2(s)ds, \tag{53}$$

and by using the relations (29) and (52), we deduce

$$|\mathcal{I}(r,t)| \leq \left(t\int_0^t \int_{L_r} e^{-\sigma s}\dot{\psi}_i(s)\dot{\psi}_i(s)d\sigma ds\right)^{\frac{1}{2}} \left(t\int_0^t \int_{L_r} e^{-\sigma s}S_{\alpha i}(s)S_{\alpha i}(s)d\sigma ds\right)^{\frac{1}{2}} \tag{54}$$
$$+\left(\sqrt{t}\int_0^t \int_{L_r} e^{-\sigma s}Au_4^2(s)d\sigma ds\right)^{\frac{1}{2}} \left(\sqrt{t}\int_0^t\int_0^s \int_{L_r} e^{-\sigma z}Au_{4,\alpha}(z)u_{4,\alpha}(z)d\sigma dz ds\right)^{\frac{1}{2}},$$

so that, by means of the arithmetic-geometric mean inequality and the estimate (35), we deduce

$$|\mathcal{I}(r,t)| \leq \sqrt{t}\int_0^t \int_{L_r} e^{-\sigma s}\left\{\sqrt{t}\varepsilon_1\frac{1}{2}\left[\rho h^2\dot{u}_\alpha(s)\dot{u}_\alpha(s)+\rho\dot{u}_3^2(s)\right]\right. \tag{55}$$
$$\left.+\left[\frac{K}{\varepsilon_2}+\frac{2K\sqrt{t}}{A\varepsilon_1\rho h^2}\left(1+\frac{1}{\varepsilon}\right)\right]\frac{A}{2K}u_4^2(s)+\frac{\sqrt{t}}{\varepsilon_1\rho}(1+\varepsilon)\mu_M W^*(U(s))\right\}d\sigma ds$$
$$+\sqrt{t}\int_0^t\int_0^s\int_{L_r} e^{-\sigma z}\frac{\varepsilon_2}{2}Au_{4,\alpha}(z)u_{4,\alpha}(z)d\sigma dz ds,\ \forall\varepsilon,\ \varepsilon_1,\ \varepsilon_2>0.$$

Now we set

$$\sqrt{t}\varepsilon_1 = \frac{K}{\varepsilon_2}+\frac{2K\sqrt{t}}{A\varepsilon_1\rho h^2}\left(1+\frac{1}{\varepsilon}\right)=\frac{\sqrt{t}\mu_M(1+\varepsilon)}{\varepsilon_1\rho}=\frac{\varepsilon_2}{2}, \tag{56}$$

that is, we choose

$$\varepsilon_1=\mathcal{K},\quad \varepsilon_2=2\sqrt{t}\mathcal{K},\quad \mathcal{K}=\sqrt{\frac{\mu_M[1+\delta(t)]}{\rho}}, \tag{57}$$

where δ is the positive root of the algebraic equation

$$\varepsilon^2+\varepsilon\left(1-\frac{K\rho}{2t\mu_M}-\frac{2K}{Ah^2\mu_M}\right)-\frac{2K}{Ah^2\mu_M}=0. \tag{58}$$

With these choices and by using the relations (28) and (55) we deduce for $\mathcal{I}(r,t)$ the following first-order partial differential inequality

$$t\mathcal{K}(t)\frac{\partial\mathcal{I}}{\partial r}(r,t)+|\mathcal{I}(r,t)|\leq 0,\quad r\geq 0,\quad 0\leq t\leq T, \tag{59}$$

which is similar to the relation (41).

Thus, by using a discussion entirely similar with that in the above analysis we deduce that for a bounded thermoelastic plate we have

$$\mathcal{F}(r,t)\leq\mathcal{F}(0,t)\exp\left(-\frac{1}{t\mathcal{K}(t)}r\right),\quad r\in[0,\Delta],\quad t\in[0,T], \tag{60}$$

where

$$\mathcal{F}(r,t)=\int_0^t\mathcal{E}(r,s)ds. \tag{61}$$

For the case of an unbounded thermoelastic plate either $\mathcal{I}(r,t)\geq 0$ for all $r\geq 0$ and then we have

$$\mathcal{I}(r,t) \leq \mathcal{I}(0,t)\exp\left(-\frac{1}{t\mathcal{K}(t)}r\right), \quad r \in [0,\infty), \quad t \in [0,T], \tag{62}$$

or there exists $r_t^* \geq 0$ so that $\mathcal{I}(r_t^*,t) < 0$ and then we have $\mathcal{I}(r,t) < 0$ for all $r \geq r_t^*$ and

$$-\mathcal{I}(r,t) \geq -\mathcal{I}(r_t^*,t)\exp\left[\frac{1}{t\mathcal{K}(t)}\left(r - r_t^*\right)\right], \quad r \geq r_t^*, \quad t \in [0,T]. \tag{63}$$

As it can be seen, the estimates (60), (62), and (63) give a good description for the spatial behaviour of the transient solutions for appropriate short values of time, while the estimates (48), (49), and (51) are convenient to use for large values of time. Thus, by combining the estimates (48), (49), and (51) with the estimates (60), (62), and (63), respectively, we can obtain a complete description for the spatial behaviour of the transient solutions in bounded and unbounded thermoelastic plates.

5. Other estimates. In this section we try to improve the above spatial decay estimates. In this aim we use an idea used by R. Quintanilla [7] in order to prove that an appropriate measure derived from $I(r,t)$ can be bounded above by the solution to a related initial-boundary value problem for the one-dimensional heat equation. Such a solution is well-known in standard textbooks and it allows to obtain various estimates for the measure in question. It was first used by Horgan, Payne and Wheeler [8] to study the spatial decay of solutions to initial-boundary value problems for the transient heat equation in a three-dimensional cylinder, subject to nonzero boundary conditions only on the ends. See Horgan and Quintanilla [9] for a generalization to functionally graded materials.

In what follows we consider that S is a semi-infinite strip of width l, and choose a Cartesian frame of reference such that $S_0 \equiv S_{(r=0)}$ is defined by

$$S_0 \equiv \left\{x = (x_1, x_2) \in R^2 : x_1 \in [0,\infty), \quad x_2 \in [0,l]\right\}, \tag{64}$$

where l is a positive constant. Thus, we have

$$S_{x_1} = S_{(r=x_1)} = [x_1,\infty) \times [0,l], \quad L_{x_1} = L_{(r=x_1)} = [0,l].$$

Because we have in our mind the above decay estimates, we assume that $\{u_\alpha, u_3, u_4\}$ resides in the class of solutions for which

$$\begin{aligned}\mathcal{E}(x_1,t) = &\int_{S_{x_1}} e^{-\sigma t}\frac{1}{2}\left[\rho h^2\dot{u}_\alpha(t)\dot{u}_\alpha(t) + \rho\dot{u}_3^2(t) + \frac{A}{K}u_4^2(t) + 2W^*(U(t))\right]da \\ &+\int_0^t\int_{S_{x_1}} e^{-\sigma s}\{\frac{\sigma}{2}[\rho h^2\dot{u}_\alpha(s)\dot{u}_\alpha(s) + \rho\dot{u}_3^2(s) + \frac{A}{K}u_4^2(s) + 2W^*(U(s))] \\ &+Au_{4,\rho}(s)u_{4,\rho}(s)\}dads\end{aligned} \tag{65}$$

is bounded and therefore, by means of (49), we have the decay estimate

$$I(x_1,t) \leq I(0,t)\exp\left[-\frac{\sigma}{k}x_1\right], \quad t \in [0,T], \quad x_1 \in [0,\infty), \tag{66}$$

where $I(x_1,t)$ is defined now by

$$I(x_1,t) = -\int_0^t \int_{L_{x_1}} e^{-\sigma s}[h^2\tau_{1\beta}(s)\dot{u}_\beta(s) + \tau_{13}(s)\dot{u}_3(s) + Au_4(s)u_{4,1}(s) \\ -u_4(s)\dot{u}_1(s)]dx_2ds, \quad x_1 \in [0,\infty), \quad t \in [0,T]. \tag{67}$$

On this basis we can use the idea of [7] and define the following measure:

$$I^*(x_1,t) = \int_{x_1}^{\infty} I(\eta,t)d\eta, \quad x_1 \in [0,\infty), \quad t \in [0,T]; \tag{68}$$

that is,

$$I^*(x_1,t) = -\int_0^t \int_{S_{x_1}} e^{-\sigma s}[h^2\tau_{1\beta}(s)\dot{u}_\beta(s) + \tau_{13}(s)\dot{u}_3(s) + Au_4(s)u_{4,1}(s) \\ -u_4(s)\dot{u}_1(s)]dads, \quad x_1 \in [0,\infty), \quad t \in [0,T]. \tag{69}$$

Further, we note that

$$I(x_1,t) = \mathcal{E}(x_1,t), \tag{70}$$

$$\frac{\partial I^*}{\partial x_1}(x_1,t) = -I(x_1,t) = -\mathcal{E}(x_1,t), \tag{71}$$

$$\frac{\partial^2 I^*}{\partial x_1^2}(x_1,t) = \int_{L_{x_1}} e^{-\sigma t}\frac{1}{2}[\rho h^2\dot{u}_\alpha(t)\dot{u}_\alpha(t) + \rho\dot{u}_3^2(t) + \frac{A}{K}u_4^2(t) + 2W^*(U(t))]dx_2 \\ + \int_0^t \int_{L_{x_1}} e^{-\sigma s}\{\frac{\sigma}{2}[\rho h^2\dot{u}_\alpha(s)\dot{u}_\alpha(s) + \rho\dot{u}_3^2(s) + \frac{A}{K}u_4^2(s) + 2W^*(U(s))] \\ +Au_{4,\rho}(s)u_{4,\rho}(s)\}dx_2ds, \tag{72}$$

$$\frac{\partial I^*}{\partial t}(x_1,t) = -\int_{S_{x_1}} e^{-\sigma t}[h^2\tau_{1\beta}(t)\dot{u}_\beta(t) + \tau_{13}(t)\dot{u}_3(t) - u_4(t)\dot{u}_1(t)]da \\ +\frac{1}{2}\int_{L_{x_1}} e^{-\sigma t}Au_4^2(t)dx_2. \tag{73}$$

By using the Schwarz's inequality, the arithmetic-geometric mean inequality, and the relations (29), (30), (31), (35), and (72), from (73) we deduce that

$$\frac{\partial I^*}{\partial t}(x_1,t) \le \frac{1}{2}\int_{S_{x_1}} e^{-\sigma t}\{\frac{1}{\varepsilon_1\rho}\left[2(1+\varepsilon)\mu_M W^*(U(t)) + 2(1+\frac{1}{\varepsilon})h^{-2}u_4^2(t)\right] \\ +\varepsilon_1\rho\left[h^2\dot{u}_\alpha(t)\dot{u}_\alpha(t) + \dot{u}_3^2(t)\right]\}da + K\frac{\partial^2 I^*}{\partial x_1^2}(x_1,t), \quad \forall\varepsilon,\ \varepsilon_1 > 0. \tag{74}$$

If we set

$$\varepsilon_1 = \omega, \quad \omega = \sqrt{\frac{(1+\varepsilon)\mu_M}{\rho}}, \tag{75}$$

where ε is the positive root of the algebraic equation

$$\varepsilon^2 + \varepsilon\left(1 - \frac{2K}{Ah^2\mu_M}\right) - \frac{2K}{Ah^2\mu_M} = 0, \tag{76}$$

then, from (65), (71), and (74), we deduce that

$$\frac{\partial I^*}{\partial t}(x_1, t) \leq -\omega\frac{\partial I^*}{\partial x_1}(x_1, t) + K\frac{\partial^2 I^*}{\partial x_1^2}(x_1, t). \tag{77}$$

Further, we set

$$I^*(x_1, t) = e^{\frac{\omega}{2K}x_1 - \frac{\omega^2}{4K}t}\tilde{I}(x_1, t) \tag{78}$$

in (77), so that we get

$$\frac{\partial \tilde{I}}{\partial t}(x_1, t) \leq K\frac{\partial^2 \tilde{I}}{\partial x_1^2}(x_1, t). \tag{79}$$

Moreover, we use the change of variable

$$x_1 = \sqrt{K}z \tag{80}$$

in order to write the relation (79) under the form

$$\frac{\partial \tilde{I}}{\partial t}(z, t) \leq \frac{\partial^2 \tilde{I}}{\partial z^2}(z, t), \tag{81}$$

which can be used as in [8] in order to obtain an estimate for $\tilde{I}(z,t)$. Thus, an upper bound for $\tilde{I}(z,t)$ can be obtained in terms of the solution of the initial-boundary value problem for the one-dimensional heat equation

$$\begin{gathered}\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial z^2} = 0, \quad z > 0, \quad t > 0, \\ w(z,0) = 0, \quad z \geq 0, \\ w(0,t) = \tilde{I}(0,t) = I^*(0,t)e^{\frac{\omega^2}{4K}t}, \quad t \geq 0, \\ w(z,t) \to 0 \ \text{(uniformly in } t\text{) as } z \to \infty.\end{gathered} \tag{82}$$

Now we use the procedure first suggested in [8]. Thus, by the maximum principle for the heat equation, it follows that [10]

$$\tilde{I}(z,t) \leq w(z,t), \quad z \geq 0, \quad t \geq 0, \tag{83}$$

where [8, 11]

$$\begin{aligned} w(z,t) &= \frac{1}{2\sqrt{\pi}}\int_0^t z(t-s)^{-3/2}e^{-z^2/4(t-s)}e^{\frac{\omega^2}{4K}s}I^*(0,s)ds \\ &= \frac{1}{2\sqrt{\pi}}e^{\frac{\omega^2}{4K}t}\int_0^t zs^{-3/2}e^{-\left(z^2/4s + \frac{\omega^2}{4K}s\right)}I^*(0,t-s)ds. \end{aligned} \tag{84}$$

By means of the relations (78), (83), and (84), we deduce that

$$I^*(x_1,t) \leq I^*(0,t)e^{\frac{\omega}{2K}x_1}G(z,t), \tag{85}$$

where

$$G(z,t) = \frac{1}{2\sqrt{\pi}}\int_0^t zs^{-3/2}e^{-\left(z^2/4s + \frac{\omega^2}{4K}s\right)}ds. \tag{86}$$

We note that estimates for $I^*(x_1, t)$ may be obtained by using various upper bounds for $G(z, t)$. A useful bound is given in [8]

$$G(z,t) \leq \frac{2z(t/\pi)^{1/2} e^{-\frac{\omega^2}{4K}t}}{z^2 - \frac{\omega^2}{K}t^2} e^{-\frac{z^2}{4t}}, \quad \text{for } z > \frac{\omega}{\sqrt{K}} t. \tag{87}$$

Thus, from (85) and (87), we get

$$I^*(x_1,t) \leq I^*(0,t) \frac{2x_1 (Kt/\pi)^{1/2} e^{-\frac{\omega^2}{4K}t}}{x_1^2 - \omega^2 t^2} e^{\frac{\omega}{2K}x_1 - \frac{x_1^2}{4Kt}}, \quad \text{for } x_1 > \omega t, \quad t \in [0,T]. \tag{88}$$

As can be seen from (88), for all fixed $t \in [0, T]$, the spatial decay at infinity is controlled by the factor $e^{-\frac{x_1^2}{4Kt}}$.

6. Uniqueness results. The analysis in this paper allows us to study the uniqueness of solutions of the boundary-initial value problem $(\mathcal{P})$. Because of the linearity of the problem $(\mathcal{P})$ it is sufficient to prove that null given data imply that the corresponding solution vanishes on $\bar{S} \times [0, \infty)$. In fact, this means that if the support $\hat{D}_T$ is empty for any $T > 0$, then we can conclude that $\{u_\alpha, u_3, u_4\} = \{0, 0, 0\}$ in $\bar{S} \times [0, \infty)$.

Thus, if we assume $\hat{D}_T$ to be empty for any $T > 0$, then we can choose the set $\hat{D}_T^*$ in the above analysis to be any regular subcurve of ∂S.

Let us first consider the case of a bounded thermoelastic plate. Since $\{u_\alpha, u_3, u_4\} = \{0, 0, 0\}$ on $\partial S \times [0, T]$, it follows from the relations (23), (45), (52), and (61) that we have $I(0, t) = \mathcal{I}(0, t) = 0$ and $\mathcal{E}(0, t) = \mathcal{F}(0, t) = 0$ for all $t \in [0, T]$. Then the relation (46) implies that $\{u_\alpha, u_3, u_4\} = \{0, 0, 0\}$ on $\bar{S} \times [0, T]$. Since $T > 0$ can be chosen arbitrarily, it follows that $\{u_\alpha, u_3, u_4\} = \{0, 0, 0\}$ in $\bar{S} \times [0, \infty)$ and therefore we have obtained the uniqueness result.

Let us now consider the case of an unbounded thermoelastic plate. Let us assume that the solution $\{u_\alpha, u_3, u_4\}$ is in the class of functions for which $\mathcal{E}(r, t)$ or $\mathcal{F}(r, t)$ is bounded. In this case we have the estimates (49) and (62). Then a discussion like that above gives a uniqueness result in the class of solutions for which $\mathcal{E}(r, t)$ or $\mathcal{F}(r, t)$ are bounded.

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